Unit 7 : **Circular motion**.

Introduction

Circular motion is a kind of motion in which an object goes in a circular path periodically returning to its starting point. It is a two dimensional motion. Common examples are that of the earth; it goes around the sun once every 365.25 days or a car going in a round-about etc. Linear motion may be super-imposed on an object executing circular motion, for example a wheel of a car in motion.

Angular velocity

The rate at which an object executing circular motion rotates per second is called its angular velocity, ω. It is related to linear velocity, v through the equation

$$v = \omega r \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7.3.1)$$

Where r is the radius of the path described by the object, figure 7.3.1.

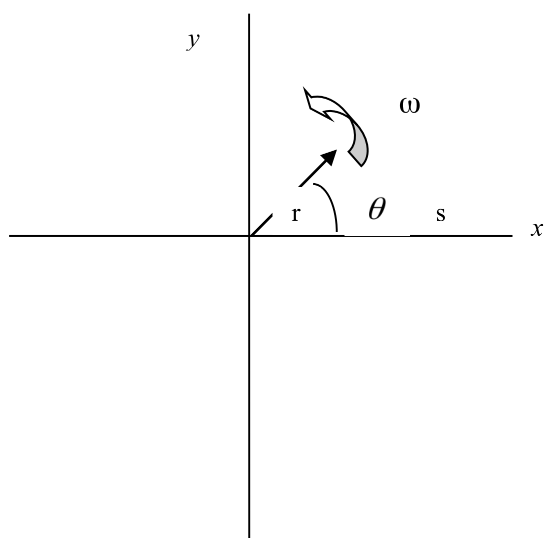


Figure 7.3.1

Let the wheel rotate through an angle θ in time t. The ratio $\frac{\Delta\theta}{\Delta t}$ is called the angular velocity, ω of the object. Its units are radians per second.

The arc s is related to the angle θ in radians through the relation $s = r\theta$.

Now $\frac{\Delta s}{\Delta t} = \frac{r\Delta\theta}{\Delta t} = r\omega = v$, where v is the linear velocity. This shows that a particle's linear velocity increases with the radius of the circle it is describing as it rotates.

Angular acceleration, α.

An object may not rotate at the same angular velocity all the time. It may increase or decrease depending on the circumstances. Suppose an object rotates at an angular velocity, ω_i. As time progresses the angular velocity may change to ω_f where $\omega_1 > \omega_0$ or vice-versa. The rate of change can be written as

$$\frac{\omega_f - \omega_i}{\Delta t} = \frac{\Delta\omega}{\Delta t} \quad \ldots\ldots\ldots\ldots\ldots \quad (7.4.1)$$

This ratio is termed the angular acceleration of the object, α. Therefore

$$\alpha = \frac{\Delta\omega}{\Delta t} \quad \ldots\ldots\ldots\ldots \quad (7.4.2)$$

measured in rad.s^{-2}. Angular acceleration is related to linear or tangential acceleration, a through the expression

$$\alpha r = a \quad \ldots\ldots\ldots\ldots \quad (7.4.3)$$

Remember that $\frac{\Delta v}{\Delta t} = r\frac{\Delta w}{\Delta t}$. This shows that the tangential acceleration of a point on a rotating object equals the distance of that point from the axis of rotation multiplied by the angular acceleration.

Example 7.4.1

A gramophone record rotates at a rate of 45 rpm. It is then speeded up to 78 rpm within 3 seconds. Find the magnitude of its angular acceleration, α.

Solution

Note that 1revolution has 2π radians or 360 degrees. Therefore a radian is equivalent to about 1.57^{o} The rotational speeds must be converted to radians/second. Thus 45 rpm = $45\times2\pi/60$ = 4.72 rad/sec. and 78 rpm = $78\times2\times\pi/60$ = 8.17 rad/sec. Using the above relation, $\alpha = (8.17 - 4.72)/3$ = 1.15 rad/s^{-2}.

Angular displacement, θ.

This is the total angle the rotating object turns through. For example if a wheel fixed at its pivot rotates with angular velocity 6 rad/sec., then in 5 seconds, it will have turned through an angle $\theta = \omega t$ = (6 rad/sec.)(5sec.) = 30 radians. Angular displacement $\Delta\theta$ is related to linear displacement s through the relation $s = r\theta$ where r is the radius of the circle described by the rotating object. The angle θ through which a rotating object turns can be worked out using the relation similar to that in linear motion i.e. $\omega_f^2 = \omega_i^2 + 2\alpha\theta$ where ω_f^2, ω_i^2 and α are the final angular velocity, initial angular velocity and angular acceleration respectively.

