MAT1100 LECTURE NOTES

Term 1

1. Sets

1.1 Set Theory

Sets make a foundation of mathematics. The concept of a set appears in all branches of mathematics. It formalizes the idea of grouping objects together and viewing them as a single entity.

Basic Definitions

Definition 1.1 A set is any well defined list or collection of objects, called *elements* or *members*.

Sets are denoted by capital letters, say A, B, $C,\cdots$. The elements or members of a set are denoted by lowercase letters a, b, c, $\cdots$.

Example 1.1If P is a set of all vowels then it is written as $P = \{a, e, i, o, u\}$.

If A is the set of all even integers between 1 and 100, it may be written in set builder notation as $A = \{x: 1 < x < 100 \ and \ \ x \ is \ an \ even \ number\}$. In this notation x represent even numbers between 1 and 100.

If p is an element of set A we write $p \in A$. If p does not belong to A we write $p \notin A$.

For example, if B is a set of positive numbers then $5 \in B$. $-2 \notin B$ since -2 is a negative integer.

Definition 1.2 Two sets A and B are said to be **identical** or **equal**, written $A = B$, if each member of A is also a member of B and vice versa. Thus $A = B$ if $x \in A \Rightarrow x \in B$ and if $x \in B \Rightarrow x \in A$.

For example, the sets $A = \{a, u, e, i, o\}$ and $B = \{i, u, e, o, a\}$ are identical or equal. Note that the order in which the elements are written does not matter.

Definition 1.3 If all the elements of A are in B and $A \neq B$ (A is not equal to B), then A is said to be **proper subset** of B, and we write $A \subset B$.

If all the elements of A belong to B and we are not sure whether A and B are identical, we simply say that A is a **subset** of B, and we write $A \subseteq B$.

For example, if A is the set of all rain days in March and B is the set of all days in March, then clearly $A \subseteq B$.

Note: 1. A set A is a subset of B if and only if every element of A is an element of B.

2. If $A \subseteq B$ and $B \subseteq A$ then $A = B$.

Definition 1.4 All sets under investigation are subsets of a fixed set called the **universal set**. In this course, we shall denote the universal set by *U*.
On the other hand it is also possible to have a set which has no elements. This set is called an **empty set** or a **null set**, and it is denoted by $\emptyset$.

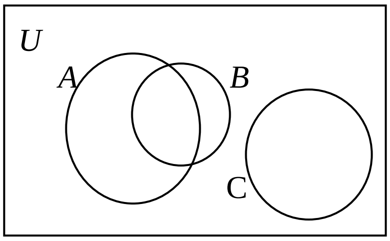
For example, the set $P = \{x: x^2 = -1 \; and \; x \; is \; a \; real \; number\} = \emptyset$.

By convention, $\emptyset$ is a subset of every set.

The number of elements in a set, say A, is denoted by $n(A)$. For example, if $A = \{a, b, c\}$, the $n(A) = 3$.

Sets can also be represented pictorially in a diagram called a **Venn diagram**, in which the sets are depicted by a enclosed areas in a plane.

For example, the universal set is a rectangle and it contains the circular subsets *A*, *B* and *C*.



Definition 1.5 A set of sets is called a **collection** or **class** or **family**.

For example, $\mathcal{B} = \big\{\{1,2\}, \{3\}, \{1,2,3\}\big\}$ is a class and the members of $\mathcal{B}$ are the sets $\{1,2\}, \{3\}$, and $\{1,2,3\}$.

Definition 1.6 The power set of set *A*, denoted by $\mathbb{P}(A)$ or 2^A, is the class of all subsets of *A*. In particular, if $A = \{a, b, c\}$, then

$$\mathbb{P}(A) = \{A, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \emptyset\}.$$

The number of elements in the power set of set, say *A* is given by $n(\mathbb{P}(A)) = 2^{n(A)}$.

For example, $n(\mathbb{P}(A)) = 2^{n(A)} = 2^3 = 8$.

Definition 1.7 A set is said to be **finite** if it has exactly *m* elements, where *m* is a positive integer. Otherwise it is said to be **infinite**.

For example, the set $A = \{a, e, i, o, u\}$ is finite while the set $\mathbb{N} = \{1,2,3, \cdots\}$ the set of all positive is infinite.

By conversion a null set $\emptyset$ is finite.

Set Operations

There are basically four set operations, namely the union, intersection ,relative complement and absolute complement or simply complement.

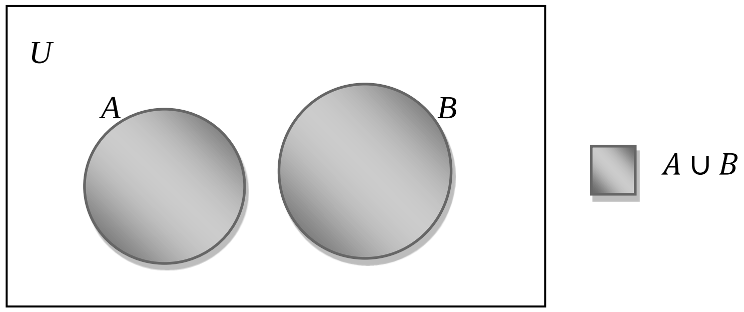
<u>Definition</u> 1.8 The **union** of two sets *A* and *B*, denoted by $A \cup B$, is the set of all elements which belong to *A* or to *B*, i.e.,

$$A \cup B = \{x : x \in A \; or \; x \in B\}.$$

<u>Example</u> 1.2 Let $A = \{a, b, c, d, e\}$ and $B = \{a, e, i, o, u\}$. Then

$$A \cup B = \{a, b, c, d, e, i, o, u\}$$

In general, $A \cup B$ can be represented in a diagram called a vein diagram, as follows:



Clearly, if *A* and *B* are finite disjoint sets, then $A \cup B$ is finite and

$$n(A \cup B) = n(A) + n(B).$$

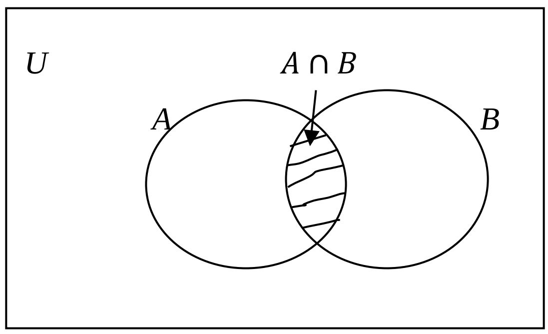
<u>Definition</u> 1.9 The **intersection** of two sets *A* and *B*, denoted by $A \cap B$, is the set of all elements which belong to *A* and to *B*, i.e.,

$$A \cap B = \{x : x \in A \; and \; x \in B\}.$$

<u>Example</u> 1.3 Let $A = \{a, b, c, d, e\}$ and $B = \{a, e, i, o, u\}$. Then

$$A \cap B = \{a, e\}$$

In a vein diagram, $A \cap B$ can be represented as follows:



Note that $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

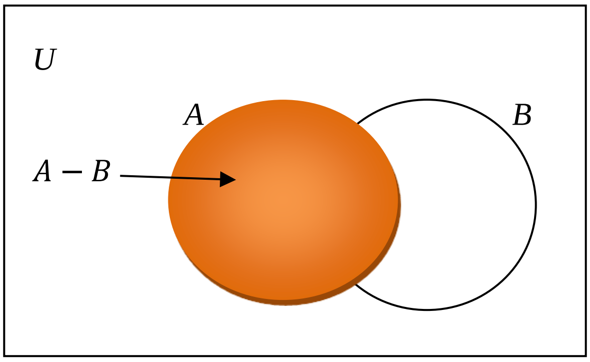
Definition 1.10 The **relative complement** of a set B with respect to set A or simply the difference of A and B, denoted by $A - B$ or $A \backslash B$, is the set of elements which belong to A and which do not belong to B, i.e.,

$$A - B = \{x: x \in A \text{ and } x \notin B\}.$$

Example 1.1.3 Let $A = \{a, b, c, d, e\}$ and $B = \{a, e, i, o, u\}$. Then

$$A - B = \{b, c, d\}$$

In a vein diagram, this represented as follows:



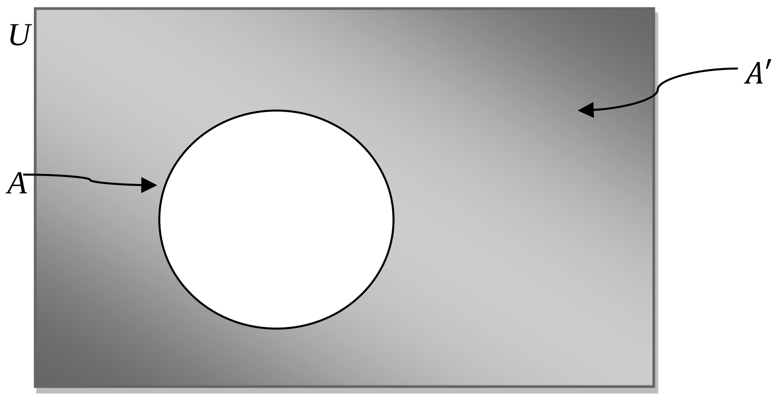
Note: 1. $A - B$ and B are disjoint sets i.e. $(A - B) \cap B = \emptyset$.

2. $A - B \subseteq A$.

Definition 1.11 The **absolute complement** or simply the complement of set A, denoted by A', is the set of elements which do not belong to A but belong to the universal set U, i.e.,

$$A' = \{x: x \in U \text{ and } x \notin A\}.$$

Note: $A' = U - A$.



Laws of Algebra of Sets

Sets under the above operations satisfy various laws which are listed below.

1. The idempotent laws:

   (a) $A \cup A = A$

   (b) $A \cap A = A$

   The proof of this is trivial.

2. The Associative laws:

   (a) $A \cup (B \cup C) = (A \cup B) \cup C$

   Proof: Let $x \in A \cup (B \cup C)$. Then $x \in A$ or $x \in B \cup C$

   $\Rightarrow x \in A$ or $x \in B\ or\ x \in C \Rightarrow x \in A \cup B\ or\ x \in C$

   $\Rightarrow x \in (A \cup B) \cup C \Rightarrow x \in A \cup B$ or x

   This means that $A \cup (B \cup C) \subset (A \cup B) \cup C$.

   Conversely, Let $x \in (A \cup B) \cup C$. Then $x \in (A \cup B)$ or $x \in C$

   $\Rightarrow x \in A$ or $x \in B\ or\ x \in C \Rightarrow x \in A\ or\ x \in (B \cup C)$

   $\Rightarrow x \in (A \cup B) \cup C$

   This means that $(A \cup B) \cup C \subset A \cup (B \cup C)$.

   Now since $A \cup (B \cup C) \subset (A \cup B) \cup C$ and $(A \cup B) \cup C \subset A \cup (B \cup C)$, it follows that $A \cup (B \cup C) = (A \cup B) \cup C$.

   (b) $A \cap (B \cap C) = (A \cap B) \cap C$

   Prove (b) the same way as an exercise.

3. The Commutative laws:

   (a) $A \cup B = B \cup A$

   (b) $A \cap B = B \cap A$

   The proof of this is trivial.

4. The Distributive laws:

   (a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

   <u>Proof</u>: (a) To prove the theorem, you must show that

   $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$ and $(A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$.

   i.e. you take an arbitrary element $x \in A \cup (B \cap C)$ and show that it is also an element of the set $(A \cup B) \cap (A \cup C)$.

   Now, let $x \in A \cup (B \cap C)$. Then

$x \in A$ or $x \in B \cap C$

$\Rightarrow$ $x \in A$ or $x \in B$ and $x \in C$

$\Rightarrow$ $x \in A$ or $x \in B$ and $x \in A$ or $x \in C$

$\Rightarrow$ $x \in A \cup B$ and $x \in A \cup C$

$\Rightarrow$ $x \in (A \cup B) \cap (A \cup C)$

Thus,

$A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$

Conversely, let $x \in (A \cup B) \cap (A \cup C)$. Then $x \in A \cup B$ and $x \in A \cup C$

$\Rightarrow$ $x \in A$ or $x \in B$ and $x \in A$ or $x \in C$

$\Rightarrow$ $x \in A$ or $x \in B$ and $x \in C$

$\Rightarrow$ $x \in A$ or $x \in B \cap C$

$\Rightarrow$ $x \in A \cup (B \cap C)$.

Thus, $(A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$.

Since $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$ and

$(A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$ it follows that

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

(b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof: Complete the proof by filling in the blanks.

Let $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$

$\Rightarrow$ $x \in A$ and _______(1)___________

$\Rightarrow$ $x \in A$ and $x \in B$ _____(2)_____ $x \in A$ and $x \in C$

$\Rightarrow$ ______(3)_______ or ______(4)_______

$\Rightarrow$ $x \in (A \cap B) \cup (A \cap C)$.

Thus, $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$.

Conversely, let $x \in$_______(5)_________. Then

$x \in A \cap B$ or $x \in A \cap C$

$\Rightarrow$ _____(6)______ or _____(7)_____

$\Rightarrow$ $x \in A$ and $x \in B$ or $x \in C$

$\Rightarrow$ $x \in A$ and _____(8)_____

$\Rightarrow \qquad x \in A \cap (B \cup C)$.

Thus, $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$.

Therefore since $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$ and

$(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$ it follows that

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

5. The Identity laws:

(a) $A \cup \emptyset = A$

(b) $A \cap \emptyset = \emptyset$

(c) $A \cup U = U$

(d) $A \cap U = A$

Proof: Exercise

6. The Complement laws:

(a) $A \cup A' = U$

(b) $A \cap A' = \emptyset$

(c) $(A')' = A$

(d) $U' = \emptyset$

(e) $\emptyset' = U$

Proof: Exercise

7. De Morgan's laws:

(a) $(A \cup B)' = A' \cap B'$

Proof: Let $x \in (A \cup B)'$. Then $x \notin A \cup B \Rightarrow x \notin A$ and $x \notin B$

$\Rightarrow x \in A'$ and $x \in B' \Rightarrow x \in A' \cap B'$.

Thus, $(A \cup B)' \subset A' \cap B'$.

Conversely, let $x \in A' \cap B'$. Then $x \in A'$ and $x \in B'$.

$\Rightarrow x \notin A$ and $x \notin B \Rightarrow x \notin A \cup B \Rightarrow x \in (A \cup B)'$.

Thus, $A' \cap B' \subset (A \cup B)'$.

Therefore, since $(A \cup B)' \subset A' \cap B'$ and $A' \cap B' \subset (A \cup B)'$ we conclude that $(A \cup B)' = A' \cap B'$.

(b) $(A \cap B)' = A' \cup B'$

Proof: Complete the proof by filling in the blanks.

Let $x \in (A \cap B)'$. Then ___________(1)____________

$\Rightarrow x \notin A$ or $x \notin B \Rightarrow$ _________(2)__________ $\Rightarrow x \in A' \cup B'$.

Thus, $(A \cap B)' = A' \cup B'$.

Conversely, let $x \in A' \cup B'$. Then ________(3)_________

$\Rightarrow\ x \notin A$ or $x \notin B\ \Rightarrow$ ________(4)_________ $\Rightarrow x \in (A \cap B)'$.

Thus, $(A \cap B)' \subset (A \cap B)'$.

Therefore, since $A' \cap B' \subset (A \cup B)'$ and $(A \cup B)' \subset A' \cap B'$ we conclude that $(A \cap B)' = A' \cup B'$.

The De Morgan's laws can be generalized as follows:

(a) $(A_1 \cup A_2 \cup \cdots \cup A_n)' = A'_1 \cap A'_2 \cap \cdots \cap A'_n$

(b) $(A_1 \cap A_2 \cap \cdots \cap A_n)' = A'_1 \cup A'_2 \cup \cdots \cup A'_n$

8. The Difference law:

   $A - B = A \cap B'$

Example 1.4 Let $U = \{1,2,3,\cdots,8,9\}$, $A = \{1,2,3,4\}$, $B = \{2,4,6,8\}$ and $C = \{3,4,5,6\}$. Find

(i) A' (ii) $(A \cap C)'$ (iii) $B - C$ (iv) $(A \cup B)'$

Solution: (i) $A' = \{5,6,7,8,9\}$

(ii) $A \cap C = \{3,4\} \Rightarrow (A \cap C)' = \{1,2,5,6,7,8,9\}$

NOTE: $A' \cup C' = \{5,6,7,8,9\} \cup \{1,2,7,8,9\} = \{1,2,5,6,7,8,9\} = (A \cap C)'$

(iii) $B - C = \{2,8\}$

NOTE: $B \cap C' = \{2,4,6,8\} \cap \{1,2,7,8,9\} = \{2,8\} = B - C$

(iv) $A \cup B = \{1,2,3,4,6,8\} \Rightarrow (A \cup B)' = \{5,7,9\}$

NOTE: $A' \cap B' = \{5,6,7,8,9\} \cap \{1,3,5,7,9\} = \{5,7,9\} = (A \cup B)'$

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1. Let the universal set $U = \{1,2,3,4,5,6,7,8,9,10\}$, the sets $A = \{1,3,6,8,10\}$, $B = \{2,4,5,6,8\}$ and $C = \{1,4,6,10\}$. Find the following sets:

   (a) $(A \cap B')'$ (d) $(A' \cup B)'$ (e) $(A \cap B) \cup C$

   (f) $(A \cup B) \cap C$.

2. In the problems below, one of the following relations is true: $A \subset B$, $A = B$, $A \neq B$, $B \subset A$. Write the correct relation in each case.

   A B

(a) $\{x|2x+3=11-2x\}$ $\{x|5x+4=x+12\}$

(b) $\{x|x^2+4=40-5x\}$ $\{x|3+2x=11\}$

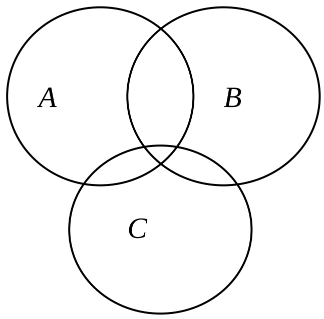
(c) $\{x|x+4=x(x+4)\}$ $\{x|x^2+3x-4=0\}$

(d) $\{x|x-1=0\} \cup \{x|x-2=0\}$ $\{x|x^2-3x+2=0\}$

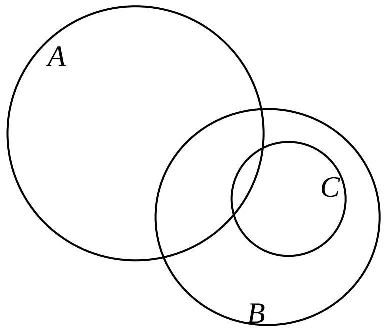
(e) $\{x|x+3=4\}$ $\{x|(x+3)^2=16\}$

3. In each of the Venn diagrams below shade: (i) $A \cap (B \cup C)$ (ii) $C-(A \cap B)$

(a)



(b)



4. Illustrate the given identities by drawing Venn diagrams, given that *A*, *B*, *C* are subsets of a universal set *U*:

(a) De Morgan's laws:

(i)$(A \cup B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$

(b) Distributive laws:

(i) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (ii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(c) $A \cup A' = U$ (d) $A \cap A' = \emptyset$

(d) Commutative laws:

(i) $A \cup B = B \cup A$ (ii) $A \cap B = B \cap A$

(e) Associative laws:

(i) $A \cup (B \cup C) = (A \cup B) \cup C$ (ii) $A \cap (B \cap C) = (A \cap B) \cap C$

(f) $A - B = A \cap B'$

5. Express each of the following in its simplest form:

(a) $[P' \cup (Q - R)]'$ (b) $X \cup (Y \cap X)$ (c) $(M \cap N) \cup (M \cap N')$

(d) $A - (A - B)$ (e) $A \cup (B - C)$(f) $[(A \cap B)' \cup (A - B)]'$

Sets of Numbers

There are special symbols used for sets of numbers. These are

$\mathbb{Z}$ = the set of integers: $\{\cdots, -2, -1, 0, 1, 2, \cdots\}$.

$\mathbb{N}$ = the set of natural numbers (positive integers or counting numbers): $\{1,2,3,\cdots\}$.

$\mathbb{Q}$ = the set of rational numbers: numbers which can be expressed as ratios of Integers.

$\mathbb{Q}'$= the set of irrational numbers: numbers which cannot be expressed as ratios of integers.

$\mathbb{R}$ = the set of real numbers.

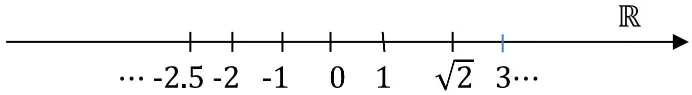
$\mathbb{C}$ = the set of complex numbers.

NOTE: 1. $N \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.

2. $\mathbb{Q}'$= the set of irrational numbers, like the numbers $\sqrt{2}, \sqrt{3}, \pi$, etc.

3. $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}'$.

4. The set of real numbers can be represented graphically by points on a straight line as follows:



Intervals

Lat a and b be distinct real numbers with, say, $a < b$. Then intervals with endpoints a and b are denoted and defined as follows:

(i) $(a, b) = \{x \in \mathbb{R}: a < x < b\}$, open interval from a to b.

Note: 1. Here the numbers $a, b \notin (a, b)$

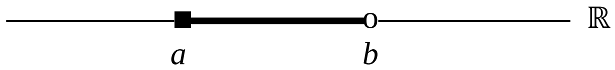
2. (a, b) is a set which contains all the numbers between a and b, and it is an infinite set. It contains an infinite number of elements. For example, for the set $(0,2)$, $0,2 \notin (0,2)$. However it contains all the numbers between 0 and 2, like $\frac{1}{10}, \frac{1}{2}, 1, \sqrt{2}, \sqrt{3}, \frac{10}{9}$, etc.

(ii) $[a, b] = \{x \in \mathbb{R}: a \leq x \leq b\}$, closed interval from a to b.

Note: 1. Here the numbers $a, b \in [a, b]$

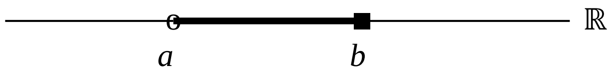
2. $[a, b]$ is a set which contains the numbers a, b and the other numbers between a and b. It is also an infinite set. It contains an infinite number of elements.

(iii) $(a, b] = \{x \in \mathbb{R}: a < x \leq b\}$, open- closed interval from a to b.



Note: 1. Here the number $a, \notin (a, b]$ but $b \in (a, b]$

(iv) $[a, b) = \{x \in \mathbb{R}: a \leq x < b\}$, closed-open interval from a to b.



Similarly, here the number $a \in [a, b)$ but $b \notin [a, b)$.

It should also be noted that

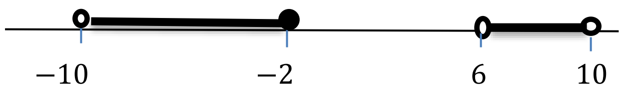
1. An interval is a subset of ℝ and it is an infinite set.
2. An interval is open if it does not include its end points and is closed if it includes its endpoints.
3. A parenthesis "("or ")" is used to indicate that an endpoint does not belong to the interval, and the bracket "["or "]" is used to indicate that an endpoint does belong to the interval.

Example 1.5 Given the sets $A = (-2, 6]$, $B = [-5, 3]$, $C = [-1, 8)$ and $X = (-10, 10)$ is the universal set. Find each of the following sets and display it on the number line:

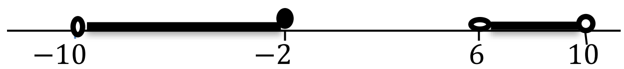
(i) A' (ii) $X - A$ (iii) $(A \cap C)'$ (iv) $(B - A) \cap C$

Solutions:

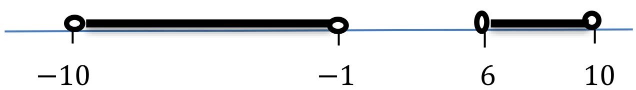
(i) $A' = (-10, -2] \cup (6, 10)$



(ii) $X - A = (-10, -2] \cup (6, 10)$



(iii) $A \cap C = [-1, 6] \Rightarrow (A \cap C)' = (-10, -1) \cup (6, 10)$



(iv) $B - A = [-5, -2]$, $(B - A) \cap C = \emptyset$

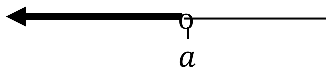
Now we shall introduce the symbol ∞, called infinity. It is not a number itself. Thus $\infty \notin \mathbb{R}$.
However, it is perceived to be greater than any real number, whereas the symbol $-\infty$ (minus infinity) is perceived to be less than any real number.

