Unit 8: Rotational Motion

Introduction

What causes material objects to rotate? From our studies on equilibrium, we saw that when a torque is applied to an object, it may rotate about a point commonly referred to as the pivot. A torque so applied causes rotational acceleration just as a force causes acceleration in linear motion according to Newton's 2nd law.

Rotational motion is important in Engineering applications. For example most vehicles (such as cars) are able to move because of the rotation of their wheels.

Torque and acceleration, moment of inertia

Material objects show reluctance against being set into rotational motion due to **rotational inertia.** In linear motion, inertia arises on account of the fact that material objects have mass. In order to apply Newton's laws to rotational motion, we seek an expression that is analogous to **F = ma**.

Consider the situation depicted in figure 8.3.1 where a force **F** is applied to a bob of mass *m* attached to a light rod of length *r* pivoted at point O.

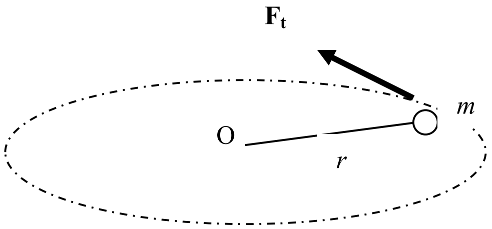

Figure 8.3.1

Since the rod is rigidly pivoted at O, it is confined to a rotation in a horizontal plane. The tangential force $\mathbf{F_t}$ acting on *m* causes the mass to undergo a tangential acceleration a_t according to Newton's second law:

$$F_t = m\,a_t \quad \ldots\ldots\ldots\ \ldots\ldots\ldots (8.3.1)$$

Multiplying the left and right hand sides of this equation by *r* gives

$$Fr = mr\,a_t \quad \ldots\ldots\ldots\ldots\ldots\ldots (8.3.2)$$

Now $a_t = \alpha r$ (see unit 7) so that the above expression becomes

$$F_t r = mr^2\alpha \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (8.3.3)$$

The left side $F_t r$ is the torque, τ acting on the mass. Hence we can write

$$\tau = mr^2\alpha \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (8.3.4)$$

showing that the torque is proportional to the angular acceleration with mr^2 as the constant of proportionality. This constant is called the moment of inertia, symbol I of the mass *m*. The above equation is normally written as

$$\tau = \mathrm{I}\alpha\,. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad \ldots. \quad (8.3.5)$$

The moment of inertia of the rod is neglected since the latter is assumed to be light.

Torque and rotating objects.

When a torque is applied to a rotating object, the angular velocity of the object, **ω** changes. Consider a solid disk of radius *R* pivoted on an axis that is perpendicular to its plane passing through its center, figure 8.4.1. It rotates about this axis with an angular velocity ω_o.

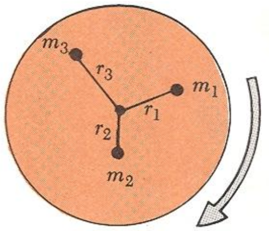

Figure 8.4.1

The disk is made up of a large number of particles which can be labeled m_1, m_2, m_3, etc. situated at radii r_1, r_2, r_3, etc. Their respective moments of inertia are

$$I_1 = m_1 r_1^2,\ I_2 = m_2 r_2^2,\ I_3 = m_3 r_3^2 \quad . \quad ..(8.4.1)$$

The torque for each particle is given by $\tau_1 = I_1\alpha, \tau_2 = I_2\alpha,\ \tau_3 = I_3\alpha$, etc.

The total torque on the disk is

$$\sum \tau = (I_1 + I_2 + I_3)\alpha \quad \ldots\ldots\ldots\ldots.(8.4.3)$$

The moment of inertia for the disk is therefore the sum of the various moments of inertia of the particles.

Therefore,

$$\sum \tau = I\alpha \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (8.4.4)$$

This equation says that the angular acceleration of an object is proportional to the net torque acting on it. The proportionality constant is the moment of inertia. This equation is the rotational counterpart to Newton's second law, **F**=ma.