Centripetal acceleration, a_c

In unit 2 we saw that an object is said to be accelerating if its velocity changes i.e. if either its speed or its direction or both change. A body executing uniform circular motion is under constant acceleration due to the fact that even if its speed remains constant, its direction is constantly changing. As a consequence of Newton's second law, it is acted upon by a constant force directed towards the center of the circle. This force is called the centripetal force, F_c. The acceleration produced by such a force is called centripetal acceleration, a_c. The expression for centripetal acceleration is derived using figure 7.6.2.

In figure 7.6.2 the velocity change that occurs in time Δt is $v_2 - v_1$. This change is shown in figure 7.6.2b. If the tangential speed is v both v_1 and v_2 have this value. From the diagrams we observe that $\Delta\theta = \frac{v\Delta t}{r}$ and $\Delta\theta = \frac{\Delta v}{v}$ from small angle approximations. Equating the two, we get $\frac{\Delta v}{v} = \frac{v\Delta t}{r}$. Re-arranging, $\frac{v^2}{r} = \frac{\Delta v}{\Delta t}$. Taking the time interval Δt very small, this becomes $\frac{\Delta v}{\Delta t} = a_c = \frac{v^2}{r}$ which is the expression for the centripetal acceleration. Using $F = ma$ obtains the expression for the centripetal force,

$$F_c = \frac{mv^2}{r} \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (7.6.1).$$

Using the relation $v = \omega r$, the same expression can be written as

$$F_c = m\omega^2 r. \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (7.6.2)$$

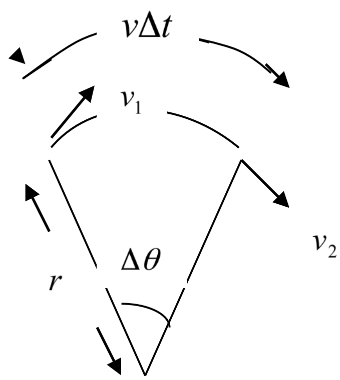


Figure 7.6.1a

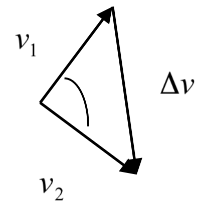


Figure 7.6.1b

Example 7.6.1

What is the centripetal acceleration of a 40 kg child sitting 2 m from the center of a round-about which turns once in 5.0 seconds? What is the resultant horizontal force acting on the child?

Solution:

$$a = \omega^2 r = \left(\frac{2\pi}{T}\right)^2 r = \frac{4\pi^2 r}{T^2} = \frac{4\pi^2 \times 2.0}{(5.0)^2} = 3.2\text{ms}^{-2}.$$

From $F = ma$ giving $F = 40$ kg × 3.2 ms^{-2} = 130 N. This force is provided by friction between the child and the seat and from the child holding onto part of the round-about.

Tangential acceleration

In section 7.6, we saw that an object executing circular motion has a centripetal acceleration associated with it. This is true as long as the direction of the object is changing whether the speed of the object in the orbit is changing or not. However, a situation sometimes arises where the speed of the object is also changing in addition to the change in direction. In this case, the object also has what is known as a **tangential acceleration**, a_t. As the name suggests, this is directed at a tangent to the path of the object, figure 7.8.1.

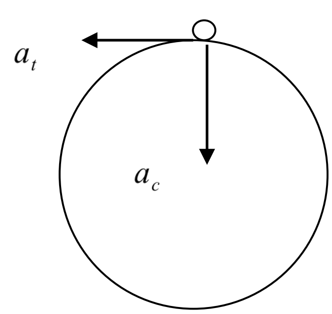


Figure 7.7.1

The expression for the linear velocity is

$$v_t = r\frac{\Delta\theta}{\Delta t} = r\omega$$

$$\frac{\Delta v_t}{\Delta t} = a_t = r\frac{\Delta\omega}{\Delta t} = r\alpha$$

$$a_t = r\alpha \quad \ldots\ldots\ldots\ldots\ldots \text{(7.7.1)}$$

In the event where both accelerations are present at the same time, one can work out the total acceleration, a using Pythagoras Theorem since the two accelerations are at right angles. Thus

$$a = \sqrt{a_c^2 + a_t^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(7.7.2)}$$

Example 7.7.1

A pendulum bob swings on a flexible string 0.5 m long in a vertical arc under the influence of gravity as in figure 7.7.1. When the string makes an angle of 25^{o} with the vertical, the bob has a speed of 200 cms^{-1}.