Definition 1.12 Let a be any real number. Then the set of real numbers x satisfying $x < a,\ x \le a, x > a\ or\ x \ge a$, is called an **infinite interval** with endpoint a.
The infinite interval is said to be open or closed according as whether the endpoint a does or does not belong to the interval.

The four intervals may also be denoted and defined as follows:

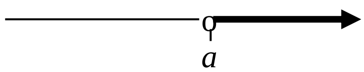
$(-\infty, a) = \{x : x < a, x \in \mathbb{R}\}$, $(-\infty, a] = \{x : x \le a, x \in \mathbb{R}\}$,
$(a, \infty) = \{x; : a > x, x \in \mathbb{R}\}$ and $[a, \infty) = \{x : a \ge x, x \in \mathbb{R}\}$.

Graphically, the infinite intervals $x < a,\ x \le a, x > a\ and\ x \ge a$ are represented as follows:

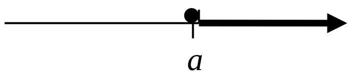


$\{x \in \mathbb{R} : x < a\} = (-\infty, a)$

$\{x \in \mathbb{R} : x \le a\} = (-\infty, a]$



$\{x \in \mathbb{R} : x > a\} = (a, \infty)$



$\{x \in \mathbb{R} : x \ge a\} = [a, \infty)$

Note that the set of real numbers,$\mathbb{R}$, as an infinite interval can be written as $(-\infty, \infty)$.

By convention, this interval is both open and closed.

Graphically, $\mathbb{R}$ can be represented as follows:

$\mathbb{R} = (-\infty, \infty)$

Rational Numbers

A rational number is a real number which can be expressed in the form $\frac{p}{q}$ where p and q are integers and $q \neq 0$. In view of this, any rational number can be represented as a decimal. Some representations terminate at after a finite number of steps, i.e. all later terms in the expansion are zero. For example,

$$\frac{1}{2} = 0.5000 \cdots$$

$$\frac{1}{4} = 0.2500 \cdots$$

But other expansions never terminate, such as

$$\frac{1}{3} = 0.3333\cdots$$

$$\frac{8}{7} = 1.14285714257\cdots$$

In the expansion of $\frac{1}{3}$, 3 is repeated after the decimal point and in $\frac{8}{7}$, 142857 is repeating after the decimal point. This is always true for rational numbers.
Now it is awkward to express non terminating decimals such as $\frac{1}{3}$ and $\frac{8}{7}$ in the form given above. To remove this ambiguity, we place a bar over the set of numbers which is to be repeated indefinitely. In this notation we write, for example,

$$0.5\overline{0}$$

$$0.25\overline{0}$$

$$\frac{1}{3} = 0.\overline{3}$$

$$\frac{8}{7} = 1.\overline{142857}$$

NOTE: Every repeating decimal expansion is a rational number.

<u>Example</u> 1.6 Show that each of the following numbers is a rational number:

(a) $3.\overline{3}$ (b) $25.\overline{12}$ (c) $0.29\overline{432}$

<u>Solution</u>: (a) Let $a = 3.\overline{3}$. Multiplying by 10 both sides we have

$$10a = 33.\overline{3}$$

$$\Rightarrow 10a - a = 33.\overline{3} - 3.\overline{3}$$

$$\Rightarrow 9a = 30 \quad \Rightarrow a = \frac{30}{9}$$

$$\therefore \quad 3.\overline{3} = \frac{30}{9}$$, which is a rational number.

(b) Let $p = 25.\overline{12}$. Multiplying both sides by 100, yields

$$100p = 2512.\overline{12}$$

$$\Rightarrow 100p - p = 2512.\overline{12} - 25..\overline{12}$$

$$\Rightarrow 99p = 2487 \quad \Rightarrow p = \frac{2487}{99}$$

$$\therefore \quad 25.\overline{12} = \frac{2487}{99} = \frac{829}{33}$$, which is a rational number.

(c) Let $t = 0.29\overline{432}$. Multiplying both sides by 100, yields

$$100t = 29.\overline{432} \qquad \text{(I)}$$

Multiplying both sides of (I) by 10000, yields

$10000t = 29432.\overline{432}$

$\Rightarrow 100000t - 100t = 29432.\overline{432} - 29.\overline{432}$

$\Rightarrow 99900t = 29403 \quad \Rightarrow t = \frac{29403}{99900} = \frac{3267}{11100} = \frac{1089}{3700}$, which is a rational number.

1.1.7 Irrational Numbers

An irrational number cannot be expressed as a ratio of two integers.
For example, $\sqrt{2}$ is an irrational number and it cannot be express in the form $\frac{p}{q},\ p, q \in \mathbb{Z}$ and $q \neq 0$.

To prove this we shall require the following preliminary result:

<u>Theorem</u> 1.1.1 If a^2 is divisible by 2, then a is also divisible by 2.

<u>Proof</u>: Every integer a can be written in one of the forms

$$a = \begin{cases} 2n \\ 2n+1 \end{cases}, \text{ where } n \text{ is an integer.}$$

Hence, $$a^2 = \begin{cases} 4n^2 \\ 4n^2 + 4n + 1 \end{cases}$$

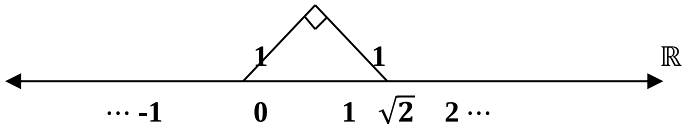
Since a^2 is divisible by 2, $a^2 = 4n^2$, since $4n^2 + 4n + 1$ is not a multiple of 2 for all $n \in \mathbb{Z}$. This means that $a = 2n$, which is divisible by 2. Hence the theorem.

<u>Theorem</u> 1.1.2 $\sqrt{2}$ is not a rational number.

<u>Proof</u>: We shall prove this by contradiction. Suppose that $\sqrt{2}$ is a rational number. Then it can be expressed in the form $\frac{p}{q}$, where $p, q \in \mathbb{Z}$, $q \neq 0$ such that p and q have no common factor. Now, if $\sqrt{2} = \frac{p}{q}$, then $2 = \frac{p^2}{q^2}$ or $p^2 = 2q^2$. Thus, p^2 is divisible by 2. From theorem 1.1.1 it follows that p is also divisible by 2, i.e. $p = 2r$, where r is an integer. When we replace $p = 2r$ in $p^2 = 2q^2$ we have $4r^2 = 2q^2$ or $q^2 = 2r^2$. Hence q^2 is divisible by 2 $\Rightarrow q$ is also divisible by 2. This means that p and q have common factor 2, contrary to our assumption. Therefore, $\sqrt{2}$ cannot be expressed as a ratio of two integers, implying that it is not a rational number. Therefore it is an irrational number.

Other examples of irrational numbers are $\sqrt{3}, \quad \pi, \quad e, \quad \sqrt{5}$ etc.
Irrational numbers are also real numbers. For example, $\sqrt{2}$ is a real number and has a

specific position on the number line.



By construction above, $\sqrt{2}$ lies on the number line. Hence $\sqrt{2} \in \mathbb{R}$.

In general, every irrational number is also a real number.

Note that irrational numbers are non - repeating, non - terminating decimals.

Complex Numbers

Some problems cannot be solved using real numbers alone. For example, we cannot find a real number x such that $x^2 = -1$. To handle such problems, the new symbol i had to be introduced with the property $i = \sqrt{-1}$ or $i^2 = -1$. i is called an **imaginary number**.

For an example to solve the quadratic equation $x^2 - 2x + 2 = 0$, we need to use the imaginary number $i = \sqrt{-1}$. Note that by using the quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm \sqrt{4 \times (-1)}}{2} = \frac{2 \pm 2\sqrt{-1}}{2} = \frac{2(1 \pm \sqrt{-1})}{2}$$

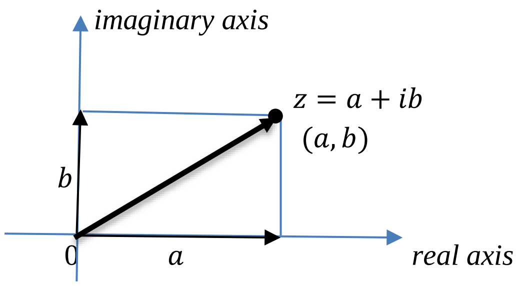
$= 1 \pm i$, since $i = \sqrt{-1}$.

Thus, the equation $x^2 - 2x + 2 = 0$ has two solutions which in this case are complex numbers $1 + i$ and $1 - i$.

Definition 1.13 A **complex number** is an ordered pair $(a, b)$of real numbers a and b, and is written $a + ib$. The number a is called the *real part* of $a + ib$, and b is its *imaginary part*.

Note that a complex number at the point (a, b) is the vector sum of the two numbers a and bj.

Usually a complex number is denoted by the letter z. Thus, $z = a + ib$.



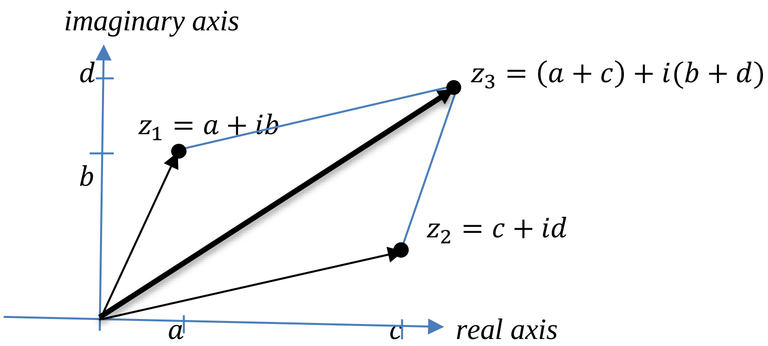
As opposed to a real number which lies on the real line, a complex number lies on the Cartesian plane.

Definition 1.14 The arithmetic operations on complex numbers are defined as follows:

(a) Two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ are equal if and only if $a = c$ and $b = d$.

(b) Addition: By vector addition in the diagram,

$$z_1 + z_2 = (a + ib) + (c + id) = (a + c) + i(b + d)$$



(c) Multiplication: Multiplication of two complex numbers is just ordinary algebraic expansion of brackets, and replacing $i^2$by -1.

$$\begin{aligned}(a+ib)\times(c+id) &= a(c+id)+ib(c+id)\\ &= ac+iad+ibc+i^2bd\\ &= ac+iad+ibc-bd\\ &= (ac-bd)+i(bc+ad).\end{aligned}$$

Example 1.7 Evaluate each of the following:

(a) $(4+i)+(3-5i)$

Solution: $(4+i)+(3-5i)=(4+3)+(i-5i)=7-4i.$

(b) $(2+3i)-(7+4i)$

Solution: $(2+3i)-(7+4i)=2+3i-7-4i=2-7+3i-4i=-5-i.$

(c) $(2-3i)(2+5i)$

Solution: $(2-3i)(2+5i)=2(2+5i)-3i(2+5i)=4+10i-6i-15i^2$

$$=4+10i-6i-15i^2=4+10i-6i-15(-1)$$

$$=4+15+10i-6i=19+4i.$$

Definition 1.15 The complex numbers $a+ib$ and $a-ib$ are said to be *conjugates* of each other. The conjugate of a complex number z is denoted by $\bar{z}$. i.e. if $z=a+ib$ then $\bar{z}=a-ib$.

Note that $z\bar{z}=(a+ib)(a-ib)=a(a-ib)+ib(a-ib)=a^2-abi+abi-i^2b^2$

$$=a^2-(-1)b^2=a^2+b^2.$$

i.e. $z\bar{z}=(a+ib)(a-ib)=a^2+b^2=\bar{z}z.$

Example 1.8 The conjugate of $-3+7i$ is $-3-7i$ and the conjugate of $-3-7i$ is $-3+7i$ and $(-3+7i)(-3-7i)=(-3)^2+(7)^2=9+49=58$

Definition 1.16 To divide a complex number $a+ib$ by $c+id$ i.e. to evaluate $\dfrac{a+ib}{c+id}$, we multiply both the numerator and the denominator by the conjugate $c-id$ and obtain

$$\frac{a+ib}{c+id}\times\frac{c-id}{c-id}=\frac{(ac+bd)+(bc-ad)i}{c^2+d^2}=\frac{ac+bd}{c^2+d^2}+i\frac{bc-ad}{c^2+d^2}.$$

Example 1.9 Express the given complex numbers in the form $a+id$, where $a,b\in\mathbb{Q}$.

(a) $\frac{4+7i}{5-i}$ (b) $\frac{-4+2i}{-6-5i}$ (c) $\frac{6-i}{2i}$

Solutions: (a) $\frac{4+7i}{5-i}=\frac{4+7i}{5-i}\times\frac{5+i}{5+i}=\frac{4(5+i)+7i(5+i)}{(5-i)(5+i)}=\frac{20+4i+35i-7}{5^2+1^2}$

$$=\frac{13+39i}{25+1}=\frac{13+39i}{26}=\frac{13}{26}+\frac{39}{26}i=\frac{1}{2}+\frac{3}{2}i.$$

(b) $\frac{-4+2i}{-6-5i}=\frac{-4+2i}{-6-5i}\times\frac{-6+5i}{-6+5i}=\frac{-4(-6+5i)+2i(-6+5i)}{(-6-5i)(-6+5i)}=\frac{24-20i-12i-10}{(-6)^2+5^2}$

$$=\frac{14-32i}{36+25}=\frac{14-32i}{61}=\frac{14}{61}-\frac{32}{61}i.$$

(c) $\frac{6-i}{2i}=\frac{6-i}{0+2i}\times\frac{0-2i}{0-2i}=\frac{6(0-2i)-i(0-2i)}{(0+2i)(0-2i)}=\frac{0-12i+0-2}{0^2+2^2}=\frac{-2-12i}{4}$

$$=-\frac{2}{4}-\frac{12}{4}i=-\frac{1}{2}-3i.$$

Surds and Manipulation of Surds

A surd is a number that cannot be simplified to remove a square root or cube root etc. Surds are used to write numbers exactly. For example, $\sqrt{2},\sqrt{3}+4,\sqrt{11}$ are exact the way they are and cannot be simplified. Surds cannot be expressed exactly as decimal fractions because they give never-ending, non repeating decimal fractions, for example, $\sqrt{2}=1.414213562\cdots$.

However, we can manipulate surds using the following rules:

1. $\sqrt{ab}=\sqrt{a}\times\sqrt{b}$
2. $\sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}}$

Note: $\sqrt{a}\times\sqrt{a}=a$, since $\left(\sqrt{a}\right)^2=a$.

Example 1.10 Simplify:

$\sqrt{28}$ 2. $\frac{\sqrt{72}}{2}$ 3. $5\sqrt{6}-2\sqrt{24}+\sqrt{294}$.

Solutions:

1. $\sqrt{28}=\sqrt{4\times 7}=\sqrt{4}\times\sqrt{7}=2\times\sqrt{7}=2\sqrt{7}$.

Note: $\sqrt{4}=2$ and **not** -2 i.e. $\sqrt{4}\neq\pm 2$.

The square root of a positive real number is positive.

2. $\frac{\sqrt{72}}{2} = \frac{\sqrt{36\times2}}{2} = \frac{\sqrt{36}\times\sqrt{2}}{2} = \frac{6\sqrt{2}}{2} = 3\sqrt{2}.$

3. $5\sqrt{6} - 2\sqrt{24} + \sqrt{294} = 5\sqrt{6} - 2\sqrt{4\times6} + \sqrt{49\times6} = 5\sqrt{6} - 2(2\sqrt{6}) + 7\sqrt{6}$
$= 5\sqrt{6} - 4\sqrt{6} + 7\sqrt{6} = 8\sqrt{6}.$

Example 1.11 Evaluate:

1. $(2\sqrt{7})(3\sqrt{7})$ 2. $(\sqrt{5}+\sqrt{2})^2$ 3. $(2\sqrt{3}-\sqrt{5})(2\sqrt{3}+2\sqrt{5})$
4. $(2\sqrt{5}-3\sqrt{2})(2\sqrt{5}+3\sqrt{2})$

Solutions:
1. $(2\sqrt{7})(3\sqrt{7}) = 2\times3\times(\sqrt{7})^2 = 2\times3\times7 = 42.$
1. $(\sqrt{5}+\sqrt{2})^2 = (\sqrt{5})^2 + 2\sqrt{5}\sqrt{2} + (\sqrt{2})^2 = 5 + 2\sqrt{10} + 2 = 7 + 2\sqrt{10}.$
2. $(2\sqrt{3}-\sqrt{5})(2\sqrt{3}+2\sqrt{5}) = 2\sqrt{3}(2\sqrt{3}+2\sqrt{5}) - \sqrt{5}(2\sqrt{3}+2\sqrt{5})$

$$= 4(\sqrt{3})^2 + 4\sqrt{3}\sqrt{5} - 2\sqrt{5}\sqrt{3} + 2(\sqrt{5})^2$$
$$= 12 + 4\sqrt{15} - 2\sqrt{15} + 10$$
$$= 22 + 2\sqrt{15}.$$

Rationalization of the Denominator of a Fraction involving Surds

To rationalize the denominator of a fraction involving surds is to make the denominator free of the root sign.

The rules to rationalize the denominator are as follows:

1. Fractions in the form of $\frac{1}{\sqrt{a}}$, multiply the numerator and the denominator by $\sqrt{a}$ so that the fraction become $\frac{\sqrt{a}}{a}$.
2. Fractions in the form $\frac{1}{\sqrt{a}+\sqrt{b}}$, multiply the numerator and the denominator by the conjugate $\sqrt{a}-\sqrt{b}$ of the denominator. Thus
$$\frac{1}{\sqrt{a}+\sqrt{b}} = \frac{1}{\sqrt{a}+\sqrt{b}} \times \frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}-\sqrt{b}} = \frac{\sqrt{a}-\sqrt{b}}{(\sqrt{a})^2-(\sqrt{b})^2} = \frac{\sqrt{a}-\sqrt{b}}{a-b}.$$

Example 1.12 Rationalize the denominator of each of the following:

1. $\frac{3}{\sqrt{5}+\sqrt{2}}$ 2. $\frac{1+\sqrt{2}}{1-\sqrt{2}}$ 3. $\frac{\sqrt{3}-2}{\sqrt{3}-2\sqrt{5}}$.

Solutions:

1. $\frac{3}{\sqrt{5}+\sqrt{2}} = \frac{3}{\sqrt{5}+\sqrt{2}} \times \frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}-\sqrt{2}} = \frac{3\sqrt{5}-\sqrt{2)}}{(\sqrt{5})^2-(\sqrt{2})^2} = \frac{(3\sqrt{5}-\sqrt{2)}}{5-2} = \sqrt{5}-\sqrt{2}.$

   Note that in the form $p\sqrt{5}+q\sqrt{2}$ where p and q are integers,

   $\frac{3}{\sqrt{5}+\sqrt{2}} = \frac{1}{3}\sqrt{5} - \frac{1}{3}\sqrt{2}.$

2. $\frac{1+\sqrt{2}}{1-\sqrt{2}} = \frac{1+\sqrt{2}}{1-\sqrt{2}} \times \frac{1+\sqrt{2}}{1+\sqrt{2}} = \frac{(1+\sqrt{2})+\sqrt{2}(1+\sqrt{2})}{1^2-(\sqrt{2})^2} = \frac{1+\sqrt{2}+\sqrt{2}+2}{1-2} = \frac{3+2\sqrt{2}}{-1}$

   $= -(3+2\sqrt{2}) = -3-2\sqrt{2}.$

3. $\frac{\sqrt{3}-2}{\sqrt{3}-2\sqrt{5}} = \frac{\sqrt{3}-2}{\sqrt{3}-2\sqrt{5}} \times \frac{\sqrt{3}+2\sqrt{5}}{\sqrt{3}+2\sqrt{5}} = \frac{\sqrt{3}(\sqrt{3}+2\sqrt{5})-2(\sqrt{3}+2\sqrt{5})}{(\sqrt{3})^2-(2\sqrt{5})^2}$

   $= \frac{(\sqrt{3})^2+2\sqrt{3}\sqrt{5}-2\sqrt{3}-4\sqrt{5}}{3-4(5)} = \frac{3+2\sqrt{15}-2\sqrt{3}-4\sqrt{5}}{-17}.$

Modulus or (Absolute Value) of a Number

The modulus (or absolute value) of a number k, written $|k|$ is defined by

$$|k| = \begin{cases} k \text{ if } k \geq 0 \\ -k \text{ if } k < 0. \end{cases}$$

For example, $|3| = 3, |-3| = -(-3) = 3, |0| = 0$.

The modulus (absolute value) of a number can be interpreted as the distance of between the number and zero on the number line. For example, $|6| = 6$ because the distance between 6 and 0 is 6 units. Also $|-10| = 10$ because the distance between -10 and 0 is 10 units.

Binary Operations

The term 'binary' emanates from the fact that the operation acts on two elements. The usual operations of arithmetic $+$, $-$ and $\times$ are some of the examples of binary operations because when we choose any **two** numbers, each operation will generate a third number.

Other examples of binary operations are the set union and intersection, $A \cup B = C$ and $A \cap B = D$.

We shall use the symbol $*$ to represent a generic binary operation.

<u>Definition</u> 1.17 Let $S = \{a, b, c, \cdots\}$ be any set. The operation $*$ is a binary operation on S if and only if to every ordered pair (a, b), where $a, b \in S$, there is assigned unique element $a * b \in S$. We indicate this assignment using the notation $(a, b) \to a * b$.

<u>Example</u> 1.13 Let the binary operation $*$ be defined on the set $\mathbb{R}$ as

$$a * b = a + b + 2ab.$$

Find (a) $2 * 5$ (b) $-5 * 3$

<u>Solutions</u>:

(a) $2 * 5 = 2 + 5 + 2(2)(5) = 7 + 20 = 27$.

(b) $-5 * 3 = -5 + 3 + 2(-5)(3) = -2 - 30 = -32$.

The following are the delicate points we need to observe in the definition of binary operation:

1. The order of a and b may be important, for (a, b) is an ordered pair and it may happen that $a * b \neq b * a$. For example, if A and B are matrices, then $A \times B \neq B \times A$.
2. For $a, b \in S$ the operation $*$ must be defined for every pair (a, b).
3. The output $a * b$ must be an element of S.

Why is the operation $\div$ not a binary operation on $\mathbb{Z}$.

<u>Definition</u> 1.18 The binary operation $*$ on a set is called **commutative** if and only if for every ordered pair (a, b) of elements in S $a * b = b * a$.

<u>Example</u> 1.14 Let S be the set of real numbers, $\mathbb{R}$. Then the binary operations $+$ and $\times$ are commutative on $\mathbb{R}$, since for every $a, b \in S$

$$a + b = b + a \text{ and } a \times b = b \times a,$$

but the operation $-$ is not commutative, since real numbers e.g. $7, 5 \in \mathbb{R}$,

$$7 - 5 \neq 5 - 7.$$

<u>Exercise</u> 1.2.1 Are set union and intersection commutative?

<u>Definition</u> 1.19 The binary operation $*$ on a set S is **associative** if and only if for every triple $a, b, c \in S$

$$a * (b * c) = (a * b) * c.$$

<u>Example</u> 1.15 (a). Addition and multiplication are associative binary operations on $\mathbb{R}$.

(b) Subtraction is not associative on $\mathbb{R}$, since for example,

$$12 - (8 - 2) = 6 \neq (12 - 8) - 2 = 2.$$

(c) Division is not associative on $\mathbb{R}$, since, for example,

$$24 \div (6 \div 2) = 8 \neq (24 \div 6) \div 2 = 2$$

<u>Example</u>: 1.16 The binary operation * on $\mathbb{R}$ is defined by

$$a * b = (a - b)^2 - ab.$$

(a) Determine whether the operation $*$ is commutative.