More on moment of inertia

Moment on inertia I for a given object depends on its geometrical construction. The location of the axis of rotation and the manner in which the mass is distributed are also important. For example, a simple mass m hooked to a light string of length l rotating in a circle has $I = ml^2$. For an object such as a hoop of mass M and radius R, $I = MR^2$ where M is the sum total ($M = \sum m_i$) of the masses of all the particles of which the hoop is composed. Since each particle is the same distance from the center of the hoop with moment of inertia $I = mR^2$, the total value of the hoop's moment of inertia is as given.

For a more complicated assembly of masses, such as that in figure 8.5.1, $I = (0.2kg)(0)^2+(0.3kg)(0.5m)^2+(0.2kg)(0)^2+(0.3kg)(0.5m) = 0.15kgm^2$, where the vertical rod is taken as the axis of rotation.

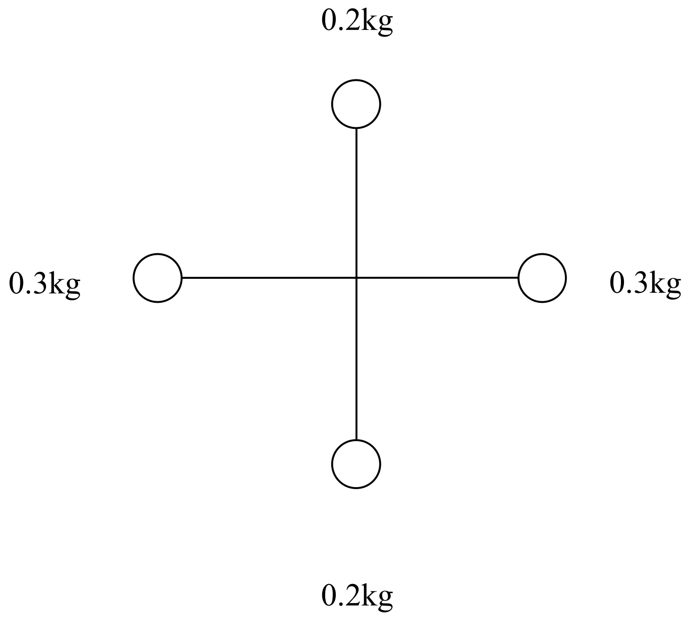

Figure 8.5.1

The situation becomes difficult when the object is more complex such as a solid disk whose moment of inertia is given by

$$I = \frac{1}{2}MR^2 \quad \ldots\ldots \quad \ldots\ldots (8.5.1)$$

where R is the radius of the disk and M its mass. This expression can be derived using the methods of a branch of mathematics called integral calculus which is beyond the scope of this module. It is, however, important to know the moment of inertia for a number of important objects as given in figure 8.5.2.

Student Activity 8.5.1

If you happen to have an opportunity to visit a garage where mechanics overhaul car engines ask them to show you the flywheel of an engine. It is a solid disk made out of steel. Try to lift it. Is its weight large? Can you account for this design. Ask the mechanic about its function in the engine.

Example 8.5.1

A solid cylinder of mass 2kg and radius 10 cm has a mass of 5 kg hanging from it. If the cylinder-mass system is started from rest, how fast will it be rotating at the instant the hanging mass falls through a distance of 3m, figure 8.5.3

Solution

We first calculate the angular acceleration, α by considering the motions of the hanging mass and the cylinder respectively. Equation of motion for the hanging mass:

$$\mathrm{mg} - \mathrm{T} = \mathrm{m}a \quad \ldots\ldots. \quad \ldots . . .(8.5.2)$$

Equation of motion for the cylinder:

$$\tau = TR = I\alpha \quad \ldots\ldots\ldots\ldots\ldots (8.5.3)$$

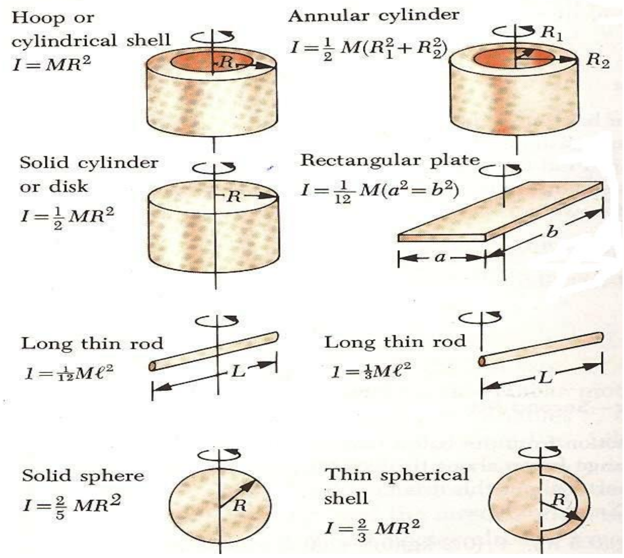

Figure 8.5.2

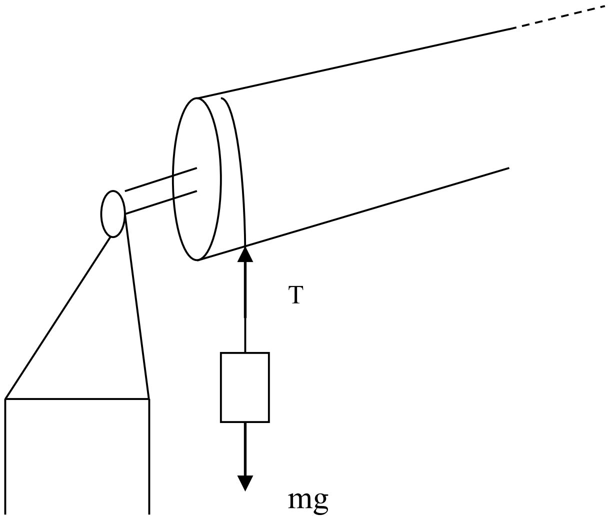

Figure 8.5.3

$$T = \frac{I\alpha}{R} \quad \ldots\ldots\ldots\ldots\ldots\ \ldots \quad (8.5.4)$$

Combining the two, we get

$$\text{mg} - \frac{I\alpha}{R} = \text{ma} \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (8.5.5)$$

For the cylinder, $I = \frac{1}{2}\text{MR}^2$. Putting this in (8.5.5), we get

$$\text{mg} - \frac{1}{2}\text{MR}\alpha = \text{m} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (8.5.6)$$

But $a = \alpha$ R so that

$$\text{mg} - \frac{1}{2}\ \text{MR}\alpha = \text{MR}\alpha \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (8.5.7)$$

and

$$\alpha = \frac{mg}{(\frac{1}{2}M + m)R} \quad \ldots\ldots\ \ldots\ \ldots\ \ldots \quad (8.5.7)$$

$$\alpha = \frac{5 \times 9.8}{\left(\frac{1}{2} \times 2 + 5\right) \times 0.1} = \frac{49}{6} = 8.17 \text{ rad-sec}^{-2}.$$

Therefore $a_t = 8.2$ m-sec^{-2}.

As the mass falls through the three meters, the cylinder turns through the angle θ given by θ = (3m/ 0.2π)×2π = 30 radians, where 0.2π is the circumference of the cylinder. The final angular velocity, ω_f^2 is obtained from the relation

$$\omega_f^2 = \omega_i^2 + 2\alpha\theta \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8.5.8)$$

where ω_i^2 is its initial angular velocity. Since the system starts from rest, $\omega_i^2 = 0$ so that $\omega_f^2 = 2\alpha\theta$ = 2×8.17 ×30 = 4900 giving ω_f = 22.14 rad-sec^{-1}. The cylinder will be rotating at the rate of 22.17 radians per second.

Example 8.5.2: coupled masses

A mass m_1 = 4kg is connected by a light inextensible and light cord to a mass m_2 =3kg which slides on a smooth surface, figure 8.5.4. The cord passes over a rough pulley which rotates about a frictionless axle and has a moment of inertia I = 0.5 kg.m^2. If the pulley has a radius of 0.3m, find