Find

(a) its centripetal acceleration at this instant;

(b) its tangential acceleration and

(c) the resultant (or total)acceleration

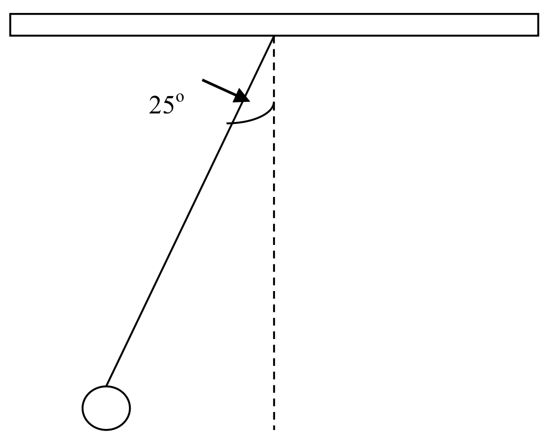


Figure 7.7.1

Solution

(i) Centripetal acceleration $a_c = \frac{v^2}{r} = \frac{(2)^2}{0.5} = 8ms^{-2}$

(ii) Tangential acceleration $a_t = g\sin 25^{o}$ =4.14 ms^{-2}.

(iii) The total acceleration $a = \sqrt{a_t^2 + a_c^2} = (8^2 + (4.14)^2)^{\frac{1}{2}} = 9.0\,\text{ms}^{-2}$

Student Activity 7.7.1

If you live in a locality where there is a round-about on some road, visit such a round-about while riding a bus or your own car if you have any. Observe what happens to you as the bus negotiates the round about at constant speed. Check if you find it necessary to cling to any part of the bus to keep yourself from being thrown to the outer part of the bus. Do you feel any force acting on you? Would you account for any of these effects using any of Newton's laws of motion?

Example 7.7.2

Justin is driving his 1500-kg Camaro through a horizontal curve on a level roadway at a speed of 23 ms^{-1}. The turning radius of the curve is 65m. determine the minimum value of the coefficient of friction which would be required to keep Justin's car on the curve.

Solution

The force that keeps the car moving in a circle is the centripetal force, F_c given by

$$F_c = \frac{mv^2}{r} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(7.7.3)$$

From the problem $v = 23\,\text{ms}^{-1}$

$r = 65\,\text{m}$

and the mass $m = 1500\,\text{kg}$

$$F_c = \frac{mv^2}{r}\ \text{N}$$

This force must be equated to the force of friction given by

$$F = \mu N = \mu mg.$$

Therefore,

$$\frac{mv^2}{r} = \mu mg$$

From which expression we get

$$\mu = \frac{v^2}{rg} = \frac{23 \times 23}{65 \times 9.8} = \frac{529}{637} = 0.83.$$

Relation between angular and linear quantities

Angular motion equations have their analogue in linear motion. For example, the equation

$$v_f = v_i + at \quad \ldots\ldots\ldots\ldots\ldots\ldots (2.6.7)$$

can be written as

$$r\omega_f = r\omega_i + \alpha rt \quad \ldots\ldots\ldots\ldots\ldots (7.8.1)$$

Where the relations $v = \omega r, a = \alpha r$ are used. Dividing throughout by r

$$\omega_f = \omega_i + \alpha t \ldots\ldots\ldots\ldots\ldots\ldots (7.8.2)$$

Similarly, the second and third equations can be written

$$\left.\begin{aligned} \theta &= \omega_i t + \frac{1}{2}\alpha t^2 \\ \omega_f^2 &= \omega_i^2 + 2\alpha t^2 \end{aligned}\right\} \quad \ldots\ldots\ldots (7.8.3)$$

where θ is the angular displacement.