(b) Evaluate (i) $(-1*3)*2$ (ii) $-1 * (3 * 2)$

Hence, state whether the binary operation $*$ is associative or not.

Solution:

(a) $a * b = (a-b)^2 - ab = (-b+a)^2 - ba = [-(b-a)]^2 - ba$

$= (b-a)^2 - ba = b * a$

Hence, the operation $*$ as defined is commutative.

(b) (i) $(-1 * 3) * 2 = \left[\left((-1)-3\right)^2 - (-1)(3)\right] * 2$

$= [(-4)^2 + 3] * 2$

$= [16+3] * 2 = 19 * 2 = (19-2)^2 - (19)(2)$

$= (17)^2 - (19)(2) = 289 - 38 = 251$

(ii) $-1 * (3 * 2) = -1 * [(3-2)^2 - 3(2)] = -1 * [1^2 - 6] = -1 * (-5)$

$= (-1-(-5))^2 - (-1)(-5) = (-1+5)^2 - 5 = 16 - 5$

$= 11 \neq 251 = (-1 * 3) * 2$

Therefore the operation $*$ is not associative.

Exercise 1.2.2 (a) Let $*$ be a binary operation on the set of integers $\mathbb{Z}$ defined by

$$a * b = a + b - ab.$$

Is the operation $*$ is associative on $\mathbb{Z}$?

(b) Evaluate (i) $(-3 * 2) * 5$ (ii) $-3 * (2 * 5)$

Definition 1.20 A binary operation $*$ on a set *S* has an **identity** element, denoted by *e*, if and only if *e* is an element of *S*, and for all elements *a* of *S*,

$$a * e = a = e * a.$$

For example, the identity element for $+$ on a set of real numbers is 0, since

$$a + 0 = a = 0 + a,$$

and the identity element for $\times$ on a set of real numbers is 1, since

$$a \times 1 = a = 1 \times a.$$

Definition 1.21 If $*$ is a binary operation on *S* which has an identity element *e*, and if *a* is any given element of *S*, then the element, denoted by a^{-1}, of *S* is called the **inverse** of *a* if and only if

$$a * a^{-1} = e = a^{-1} * a.$$

For example, the additive inverse of any real number *a* is $-a$, since

$$a + (-a) = 0 = -a + a.$$

The multiplicative inverse of any real number *a* is $\frac{1}{a}$, since

$$a \times \frac{1}{a} = 1 = \frac{1}{a} \times a.$$

NOTE: For an operation $*$ on a set *S*, the inverse a^{-1} of a in *S* in not necessarily $\frac{1}{a}$.

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1. Rewrite each interval in set builder form:

   (a) $A = (-10,10]$ (b) $B = [4,9]$ (c) $C = (-1,5)$ (d) $[0,-2)$.

2. Let $\mathbb{R}$ be the universal set and $A = (-1,6]$, $B = (0,4)$, $C = \{x: x \geq 4, x \in \mathbb{R}\}$ be its subsets. Illustrate each set on the same number line. Hence, find each of the following sets:

   (a) $A - C$ (b) $(B' \cap C)'$ (c) $(C - A) \cap B$.

3. Express the each of the following numbers as a fraction in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$, in its lowest terms:

   (a) $4.\overline{3}$ (b) $-0.2\overline{55}$ (c) $12.34\overline{11}$.

4. Prove that each of the following is an irrational number:

   (a) $\sqrt{3}$ (b) $\sqrt{2} - 1$.

5. Simplify each of the following:

   (a) $2\sqrt{27} - 3\sqrt{48} + \sqrt{75}$ (b) $\sqrt{80} - \sqrt{20} - 2\sqrt{45}$ (c) $\frac{\sqrt{28}}{\sqrt{175}}$.

6. Rationalize the denominator of each of the following:

   (a) $\frac{\sqrt{2}-1}{1+\sqrt{2}}$ (b) $\frac{\sqrt{3}-2}{\sqrt{3}-1}$ (c) $\frac{2+\sqrt{2}}{(\sqrt{2}-1)^2}$.

7. Simplify each of the following, giving your answer in the form $a + bi$, where $a, b \in \mathbb{R}$:

   (a) $(2 - 3i) - (1 + 2i)$ (b) $(3 - i)^2$ (c) $\frac{(1+2i)^2}{2+i}$.

8. Given that $z_1 = 8 + 2i$, $z_2 = 2 + i$, $z_3 = 3 + i$, find the answer to each of the following in the form $a + bi$, where $a, b \in \mathbb{R}$:

   (a) $\frac{z_1 z_2}{z_3}$ (b) $\frac{z_1 - z_2}{z_1 + z_3}$ (c) $\frac{(z_1 - 2z_3)^2}{(3z_2)^2}$.

9. Express $z = \sqrt{4 + 3i}$ in the form of $p + qi$, where $p, q \in \mathbb{R}$.

10. State whether each of the following operation is a binary operation on $\mathbb{Z}$, the set of integers:

    (a) $a * b = a - 2b$ (b) $a * b = \sqrt{a + b}$ (c) $a * b = (a - b)^2$ (d) $a * b = a^2 - b^2$.

11. Let $*$ be a binary operation on the set of real numbers and a, b, c be real numbers. Given that the operation is defined by $a * b = (a - b)^2 - 3ab$.

    (a) Does the operation posses the commutative and/or associative?

    (b) For this binary operation calculate (i) $-2 * (3 * 4)$ (ii) $(-4 * 3) * 2$.

12. The binary operation $*$ is defined on the set of real numbers by

$a * b = 3(a - b)^2$.

Show that the binary operation is commutative.

MAT1100 LECTURE NOTES

2.1 FUNCTIONS

Relations

Definition 2.1.1 Let A and B be two sets. Then the product (or Cartesian) product of A and B, written $A \times B$ and read "A cross B", is the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$. i.e.

$$A \times B = \{(a, b): a \in A, b \in B\}.$$

Example 2.1.1 Let $A = \{1,2\}$ and $B = \{a, b, c\}$. Then

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}.$$

$$\mathrm{B}\times A = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}.$$

$$A \times A = A^2 = \{(1,1), (1,2), (2,1), (2,2)\}.$$

NOTE: $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is the set of ordered pairs of real numbers.

Definition 2.1.2 Let A and B be two sets. Then a binary relation or, simply a relation from A to B is a subset of $A \times B$. i.e. R is a relation from A to B if it is the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$. i.e.

$$R = \{(a, b): a \in A, b \in B\}$$

When $(a, b) \in R$ we say a is R-related to b and we write aRb.

Example 2.1.2 Let $A = \{1,2,3\}$ and $B = \{x, y, z\}$ and let $R = \{(1, y), (1, z), (3, y)\}$. Then R is a relation from A to B since

$$R \subseteq A \times B = \{(1, x), (1, y), (1, z), (2, x), (2, y), (2, z), (3, x), (3, y), (3, z)\}.$$

The set of all the first components of the ordered pairs is called the **domain** of the relation and the set of all the second components of the ordered pairs is called the **range** of the relation.

The domain of R in example 2.2.2 is $\{1,3\}$ and the range is $\{y, z\}$.

NOTE: The domain of R is a subset of A and the range of R is the subset of B.

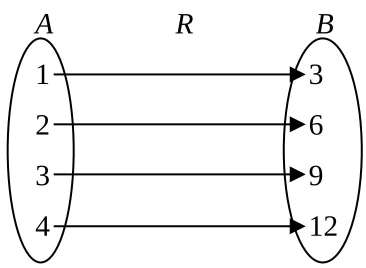
Relations can be defined by an equation or a rule or a table or an arrow diagram.

Example 2.1.3 Let the relation $R: A \to B$ be defined by

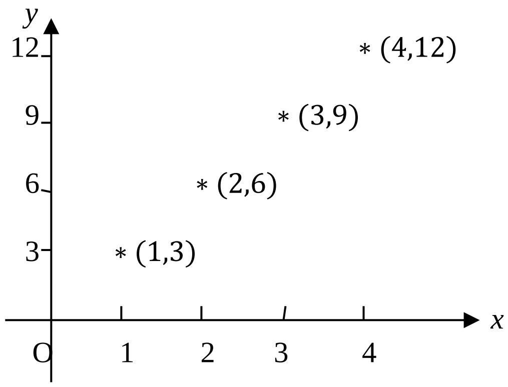
$$R = \{(1,3), (2,6), (3,9), (4,12)\}.$$

Then R can be defined by $y = 3x$ where $x \in A = \{1,2,3,4\}$ and $y \in B = \{3,6,9,12\}$.

It can also be defined using an arrow diagram



The relation can also be defined using the Cartesian coordinate system.

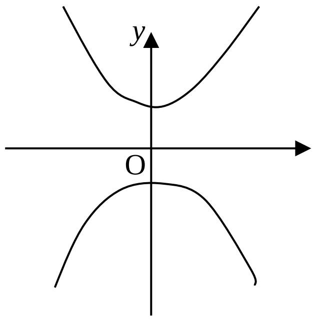


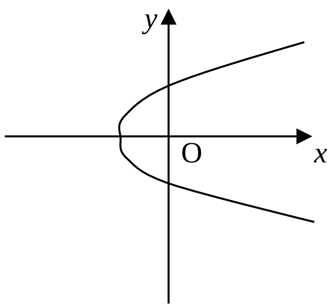
The domain of R is the set $\{1,2,3,4\}$ and its range is $\{3,6,9,12\}$.

Example 2.1.4 Find the domain and range of each relation whose defining rule and graph is given below:

(a) $\frac{y^2}{16} - \frac{x^2}{9} = 1$

(b) $x = y^2 - 3$





Solution: (a) The domain of R is $\mathbb{R}$ and the range is $(-\infty, -4] \cup [4, \infty)$

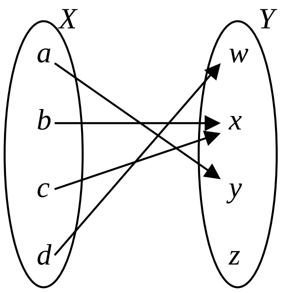
(b) The domain of $\mathbb{R}$ is $[-3, \infty)$ and the range is $\mathbb{R}$.

Functions

Definition 2.1.3 Let X and Y be two sets. Then a **function** f from X into Y is a rule that assigns each element $x \in X$ to unique (one and only one) element $y \in Y$. The notation for the function is $f: X \to Y$. This is read as f maps X into Y.

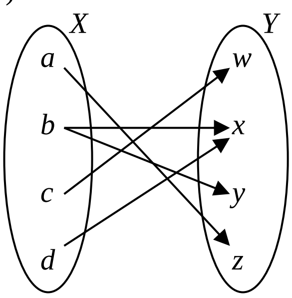
Example 2.5 Let $X = \{a, b, c, d\}$and $B = \{w, x, y, z\}$. Then the relation defined by the arrow diagram

(a)



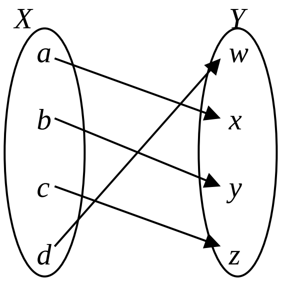
is a function since each element in X is related to only one element in Y

(b)



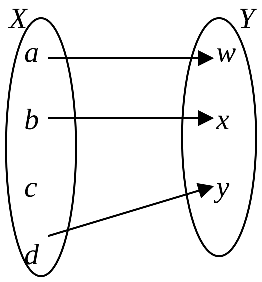
is not a function since there is an element b in X which is related to more than one element in Y.

(c)



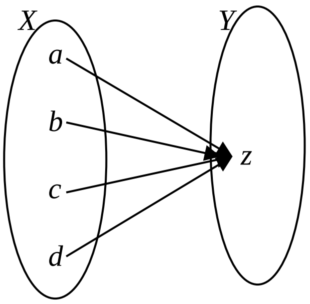
is a function since each element in X is related to only one element in Y

(d)



is not a function since there is an element c in X which is not related to any element in Y.

(e)



is a function since each element in X is related to only one element in Y.

For the function f the unique element $y \in Y$ related to $x \in X$ is called the **image** of x and it is written $f(x)$. The set of images is called the **range** of (or **image)** of f and is denoted by Ran(f) (or Im(f)). The **domain** of f is X. The elements of the domain corresponding to the images are called the **pre-images**. If X and Y are sets of real numbers, $f(x) \in \mathbb{R}$ and is the value of the function f at x.

NOTE: 1. To every function $f: X \to Y$ there corresponds the relation $\{(x, f(x)): x \in X\}$ in $A \times B$ i.e. $\{(x, f(x)): x \in X\} \subseteq A \times B$.

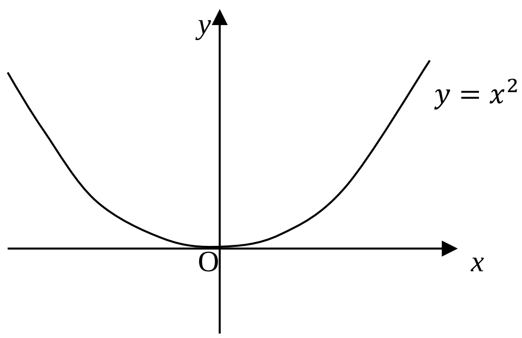
2. $f: X \to Y$ is a function if each $x \in X$ appears as the first coordinate in exactly one ordered pair (x, y) in f.

3. The range of f is denoted by $f(X)$ and is equal to $f(X) = \{f(x): x \in X\}$.

<u>Example</u> 2.1.6 Let $f: \mathbb{R} \to \mathbb{R}$ be a function which relates to each real number its square.

1. This function can be presented as an equation as: For each $\in \mathbb{R}, f(x) = x^2$. i.e. $\{(x, x^2): x \in \mathbb{R}\}$. It is said to be a real valued function.

2. The function $f(x) = x^2$ can also be represented as a graph as follows:

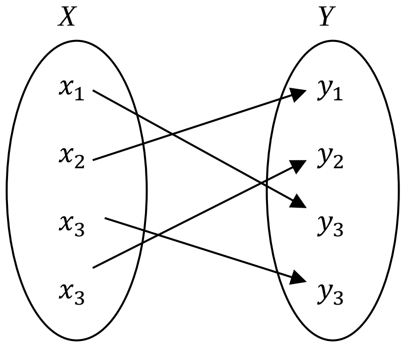


3. The domain of the function is $\mathbb{R}$ and its range is $f(\mathbb{R}) = \{x^2 : x \in \mathbb{R}\}$.

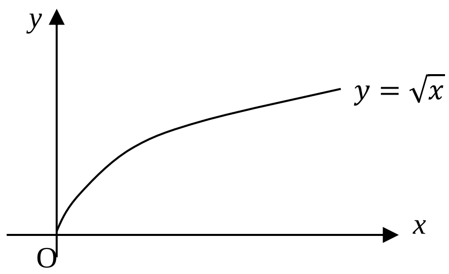
<u>**Definition**</u> **2.1.4** A function $f: X \to Y$ is said to be **one-to-one** (or one-one or 1-1) if each element in X corresponds to a distinct image in Y. i.e. f is one-to-one if

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2.$$

<u>Example</u> 2.7 (a) The function $f: X \to Y$ defined by an arrow diagram shown below is one-one since there is a one to one correspondence between elements of set X and those of set Y.



(b) The function $f: [0, \infty) \to [0, \infty)$ defined by $f(x) = \sqrt{x}$ is one-one.



<u>Example</u> 2.1.8 Prove that the function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$ is one to-to-one.

<u>Proof</u>: Let $x_1, x_2 \in \mathbb{R}$. We need to show that $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

Now,

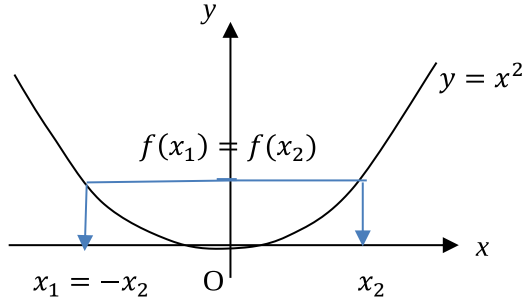
$$f(x_1) = f(x_2) \Rightarrow \sqrt{x_1} = \sqrt{x_2}$$

Squaring both sides we have

$$(\sqrt{x_1})^2 = (\sqrt{x_2})^2 \;\Rightarrow\; x_1 = x_2.$$

Hence, the function as defined is one-to-one.

Example 2.1.9 Prove that the function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$



is not one-to-one.

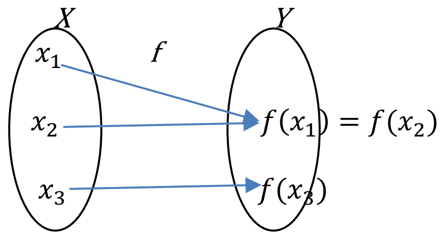
Proof: Let $x_1, x_2 \in \mathbb{R}$. Then

$$f(x_1) = f(x_2) \Rightarrow (x_1)^2 = (x_2)^2 \;\Rightarrow\; x_1 = \pm\sqrt{(x_2)^2} = \pm x_2.$$

i.e. $x_1 = +x_2$ and $x_1 = -x_2 \Rightarrow$ two different element in the domain are mapped to the same element in the range. Hence the function is not one-to-one.

Definition 2.1.5 A function $f: X \to Y$ is said to be many to one if there are at least two distinct elements $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$.

For example,



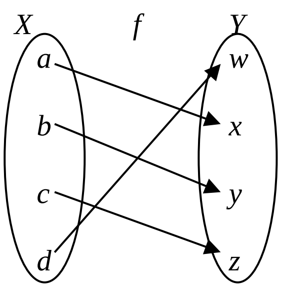
is a many to one function.

Definition 2.1.6 A function $f: X \to Y$ is said to be **onto** if every $y \in Y$ is the image of some $x \in X$, i.e. if $y \in Y \Rightarrow$ there exists $x \in X$ for which $f(x) = y$.

Note that if f is onto, then $f(X) = Y$.

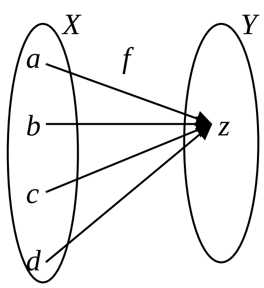
Example 2.1.10 The following functions as defined are onto functions:

(a)



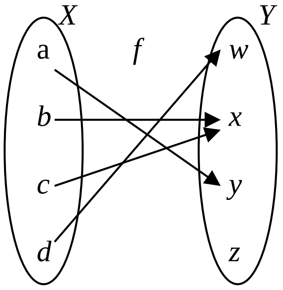
is an onto function since each element in Y is related to some element in X or $f(X) = Y$.

(b)



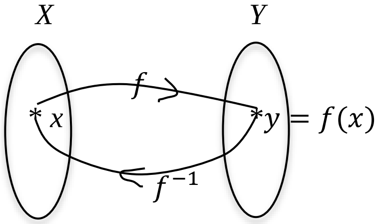
is an onto function since each element in Y is related to some element in X or $f(X) = Y$.

But the function defined below is not onto since there is an element in $z \in Y$ which is not related to any of the elements in *X*.



Inverse functions

The inverse of the function $f: X \to Y$ is the function which maps the elements of *Y* into the elements of *X* and it is denoted by $f^{-1}: Y \to X$, as shown in the arrow diagram below:



i.e. if $x \in X$, then $y = f(x) \in Y$, and for $y \in Y$, $f^{-1}(y) = f^{-1}\big(f(x)\big) = x \in X$.

NOTE: 1. A function f has an inverse function f^{-1} if and only if it is one-to-one and onto, and

2. The domain of the inverse function f^{-1} is the range of f and the range of f^{-1} is the domain of f.

To find the inverse of a given function $y = f(x)$, interchange *x* and *y* so that $x = f(y)$, and change the subject of the formula back to *y* and obtain $y = f^{-1}(x)$.

<u>Example</u> 2.1.11 Find the inverse of the function

$$f(x) = \frac{2-x}{3x+2},\ x \neq -\frac{2}{3}.$$

<u>Solution</u>: Let $y = \frac{2-x}{3x+2}$. Then interchange *x* and *y* to obtain $x = \frac{2-y}{3y+2}$.

Make *y* the subject of the formula:

$$x(3y+2) = 2 - y$$
$$3xy + 2x = 2 - y$$
$$3xy + y = 2 - 2x$$
$$y(3x+1) = 2 - 2x$$
$$y = \frac{2-2x}{3x+1}$$

Therefore, $$f^{-1}(x) = \frac{2-2x}{3x+1},\ x \neq -\frac{1}{3}.$$

NOTE: The domain of the inverse function $f^{-1}$of f in Example 2.11 is

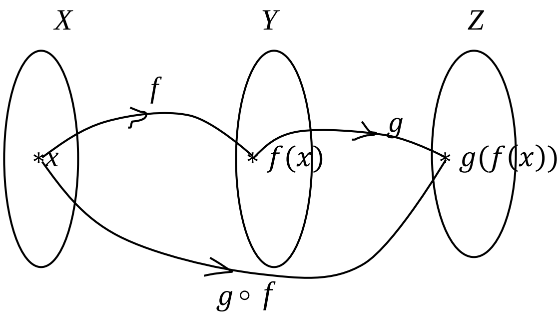
$$D_{f^{-1}} = \left\{x \in \mathbb{R}: x \neq -\frac{1}{3}\right\}.$$

Composite functions

<u>**Definition**</u> **2.7** Consider functions $f: X \to Y$ and $g: Y \to Z$ i.e. where the range of the of *f* is the domain of g. Pictorially is shown below:

$$X \xrightarrow{f} Y \xrightarrow{g} Z .$$

or



Let $x \in X$. Then the image of *x* under *f* is $f(x) \in Y$ (the domain of g). Accordingly, we can find the image of $f(x)$ under g, which is $g\big(f(x)\big) \in Z$. Thus the rule which assigns each element *x* in *X* an element $g(f(x))$ in *Z* is called **the composition function** of *f* and g, and it is denoted by $g \circ f$. Briefly, $g \circ f : X \to Z$ and it is defined by

$$(g \circ f)(x) = g[f(x)]$$

The function $f \circ g$ is defined by

$$(f \circ g)(x) = f[g(x)]$$

<u>Example</u> 2.1.12 Let the function *f* be defined by $f(x) = 3x - 5$ and the function g by $g(x) = x^2$. Find (a) $(g \circ f)(x)$ (b) $(g \circ f)(-2)$ (c) $(f \circ g)(x)$ (d) $(f \circ g)(-2)$.

<u>Solutions</u>:

(a) $(gof)(x) = g[f(x)] = g(3x - 5) = (3x - 5)^2$

(b) $(gof)(-2) = (3(-2) - 5)^2 = (-11)^2 = 121$

(c) $(fog)(x) = f[g(x)] = f(x^2) = 3(x^2) - 5 = 3x^2 - 5$

(d) $(fog)(-2) = 3(-2)^2 - 5 = 12 - 5 = 7$

Note that $(g \circ f)(x) \neq (f \circ g)(x)$ i.e. the composition of function is not commutative.

<u>Exercise</u> Let $f(x) = 3x - 4$. Show that

(a) $(f \circ f^{-1})(x) = x$

(b) $(f^{-1} \circ f)(x) = x$

In general for all functions *f*, $(f \circ f^{-1})(x) = x = (f^{-1} \circ f)(x) = x$.

The composition of functions can be extended to a composite of more than two functions.

For example, if $f: W \to X$, $g: X \to Y$ and $h: Y \to Z$, then $h \circ (g \circ f)$ is defined by

$$h \circ (g \circ f)(x) = h\{g[f(x)]\}.$$

Example 2.1.13 Let the function f be defined by $f(x) = 5 - 3x$, the function g by $g(x) = x + 2$ and h by $h(x) = 2x^2$. Find (a) $[h \circ (g \circ f)](x)$ (b) $[(h \circ g) \circ f](x)$.

Solutions:

(a) $(gof)(x) = g[f(x)] = (5 - 3x) + 2 = 7 - 3x$

$[ho(gof)](x) = h[(gof)(x)] = h(7 - 3x) = 2(7 - 3x)^2$

(b) $(hog)(x) = h[g(x)] = h(x + 2) = 2(x + 2)^2$

$[(hog)of](x) = (hog)[f(x)] = (hog)[5 - 3x] = [2(5 - 3x + 2)^2]$

$= 2(7 - 3x)^2.$

Note that $h \circ (g \circ f)(x) = [(h \circ g) \circ f](x)$.