(a) the acceleration of the two masses and

(b) the tensions T_1 and T_2

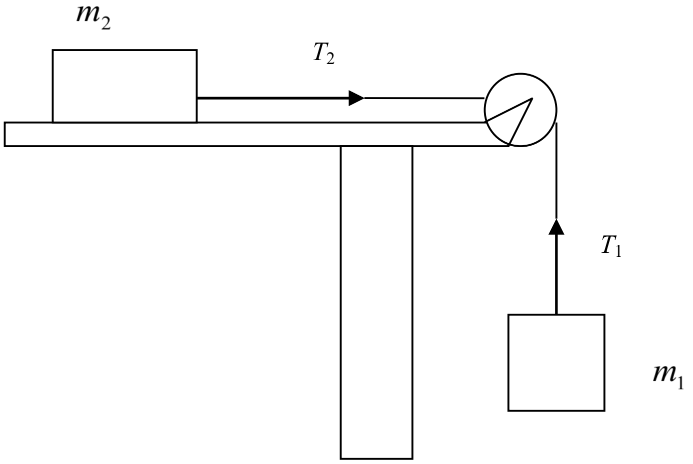

Figure 8.5.4

Solution

To solve this problem, the motions of the masses are considered one by one and the equations so obtained are solved simultaneously.

Mass m_1 = 4 kg:

$$m_1 g - T_1 = m_1 a_t. \;\ldots\ldots\ldots\ldots (i)$$

Mass m_2 = 3kg

$$T_2 = m_2 a_t \mathrm{T} \quad \ldots\ldots\ldots \quad \text{(ii)}$$

Pulley $I = 0.5\,\text{kg.m}^2$

$$(T_1 - T_2)R = I\alpha \qquad \ldots\ldots\ldots \text{(iii)}$$

Where a_t and α are the tangential and angular accelerations respectively.

Using (i) and (ii) in (iii), we get

$$(m_1 g - m_1 a_t - m_2 a_t)R = I\frac{a_t}{R} \qquad \ldots\ldots \quad \text{(iv)}$$

Or $m_1 gR^2 - m_1 a_t R^2 - m_2 a_t R^2 = Ia_t \qquad \ldots\ldots\ldots \quad \text{(v)}$

From (v) we get

$$a_t = \frac{m_1 gR^2}{(m_1 R^2 + m_2 R^2 + I)} \qquad \ldots\ldots\ldots \quad \text{.(vi)}$$

Using the given values, $a_t = \frac{4 \times 9.8 \times (0.3)^2}{\left[(4+3)(0.3)^2 + 0.5\right]} = \frac{3.528}{1.13} = 3.12\,\text{ms}^{-2}$

From (i) $T_2 = 3 \times 3.12 = 9.36\text{N}$ and $T_1 = 26.72\text{N}$

Student Activity 8.5.2

The moment of inertia of a solid flywheel about its axis is 0.1 kgm^2. It is set in rotation by applying a tangential force of 20N with a rope wound around its circumference, the radius of the wheel being 0.1m. Calculate the angular acceleration of the flywheel. What would be the angular acceleration if a mass of 2 kg were hung from the end of the rope?

Radius of gyration

The radius of gyration, *K* is sometimes referred to as the effective radius. For a body of mass *M* and moment of inertia *I* the radius of gyration is defined as

$$I = MK^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (8.6.1)$$

Take a disk for example whose moment of inertia is

$$I = \frac{1}{2}MR^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (8.6.2)$$

We can get *K* by equating

$$I = MK^2 = \frac{1}{2}MR^2, \quad \ldots\ldots\ldots\ldots \quad (8.6.3)$$

From which

$$K^2 = \frac{R^2}{2} \quad \ldots\ldots\ldots\ldots .(8.6.4)$$

Or $$K = \frac{R}{\sqrt{2}} \quad \ldots\ldots\ldots\ldots (8.6.5)$$

Rotational kinetic energy

In linear motion, we defined the kinetic energy, k.e. of a particle of mass *m* moving with a velocity *v* as a product of the mass and the square of the velocity divided by k.e. = $\frac{1}{2}mv^2$. Similarly, a body rotating about an axis with an angular velocity ω is said to have rotational kinetic given by

$$\text{k.e.}_{\text{rot.}} = \frac{1}{2}I\omega^2 \ldots\ldots\ldots\ldots .(8.7.1)$$