Example 7.8.1

A wheel rotates with a constant angular acceleration of 3.5 rads^{-2}. If the angular velocity of the wheel at $t = 0$ is 2.0 rads^{-1},

(a) through what angle does the wheel rotate in 2 secs?

(b) What is the wheel's angular velocity at t = 2secs?

Solution

(a) From the relation $\theta = \omega_i t + \frac{1}{2}\alpha t^2 = 2.0 \times 2 + \frac{1}{2} \times 3.5 \times 2^2 = 11$ radians = 630°

(b) Here we make use of the relation $\omega_f = \omega_i + \alpha t = 2.0 + 3.5 \times 2 = 9.0$

Student Activity 7.8.1

An object of mass 0.5 kg is rotated in a horizontal circle by a string 1 m long. The maximum tension in the string before it breaks is 50N. What is the greatest number of revolutions per second the object can do without the string breaking? Answer: 1.6 revs^{-1}

Newton's law of Universal gravitation

This law states that there exists a force of attraction between any two bodies separated by a distance *r*. The force is directly proportional to the product of the masses and is inversely proportional to the square of the distance of separation. Thus it is an inverse square law. In mathematical terms, the same is stated as

$$F = G\frac{Mm}{r^2} \quad \ldots\ldots\ldots\ldots\ldots \quad .(7.9.1)$$

where r is the separation distance and G a constant called the universal gravitation constant. It has a numerical value of $6.67\times10^{-11}\,\mathrm{m^3Kg^{-1}s^{-2}}$. Al objects situated in the earth's gravitational field such a communication satellites and human beings are subject to this force. It provides the centripetal force required for the motion of satellites in near circular orbits.

Example 7.9.1: Communication satellite.

A communication satellite is to be sent into orbit about the earth in an equatorial plane such that it will appear to be stationary relative to an observer on earth. Find the radius of its orbit (r) and its height (H) above the surface of the Earth. Take radius of the Earth $R = 6400\,\mathrm{Km}$. Mass of the Earth $M_E = 5.98\times10^{24}\,\mathrm{Kg}$. Gravitational constant $G = 6.67\times10^{-11}\,\mathrm{m^3Kg^{-1}s^{-2}}$.

Solution

To appear to remain stationary, the satellite must have the same angular velocity as that of the earth. In other words its rotation about the Earth must take 24 hrs per rotation.

Angular velocity of the earth $\omega = \frac{2\pi}{T}$ where T = (24hrs)(60min/hr)(60secs/min) = 7.27×10^{-5} rad/sec.

The centripetal force, $F_c = \frac{m_s v^2}{r}$, where v is the linear speed of the satellite in its orbit, r the distance from the center of the Earth to the satellite, is provided by the gravitational force of attraction between the satellite and the Earth i.e. $F_c = \frac{GM_E m_s}{r^2}$. Equating the two, we get

$$F_c = \frac{GM_E m_s}{r^2} = \frac{m_s v^2}{r}$$

$$\frac{GM_E}{r} = v^2 = \omega^2 r^2$$

where the relation $\boldsymbol{\omega} r$ is used. Re-arranging, we get

$$r^3 = \frac{GM_E}{\omega^2}$$

$$r^3 = \frac{(6.67\times10^{-11})(5.98\times10^{24})}{(7.27\times10^{-5})^2}$$

$$r^3 = \frac{39.89\times10^{13}}{52.8\times10^{-10}} = 7.55\times10^{22}\text{m. or } 42.26\times10^3 \text{ km.}$$

Subtracting the radius of the Earth, we find that the satellite is 35 860 km above the Earth's surface! Are we justified to use $g = 9.8\text{ms}^{-2}$ at this height? If not can you identify the reason why?