Domain of a composite function

Example 2.1.14 Let $f(x) = \frac{x-3}{x}$ and $g(x) = x + \frac{x-4}{x-1}$. Find the domain of the following composite functions:

(i) fog (ii) gof.

Solution: (i)

$$(fog)(x) = f[g(x)] = f\left(x + \frac{x-4}{x-1}\right) = \frac{\left(x + \frac{x-4}{x-1}\right) - 3}{x + \frac{x-4}{x-1}}$$

$$= \frac{x^2 - x + x - 4 - 3(x-1)}{x^2 - x + x - 4}$$

$$= \frac{x^2 - 3x - 1}{x^2 - 4}$$

$$= \frac{x^2 - 4x + 2}{(x+2)(x-2)}$$

Now, fog is not defined at $x = -2$, $x = 2$ and $g(x)$ is not defined at $x = 1$. This means that the domain of fog is $\{x \in \mathbb{R}: x \neq -2, 1, 2\}$.

(ii) $$(gof)(x) = g[f(x)] = g\left(\frac{x-3}{x}\right) = \frac{x-3}{x} + \frac{\left(\frac{x-3}{x}\right) - 4}{\left(\frac{x-3}{x}\right) - 1}$$

$$= \frac{x-3}{x} + \frac{\frac{x-3-4x}{x}}{\frac{x-3-x}{x}} = \frac{x-3}{x} + \frac{3x+3}{3}$$

$$= \frac{x^2+2x-3}{x}$$

Now, gof is not defined at $x=0$, and $f(x)$ is also not defined at $x = 0$. Therefore, the domain of gof is $\{x \in \mathbb{R}: x \neq 0\}$.

Definition 2.1.8 A **peicewise** function is a function defined by at least two equations each of which applies to a different part of the domain.

Piecewise defined functions can take on a variety of forms. Their pieces may be all linear or a combination of functional forms (such as constant, linear, quadratic, cubic, square roots, cube roots etc.).

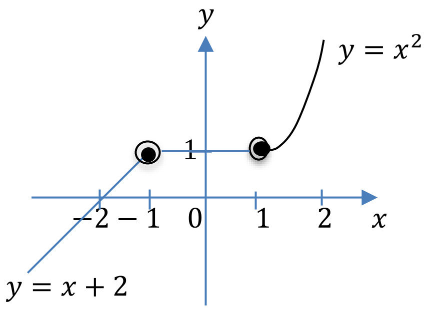
Example 2.1.15 For each of the following functions sketch and find

(a) $f(-1)$ (b) $f(0)$ (c) $f(3)$ (d) its domain and range:

1. $f(x) = \begin{cases} x+2, & x \leq -1 \\ 1, & -1 < x < 1 \\ x^2, & x \geq 1 \end{cases}$

2. $f(x) = \begin{cases} 0, & -5 \leq x < -2 \\ -x^2+4, & -2 \leq x \leq 1 \\ -x+3, & 2 < x \leq 5 \end{cases}$

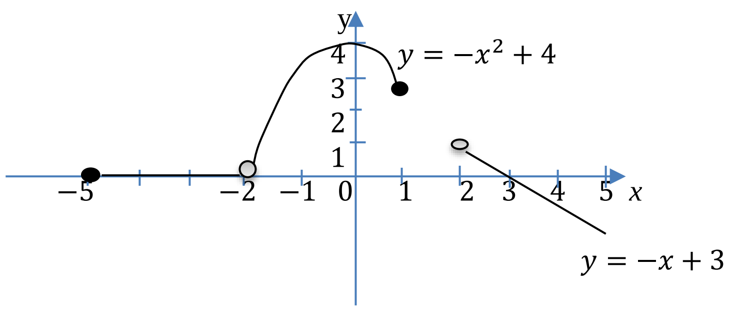
Solution: 1.



(a) $f(-1) = -1+2 = 1$ (b) $f(0) = 1$ (c) $f(3) = (3)^2 = 9$

(d) Domain of $f = \mathbb{R}$, Range of $f = \mathbb{R}$.

2.



(a) $f(-1) = -(-1)^2 + 4 = -1+4 = 3$

(b) $f(0) = -(0)^2 + 4 = 0+4 = 4$

(c) $f(3) = -3 + 3 = 0$

(d) Domain of $f = (-\infty, 1] \cup (2, \infty)$, Range of $f = (-\infty, 4]$.

Definition 2.1.9 Let $f: X \to Y$ be a function. Then f is said to be an **even** function if for each $x \in X$, $f(-x) = f(x)$.

Example 2.1.16 Show that the function $f(x) = 3x^2 - 4x^4$ is even.

Solution: $f(-x) = 3(-x)^2 - 4(-x)^4 = 3x^2 - 4x^4 = f(x)$. Since $f(-x) = f(x)$, f is an even function.

Definition 2.1.10 Let $f: X \to Y$ be a function. Then f is said to be an **odd** function if for each $x \in X$, $f(-x) = -f(x)$.

Example 2.1.17 Show that the function $f(x) = 6x^3 - 5x$ is odd.

Solution: $f(-x) = 6(-x)^3 - 5(-x) = -6x^3 + 5x = -(6x^3 - 5x) = -f(x)$. Since $f(-x) = -f(x)$, f is an odd function.

However, a function which is not even may not necessarily be odd and a function which is not odd may not necessarily be even. Some functions are neither even nor odd.

Example 2.18 Determine whether the function $f(x) = x^3 - 7x^2$ is even or odd or neither even nor odd.

Solution: $f(-x) = (-x)^3 - 7(-x)^2 = -x^3 + 7x^2 = -(x^3 - 7x^2) \neq -f(x)\ or\ f(x)$. Therefore, f is neither even nor odd.

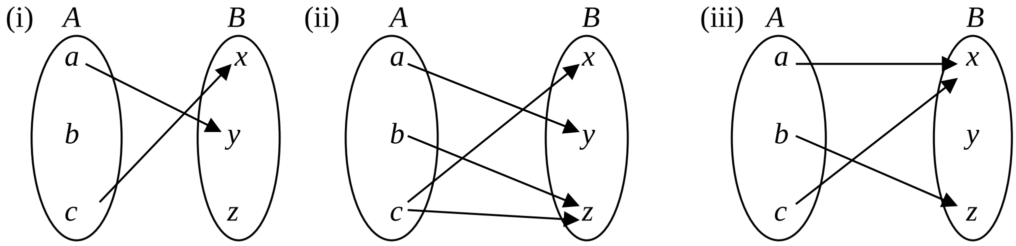
TUTORIAL SHEET 3

1. Which of the following sets of ordered pairs represent a functions:

(a) $\{(1,4),(3,4),(7,3)\}$ (b) $\{(1,2),(1,3),(2,3)\}$ (c) $\{(4,3),(4,7),(3,4)\}$

(d) $\{(1,2),(2,3),(3,4)\}$

2. State whether or not each of the diagrams defines a function from $A = \{a,b,c\}$ into $B = \{x,y,z\}$.



3. Let $X = \{1,2,3,4\}$ and $Y = \{1,2,3,4\}$. Illustrate each of the following in an arrow diagram and state whether or not each relation from *X* into *Y* is a function:

(a) $f = \{(2,3),(1,4),(2,1),(3,2),(4,4)\}$ (b) $g = \{(3,1),(4,2),(1,1)\}$

(c) $h = \{(2,1),(3,4),(1,4),(2,1),(4,4)\}$.

4. Verify that the two given functions are inverses of each other.

(a) $f(x) = 5x - 9, \quad and \quad g(x) = \dfrac{x+9}{5}$

(b) $f(x) = x^3 + 1 \quad and \quad g(x) = \sqrt[3]{x-1}$

(c) $f(x) = \dfrac{1}{x-1} \; for \; x > 1 \quad and \quad g(x) = \dfrac{x+1}{x} \; for \; x > 0$.

5. Let *f* and *g* be two functions. Find $(f \circ g)(x)$ and $(g \circ f)(x)$. Also specify the domain for each.

(a) $f(x) = 3x + 4, \; g(x) = x^2 + 1$ (b) $f(x) = 2x^2 - x - 1, \; g(x) = x + 4$

(c) $f(x) = \sqrt{x-2}, \; g(x) = 3x - 1$ (d) $f(x) = \dfrac{1}{x-1}, \; g(x) = \dfrac{2}{x}$.

6. If $f(x) = \sqrt{x}, \quad g(x) = 3x - 1$, find $(f \circ g)(4)$ and $(g \circ f)(4)$.

7. For each given function, find f^{-1} and verify that $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$: (a) $f(x) = \dfrac{2}{x-1}$ for $x > 1$ (b) $f(x) = \dfrac{1-x}{x}$

8. If $f(x) = 2x + 3$ and $g(x) = 3x - 5$, find

(a) $(f \circ g)^{-1}(x)$ (b) $(f^{-1} \circ g^{-1})(x)$ (c) $(g^{-1} \circ g^{-1})(x)$.

9. Let the function be defined by $f(x) = \begin{cases} 2 & for \quad x < 0 \\ x^2 + 1 & for \; 0 \le x \le 4 \\ -1 & for \quad x > 4 \end{cases}$.

Compute $f(3), f(6)$ and $f(-3)$ and sketch the graph of the function.

10. Determine which of the following function are even or odd or neither even nor odd.

(a) $f(x) = 4x - 7x^3$ (b) $f(x) = 3 + 5x - x^2$ (c) $f(x) = 5x^2 - 2x^4$

11. Prove that each of the following functions is one – to – one:

(a) $f(x) = 3 - 4x$ (b) $f(x) = \frac{x+2}{3x}$.

2.2 Linear and Quadratic Functions

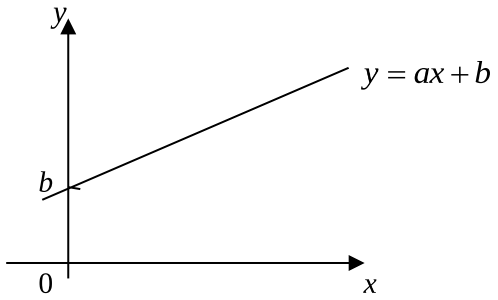
Linear Functions

<u>Definition</u> 2.2.1 A function is of the form

$$f(x) = ax + b$$

where *a* and *b* are constants is called a *linear function*.

The graph of a linear function is simply a straight line.



In the Cartesian plane, the constant *a* is the *gradient* or *slope* of the straight line and *b* is the *y*- intercept.

Quadratic Function

<u>Definition</u> 2.2.2 A quadratic function is of the form

$$f(x) = ax^2 + bx + c\ ,$$

where *a*, *b* and *c* are constants and $a \neq 0$.

Note that when $a = 0$, the function becomes a linear function.

A quadratic function can also be expressed in the form

$$f(x) = a(x + p)^2 + q\ ,$$

where *a*, *p* and *q* are constants. This is done by ***completing the square***.

<u>Example</u> 2.2.1 $f(x) = ax^2 + bx + c$

$$= a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right]$$, by factoring out the coefficient of x^2.

Dividing the coefficient of *x* by 2 and squaring the result we write the expression in the form

$$f(x) = a\left[x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right]$$

Now the expression $x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(x + \frac{b}{2a}\right)^2$, is a perfect square. Therefore

$$f(x) = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right]$$

$$= a\left[\left(x+\frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2}\right]$$

$$= a\left(x+\frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a},$$

in which $p = \frac{b}{2a}$ and $q = \frac{4ac-b^2}{4a}$.

Example 2.2.2 Complete the square of each of the quadratic functions:

(a) $f(x) = 2x^2 - 4x + 5 = 2\left[x^2 - 2x + \frac{5}{2}\right] = 2\left[x^2 - 2x + (-1)^2 - (-1)^2 + \frac{5}{2}\right]$
$= 2\left[x^2 - 2x + 1 - 1 + \frac{5}{2}\right] = 2\left[(x-1)^2 - 1 + \frac{5}{2}\right] = 2\left[(x-1)^2 + \frac{3}{2}\right]$
$= 2(x-1)^2 + 3$

(b) $f(x) = 3 - 5x - x^2 = -x^2 - 5x + 3 = -[x^2 + 5x - 3] = -\left[x^2 + 5x + \left(\frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2 - 3\right]$

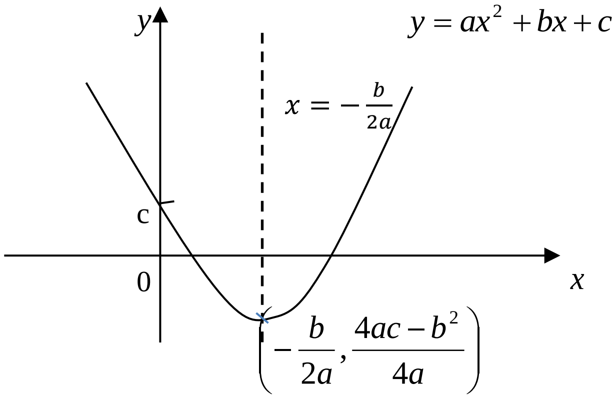
$$= -\left[\left(x+\frac{5}{2}\right)^2 - \frac{25}{4} - 3\right] = -\left[\left(x+\frac{5}{2}\right)^2 - \frac{37}{4}\right] = -\left(x+\frac{5}{2}\right)^2 + \frac{37}{4}$$

Graph of a Quadratic Function

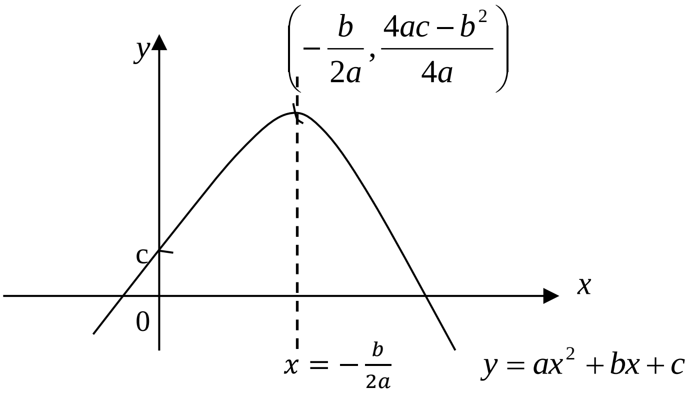
We consider an arbitrary function

$$f(x) = ax^2 + bx + c = a\left(x+\frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a}.$$

(a) If $a > 0$, the graph of the quadratic function opens upward and has a minimum turning point $\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$.



(b) If $a < 0$, the graph of the quadratic function opens downward and has a maximum turning point $\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$.



Note that in both cases, the turning point is given by $\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$ and the y - intercept is c.

(c) The equation of the line of symmetry of the graph of a quadratic function is $x = -\frac{b}{2a}$.

If the graph of the quadratic function cuts the x – axis, the x - intercepts are found by solving the quadratic equation $f(x) = 0$ i.e.

$$a\left(x+\frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a} = 0 .$$

Thus, $$a\left(x+\frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a}$$

$\Rightarrow$ $$\left(x+\frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a^2}$$

$\Rightarrow$ $$x+\frac{b}{2a} = \pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a} = \frac{-b\pm\sqrt{b^2-4ac}}{2a} .$$

This is the ***quadratic formula*** used in finding the solutions or roots of a quadratic equation

$$ax^2 + bx + c = 0.$$

One x - intercept is $x = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and the other is $x = \frac{-b-\sqrt{b^2-4ac}}{2a}$.

Example 2.2.3 Complete the square of each of the following quadratic functions. Hence sketch its graph indicating the turning point and the intercepts, and write down the equation of its line of symmetry.

1. $f(x) = 2x^2 + x - 10$ 2. $f(x) = 3 + 5x^2 - 2x^2$.

Solutions:

1. $f(x) = 2x^2 + x - 10 = 2\left(x^2 + \frac{1}{2}x - 5\right) = 2\left(x^2 + \frac{1}{2}x + \left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^2 - 5\right)$

$$= 2\left(\left(x + \tfrac{1}{4}\right)^2 - \tfrac{1}{16} - 5\right) = 2\left(\left(x + \tfrac{1}{4}\right)^2 - \tfrac{81}{16}\right) = 2\left(x + \tfrac{1}{4}\right)^2 - \tfrac{81}{8}.$$

Since $a > 0$, the function has a *minimum turning point* and it occurs at point $\left(-\frac{1}{4}, -\frac{81}{8}\right)$.

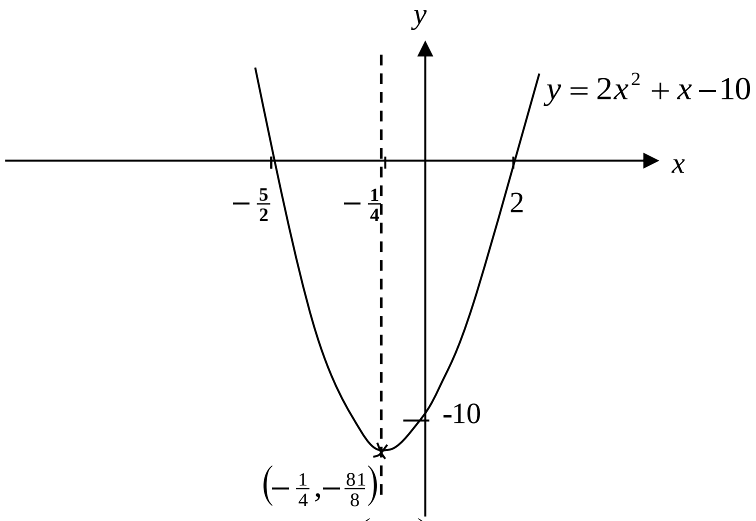
The *x*-intercepts are

$$x = \frac{-1 + \sqrt{1^2 - 4(2)(-10)}}{2(2)} \text{ and } x = \frac{-1 - \sqrt{1^2 - 4(2)(-10)}}{2(2)}$$

i.e. $x = \frac{-1 + \sqrt{81}}{4}$ and $x = \frac{-1 - \sqrt{81}}{4}$

i.e. $x = \frac{8}{4} = 2$ and $x = \frac{-10}{4} = \frac{-5}{2}$.

The *y* – intercept is the term independent of *x* in the quadratic equation, which in this case is -10.



The minimum value of the function is $f\left(-\frac{1}{4}\right) = -\frac{81}{8}$ and the line of symmetry is $x = -\frac{1}{4}$.

2. $f(x) = 3 + 5x - 2x^2 = -2\left(x^2 - \frac{5}{2}x - \frac{3}{2}\right) = -2\left(x^2 - \frac{5}{2}x + \left(-\frac{5}{4}\right)^2 - \left(-\frac{5}{4}\right)^2 - \frac{3}{2}\right)$

$$= -2\left(\left(x - \tfrac{5}{4}\right)^2 - \tfrac{25}{16} - \tfrac{3}{2}\right) = -2\left(\left(x - \tfrac{5}{4}\right)^2 - \tfrac{49}{16}\right) = -2\left(x - \tfrac{5}{4}\right)^2 + \tfrac{49}{8}.$$

The *x*-intercepts are

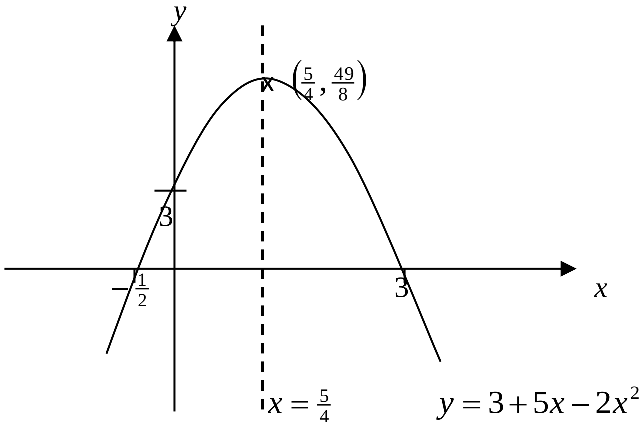
$$x = \frac{-5 + \sqrt{5^2 - 4(-2)(3)}}{2(-2)} \text{ and } x = \frac{-5 - \sqrt{1^2 - 4(-2)(3)}}{2(-2)}$$

i.e. $x = \frac{-5+\sqrt{49}}{-4}$ and $x = \frac{-5-\sqrt{49}}{-4}$

i.e. $x = \frac{2}{4} = -\frac{1}{2}$ and $x = \frac{-12}{4} = 3$.

Since $a < 0$, the function has a *maximum turning point* and it occurs at $\left(\frac{5}{4}, \frac{49}{8}\right)$.

The *y* – intercept is 3.



The maximum value of the function is $f\left(\frac{5}{4}\right) = \frac{49}{8}$ and the line of symmetry is $x = \frac{5}{4}$.

Applications of Quadratic Functions.

One method of solving a maximum or minimum problems which can be transformed into a quadratic function is the use of completion of the square.

Example 2.2.4. If the selling price *x* of an item is related to the profit *P* by the equation

$$P = 1000x - 25x^2$$

Determine the value of *x* that would yield maximum profit and state the maximum profit.

Solution: To find the value of *x* that that would yield maximum profit we have to use the method of completing the square.

$$P = 1000x - 25x^2 = -25\left(x^2 - 40x\right)$$
$$= -25\left(x^2 - 40x + (-20)^2 - (-20)^2\right)$$
$$= -25\left(x^2 - 40x + (-20)^2 - 400\right)$$
$$= -25\left((x-20)^2 - 400\right)$$
$$= -25(x-20)^2 + 10000$$

The maximum profit is attained when $x = 20$ and the maximum profit is $10\,000$.

2. A farmer wishes to enclose a rectangular lot of maximum area with a fence 400 m long. Find the dimensions of the rectangle and state its maximum area.

Solution: Suppose the length of the rectangle is *x* and the width is *y*. Then the perimeter of the rectangle is

$$2x + 2y = 400$$

$$\Rightarrow \quad x + y = 200 \ \Rightarrow y = 200 - x$$

The area of the rectangle is

$$A = xy$$

$$\Rightarrow \quad A = x(200 - x) = 200x - x^2$$

This is a quadratic function

$$\begin{aligned} A(x) &= 200x - x^2 \\ &= -(x^2 - 200x) \\ &= -(x^2 - 200x + (-100)^2 - (-100)^2) \\ &= -((x-100)^2 - 10000) \\ &= -(x-100)^2 + 10000 \end{aligned}$$

This means that the maximum area of the rectangle is attained at $x = 100$.
Therefore, the dimensions of the rectangle are $length = 100\ m\ and\ width = 100\ m$ and hence, the maximum area is $10\,000\,m^2$.

2.3 Polynomail Functions

Let *n* be a nonnegative integer and let $a_0, a_1, a_2, \cdots, a_n$ be real numbers with $a_n \neq 0$, then the function defined by

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

is called a *polynomial function* of degree *n* (or simply a *polynomial*). The numbers $a_0, a_1, a_2, \cdots, a_n$ are called the *coefficients* and a_n the leading coefficient of the function p.

We have already encountered some special polynomials like the linear function $p(x) = a_1 x + a_0$, the quadratic function $p(x) = a_2 x^2 + a_1 x + a_0$. The constant function is defined by $p(x) = a_0$. A constant function is of the degree 0, a linear function is of degree 1, and a quadratic function is of degree 2.

Note that no degree is assigned to a zero function $p(x) = 0$.

Polynomials may be added, subtracted, multiplied, or divided. Thus if *p* and *q* are polynomials of degree *m* and *n* respectively, then

(i) $p \pm q$ is a polynomial of degree less than or equal to the maximum of *m* and *n*.

(ii) $p \cdot q$ is a polynomial of degree $m + n$.

Example 2.3.1 Let $p(x) = x^3 - 3x^2 + 5$ and $q(x) = x^3 + 2x^2 - x + 3$. Then

(i) $p(x) + q(x) = \left(x^3 - 3x^2 + 5\right) + \left(x^3 + 2x^2 - x + 3\right)$

$$= (x^3 + x^3) + (-3x^2 + 2x^2) + (-x) + (5 + 3)$$

$$= 2x^3 - x^2 - x + 8,$$

a polynomial of degree 3.