Proof : Consider a solid disk of mass *M* radius *R* rotating with an angular velocity ω. The disk consists of many small particles each of mass *m*. If a particle is situated at a radius *r* from the axis of rotation, the velocity of the particle is ωr. It kinetic energy

$$\text{k.e.} = \frac{1}{2}\text{m}(\omega r)^2 = \frac{1}{2}mr^2\omega^2 \ldots\ldots\ldots (8.7.2)$$

The kinetic energies of the particles can be added up to get the total kinetic energy K.E for the disc i.e.

$$K.E = \sum\frac{1}{2}(mr^2\omega^2) = \frac{1}{2}\left(\sum mr^2\right)\omega^2 \quad ..(8.7.3)$$

Now

$$\frac{1}{2}(\sum mr^2) = I \ldots\ldots\ldots\ldots (8.7.4)$$

Where I the total moment of inertia of the disk. Therefore

$$K.E_{rotational} = \frac{1}{2}I\omega^2 \ldots\ldots .(8.7.5)$$

8.7.1 Work done by a torque

Consider a force **F** applied tangentially to a cylinder of radius R through a rope wound around the cylinder. This causes the cylinder to rotate. After a length *s* of rope has been unwound, the amount of work done by the force is

$$\text{Work done} = Fs \quad ..(8.7.6)$$

As a consequence of the work done, the cylinder accelerates from zero angular velocity to ω with kinetic energy $K.E = \frac{1}{2}I\omega^2$. The final angular velocity ω_f can be found from the relation

$\omega_f^2 = 2\alpha\theta$ since the initial angular velocity is zero. Radian measure gives $s = r\theta$. Therefore, the work done can also be expressed as $Fr\theta$. But $Fr = \tau$ so that the work done can be written as

$$W = \tau\theta = \frac{1}{2}I\omega^2 \text{. . . . . (8.7.7)}$$

Thus the work done on an object is equal to the change in the kinetic energy of the object. The energy concept is very useful in simplifying the analysis of rotational motion.

Example 8.7.1

The race between the hollow and solid cylinders.

Two cylinders, one hollow and the other solid and of the same mass, *m*, and radius *R*, are released at the same time from the top of an incline of height *h* meters. Using the law of conservation of kinetic and potential energies, find which of the two cylinders will reach the bottom of the incline first. Neglect friction.

Solution.

For the hollow cylinder, conservation of mechanical energy demands

$$\frac{1}{2}mv_h^2 + \frac{1}{2}I_h\omega_h^2 = mgh \qquad \text{. . . . . (8.7.8)}$$

where v and ω are the linear and angular velocities of the cylinder. I_h is the moment of inertia of the hollow cylinder. The same equation obtains for the solid cylinder except that its moment of inertia is I_s

$$\frac{1}{2}mv_s^2 + \frac{1}{2}I_s\omega_s^2 = mgh. \qquad \text{... ....(8.7.9))}$$

Using the relations $v = \omega R$, $I_s = mR^2$ and $I_s = \frac{1}{2}mR^2$ for both cylinders, we get

$$\frac{1}{2}mv_h^2 + \frac{1}{2}mR^2\left(\frac{v_h}{R}\right)^2 = mgh \quad \text{. . . . . (8.7.10)}$$

for the hollow cylinder and

$$\frac{1}{2}mv_s^2 + \frac{1}{2}\times\frac{1}{2}mR^2\left(\frac{v_s}{R}\right)^2 = mgh \quad \text{. . . . . . . (8.7.11)}$$

for the solid cylinder. Simplifying the expression for the hollow cylinder, we get

$$v_h^2 + v_h^2 = 2v_h^2 = 2gh \quad \text{. . . . . (8.7.12)}$$

$$v_h = \sqrt{gh}\ \text{m/s.} \qquad \text{. . . (8.7.13)}$$

For the solid cylinder, the expression simplifies to

$$v_s^2 + \frac{v_s^2}{2} = \frac{3}{2} v_s^2 = 2gh \quad \ldots\ldots\ldots\ldots\ldots \quad (8.7.14)$$

$$\text{or } v_s = \sqrt{\frac{4}{3} gh} = 2\sqrt{\frac{gh}{3}}. \quad \ldots\ldots \quad (8.7.15)$$

This analysis shows that the solid cylinder is faster; it will therefore reach the bottom of the incline earlier.