Student activity 7.9.1

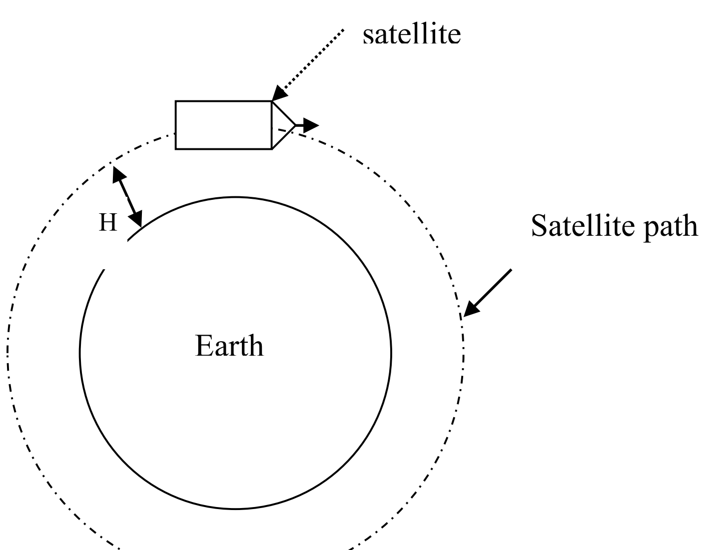
(a) Television signals relayed by DSTv are emitted by a satellite placed high up in the sky. Find out how high above the surface of the Earth such a satellite is and its rate of rotation. How come it does not go out of view of the satellite dishes placed on your roof and indeed on other people's roofs?

(b) Discuss the advantage of transmitting television signals by satellite other then Earth transmitters.

What is a *parking* orbit?

Communication satellites

Communication satellites are un-powered space-ships (literary projectiles) going around the earth at various heights above the surface of the Earth. They receive, amplify and re-transmit telephone and other electric signals from the Earth. The path followed by a satellite is its **orbit.** In this **situation** the satellites are under free fall and the only force acting on them is the force of gravity. Since they are executing circular motion, the centripetal force on them is provided by the force of gravity, mg. Therefore $\frac{mv^2}{r} = mg = m\omega^2 r$ or $\omega = \sqrt{\frac{g}{r}}$ where $r = H + R_E$. In this equation, R_E is the radius of the Earth and H is the height of the satellite above the Earth's surface, figure 7.10.1.



Example 7.10.1

What is the period of revolution of a spy satellite in a low Earth orbit (distance 700 km from the center of the Earth) where gravitational field strength is 8.0 Nkg^{-1}?

Solution

Using the relation $\omega = \sqrt{\frac{r}{g}} = 2\pi f = \frac{2\pi}{T}$, we get $T = 2\pi\sqrt{\frac{r}{g}} = 2\pi\sqrt{\frac{7.1\times10^{6}m}{8.0Nkg}} = 5900$ seconds or 1 hr 39 min.

The conical pendulum

A mass attached to the end of a string can be made to move in a horizontal circle so that the string traces out the surface of a cone. This is called a conical pendulum. Since the mass is moving in a uniform circular motion, the resultant of all the forces acting on it must act toward the center of the circle and equal to $m\omega^2 r$, figure 7.11.1

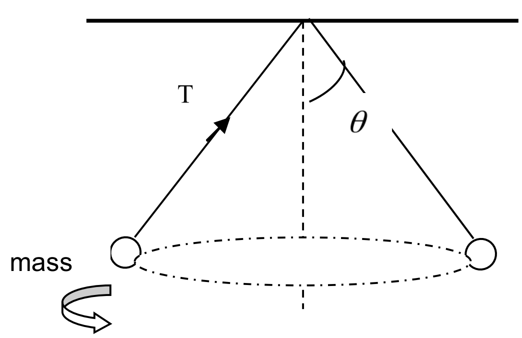
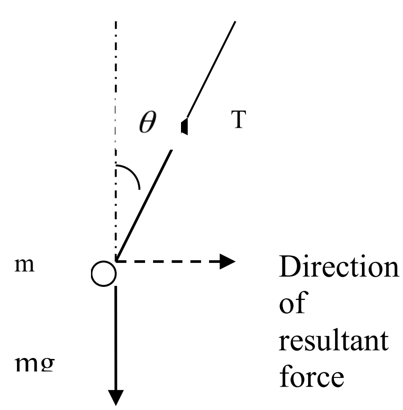


Figure 7.11.1

The forces acting on the mass are the tension T, its weight w . Thes are represented in the free body diagram of figure 7.11.2



Resolving the forces in the vertical direction, we get

Vertical equilibrium,

$$T\cos\theta - mg = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7.11.1)$$

Horizontal equilibrium

$$T\sin\theta = ma = m\omega^2 r. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7.11.2)$$

Dividing (7.10.2) by (7.10.1) gives

$$\tan\theta = \frac{\omega^2 r}{g} = \frac{v^2}{rg} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7.11.3)$$

from which

$$\theta = \tan^{-1}\left(\frac{v^2}{rg}\right) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7.11.4)$$

Example 7.11.1

A simple pendulum of mass 0.02kg suspended from a string 0.5m long is whirled in a horizontal circle (it is effectively a conical pendulum) of radius 30cm, figure 7.9.3. Calculate

(i) the tension, T in the string;

(ii) the angular frequency, ω of the motion.