(ii) $p(x) - q(x) = (x^3 - 3x^2 + 5) - (x^3 + 2x^2 - x + 3)$

$$= (x^3 - x^3) + (-3x^2 - 2x^2) + (0x - (-x)) + (5 - 3)$$

$$= -5x^2 + x + 2,$$

a polynomial of degree 2.

(iii) $p(x) \cdot q(x) = \left(x^3 - 3x^2 + 5\right) \cdot \left(x^3 + 2x^2 - x + 3\right)$

$$= x^3\left(x^3 + 2x^2 - x + 3\right) - 3x^2\left(x^3 + 2x^2 - x + 3\right) + 5\left(x^3 + 2x^2 - x + 3\right)$$

$$= x^6 + 2x^5 - x^4 + 3x^3 - 3x^5 - 6x^4 + 3x^3 - 9x^2 + 5x^3 + 10x^2 - 5x + 15$$

$$= x^6 + (2x^5 - 3x^5) + (-x^4 - 6x^4) + (3x^3 + 5x^3) + (-9x^2 + 10x^2) + (-5x) + 15$$

$$= x^6 - x^5 - 7x^4 + 11x^3 + x^2 - 5x + 15,$$

a polynomial of degree 6.

The concept of division involving polynomials is quite similar to that of integers. Thus, if *p* and *h* are polynomials, then *p* is divisible by *h* if and only if there is a polynomial *q* such that

$$\frac{p}{h} = q\ .$$

i.e.

$$\frac{p(x)}{h(x)} = q(x)$$

or

$$p(x) = q(x)h(x)\ .$$

<u>Example</u> 2.3.2 Let $p(x) = x^3 - 3x^2 + 5x - 6$ and $h(x) = x - 2$ be two polynomials. Then p is divisible by *h* if and only if there exist a polynomial $q(x) = x^2 - x + 3$ such that

$$\frac{p(x)}{h(x)} = q(x)$$

i.e.

$$\frac{x^3 - 3x^2 + 5x - 6}{x - 2} = x^2 - x + 3\ .$$

<u>Theorem</u> 2.3.1 If *p* and *h* are polynomials and *h* is of degree greater than zero, then there exists unique polynomials *q* and *r* such that

$$\frac{p(x)}{h(x)} = q(x) + \frac{r(x)}{h(x)},$$

or

$$p(x) = q(x)h(x) + r(x),$$

where *r* is either a polynomial of degree less than the degree of *h* or the zero function.

The polynomial *p* is called the *dividend*, *h* is the *divisor*, *q* is the *quotient*, and *r* is the *remainder*.

Long Division of Polynomials

Examples 2.3.3

1. Divide $2x^4 + 4x^3 - 5x^2 + 3x - 2$ by $x^2 + 2x - 3$

2. Divide $12x^3 - 6x^2 + 10$ by $2x + 1$

Solutions: 1.

$$
\begin{array}{r|l}
 & 2x^2 \qquad\qquad +1 \\
\hline
x^2+2x-3 & 2x^4+4x^3-5x^2+3x-2 \\
 & -(2x^4+4x^3-6x^2) \\
\cline{2-2}
 & \qquad\qquad\quad x^2+3x-2 \\
 & \qquad\quad -(x^2+2x-3) \\
\cline{2-2}
 & \qquad\qquad\qquad\quad x+1
\end{array}
$$

Therefore,

$$\frac{2x^4 + 4x^3 - 5x^2 + 3x - 2}{x^2 + 2x - 3} = 2x^2 + 1 + \frac{x+1}{x^2 + 2x - 3}$$

The quotient $q(x) = 2x^2 + 1$ and the remainder $r(x) = x + 1$.

2.

$$
\begin{array}{r|l}
 & 6x^2 - 6x + 3 \\
\hline
2x+1 & 12x^3 - 6x^2 + 0x + 10 \\
 & -(12x^3 + 6x^2) \\
\cline{2-2}
 & \qquad -12x^2 + 0x \\
 & \quad -(-12x^2 - 6x) \\
\cline{2-2}
 & \qquad\qquad\quad 6x + 10 \\
 & \qquad\qquad -(6x + 3) \\
\cline{2-2}
 & \qquad\qquad\qquad\quad 7
\end{array}
$$

Therefore,

$$\frac{12x^3 - 6x^2 + 10}{2x + 1} = 6x^2 - 6x + 3 + \frac{7}{2x + 1}$$

The quotient $q(x) = 6x^2 - 6x + 3$ and the remainder $r(x) = 7$

Synthetic Division

There is a shortcut called synthetic division for long division of polynomials when dividing by divisors of the form $x - k$. The procedure is given below:

To divide $ax^3 + bx^2 + cx + d$ by $x - k$, use the following procedure

k	a	b	c	d
Add		ka	$kb + k^2a$	$kc + k^2b + k^3a$
	a	$b + ka$	$c + kb + k^2a$	$d + ck + k^2b + k^3a = remainder$

coefficients of quotient $q(x)$ whose degree is 1 less than that of that of the dividend.

Vertical pattern: Add terms in columns
Diagonal pattern: Multiply results by k

Hence

$$q(x) = ax^2 + (b + ka)x + (c + kb + k^2 a)$$

and

$$r(x) = d + ck + k^2 b + k^3 a$$

Therefore,

$$\frac{ax^3 + bx^2 + cx + d}{x - k} = ax^2 + (b + ka)x + (c + kb + k^2 a) + \frac{d + ck + k^2 b + k^3 a}{x - k}$$

Example 2.3.4 Use synthetic division to divide each of the following polynomials:

1. $2x^3 - 3x^2 + 4x + 5$ by $x - 2$
2. $x^4 + 3x^3 + x^2 - 2x - 6$ by $x + 3$

Solutions: 1.

2	2	-3	4	5
Add	↓	4	2	12
	2	1	6	$17 = r$

$q(x) = 2x^2 + x + 6$
$r(x) = 17$

Therefore,

$$\frac{2x^3 - 3x^2 + 4x + 5}{x - 2} = 2x^2 + x + 6 + \frac{17}{x - 2}$$

2.

-3	1	3	1	-2	-6
Add		-3	0	-3	15
	1	0	1	-5	$9 = r$

Here the quotient is

$q(x) = x^3 + 0x^2 + x - 5$ i.e.

$q(x) = x^3 + x - 5$

$r(x) = 9$

Therefore,

$$\frac{x^4 + 3x^3 + x^2 - 2x - 6}{x + 3} = x^3 + x - 5 + \frac{9}{x + 3}$$

We have noted that when a polynomial $p(x)$ of degree n is divided by $(x - k)$ then there exists another polynomial $q(x)$ of degree $n - 1$ such that

$$p(x) = q(x)(x - k) + r,$$

for all x, where r is the remainder.

Now note that

$$p(k) = q(k)(k-k) + r = r \ ,$$

which is the remainder. This leads us to the remainder theorem.

Theorem 2.3.2 (Remainder theorem) If the polynomial $p(x)$ is divided by $(x-k)$ then the remainder is

$$p(k) = r \ .$$

Example 2.3.5 Use the remainder theorem to find the remainder when the polynomial $p(x)$ is divided by $(x-k)$:

1. $2x^3 - 3x^2 + 4x + 5$ by $x-2$
2. $x^4 + 3x^3 + x^2 - 2x - 6$ by $x+3$

Solutions: 1. Let $p(x) = 2x^3 - 3x^2 + 4x + 5$. Then, by the remainder theorem, the remainder is

$$p(2) = 2(2)^3 - 3(2)^2 + 4(2) + 5 = 16 - 12 + 8 + 5 = 17 = r \ ,$$

2. Let $p(x) = x^4 + 3x^3 + x^2 - 2x - 6$. Then the remainder is

$$p(-3) = (-3)^4 + 3(-3)^3 + (-3)^2 - 2(-3) - 6$$

$$= 81 - 81 + 9 + 6 - 6 = 9 = r \ .$$

When a polynomial $p(x)$ is divided by $(x-k)$ and the remainder is zero, i.e. $p(k) = 0$, we say that $p(x)$ is divisible by $(x-k)$ or $(x-k)$ is a factor of $p(x)$. This leads us to the Factor theorem:

Theorem 2.3.3 (Factor theorem) If $p(x)$ is a polynomial and k a real number such that $p(k) = 0$, then $(x-k)$ is a factor of $p(x)$.

Note: If $(x-k)$ is a factor of $p(x)$, then

$$p(x) = q(x)(x-k)$$

and the remainder $r = 0$.

Example 2.3.6 Show that $(x-k)$ is a factor of the given polynomial $p(x)$:

1. $p(x) = 2x^3 - x^2 - 4x + 3$; $x-1$
2. $p(x) = x^4 + 2x^3 - x^2 - x + 2$; $x+2$

Solutions: 1. Let $p(x) = 2x^3 - x^2 - 4x + 3$ .Then

$$p(1) = 2(1)^3 - (1)^2 + 4(1) + 3 = 2 - 1 - 4 + 3 = 0 \ .$$

By the factor theorem, $(x-1)$ if a factor of $2x^3 - x^2 - 4x + 3$.

Note: Dividing using synthetic division, we have

$$\begin{array}{r|rrrr} 1 & 2 & -1 & -4 & 3 \\ Add & & 2 & 1 & -3 \\ \hline & 2 & 1 & -3 & 0 = r \end{array}$$

$$2x^3 - x^2 - 4x + 3 = (x-1)(2x^2 + x - 3)$$

2. Let $p(x) = x^4 + 2x^3 - x^2 - x + 2$. Then

$$p(-2) = (-2)^4 + 2(-2)^3 - (-2)^2 - (-2) + 2$$
$$= 16 - 16 - 4 + 2 + 2 = 0$$

By the factor theorem, $(x+2)$ if a factor of $x^4 + 2x^3 - x^2 - x + 2$.

Dividing using synthetic division, we have

$$\begin{array}{r|rrrrr} -2 & 1 & 2 & -1 & -1 & 2 \\ Add & & -2 & 0 & 2 & -2 \\ \hline & 1 & 0 & -1 & 1 & 0 = r \end{array}$$

$$x^4 + 2x^3 - x^2 - x + 2 = (x+2)(x^3 - x + 1)$$

Zeros or Roots of a Polynomial

We have seen from the factor theorem that if *p* is a polynomial of degree $n \geq 1$ and *k* is a number, then $p(k) = 0$ implies that $x - k$ is a factor of *p*. The number *k* is called a zero (or root) of *p*. Geometrically, *k* represents the point where the graph of *p* intersects the *x* -axis.

Clearly, since a polynomial *p* of degree *n* cannot have more than *n* factors, then *p* has at most *n* zero.

For the rational zeros of a polynomial we have the following theorem:

<u>Theorem</u> 2.3.4 If $\frac{a}{b}$, a rational number in lowest terms, is a zero of the polynomial

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 ,$$

where the a_i's $(i = 0,1,2,\cdots,n)$ are integers and $a_n \neq 0$, the a is an integral factor of a_0 and b is an integral factor of a_n.

It must be noted that this theorem does not guarantee the existence of rational zeros of a polynomial. It merely enables us to identify the possible rational zeros. These are then checked using synthetic division or otherwise.

<u>Example</u> 2.3.7 Find all rational zeros of the polynomial

$$p(x) = 2x^3 + 5x^2 - 4x - 3 .$$

<u>Solution</u>: If $\frac{a}{b}$ is a rational zero of *p*, then by the theorem, *a* must be an integral factor of

3 and *b* must be an integral factor of 2. i.e.

$$a \in \{-1,-3,1,3\} \text{ and } b \in \{-1,-2,1,2\}$$

and the set of possible rational zeros of *p* is

$$k = \frac{a}{b} \in \{-3, -\tfrac{3}{2}, -1, -\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}, 3\}$$

Using synthetic division we check each of the possible candidates starting with -3:

$$\begin{array}{r|rrrr} -3 & 2 & 5 & -4 & -3 \\ Add & & -6 & 3 & 3 \\ \hline & 2 & -1 & -1 & 0=r \end{array}$$

Thus, 3 is a zero of *p*.

Next we check $-\frac{3}{2}$:

$$\begin{array}{r|rrrr} -\frac{3}{2} & 2 & 5 & -4 & -3 \\ Add & & -3 & -3 & \frac{21}{2} \\ \hline & 2 & 2 & -7 & \frac{15}{2}=r\neq 0 \end{array}$$

Thus, $-\frac{3}{2}$ is not a zero of *p*.

The remaining possible zeros can be checked the same way.

For this polynomial, the zeros are $-3, -\frac{1}{2}$ and 1.

Factoring a Polynomial

To factorize a polynomial we use the factor theorem sometimes combined with repeated division.

Example 2.3.8 Factorize the polynomial:

$$p(x) = 2x^4 - 7x^3 - 2x^2 + 13x + 6$$

Solution: Integral factors of 6 are $a \in \{-6,-3,-1,1,3,6\}$ and integral $p(x)$ factors of 2 are $b \in \{-2,-1,1,2\}$. The possible rational zeros of *p* are

$$k = \frac{a}{b} \in \{-6,-3,-\tfrac{3}{2},-1,-\tfrac{1}{2},\tfrac{1}{2},1,\tfrac{3}{2},3,6\}.$$

We determine one of the actual zeros of *p* by using synthetic division:

$$\begin{array}{r|rrrrr} -1 & 2 & -7 & -2 & 13 & 6 \\ Add & & -2 & 9 & -7 & -6 \\ \hline & 2 & -9 & 7 & 6 & 0=r \end{array}$$

$\Rightarrow$ -1 is a zero of *p*, and by Factor theorem $(x+1)$ is a factor of $p(x)$. Thus

$$p(x) = (x+1)(2x^3 - 9x^2 + 7x + 6).$$

Let $q(x) = 2x^3 - 9x^2 + 7x + 6$. Then again the integral factors of 6 are $a \in \{-6,-3,-1,1,3,6\}$ and the integral factors of 2 are $b \in \{-2,-1,1,2\}$.

The possible rational zeros of *q* are

$$k = \frac{a}{b} \in \{-6,-3,-\tfrac{3}{2},-1,-\tfrac{1}{2},\tfrac{1}{2},1,\tfrac{3}{2},3,6\}.$$

Again we determine one of the actual zeros of *q* by synthetic division:

$$\begin{array}{r|rrrr} 2 & 2 & -9 & 7 & 6 \\ Add & & 4 & -10 & -6 \\ \hline & 2 & -5 & -3 & 0 = r \end{array}$$

$\Rightarrow$ 2 is a zero of *q* and by Factor theorem $(x-2)$ is a factor of $q(x)$. Thus

$$q(x) = (x-2)(2x^2 - 5x - 3)$$
$$= (x-2)(2x+1)(x-3).$$

Therefore,

$$p(x) = (x+1)(x-2)(2x+1)(x-3) .$$

Clearly note that the other zeros of *p* are $-\frac{1}{2}$ and 3.

Therefore when we solve the polynomial equation $p(x) = 0$, i.e. say the equation

$$p(x) = (x+1)(x-2)(2x+1)(x-3) = 0 ,$$

we obtain $x = 2, x = -1, x = -\frac{1}{2}$ and $x = 3$, which are the zeros or roots of *p*. What this means is that the zeros or roots of a polynomial indicate where the value of the polynomial function is equal to zero, i.e. where the graph of the function cuts the *x* – axis. Using these *x*- intercepts and the *y* - intercept we can sketch the graph of polynomial.

Example 2.3.9 Sketch the graph of each of the following polynomial functions, indicating the points where the curve cuts the axes.

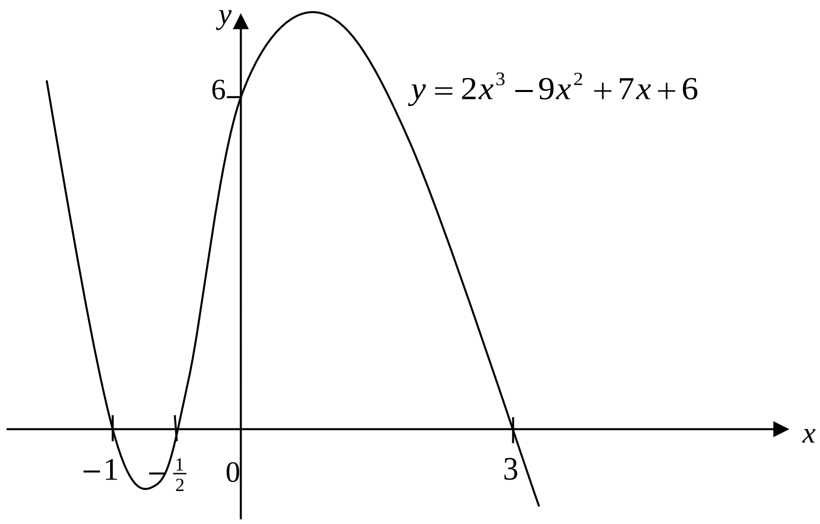
(a) $p(x) = 2x^3 - 9x^2 + 7x + 6$

(b) $p(x) = 2x^4 + 5x^3 - 5x^2 - 5x - 3$.

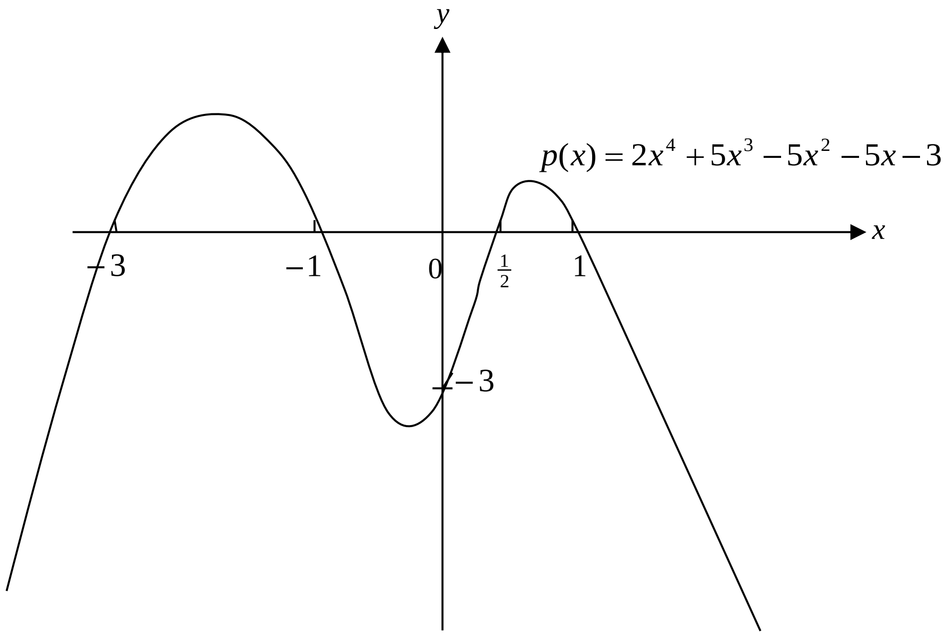
Solution: (a) The zeros of $p(x) = 2x^3 - 9x^2 + 7x + 6$ are $x = -1, x = -\frac{1}{2}, x = 3$. These are the *x* – intercepts the curve $y = 2x^3 - 9x^2 + 7x + 6$ and the *y* – intercept is 6.

Now note that a polynomial of degree 1 has no turning point, a polynomial of degree 2 has one turning point, a polynomial of degree 3 has 2 turning point, etc. This curve has 2 turning points.

Hence, we sketch the curve passing through the intercepts.



(b) The zeros of $p(x) = 2x^4 + 5x^3 - 5x^2 - 5x - 3$ are $-3, -1, \frac{1}{2}$ and 1, which are the x – intercepts of the curve. The y – intercept is $y = -3$.



We will only be able to find the exact turning points of a polynomial function of degree greater than 2 when we do differential calculus.

2.4 Rational Functions

A rational function is one that is written in the form of

$$f(x) = \frac{p(x)}{q(x)},$$

where $p(x)$ and $q(x)$ are polynomials and $q(x)$ is not the zero polynomial.

We shall assume that $p(x)$ and $q(x)$ have no common factors.

Examples of rational functions are $\frac{1}{x}$, $\frac{x}{x^2-1}$, $\frac{3x^2-4x}{2x+1}$, etc.

The **domain** of a rational function of *x* includes all real numbers except *x* – values that make the denominator zero.

Example 2.4.1

1. The function $f(x)=\frac{1}{x}$ is not defined at $x=0$ and thus the domain of the function is the set $\{x\in\mathbf{R}: x\neq 0\}$.

2. The function $f(x)=\frac{x}{x^2-1}$ is not defined at $x=\pm 1$, and thus the domain of the function is the set $\{x\in\mathbf{R}: x\neq -1,1\}$.

3. The function $f(x)=\frac{3x^2-4x}{2x+1}$ is not defined at $x=-\frac{1}{2}$ and thus the domain of the function is the set $\{x\in\mathbf{R}: x\neq -\frac{1}{2}\}$.

Recall that the range of a function is the domain of its inverse. Thus, to find the range of a rational function $y=f(x)$, we need to find the domain of the inverse function $y=f^{-1}(x)$.

Example 2.4.1 Find the range of each of the following functions:

1. $f(x)=\frac{1}{x+2}$ 2. $f(x)=\frac{x+4}{2x-1}$

Solutions: 1. We first find the inverse of the function. Let $y=\frac{1}{x+2}$ and interchange *x* and y and obtain $x=\frac{1}{y+2}$. Then we make y the subject of the formula.

$y+2=\frac{1}{x}\Rightarrow y=\frac{1}{x}-2 \; i.e. \, y=\frac{1-2x}{x}\Rightarrow f^{-1}(x)=\frac{1-2x}{x}, x\neq 0.$

Therefore, the range of the function is $\{x\in\mathbb{R}: x\neq 0\}$

2. $y=\frac{x+4}{2x-1}$ interchanging *x* and y we have $x=\frac{y+4}{2y-1}$. Making *y* the subject of the formula we have $x(2y-1)=y+4\Rightarrow 2xy-x=y+4\Rightarrow 2xy-y=x+4$

$\Rightarrow y(2x-1)=x+4\Rightarrow y=\frac{x+4}{2x-1}$. Thus $f^{-1}(x)=\frac{x+4}{2x-1}, x\neq\frac{1}{2}$.

Therefore, the range of the function is $\left\{x\in\mathbb{R}: x\neq\frac{1}{2}\right\}$

2.5 Modulus Function

A modulus function is a function of the form

$$f(x)=|g(x)|,$$

where $g(x)$ is a function.

Note: 1. When $g(x) \geq 0$, $|g(x)| = g(x)$.

2. When $g(x) < 0$, $|g(x)| = -g(x)$.

The **domain** of a modulus function is the same as that of the function $g(x)$ and its **range** is $\{y = f(x) \in \mathbf{R} : y = f(x) \geq 0\}$.

Graph of a Modulus Function

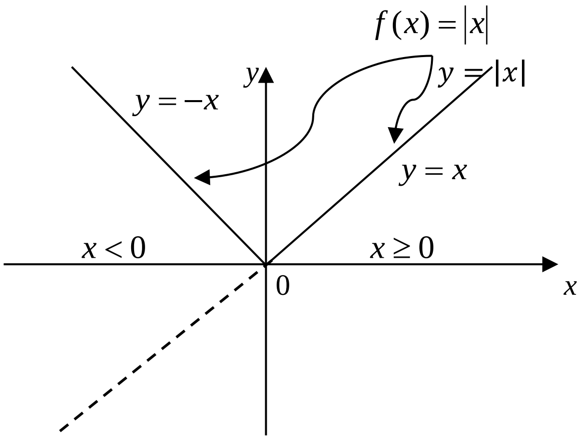
Example 2.5.1 Sketch the graph of each of the following modulus functions:

1. $f(x) = |x|$.