Angular momentum

If a particle of mass *m* located at a distance *r* from a point about which it is free to rotate is acted upon by a tangential force $\mathbf{F_t}$, it experiences a torque of magnitude $\tau = rF$ as a result of which its angular velocity increases from its initial value ω_i to a final value ω_f.

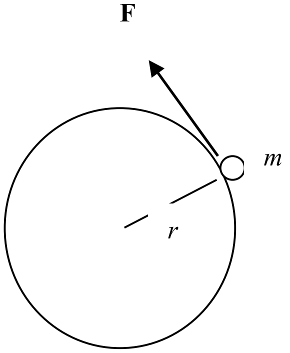

Figure 8.8.1

Now the torque τ is related to the moment of inertia *I* and angular acceleration α through

$$\tau = I\alpha = I\left(\frac{\omega_f - \omega_i}{\Delta t}\right) \quad \ldots\ldots\ldots\ldots\ldots \quad (8.8.1)$$

Expanding this expression obtains

$$\tau = \frac{I\omega_f - I\omega_i}{\Delta t} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (8.8.2)$$

We define a quantity

$$\mathbf{L} = \mathbf{I}\omega \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (8.8.6)$$

called the angular momentum of the particle. The above relation can therefore be written as follows:

$$\tau = \frac{\Delta L}{\Delta t},$$

where ΔL is the change in angular momentum over an interval of time Δt. This is the rotational analog of Newton's second law,

$$F = \frac{\Delta p}{\Delta t}.$$

It can be stated formally as follows:

The torque acting on a particle is equal to the time rate of change of its angular momentum. Alternatively, it can be stated that

When the torque acting on an object is zero, $\frac{\Delta L}{\Delta t} = 0$, the momentum of the object remains constant or

$$I\omega_i = I\omega_f \text{ . . . . (8.8.7)}$$

If

$$\sum \tau = 0. \text{ . . . .(8.8.8)}$$

Student Activity 8.8.1

One interesting demonstration of the conservation of angular momentum would involve a person kneeling or sitting on a circular platform able to rotate on a hub with minimal friction in its bearings. Initially, the person has his/her arms stretched. Let the platform be sent spinning at a certain angular speed, **ω.** Now ask the person to fold his/her arms and see what happens. Does your observation conform to the conservation of angular momentum?

8.8.1 Law of conservation of angular momentum.

The angular momentum of a system is conserved when the resultant external torque acting on the system is zero. That is the initial angular momentum equals the final angular momentum. Angular momentum is a vector quantity just like linear momentum. It has the same direction as the angular velocity. The direction can be determined using the right-hand rule.

Example 8.8.1

A bug of mass 50 grams is located at a distance of 20 cm from the center of a gramophone record (a solid disk) of radius 20 cm rotating at an angular velocity of 4 rad/s. The record has a mass of 200 grams. The bug then "decides" to walk inside the record and stays 10 cm away from the center of the record. Find the new angular velocity of the record.

Solution

Angular momentum before the bug moved to the 10 cm radius mark = angular momentum after the movement i.e.

$$I_{before}\omega_{before} = I_{after}\omega_{afetr} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (8.8.9)$$

where I_{before} is the total moment of inertia of the disk and bug before the latter moved and I_{after} is the total moment of inertia of the bug and disk after the bug moves. Now For the bug,

$$\mathbf{I}_{\boldsymbol{before}} = m_{bug}r^2 = (0.050\text{ kg})(0.20\text{ m})^2 = 0.002\text{kgm}^2.$$

For the record,

$$\mathbf{I}_{\text{before}} = \frac{1}{2}m_{\text{record}}R^2 = (0.5\times 0.200\text{kg})(0.2\text{m})^2 = 0.004\text{ kgm}^2$$

where R is the radius of the record. Total moment of inertia

$$\mathbf{I}_{\text{total}} = I_{\text{bug}} + I_{\text{record}} = 0.006\text{kgm}^2.$$

Therefore momentum of the system before

$$\mathbf{L}_{\text{before}} = I_{before}\omega_{before} = (0.006\text{kgm}^2)(4\text{ rad/sec}) = 0.024\text{ rad/kgm}^2.$$