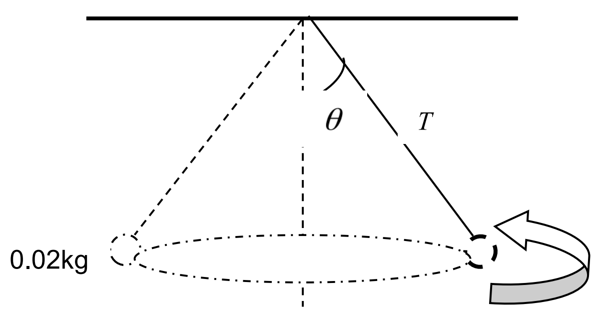


Figure 7.11.3

Solution

To start with, find the angle θ. Now $\sin\theta = \frac{0.3}{0.5} = 0.6$. This gives $\theta = \sin^{-1}(0.6) = 37^{o}$

Using eqn (7.10.1) gives

(a) $$T\cos 37^{o} = 0.02 \times 9.8 = 0.196$$

i.e. $T = 0.196/0.8 = 0.245$ N

(b)

$$\omega^2 = \frac{g}{r}\tan 37^0 = \frac{9.8}{0.3}(0.754)$$

$$\omega^2 = 24.63$$

$$\omega = 4.96 \text{ radsec}^{-1}$$

Motion in a vertical circle

The equations we derived before can be used to solve problems involving motion in a vertical circle. The example (example 7.12.1) below will help to illustrate this.

Example 7.12.1

In a test of machine-part reliability, a specimen of mass *m* is swung in a vertical circle of constant radius $R = 0.75$m. When the object is at the bottom of the circular path, the tension in the supporting wire is found to be six times the weight of the object. Determine the specimen's rotation rate in revolutions per minute.

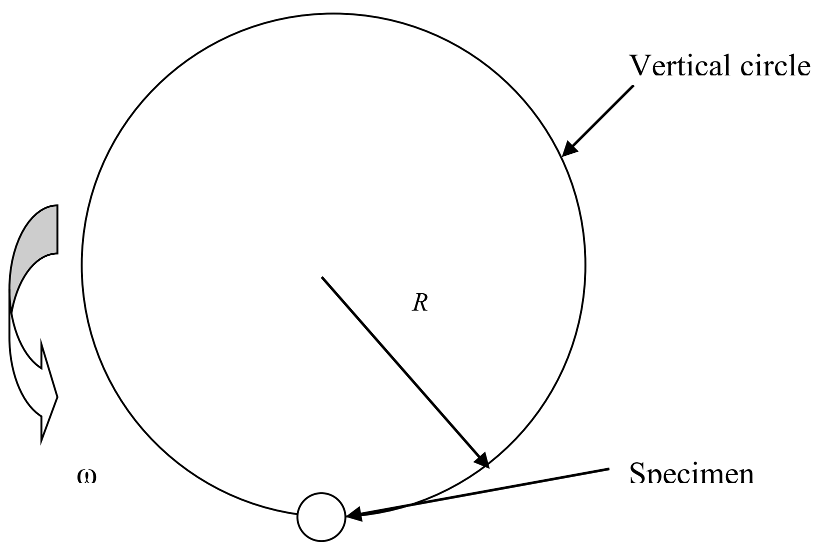


Figure 7.12.1

Solution

At the bottom of the circle, the forces acting on the specimen are: its weight, *W* and the centripetal force, F_c i.e.

$$W + m\omega^2 R = 6mg \quad \ldots\ldots\ldots\ldots\ldots(7.12.1)$$

$$mg + m\omega^2 R = 6mg$$

$$g + \omega^2 R = 6g$$

$$\omega = \sqrt{\frac{5g}{R}} = \sqrt{\frac{5 \times 9.8}{0.75}} = 8.08 rad/s$$

In revolutions per minutes, this is quoted as ω= 77.2 $rev.min^{-1}$.