Solution: $y = |x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0. \end{cases}$

Step 1. Sketch the graph of $y = x$

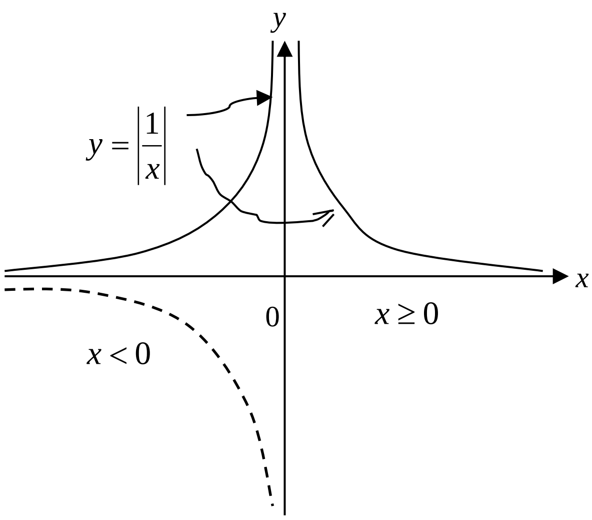
Step 2. For the part of the line below the x – axis (i.e. where $y < 0$), reflect the line in the x – axis.



Note: (a) For both $x < 0$ and $x \geq 0$, $y = |x| \geq 0$.

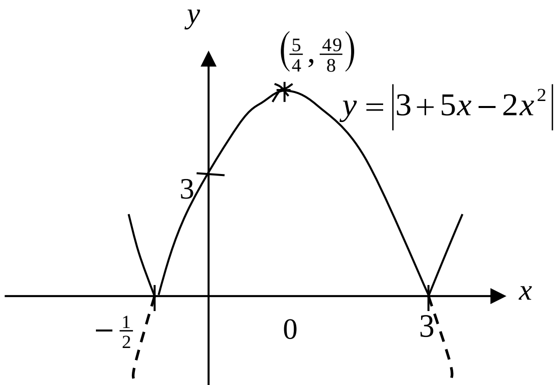
(b) Domain of $f(x) = |x|$ is $\mathbb{R}$ and its range is $\{y \in \mathbb{R}: y \geq 0\}$.

2. $f(x) = \left|\frac{1}{x}\right|$.



Domain of $f(x) = \left|\frac{1}{x}\right|$ is $\{x \in \mathbf{R} : x \neq 0\}$ and its range is $\{y \in \mathbf{R} : y > 0\}$.

3. $f(x) = |3 + 5x - 2x^2|$



Domain of $f(x) = |3 + 5x - 2x^2|$ is $\mathbf{R}$ and its range is $\{y \in \mathbf{R}: y \geq 0\}$

2.6 Radical Functions

Radical functions are functions involving roots (square roots, cube roots etc.)
For example,

$$f(x) = \sqrt{x}, \quad g(x) = \sqrt{2 - x} \quad h(x) = \sqrt[3]{x + 4} \text{ e.t.c.}$$

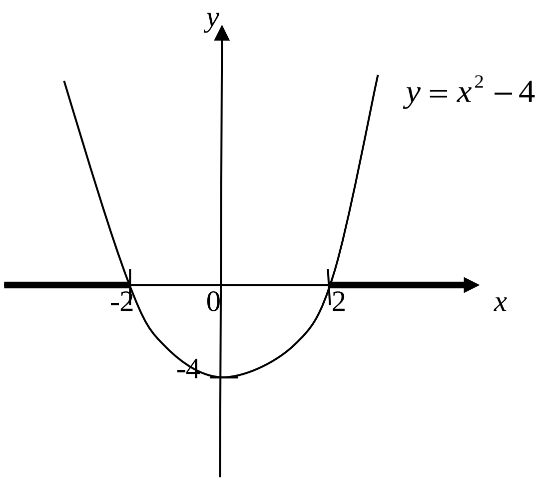
are all radical functions.

In this course we shall only consider radical functions involving the square root.

The **domain** of a radical function is the set of values of *x* for which the function is defined.

Example 2.6.1 The domain of

1. $f(x) = \sqrt{x}$ is the set $\{x \in \mathbf{R} : x \geq 0\}$.
2. $f(x) = \sqrt{2 - x}$ is the set $\{x \in \mathbf{R} : x \leq 2\}$ since for $x \leq 2$, $2 - x \geq 0$.
3. $f(x) = \sqrt{x^2 - 4}$ is the set $\{x \in \mathbf{R} : x \leq -2 \text{ or } x \geq 2\}$ since the function is defined for values of *x* for which $x^2 - 4 \geq 0$ i.e. $(x + 2)(x - 2) \geq 0$ i.e. $x \leq -2 \text{ or } x \geq 2$.



The **range** of a radical function $y = f(x)$ is the set of values *y* takes for all values of *x* within the domain of *f*.

Example 2.6.1 The range of

1. $f(x) = \sqrt{x}$ is the set $\{y = f(x) \in \mathbf{R} : x \geq 0\} = [0, \infty)$.
2. $f(x) = -\sqrt{2 - x}$ is the set $\{y = f(x) \in \mathbf{R} : x \leq 2\} = (-\infty, 0]$.
3. $f(x) = \sqrt{x^2 - 4}$ is the set $\{y = f(x) \in \mathbf{R} : x \leq -2 \text{ or } x \geq 2\} = [0, \infty)$ since the value of the function is greater or equal to zero for all $x \leq -2 \text{ or } x \geq 2$.

Graphs of a Radical Functions

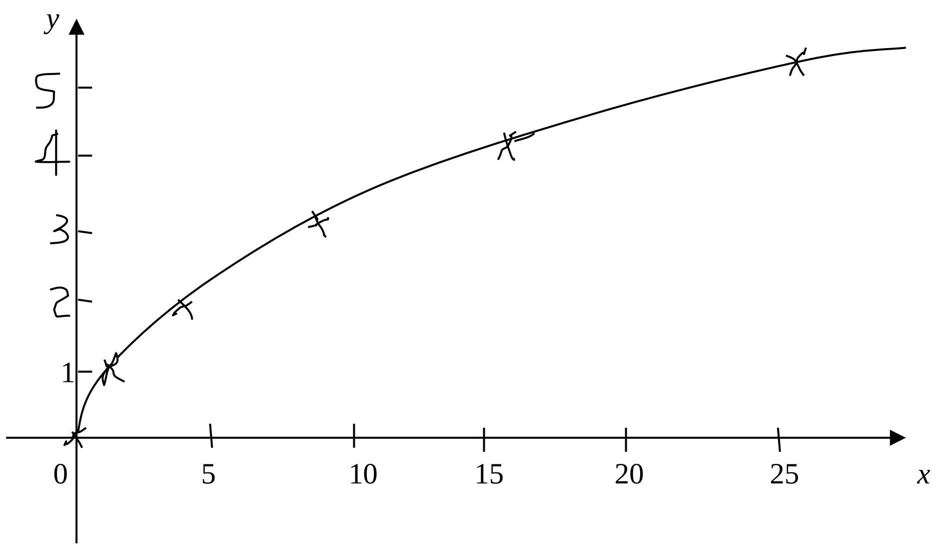
Example 2.6.2 Sketch the graph of each of the following functions:

1. $f(x) = \sqrt{x}$

Solution: Step 1: Plot the points.

x	0	1	4	9	16	25
$f(x)$	0	1	2	3	4	5

Step 2: Sketch the curve of the function passing through the plotted points.

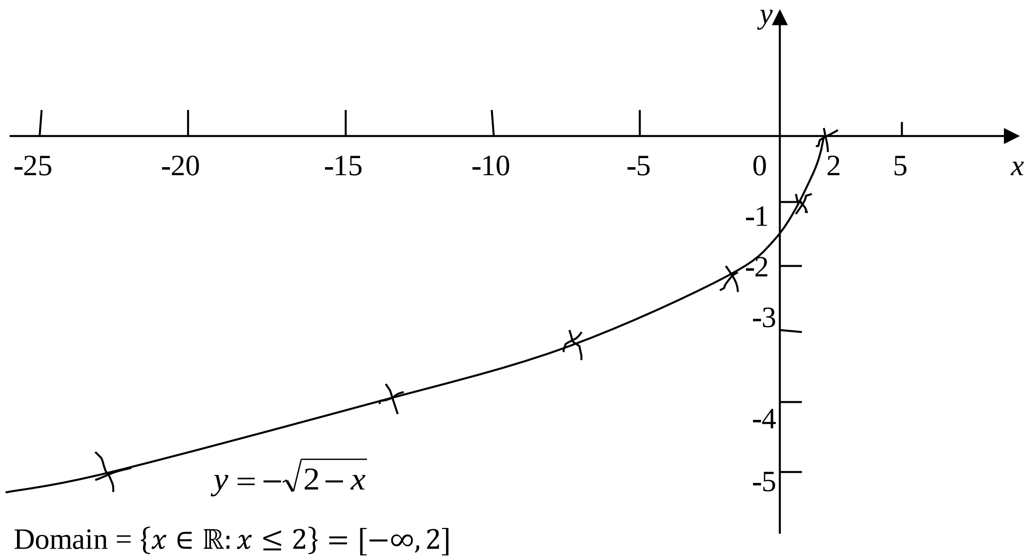


2. $f(x) = -\sqrt{2 - x}$

Solution: Step 1: Plot the points in the table.

x	-23	-14	-7	-2	1	2
$f(x)$	-5	-4	-3	-2	-1	0

Step 2: Sketch the curve of the function passing through the plotted points.



Domain = $\{x \in \mathbb{R}: x \leq 2\} = [-\infty, 2]$

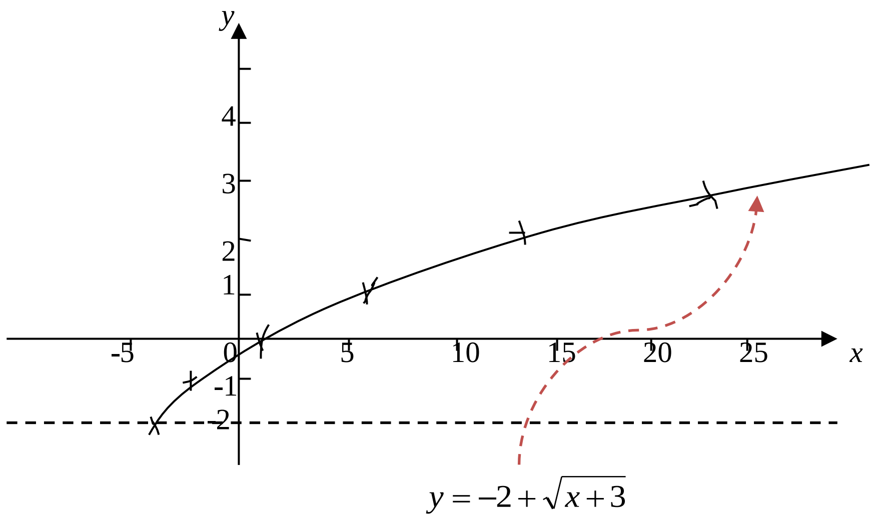
Range = $\{y \in \mathbb{R}: y \leq 0\} = [-\infty, 0]$

3. $f(x) = -2 + \sqrt{x+3}$

<u>Solution</u>: Step 1: Plot the points in the table.

x	-3	-2	1	6	13	22
$f(x)$	-2	-1	0	1	2	3

Step 2: Sketch the curve of the function passing through the plotted points.



Domain = $\{x \in \mathbb{R}: x \geq -3\} = [-3, \infty)$

Range = $\{y \in \mathbb{R}: y \geq -2\} = [-2, \infty)$

TUTORIAL SHEET 4

1. Complete the square of each of the following quadratic functions. Hence sketch the graph of the function, showing clearly the x – and y – intercepts and the turning point. State

   (i) the line of symmetry, and

   (ii) the maximum or minimum value of the function.

   (a) $f(x) = 2x^2 - 4x + 5$ (b) $f(x) = x^2 + 2x - 5$ (c) $f(x) = 4 - 3x^2$

   (d) $f(x) = 3 - 7x - 3x^2$.

2. What are the dimension of the largest rectangular field which can be enclosed by 1200 m of fencing?

3. If the profit p in the manufacture and sale of x units of a product is given by

$$p(x) = 200x - 0.001x^2,$$

   (a) Find the number x that yields the maximum profit.

   (b) Find the maximum profit if each item is sold at K2.50.

   (c) Sketch the graph of the function p.

4. A window is to be constructed in the shape of a rectangle surmounted by a semicircle. If the perimeter of the window is 540 cm, find its dimensions for maximum area.

5. Let $p(x) = 6^3 + 3x^2 - 2x - 7$ and $q(x) = x^3 + 3x^2 + x + 5$. Find

   (a) $p(x) + qx)$ (b) $p(x) - qx)$ (c) $p(x) - qx)$ (d) $p(x) \cdot qx)$

6. Use long division to divide:

Dividend	Divisor
(i) $x^3 + 8x^2 - 5x - 1$	$x - 2$
(ii) $2x^3 + 6x^2 - x + 5$	$2x^2 - 1$
(iii) $x^4 - 4x^2 + 3$	$3 + 2x - x^2$

7. Use synthetic division to divide the polynomials and write the function in the form $p(x) = (x - k)q(x) + r$, where $q(x)$ is the quotient and r is the remainder:

Dividend	Divisor
(i) $x^3 - 10x^2 + 31x - 30$	$x + 3$
(ii) $x^3 + 15x^2 + 68x + 96$	$x - 2$
(iii) $6x^3 + x^2 - 21x - 10$	$2x - 1$

8. Write the function in the form $p(x) = (x - k)q(x) + r$, where $q(x)$ is the quotient and r is the remainder:

   (i) $p(x) = x^3 + x^2 - 12x + 20, \quad k = 2$

   (ii) $p(x) = x^3 - 2x^2 - 15x + 7, \quad k = -4$

   (iii) $p(x) = x^3 + 2x^2 - 3x - 12, \quad k = \sqrt{3}$

(iv) $p(x) = 3x^3 - 19x^2 + 27x - 7, \quad k = 3 - \sqrt{2}$

9. Factorize the polynomial completely:

(i) $p(x) = x^3 - 12x - 16$ (ii) $p(x) = 3x^3 + 10x^2 - 27x - 10$

(iii) $p(x) = x^3 + 2x^2 - 3x - 6$ (iv) $p(x) = x^3 + 2x^2 - 2x - 4$

10. Given that $(x-1)$ and $(x+1)$ are factors of $px^3 + qx^2 - 3x - 7$, find the value of *p* and *q*.

11. The expression $2x^3 - ax^2 + bx + 3$ gives a remainder -15 when divided by $(x+1)$ and a remainder -46 when divided by $(x-3)$. Find the value of *a* and of *b*.

12. Find the zeros of each of the following polynomial functions. Hence sketch its graph indicating the *x* – and *y* - intercepts:

(i) $p(x) = x^3 - 2x - 7x + 12$ (ii) $p(x) = -x^3 + x^2 + 5x - 2$

(iii) $p(x) = 15 + 5x - 3x^2 - x^3$ (iv) $p(x) = x^3 + 5x^2 + 6x + 2$

13. (a) Show that $(x-2)$ is a factor of $p(x) = x^3 + x^2 - 5x - 2$.

(b) Hence, or otherwise, find the exact solutions of the equation $p(x) = 0$.

14. Sketch the graph of each of the following modulus functions:

(a) $f(x) = -|x| + 3$ (b) $f(x) = |x^3|$ (c) $f(x) = |(x+1)(2-x)|$

(d) $f(x) = |2x^2 - 7x + 3|$.

15. Sketch the graphs of the following functions and determine its domain and range:

(a) $f(x) = \sqrt{x-2}$ (b) $g(x) = -4 + \sqrt{x+3}$ (c) $h(x) = 1 + \sqrt{-x-2}$

(d) $y = -\sqrt{3x+1}$.

16. The description of body-heat loss due to convection involves a coefficient of convection K_c, which depends on wind speed *v* according to the equation:

$K_c = 4\sqrt{4v+1}$.

(a) What is the domain?

(b) What restrictions do nature and common sense put on *v*?

2.7 Equations

Quadratic Equations

Any equation of the form

$$ax^2 + bx + c = 0$$

is called a quadratic equation.

Nature of Roots of a Quadratic Equation

By completing the square of the quadratic function $f(x) = ax^2 + bx + c$ and equating to zero we have the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

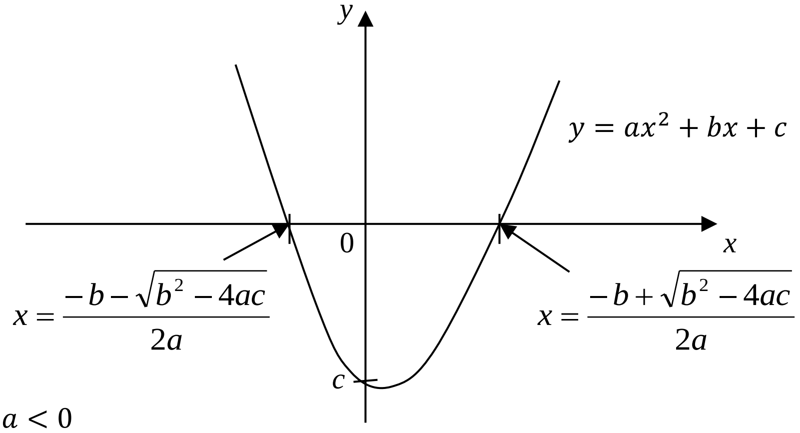
used to obtain the solutions (or the roots) of the quadratic equation.

The expression $b^2 - 4ac$, under the square root sign, is called the **discriminant,** and it determines the nature of the roots of the quadratic equation.

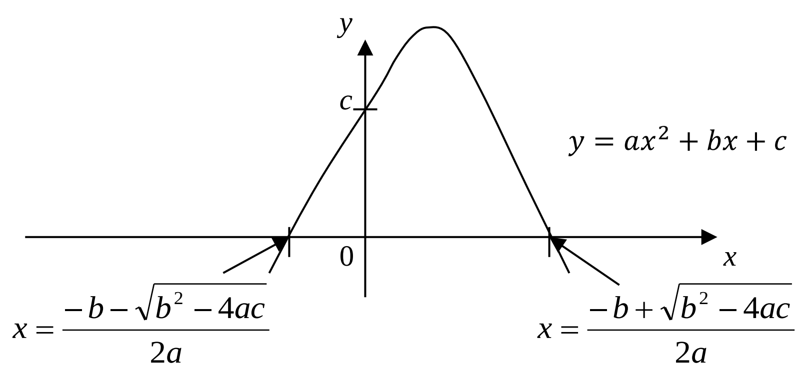
1. If $b^2 - 4ac > 0$, the equation has two and ***two distinct real*** roots

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Case 1. $a > 0$



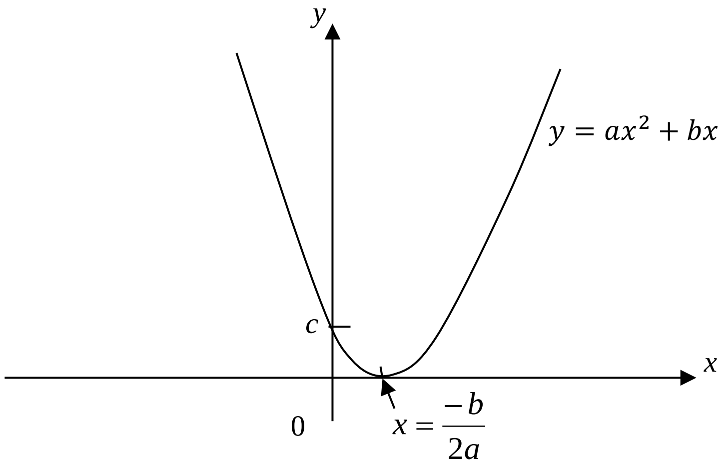
Case 2. $a < 0$



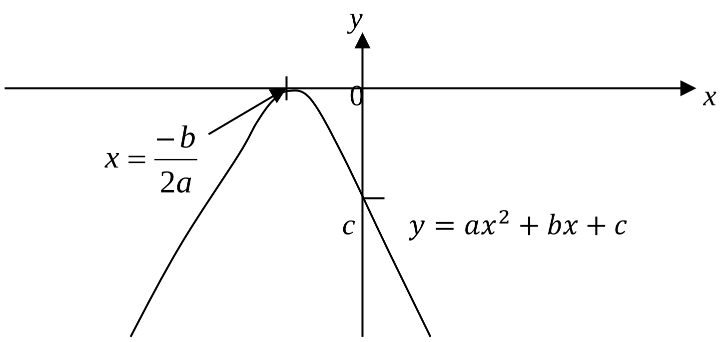
Note that when the quadratic equation has two distinct real roots the graph of the curve $y = ax^2 + bx + c$ cuts the x – axis at two distinct points.

2. If $b^2 - 4ac = 0$, the equation has ***two equal real*** roots $x = \dfrac{-b}{2a}$.

Case 1. $a > 0$



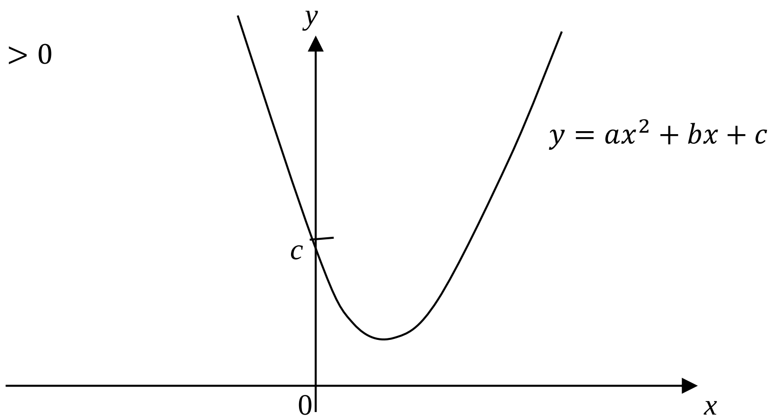
Case 2. $a > 0$



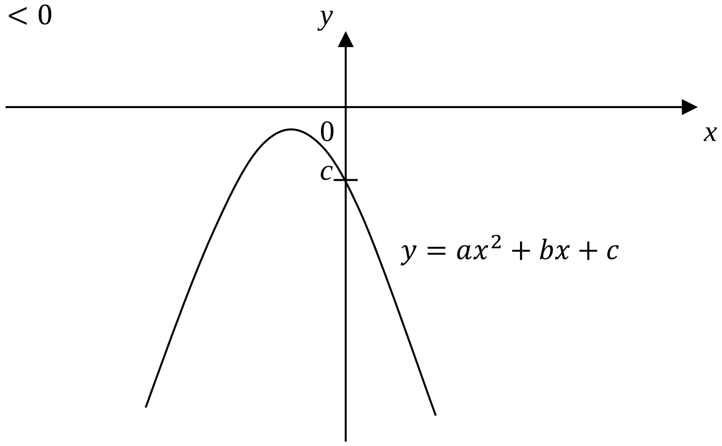
Note that when the quadratic equation has two equal real roots the graph of the curve $y = ax^2 + bx + c$ touches the x – axis at exactly one point.

3. If $b^2 - 4ac < 0$, the equation has no real roots. It has **two complex roots** which are conjugates of each other.

Case 1. $a > 0$



Case 2. $a < 0$



Note that when the quadratic equation has complex roots the graph of the curve $y = ax^2 + bx + c$ does not cut or touch the x – axis.

Example 2.7.1 Determine the nature of the roots of each of the following quadratic equations:

1. $x^2 - 6x + 9 = 0$

Solution: $a = 1, b = -6,$ and $c = 9$.

Using the discriminant, we have

$b^2 - 4ac = (-6)^2 - 4(1)(9) = 36 + 36 = 0.$

$\Rightarrow$ the equation has two equal real roots.

2. $x^2 + 4x - 8 = 0$

Solution: $a = 1, b = 4,$ and $c = -8$.

Using the discriminant, we have

$b^2 - 4ac = (4)^2 - 4(1)(-8) = 16 + 32 = 48 > 0.$

$\Rightarrow$ the equation has two distinct real roots.

3. $3x^2 + 4x + 2 = 0$

Solution: $a = 3, b = 4,$ and $c = 2$.

Using the discriminant, we have

$b^2 - 4ac = (4)^2 - 4(3)(2) = 16 - 24 = -8 < 0.$

$\Rightarrow$ the equation has two complex roots.

4. Prove that $kx^2 + 2x - (k-2) = 0$ has real roots for any value of k.

Proof: If the equation has real roots then $b^2 - 4ac \geq 0$.