The moment of inertia of the disk remains the same after the movement. However, the moment of inertia of the bug changes since it is at a smaller radius. So for the bug,

$\mathbf{I}_{\text{after}} = (0.05\text{kg})(0.1\text{m})^2 = 0.0005\text{kgm}^2$. The total moment of inertia is therefore $(0.0005 + 0.004)\text{kgm}^2 = 0.0045\text{ kgm}^2$. Since angular momentum is conserved, we have

$0.024\text{ rad/s kgm}^2 = \mathbf{I}_{\text{total}}\,\omega_f$ or $\omega_f = \dfrac{0.024}{0.0045}\text{rad/s} = 5.33\text{ rad/sec}$. The rotation rate decreases.

Student Activity 8.8.2

Conservation of angular momentum is dramatically demonstrated at a ballet dancing hall.
Make an effort to watch a ballet dancer either at a circus or in a concert hall or on TV if you can't make it to the other two. Suppose when performing the dancing antics, the dancer spins about a vertical axis at 1 revolution per second with arms stretched . Later, with the arms folded, her moment of inertia about the vertical axis decreases by almost 60%. Can you calculate the final rate of rotation in this case?

Self-assessment questions

Q1 A tangential force of 20N is applied to a uniform solid sphere of radius 30cm. The sphere is made of copper (density 8.93 g/cm^3). Calculate the angular velocity of the sphere after 5 seconds.

Q2 Calculate the rotational kinetic energy of the sphere in Q1 after 5 secs.

Q3. An orbiting satellite carries two sensors each of mass 2.5 kg at the ends of its telescopic booms, figure 8.9.1

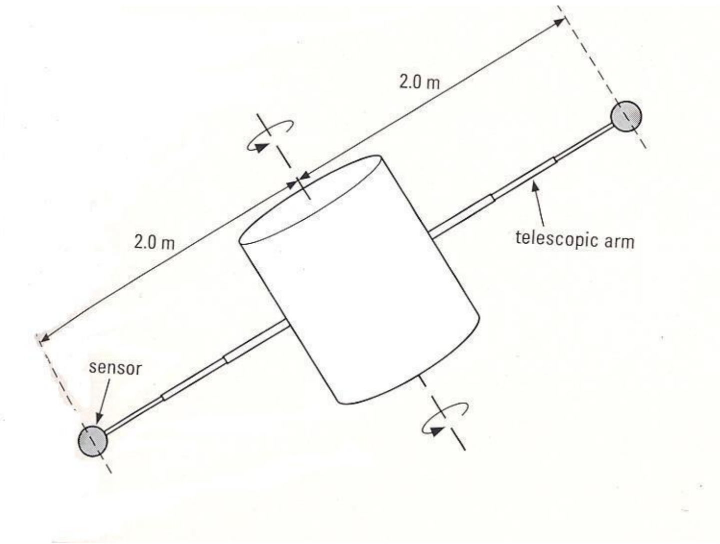

Figure 8.9.1

When the booms are fully extended, the center of mass of each sensor is 2.0 m from the axis of the satellite which spins at 0.10 rad/s about its axis as shown in the diagram. The mass of each boom is negligible compared with that of the sensor: A motor can be used to retract the boom so that the sensors lie on the axis of the satellite and make negligible contribution to the moment of inertia of the satellite. Under these conditions, the moment of inertia of the satellite about its axis is 100 kgm^2.

(a) When the booms are fully extended,
  (i) show that the moment of inertia of the satellite is 120 kgm^2;
  (ii) calculate the rotational kinetic energy of the satellite when spinning at 0.10 rad/s
(b) With the satellite spinning initially at 0.10 rad/s, the booms are fully retracted. calculate
  (i) the new rotational speed of the satellite, carefully explaining your reasons;
  (ii) the rotational kinetic energy of the satellite;
(c) With reference to your answers in (a) (ii) and (b) (ii), state whether or not the law of conservation of energy can be applied in this situation, clearly explaining your reasoning

Q4 Calculate the angular momentum of the Earth. Average radius of Earth = 6400 km. Mass of the earth, $M_E = 5.98 \times 10^{24}$kg. Rotational rate =24hrs