Student Activity 7.12.1

Clever girls and boys can energize a brazier full of charcoal and a bit of fire by rotating it in a vertical circle without dropping a single piece of charcoal. Can you apply principles of circular motion to account for this feat? Will this be possible at any rotational speed?

Banking

When an aircraft has to change direction or is to run a circle, it banks its wings so that a component of its normal contact force acts towards the centre of the turn and helps provide the necessary centripetal force. Roads are also banked on bends to allow cars to negotiate the bend at a specified speed even though there is no friction between the tires and the road surface, figure 7.13.1.

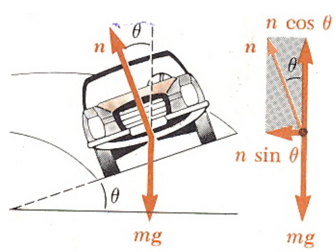


Figure 7.13.1

Vertically the weight of the car balances the component **n**cosθ where **n** is the normal reaction force provided by the road surface i.e.

$$\mathbf{n}\cos\theta = mg \quad \ldots\ldots\ldots\ldots\ldots \quad (7.13.1)$$

Horizontally, the component **n**sinθ is unbalanced, directed towards the center of the circle. It provides the necessary centripetal force that constrains the car to move in a circle. Therefore

$$\mathbf{n}\sin\theta = \frac{mv^2}{r} \quad \ldots\ldots\ldots \quad \ldots\ldots \quad (7.13.2)$$

where r is the radius of the circle. Dividing equation (7.13.2) by equation (7.13.1), gives

$$\tan\theta = \frac{v^2}{rg} \quad \ldots\ldots\ldots\ldots \quad (7.13.3).$$

The critical speed limit can therefore be found.

Example 7.13.1

A high-way curve has a radius of 150m and is designed for traffic of speed $17.9ms^{-1}$.

(a) If the curve is not banked, determine the minimum coefficient of friction between the car tires and the road to keep the car from skidding;

(b) At what angle should the curve be banked to enable that car to turn without friction?

Solution

(a) For the car to turn, the frictional force (which provides the centripetal force) that must exist between the tires and the road must be equal to the centripetal force on the car. That is

$$\mu_s mg = \frac{mv^2}{r}$$

Or

$$\mu_s g = \frac{v^2}{r}$$

$$\mu_s = \frac{v^2}{rg} = \frac{(17.9)^2}{150 \times 9.8} = 0.22$$

(c) Without friction, the road has to be banked according to equation (7.13.3), i.e.

$$\tan^{-1}\left(\frac{v^2}{rg}\right) = \tan^{-1}(0.22) = 12.41^o$$

Self-assessment questions

Q1 (a) A centrifuge in a chemical laboratory turns at a rate of 6000 revolutions per minute. Express this rotational rate in radians per second.

(b) Find the angular velocity of the Earth in radians per second as it spins about its axis. Is it the same for all people on Earth?

Q2 A gramophone record rotating at a rate of 45 rev./min suddenly has its angular velocity increased to 78 rev/min. in 10 seconds. Find its angular acceleration.

Q3 **Worked example**:

In a popular amusement park ride, a rotating cylinder of diameter 5.0 m is set into rotation at an angular velocity of 5 rad/s, as in figure 7.15.1. The floor drops away, leaving the riders suspended against the wall in a vertical position. What minimum coefficient of friction between the rider's clothing and the wall is needed to keep the rider from slipping?

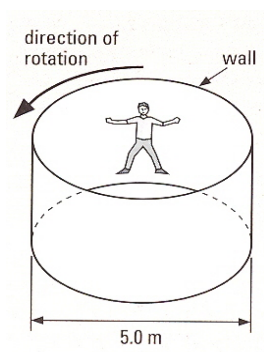


Figure 7.14.1

Solution

The maximum force of static friction is equal to $\mu\,\mathbf{n}$, where **n** is the normal force, which in this case is the centripetal force. Thus

$$\mu\,\mathbf{n} = \mu\frac{mv^2}{r} = \mu m\omega^2 r$$

This force balances the weight, mg, of the rider so that

$$mg = \mu m\omega^2 r \text{ or } \mu = \frac{g}{\omega^2 r}$$

Putting in the given values, we get $\mu = \dfrac{9.8}{(5rad/s)^2 \times 2.5m}$

$$\mu = 0.784$$