Now, $b^2 - 4ac = 2^2 - 4(k)\left(-(k-2)\right) = 4 + 4k^2 - 8k$

$= 4\left(k^2 - 2k + 1\right) = 4(k-1)^2 \geq 0$ for any value of k.

Relationships between the Roots and Coefficients of a Quadratic Equation

Let α and β be the roots of a quadratic equation

$$ax^2 + bx + c = 0.$$

Then the equations

$$(x-\alpha)(x-\beta) = 0 \qquad \text{(I)}$$

and

$$ax^2 + bx + c = 0 \qquad \text{(II)}$$

have the same roots.

But from (I)

$$(x-\alpha)(x-\beta) = x^2(\alpha+\beta)x + \alpha\beta = 0$$

i.e. $$x^2 - (\alpha+\beta)x + \alpha\beta = 0 \qquad \text{(III)}$$

Dividing (II) by a we have

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \qquad \text{(IV)}$$

Now, (II) and (IV) have the same roots.

Comparing the coefficients of (III) and (IV) we have

$$\boxed{\alpha+\beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}.}$$

Example 2.7.2 If the equation

$$2x^2 - 3x + 6 = 0$$

has roots α and β, then the sum of roots

$$\alpha+\beta = -\frac{b}{a} = -\frac{-3}{2} = \frac{3}{2}$$

and the product of roots

$$\alpha\beta = \frac{c}{a} = \frac{6}{2} = 3.$$

Example 2.7.3 The roots of the equation

$$2x^2 + x - 7 = 0$$

are α and β. Find the values of $\frac{1}{\alpha} + \frac{1}{\beta}$ and $\frac{1}{\alpha\beta}$.

Solution: sum of roots = $\alpha+\beta = -\frac{b}{a} = -\frac{1}{2}$

Product of roots = $\alpha\beta = \frac{c}{a} = \frac{-7}{2}$.

$$\frac{1}{\alpha}+\frac{1}{\beta}=\frac{\beta+\alpha}{\alpha\beta}=\frac{-\frac{1}{2}}{-\frac{7}{2}}=\frac{1}{7} \text{ and } \frac{1}{\alpha\beta}=\frac{1}{-\frac{7}{2}}=-\frac{2}{7}.$$

Example 2.7.4 If α and β are the roots of the function $x(x-3)=x+4$, find the values of $\alpha^3+\beta^3$ and $\alpha^3\beta^3$.

Solution: $x(x-3)=x+4 \Rightarrow x^2-4x-4=0$ Thus $a=1, b=-4$ and $c=-4$.

sum of roots = $\alpha+\beta=-\frac{b}{a}=-\frac{-4}{1}=4$

Product of roots = $\alpha\beta=\frac{c}{a}=\frac{-4}{1}=-4$.

But

$$(\alpha+\beta)^3=(\alpha+\beta)(\alpha+\beta)^2=(\alpha+\beta)(\alpha^2+2\alpha\beta+\beta^2)$$
$$=\alpha^3+3\alpha\beta^2+3\alpha\beta^2+\beta^3$$

$\Rightarrow \alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)=(4)^3-3(-4)(4)=64+48=112$

and $\alpha^3\beta^3=(\alpha\beta)^3=(-4)^3=-64$.

Note from equations (III) and (IV) that a quadratic equation can be written as

$$x^2-(sum\ of\ roots)x+product\ of\ roots=0.$$

Using this we consider the following Example:

Example 2.7.5 Write down the quadratic equation whose roots are $\frac{1}{3}$ and $-\frac{2}{3}$.

Solution: $x^2-(\alpha+\beta)x+\alpha\beta=0$

$$\alpha+\beta=\frac{1}{3}+\frac{-2}{3}=-\frac{1}{3} \text{ and } \alpha\beta=\frac{1}{3}\times\frac{-2}{3}=-\frac{2}{9}.$$

Therefore, the equation is

$$x^2+\frac{1}{3}x-\frac{2}{9}=0$$

or $$9x^2+3x-2=0.$$

2.8 Polynomial Equations

Polynomial equations are of the form

$$a_n x^n + a_{n-1}x^{n-1}+\cdots+a_2x^2+a_1x+a_0=0,$$

where $a_0, a_1, a_2, \cdots, a_n$ are real numbers and $a_n \neq 0$.

We solve the polynomial equations the same way we find roots or zeros of a polynomial function.

Example 2.8.1 Solve the polynomial equation

$$2x^3 - 5x^2 + x + 2 = 0 .$$

Solution: Find a zero of $f(x) = 2x^3 - 5x^2 + x + 2$

$$f(1) = 2(1)^3 - 5(1)^2 + (1) + 2 = 2 - 5 + 1 + 2 = 0$$

$\Rightarrow$ 1 is a zero of $f(x)$.

$\Rightarrow$ $(x-1)$ is factor of $f(x)$.

1	2	-5	1	2
Add		2	-3	-2
	2	-3	-2	$0 = r$

$$\Rightarrow \; f(x) = (x-1)(2x^2 - 3x - 2) = (x-1)(2x+1)(x-2)$$

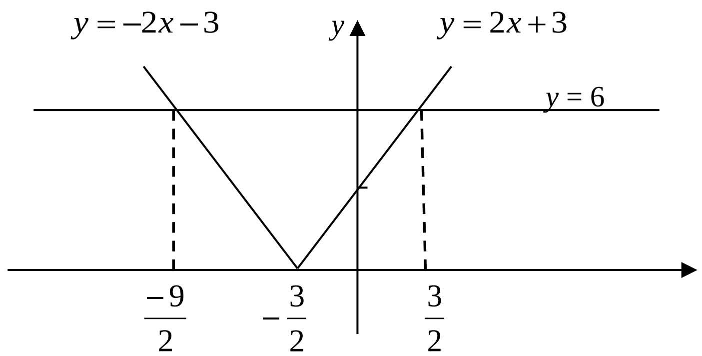
$$(x-1)(2x+1)(x-2) = 0 \quad \Rightarrow x = 1, \quad x = -\tfrac{1}{2}, \quad x = 2 .$$

2.9 Equations Involving the Absolute Value

Example 2.9.1 Solve each of the following equations:

1. $|2x+3| = 6$

Solution: Method 1: $|2x+3| = \begin{cases} 2x+3 \; if \;\; x \ge -\frac{3}{2} \\ -(2x+3) \; if \;\; x < -\frac{3}{2} \end{cases}$



$$\Rightarrow \qquad 2x+3 = 6 \quad \Rightarrow x = \frac{3}{2} \text{ and } -(2x+3) = 6 \quad \Rightarrow x = \frac{-9}{2} .$$

Method 2: It must be noted that $|x|$ can also be defined as

$$|x| = \sqrt{x^2} .$$

Thus, $|2x+3| = 6 \quad \Rightarrow \quad \sqrt{(2x+3)^2} = 6 \qquad \Rightarrow \qquad \left(\sqrt{(2x+3)^2}\right)^2 = 6^2$

$$\Rightarrow \qquad (2x+3)^2 = 36 \qquad \Rightarrow \qquad 4x^2 + 12x + 9 = 36$$

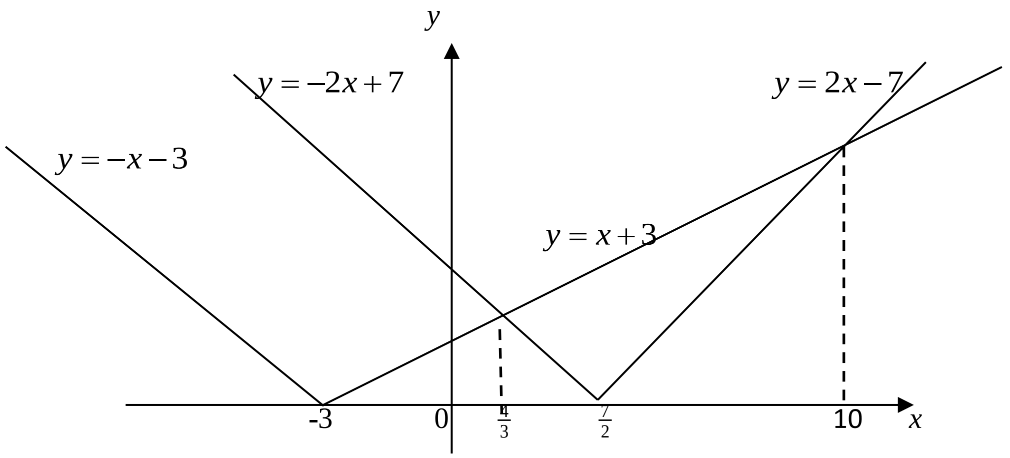
$\Rightarrow \quad 4x^2 + 12x - 27 = 0 \quad \Rightarrow \quad (2x-3)(2x+9) = 0$

$\Rightarrow \quad x = \dfrac{3}{2} \quad$ or $\quad x = -\dfrac{9}{2}$.

2. $|2x-7| = |x+3|$

Solution: Method 1: $|2x-7| = \begin{cases} 2x-7 \; if \;\; x \geq \frac{7}{2} \\ -(2x-7) \; if \;\; x < \frac{7}{2} \end{cases}$

and $|x+3| = \begin{cases} x+3 \; if \;\; x \geq -3 \\ -(x+3) \; if \;\; x < -3 \end{cases}$



$\Rightarrow \quad 2x - 7 = x + 3 \quad \Rightarrow x = 10$ and $-2x + 7 = x + 3 \quad \Rightarrow x = \dfrac{4}{3}$

Note that $2x-7$ is only defined for $x \geq \frac{7}{2}$ and $-(x+3)$ is only defined for $x < -3$, thus $2x - 7 \neq -(x+3)$ for $x < -3$ and $x \geq \frac{7}{2}$.

Method 2: $|2x-7| = |x+3| \quad \Rightarrow \quad \left(\sqrt{(2x-7)^2}\right)^2 = \left(\sqrt{(x+3)^2}\right)^2$

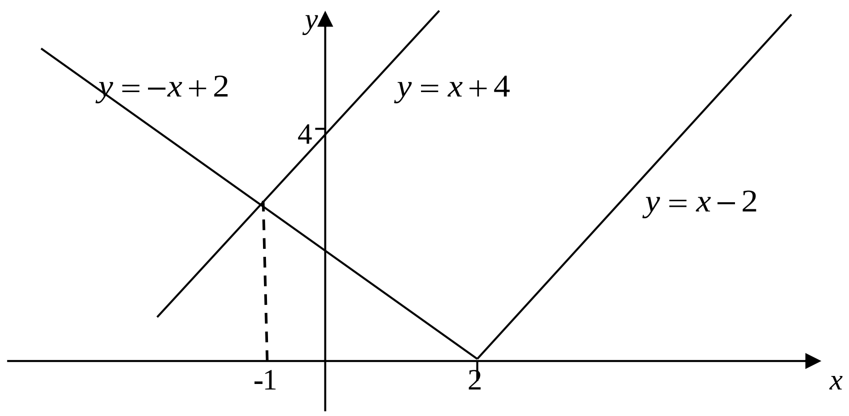
$\Rightarrow \quad (2x-7)^2 = (x+3)^2 \quad \Rightarrow \quad 4x^2 - 28x + 49 = x^2 + 6x + 9$

$\Rightarrow \quad 3x^2 - 34x + 40 = 0 \quad \Rightarrow \quad (3x-4)(x-10) = 0$

$\Rightarrow \quad x = \dfrac{4}{3} \quad$ or $\quad x = 10$.

3. $|x-2| = x + 4$.

Solution: Method 1: $|x-2| = \begin{cases} x-2 \; if \;\; x \geq 2 \\ -(x-2) \; if \;\; x < 2 \end{cases}$.



$$-(x-2)=x+4 \quad \Rightarrow -2x=2 \quad \Rightarrow x=-1$$

Note that $x-2=x+4$ has no solution because the lines $y=x-2$ and $y=x+4$ do not intersect since they have the same gradient and they are parallel.

Method 2 $|x-2|=x+4 \quad \Rightarrow \sqrt{(x-2)^2}=x+4 \quad \Rightarrow \left(\sqrt{(x-2)^2}\right)^2=(x+4)^2$

$\Rightarrow \quad (x-2)^2=(x+4)^2 \quad \Rightarrow x^2-4x+4=x^2+8x+16 \quad \Rightarrow 12x=-12 \quad \Rightarrow x=-1$

2.10 Equations Involving Radicals

Example 2.10.1 Solve each of the following equations:

1. $\sqrt{3x-8}-\sqrt{x-2}=0$
2. $\sqrt{3x+1}+\sqrt{2x+4}=3$.
3. $\sqrt{x-2}-\sqrt{2x-11}=\sqrt{x-5}$

Solution: 1. $\sqrt{3x-8}=\sqrt{x-2}$

$$\left(\sqrt{3x-8}\right)^2=\left(\sqrt{x-2}\right)^2$$

$$\Rightarrow \quad 3x-8=x-2 \quad \Rightarrow 2x=6 \quad \Rightarrow x=3$$

Test the root: When $x=3$, LHS = $\sqrt{3(3)-8}-\sqrt{3-2}=1-1=0=$ RHS

$\Rightarrow x=3$ is a root of the given equation.

2. $\sqrt{3x+1}+\sqrt{2x+4}=3$.

Solution: $\sqrt{3x+1}=3-\sqrt{2x+4}$

$$\Rightarrow \quad \left(\sqrt{3x+1}\right)^2=\left(3-\sqrt{2x+4}\right)^2$$

$$\Rightarrow \quad 3x+1=9-6\sqrt{2x+4}+2x+4$$

$$\Rightarrow \quad x-12=-6\sqrt{2x+4}$$

$$\Rightarrow \quad (x-12)^2=\left(-6\sqrt{2x+4}\right)^2$$

$$\Rightarrow \quad x^2-24x+144=36(2x+4)=72x+144$$

$$\Rightarrow \quad x^2-96x=0 \quad \Rightarrow x(x-96)=0 \quad \Rightarrow \quad x=0 \text{ or } x=96$$

Test the roots: When $x=0$, LHS = $\sqrt{3(0)+1}+\sqrt{2(0)+4}=1+2=3=$ RHS

$\Rightarrow x=0$ is a root of the given equation.

When $x=96$, LHS = $\sqrt{3(96)+1}+\sqrt{2(96)+4}=17+14\neq 3=$ RHS

$\Rightarrow x=96$ is not a root of the given equation.

Therefore, the equation on has one root $x=0$.

3. $\sqrt{x-2}-\sqrt{2x-11}=\sqrt{x-5}$

Solution: $\sqrt{x-2}=\sqrt{2x-11}+\sqrt{x-5}$

$\Rightarrow \quad \left(\sqrt{x-2}\right)^2=\left(\sqrt{2x-11}+\sqrt{x-5}\right)^2$

$\Rightarrow \quad x-2=2x-11+2\sqrt{2x-11}\cdot\sqrt{x-5}+x-5$

$\Rightarrow \quad 2x-14=2\sqrt{2x-11}\cdot\sqrt{x-5}$

$\Rightarrow \quad (2x-14)^2=\left(2\sqrt{2x-11}\cdot\sqrt{x-5}\right)^2$

$\Rightarrow \quad 4x^2-56x+196=4(2x-11)(x-5)=8x^2-84x+220$

$\Rightarrow \quad 4x^2-28x+24=0 \quad \Rightarrow x^2-7x+6=0 \quad \Rightarrow (x-1)(x-6)=0$

$\Rightarrow \quad x=1 \; or \; x=6$

Test the roots: When $x=1$, LHS = $\sqrt{1-2}+\sqrt{2(1)+4}$ is not defined.

$\Rightarrow x=1$ is not a root of the given equation.

When $x=6$, LHS = $\sqrt{6-2}-\sqrt{2(6)-11}=2-1=1$

RHS $=\sqrt{6-5}=1=$ RHS

$\Rightarrow x=6$ is a root of the given equation.

Therefore, the equation on has one root $x=6$.

2.11 System of Equations in Two unknowns

Elimination Method

Example 2.11.1 Solve the system of equations by elimination method:

1. $3x+2y=1;\ 5x-2y=23$

Solution: $3x+2y=1$

$+\ (5x-2y=23)$

$8x \quad = 24 \quad \Rightarrow x=3$ and $3(3)+2y=1 \quad \Rightarrow y=-4$.

Substitution Method

Example 3.9 Solve the system of equations by substitution method:

1. $3x+2y=1;\ 5x-2y=23$

2. $y=2x; \quad y=x^2-1$

3. $3x - 7y + 6 = 0; \quad x^2 - y^2 = 4$

Solutions:

1. $3x + 2y = 1; 5x - 2y = 23$

We make either *x* or *y* the subject of the formula of one equation and substitute in the other equation:

$x = \frac{1-2y}{3} \Rightarrow 5\left(\frac{1-2y}{3}\right) - 2y = 23 \Rightarrow 5 - 10y - 6y = 69$

$\Rightarrow y = -4$. Replacing this in the first equation we have $x = \frac{1-2(-4)}{3} = 3$.

Therefore the solution set is $\{3, -4\}$.

2. $y = 2x; \quad y = x^2 - 1$

Replacing $y = 2x$ in the other equation we have

$2x = x^2 - 1$ or $x^2 - 2x - 1 = 0 \Rightarrow x = 1 \pm \sqrt{2}$.

When $x = 1 \pm \sqrt{2}$, $y = 2 \pm 2\sqrt{2}$. Therefore the solution set is

$\{(1 + \sqrt{2}, 2 + 2\sqrt{2}), (1 - \sqrt{2}, 2 - 2\sqrt{2})\}$.

3. $3x - 7y + 6 = 0;\ x^2 - y^2 = 4$

From the first equation we have

$x = \frac{7y-6}{3}$. Replacing this in the second equation we have

$\left(\frac{7y-6}{3}\right)^2 - y^2 = 4$ or $(7y-6)^2 - 9y^2 = 36$

$49y^2 - 84y + 36 - 9y^2 = 36$ or $40y^2 - 84y = 0$

$\Rightarrow 4y(y - 21) = 0 \ \Rightarrow y = 0 \ or \ y = 21$

When $y = 0, x = -2$ and when $y = 21, x = 47$.

Therefore, the solution set is $\{(-2,0), (47,21)\}$.

TUTORIAL SHEET 5

1. Without solving the equations determine the nature of the roots of each of the following quadratic equations.

   (a) $3x^2 + 13x - 10 = 0$ (b) $2x^2 + 3x + 2 = 0$ (c) $4x^2 - 12x + 9 = 0$

2. If the roots of the quadratic equation $kx^2 + 30x + 25 = 0$ are equal, find the value of *k*.

3. Prove that $kx^2 + 2x - (k-2) = 0$ has real roots for any value of *k*.

4. Find a relationship between *p* and *q* if the roots of $px^2 + qx + 1 = 0$ are equal.

5. Without solving write down the sums and products of the roots of the following equations:

   (a) $4x^2 + 7x - 3 = 0$ (b) $\dfrac{x-1}{2} = \dfrac{3}{x+2}$ (c) $ax^2 - x(a+2) - a = 0$.

6. The roots of the quadratic equation $3x^2 + 13x - 10 = 0$ are α and β. Find the value of :

   (a) $\alpha^2 + \beta^2$ (b) $\dfrac{1}{\alpha^2+1} + \dfrac{1}{\beta^2+1}$ (d) $(\alpha - \beta)^2$.

7. The roots $x^2 + 3x - 10 = 0$ are α and β. Without finding the values of α and β, find the equations whose roots are:

   (a) $\alpha + 2, \beta + 2$ (b) $\dfrac{1}{\alpha}, \dfrac{1}{\beta}$ (c) $\dfrac{\alpha}{\beta}, \dfrac{\beta}{\alpha}$ (d) $\alpha - \beta, \beta - \alpha$.

8. (a) On the same diagram, sketch the graphs of $y = |x|$ and $y = |2x - 1|$.

   (b) Solve the equation $|x| = |2x - 1|$.

9. Solve each of the following equations:

   (a) $|2x - 5| = 7$ (b) $|2x+1| = |4x-3|$ (c) $\left|\frac{k-2}{k-1}\right| = 3$.

10. On the same diagram, sketch the graphs of $y = 24 + 2x - x^2$ and $y = |5x - 4|$.

    Solve the equation $24 + 2x - x^2 = |5x - 4|$.

11. Solve each of the following equations:

    (a) $\sqrt{2t-1} + 2 = t$ (b) $\sqrt{2x-1} - \sqrt{x+3} = 1$ (c) $\sqrt{2x-1} - \sqrt{x+3} = 1$

    (d) $\sqrt{x-2} - \sqrt{2x-11} = \sqrt{x-5}$ (e) $\sqrt{1+2\sqrt{x}} = \sqrt{x+1}$.

12. Solve each of following system of equations by the substitution method:

    (a) $\begin{cases} x + 2y = 3 \\ x - 2y = 1 \end{cases}$ (b) $\begin{cases} 3x - 5y = 2 \\ 2x + 5yy = 13 \end{cases}$.

13. Solve each of following simultaneous equations:

    (a) $\begin{cases} 2x - y + 3 = 0 \\ x^2 - 2x - y = 3 \end{cases}$ (b)) $\begin{cases} x - y = 2 \\ x^2 - y^2 = 8 \end{cases}$

MAT 1100 LECTURE NOTES

3 INEQUALITIES

An inequality is a relation which makes a comparison between two quantities or numbers or mathematical expressions. The following are different mathematical symbols used to represent different kinds of inequalities: $<$ (read as less than); $\leq$ (read as less or equal to); $>$ (read as greater than); $\geq$ (read as greater or equal to).

3.1 Linear Inequalities

Example 3.1.1 Find the set of values of *x* which satisfy each of the following inequalities:

1. $\frac{x-4}{6}-\frac{x-2}{9}\leq\frac{5}{18}$.

   Solution:

   Multiplying through by 18, we obtain

   $18\times\left(\frac{x-4}{6}-\frac{x-2}{9}\leq\frac{5}{18}\right)$

   i.e. $3(x-4)-2(x-2)\leq 5$

   $\Rightarrow\ 3x-12-2x+4\leq 5\ \Rightarrow x\leq 13.$

   In set builder notation, the solution set is $\{x\in\mathbb{R}: x\leq 13\}$.

   In interval notation, the solution set is $(-\infty, 13]$.

2. $\frac{x}{2}-\frac{x-1}{5}\geq\frac{x+2}{10}-4$.

   Solution:

   $10\times\left(\frac{x}{2}-\frac{x-1}{5}\geq\frac{x+2}{10}-4\right)$

   $5x-2(x-1)\geq(x+2)-(10\times 4)$

   $5x-2x+2\geq x+2-40$

   $5x-2x-x\geq 2-2-40 \Rightarrow 2x\geq -40\Rightarrow x\geq -20.$

   Therefore the solution set is $\{x\in\mathbb{R}: x\geq -20\}$ or $[-20,\infty)$.

3. $-3<\frac{4x+3}{2}\leq 1$.

   Solution:

   This can be written as

   $-3<\frac{4x+3}{2}$ **and** $\frac{4x+3}{2}\leq 1$ (*)

   $2\times\left(-3<\frac{4x+3}{2}\right)$ **and** $2\times\left(\frac{4x+3}{2}\leq 1\right)$

   $\Rightarrow -6<4x+3$ **and** $4x+3\leq 2$

   $\Rightarrow -6<4x+3$ **and** $4x+3\leq 1$

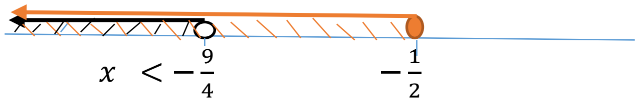
$\Rightarrow -6 - 3 < 4x$ **and** $4x \leq 1 - 3$

$\Rightarrow -9 < 4x$ **and** $4x \leq -2$

$\Rightarrow \frac{-9}{4} < x$ **and** $x \leq \frac{-2}{4} = -\frac{1}{2}$

The required values of *x* of the given inequality must satisfy the two inequalities marked (*). Thus we need to take the intersection of the sets

$-\frac{9}{4} < x$ **and** $x \leq -\frac{1}{2}$.



Therefore the solution set is $\left\{x \in \mathbb{R}: x < -\frac{9}{4}\right\}$ or $\left(-\infty, -\frac{9}{4}\right]$.

4. $3 \geq \frac{7-2x}{2} \geq -5$

<u>Solutions</u>:

This also can be written as

$3 \geq \frac{7-2x}{2}$ **and** $\frac{7-2x}{2} \geq -5$ (**)

$2 \times \left(3 \geq \frac{7-2x}{2}\right)$ **and** $2 \times \left(\frac{7-2x}{2} \geq -5\right)$

$\Rightarrow 6 \geq 7 - 2x$ **and** $7 - 2x \geq -10$

$\Rightarrow 2x \geq 7 - 6 = 1$ **and** $-2x \geq -10 - 7 = -17$

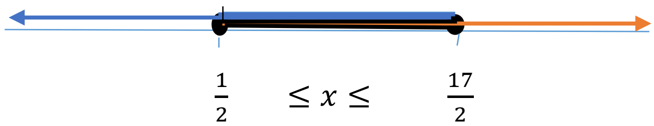
$\Rightarrow x \geq \frac{1}{2}$ **and** $\leq \frac{17}{2}$.

Note that when you multiply or divide an inequality by a negative number the inequality sign changes direction, either from " $\leq$ *to* $\geq$ "

(or from " $<$ "*to*" $>$ ") or from " $\geq$ *to* $\leq$ " (or from " $<$ "*to*" $>$ ").

Thus the required values of *x* of the given inequality must satisfy the two inequalities marked (**). Thus we need to take the intersection of the sets

$x \geq \frac{1}{2}$ and $x \leq \frac{17}{2}$.



Therefore the solution set is $\left\{x \in \mathbb{R}: \frac{1}{2} \leq x \leq \frac{17}{2}\right\}$ or $\left[\frac{1}{2}, \frac{17}{2}\right]$.

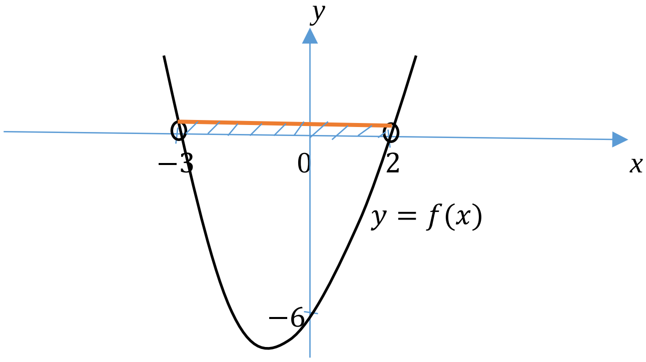
3.2 Quadratic Inequalities

Example 3.2.1 Solve each of the following inequalities:

1. $x^2 + x - 6 < 0$

Solution:

Graphical method: Using this method you need first to sketch the graph of the quadratic function $f(x) = x^2 + x - 6$. Then from the graph determine the values of x for which $f(x) < 0$.



From the graph we see that for all values of x between -3 and 2, $y = f(x) < 0$. Therefore the solution set for the inequality is $\{x \in \mathbb{R}: -3 < x < 2\}$ or $(-3,2)$.

Table Method: In this method you first need to factorize the quadratic expression $x^2 + x - 6$ and find the critical points by solving the equation $(x+3)(x-2) = 0$. Here the critical points are -3 and 2. Then draw the table and label it as follows:

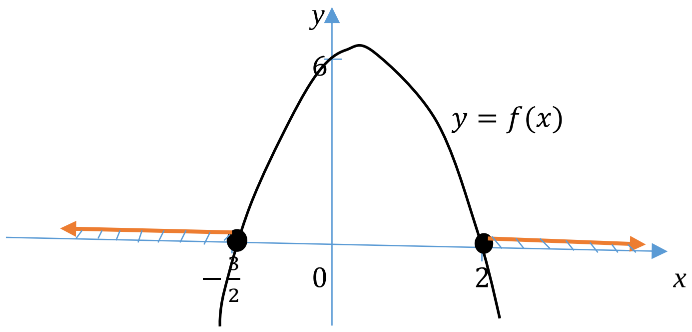
	-3		2
Factors	$x < -3$	$-3 < x < 2$	$x > 2$
$x + 3$	$-$	$+$	$+$
$x - 2$	$-$	$-$	$+$
$(x+3)(x-2) < 0$	$+$ ✗	$-$ ✓	$+$ ✗

Now, note that for $x < -3$ the factor $(x+3)$ is negative, for $-3 < x < 2$ it is positive and for $x > 2$ it is positive. For $x < -3$ the factor $(x-2)$ is negative, for $-3 < x < 2$ it is negative and for $x > 2$ it is positive. Then from the table, for $x < -3$ the product $(x+3)(x-2)$ is positive, for $-3 < x < 2$ it is negative and for $x > 2$ it is positive. But we are looking for the values of x for which $(x+3)(x-2) = x^2 + x - 6 < 0$. We note that the required values of x which satisfy the given inequality lie between -3 and 2.

Therefore the solution set is again $\{x \epsilon \mathbb{R}: -3 < x < 2\}$ or $(-3,2)$.

2. $6 + x - 2x^2 \leq 0$.

Graphical method: Just like in (1) we first sketch the graph of $f(x) = 6 + x - 2x^2$.



From the graph we see that for all values of x less or equal to $-\frac{3}{2}$ or greater or equal to 2, $y = f(x) = 6 + x - 2x^2 \leq 0$. Therefore the solution set for the inequality is $\left\{x \in \mathbb{R}: x \leq -\frac{3}{2} \text{ or } x \geq 2\right\}$ **or** $\left(-\infty, -\frac{3}{2}\right] \cup [2, \infty)$.

<u>Table Method</u>: Here the critical points are $-\frac{3}{2}$ and 2. The table and its contents is as follows:

Factors	$x < -\frac{3}{2}$	$-\frac{3}{2} < x < 2$	$x > 2$
$2x + 3$	$-$	$+$	$+$
$2 - x$	$+$	$+$	$-$
$(2x + 3)(2 - x) \leq 0$	$-$ ✓	$+$ ✗	$-$ ✓

Therefore, from the table the solution set is again

$$\left\{x \in \mathbb{R}: x \leq -\frac{3}{2} \text{ or } x \geq 2\right\} \textbf{ or } \left(-\infty, -\frac{3}{2}\right] \cup [2, \infty).$$

<u>Method of considering Cases</u>

In this method we use the fact that if a and b are real numbers, then either $ab > 0$ or $ab < 0$ or $ab = 0$.

If $ab < 0$, then we have two cases, either

1. $a > 0$ and $b < 0$ or

2. $a < 0$ and $b > 0$.

Similarly, if $ab > 0$, then either

1. $a > 0$ and $b > 0$ or

2. $a < 0$ and $b < 0$.

If $ab = 0$, then either

1. $a = 0$ and $b \neq 0$ or

2. $a \neq 0$ and $b = 0$ or

3. both $a = 0$ and $b = 0$.

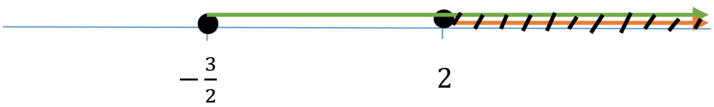
Now, if we consider the inequality $(2x + 3)(2 - x) \leq 0$, we shall have the following cases:

CASE 1: $(2x + 3) \leq 0$ **and** $(2 - x) \geq 0$. Note that since one factor is negative and the other is positive, their product will be negative. From the inequalities $x \leq -\frac{3}{2}$ and $x \leq 2$. But the required values of x must satisfy both inequalities $(2x + 3) \leq 0$ **and** $(2 - x) \geq 0$. We determine the required values of x by finding the intersection of the two sets using the number line.

From the number line we see that the intersection set is $x \leq -\frac{3}{2}$. Therefore the set $\left\{x \in \mathbb{R}: x \leq -\frac{3}{2}\right\}$ is part of the solution set.

We also need to consider the other case.

CASE 2: $(2x + 3) \geq 0$ **and** $(2 - x) \leq 0$. Even here, since one factor is positive and the other is negative, their product will be negative. From the inequalities, $x \geq -\frac{3}{2}$ and $x \geq 2$. Again from the number line,



we see that the intersection set is $\{x \in \mathbb{R}: x \geq 2\}$, and it forms part of the solution set.

Combining the two intersection sets by taking their union we get the required solution set, which is

$$\left\{x \in \mathbb{R}: x \leq -\frac{3}{2} \ \boldsymbol{or}\ x \geq 2\right\} = \left(-\infty, -\frac{3}{2}\right] \cup [2, \infty).$$

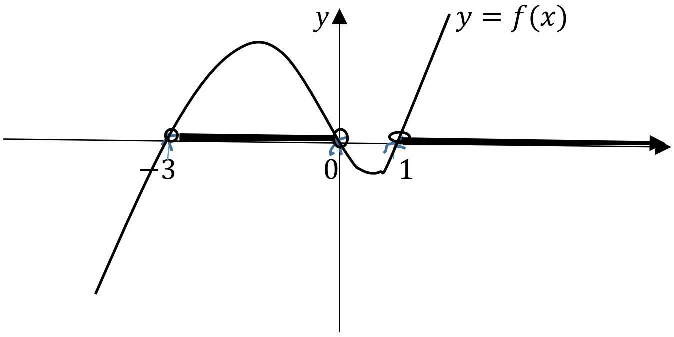
3.3 Inequalities Involving Polynomials

Example 3.3.1 Solve each of the following inequalities:

1. $x^3 + 2x^2 - 3x > 0$.

Solution:

Graphical method: $y = x^3 + 2x^2 - 3x = x(x + 3)(x - 1) > 0$



From the graph note that $y > 0$ for values of x beween -3 and 0 **or** values of x greater than 1

Therefore the solution set is $\{x \in \mathbb{R}: -3 < x < 0 \ \boldsymbol{or} \ x > 1\} = (-\infty, 0) \cup (1, \infty)$.

<u>Table method</u>:

The critical points are $-3, 0$ and 1.

-3 $\qquad$ 0 $\qquad$ 1

Factors	$x < -3$	$-3 < x < 0$	$0 < x < 1$	$x > 1$
x	$-$	$-$	$+$	$+$
$x+3$	$-$	$+$	$+$	$+$
$x-1$	$-$ ✗	$-$ ✓	$-$ ✗	$+$ ✓
$x(x+3)(x-1) > 0$	$-$	$+$	$-$	$+$

From the table note that $x^3 + 2x^2 - 3x > 0$ for $-3 < x < 0$ or $x > 1$,

Therefore the solution set is $\{x \in \mathbb{R}: -3 < x < 0 \ \boldsymbol{or} \ x > 1\} = (-\infty, 0) \cup (1, \infty)$.

Using the cases method, note that for $a, b, c \in \mathbb{R}$, $abc > 0$ if either

1. $a > 0, b > 0 \ and \ c > 0$ or
2. $a > 0, b < 0 \ and \ c < 0$ or
3. $a < 0, b < 0 \ and \ c > 0$ or
4. $a < 0, b > 0 \ and \ c < 0$

<u>Exercise</u>: Use the method for considering cases to find the solution set for the inequality

$x^3 + 2x^2 - 3x > 0$.

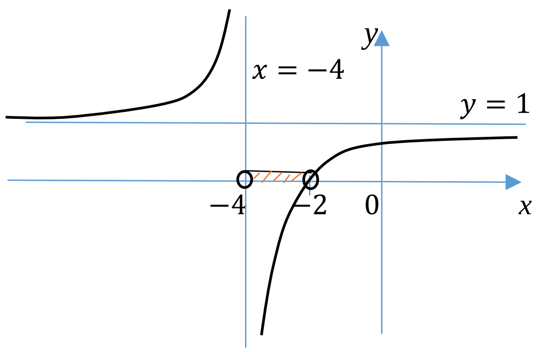
3.4 <u>Inequalities Involving Rational expressions</u>

<u>Example</u> 3.4.1 Solve each of the following inequalities:

1. $\dfrac{x+2}{x+4} < 0$

<u>Solution</u>: It is possible to find the solution set of inequalities involving rational functions using the graphical method.

For example, the graph of the function $y = \frac{x+2}{x+4}$ is given below.



From the graph we see that $\frac{x+2}{x+4} < 0$ for values of x between -4 and -2. Therefore the solution set is $\{x \in \mathbb{R}: -4 < x < -2\}$ or $(-4, -2)$.

But for now, since we have not yet done the graphing of rational functions we shall not use the graphical method to solve inequalities involving rational functions.

Table method:

The critical points are -4 and -2.

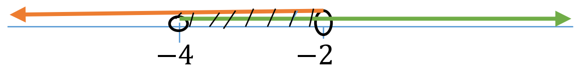
-4 $\quad\quad$ -2

Factors	$x < -4$	$-4 < x < -2$	$x > -2$
$x+2$	$-$	$-$	$+$
$x+4$	$-$	$+$	$+$
$\frac{x+2}{x+4} < 0$	$+$ ✗	$-$ ✓	$+$ ✗

From the table, the solution set is $\{x \in \mathbb{R}: -4 < x < -2\}$ or $(-4, -2)$.

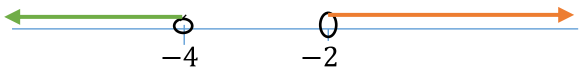
Method of considering cases: Here $\frac{x+2}{x+4} < 0$ when either case 1: $x+2 < 0$ and $x+4 > 0$ or case 2: $x+2 > 0$ and $x+4 < 0$.

CASE 1: $x+2 < 0$ and $x+4 > 0$ i.e. $x < -2$ **and** $x > -4$. From the number line



we see that x lies between -4 and -2. Thus the set $\{x \in \mathbb{R}: -4 < x < -2\}$ or $(-4, -2)$ is part of the solution set.

CASE 2: $x+2 > 0$ and $x+4 < 0$ i.e. $x > -2$ **and** $x < -4$. From the number line



we see that the two lines do **not** intersect, thus we discard this case. Therefore we conclude that the solution set is $\{x \in \mathbb{R}: -4 < x < -2\}$ or $(-4, -2)$.

2. $\frac{x+3}{x-7} \leq 1$

To solve this inequality, we first need to express it in the form $\frac{ax+b}{cx+d} \leq 0$, where a, b, c and d are real numbers and then use the method of our choice.

DO NOT CROSS MULTIPLY WHEN YOU ARE DEALING WITH INEQUALITIES INVOLVING RATIONAL FUNCTIONS.

3. In most cases, when you cross multiply by an algebraic expression you will get a wrong solution. For example, in this case, if we multiply both sides of the inequality $\frac{x+3}{x-7} \leq 1$

by $(x-7)$ we obtain $x+3 \leq x-7 \Rightarrow 3 \leq -7$ or $0 \leq -10$. This doesn't make sense because 0 cannot be less than -10.

4. Therefore in this case, from $\frac{x+3}{x-7} \leq 1$we need to subtract -1 from both sides so that

$\frac{x+3}{x-7} - 1 \leq 0 \Rightarrow \frac{x+3-(x-7)}{x-7} \leq 0 \Rightarrow \frac{10}{x-7} \leq 0.$

Now, using the table, we see that the solution set

7

	$x < 7$	$x > 7$
10	$+$	$+$
$x-7$	$-$	$+$
$\frac{10}{x-7} \leq 0$	$-$ ✓	$+$ ✗

is $\{x \in \mathbb{R}: x < 7\}$ or $(-\infty, 7)$. Note that for $\boldsymbol{x = 7}$ the inequality does not hold.

When we look at the inequality $\frac{10}{x-7} \geq 0$ and use the cases method, we see that $10 > 0$ for all values of $x \in \mathbb{R}$. Thus $(x-7)$ must also be negative i.e. $(x-7) < 0$ or $x < 7$. Therefore the solution set is simply $\{x \in \mathbb{R}: x < 7\}$ or $(-\infty, 7)$.

5. $\frac{2}{x-1} \geq \frac{3}{x+4}$

Similarly here don't cross multiply. Just take one term to the other side of the inequality. $\frac{2}{x-1} - \frac{3}{x+4} \geq 0 \Rightarrow \frac{2(x+4)-3(x-1)}{(x-1)(x+4)} \geq 0 \Rightarrow \frac{2x+8-3x+3}{(x-1)(x+4)} \geq 0 \Rightarrow$

$\frac{-x+11}{(x-1)(x+4)} \geq 0$ or $\Rightarrow \frac{x-11}{(x-1)(x+4)} \leq 0$ $\left(\substack{\textit{Here the inequality sign has changed} \\ \textit{direction because we multiplied both sides } -1}\right)$

By table method,

−4 1 11

Factors	$x < -4$	$-4 < x < 1$	$1 < x < 11$	$x > 1$
$x-11$	$-$	$-$	$-$	$+$
$x-1$	$-$	$-$	$+$	$+$
$x+4$	$-$	$+$	$+$	$+$
$\frac{x-11}{(x-1)(x+4)} < 0$	$-$ ✓	$+$ ✗	$-$ ✓	$+$ ✗

From the table note that the solution set is $\{x \in \mathbb{R}: x < -4 \text{ or } 1 < x \leq 11\}$ or $(-\infty, -4) \cup (1, 11]$.

Note that -4 and 11 are not part of the solution set because the inequality is not defined at these points.

To use the cases methods we need to considered the following cases:

Case 1: $x-11 \geq 0, x-1>0$ *and* $x+4<0$;

Case 2: $x-11 \geq 0, x-1<0$ *and* $x+4>0$;

Case 3: $x-11 \leq 0, x-1>0$ *and* $x+4>0$;

Case 4: $x-11 \leq 0, x-1<0$ *and* $x+4<0$;

<u>Exercise</u>**:** Solve the inequality using the cases method and compare the solutions.

6. $-2 \leq \frac{x-2}{x+3} < 4$

To solve this inequality, we first need to break the inequality into two.

$-2 \leq \frac{x-2}{x+3}$ **and** $\frac{x-2}{x+3} < 4.$

Then $\frac{x-2}{x+3} \geq -2$ **and** $\frac{x-2}{x+3} < 4 \Rightarrow \frac{x-2}{x+3} + 2 \geq 0$ **and** $\frac{x-2}{x+3} - 4 < 0$

$\Rightarrow \frac{x-2+2(x+3)}{x+3} \geq 0$ **and** $\frac{x-2-4(x+3)}{x+3} < 0$

$\Rightarrow \frac{3x-4}{x+3} \geq 0$ **and** $\frac{-3x-14}{x+3} < 0 \Rightarrow \frac{3x-4}{x+3} \geq 0$ **and** $\frac{3x+14}{x+3} > 0$

Here we shall have two tables:

-3 $\qquad\qquad$ $\frac{4}{3}$

Factors	$x<-3$	$-3<x<\frac{4}{3}$	$x>\frac{4}{3}$
$3x-4$	$-$	$-$	$+$
$x+3$	$-$	$+$	$+$
$\frac{3x-4}{x+3} \geq 0$	$+$	$-$	$+$

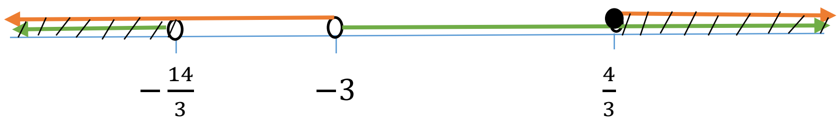
The solution for the inequality $\frac{3x-4}{x+3} \geq 0$ is $\left\{x \in \mathbb{R}: x<-3 \text{ or } x \geq \frac{4}{3}\right\}$ (I)

and $\qquad$ $-\frac{14}{3}$ $\qquad\qquad$ -3

Factors	$x<-\frac{14}{3}$	$-\frac{14}{3}<x<-3$	$x>-3$
$3x+14$	$-$	$+$	$-$
$x+3$	$-$	$-$	$-$
$\frac{3x+14}{x+3} > 0$	$+$	$-$	$+$

The solution for the inequality $\frac{3x+14}{x+3} > 0$ is $\left\{x \in \mathbb{R}: x < -\frac{14}{3} \ or\ x > -3\right\}$ (II)

To find the solution set for the given inequality we need to find the intersection of the set (I) and the set (II) using the number line.



Therefore the solution set is $\left\{x \in \mathbb{R}: < -\frac{14}{3} \ or\ x \geq \frac{4}{3}\right\}$ or $\left(-\infty, -\frac{14}{3}\right) \cup \left[\frac{4}{3}, \infty\right)$.

3.5 Inequalities Involving the Modulus functions

Definition 3.5.1 For any real number $k > 0, |x| \leq k \Leftrightarrow -k \leq x \leq k$.

Definition 3.5.2 For any real number $k > 0, |x| \geq k \Leftrightarrow x \leq -k$ or $x \geq k$.

To solve inequalities involving modulus functions, we use either Definition 3.5.1 or Definition 3.5.2.

Example 3.5 Solve each of the following inequalities:

1. $|2x-1| \geq 7$.

This inequality is of the form in Definition 3.5.2. Thus $|2x-1| \geq 7$

$\Leftrightarrow 2x - 1 \leq -7$ or $2x - 1 \geq 7$

$\Leftrightarrow 2x \leq -6$ or $2x \geq 8$

$\Leftrightarrow x \leq -3$ or $x \geq 4$.

Therefore the solution set is $\{x \in \mathbb{R}: x \leq -3$ or $x \geq 4\}$ or $(-\infty, -3] \cup [4, \infty)$.

2. $\left|\frac{x-1}{x+2}\right| \leq 3$.

This inequality is of the form in Definition 3.5.1. Thus $\left|\frac{x-1}{x+2}\right| \leq 3$

$\Leftrightarrow -3 \leq \frac{x-1}{x+2} \leq 3$

$\Leftrightarrow -3 \leq \frac{x-1}{x+2}$ **and** $\frac{x-1}{x+2} \leq 3$

$\Leftrightarrow 0 \leq \frac{x-1}{x+2} + 3$ **and** $\frac{x-1}{x+2} - 3 \leq 0$

$\Leftrightarrow 0 \leq \frac{x-1+3(x+2)}{x+2}$ **and** $\frac{x-1-3(x+2)}{x+2} \leq 0$

$\Leftrightarrow 0 \leq \frac{x-1+3x+6)}{x+2}$ **and** $\frac{x-1-3x-6}{x+2} \leq 0$

$\Leftrightarrow 0 \leq \frac{4x+5}{x+2}$ **and** $\frac{-2x-7}{x+2} \leq 0 \ or\ \frac{2x+7}{x+2} \geq 0$

Here we shall have two tables:

-2 $\qquad\qquad$ $-\frac{5}{4}$

Factors	$x < -2$	$-2 < x < -\frac{5}{4}$	$x > -\frac{5}{4}$
$4x + 5$	$-$	$-$	$+$
$x + 2$	$-$	$+$	$+$
$\frac{4x+5}{x+2} \geq 0$	$+$	$-$	$+$

The solution for the inequality $\frac{4x+5}{x+2} \geq 0$ is $\left\{x \in \mathbb{R}: x < -2 \text{ } or \text{ } x \geq -\frac{5}{4}\right\}$ (I)

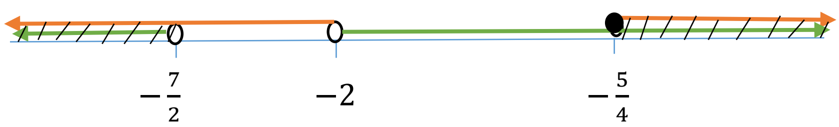
and

$-\frac{7}{2}$ $\qquad\qquad$ -2

Factors	$x < -\frac{7}{2}$	$-\frac{7}{2} < x < -2$	$x > -2$
$2x + 7$	$-$	$+$	$+$
$x + 2$	$-$	$-$	$+$
$\frac{2x+7}{x+2} \geq 0$	$+$	$-$	$+$

The solution for the inequality $\frac{2x+7}{x+2} \geq 0$ is $\left\{x \in \mathbb{R}: x < -\frac{7}{2} \text{ } or \text{ } x \geq -2\right\}$ (II)

To find the solution set for the given inequality we need to find the intersection of the set (I) and the set (II) using the number line.



Therefore the solution set is $\left\{x \in \mathbb{R}: < -\frac{7}{2} \text{ } or \text{ } x \geq -\frac{5}{4}\right\}$ or $\left(-\infty, -\frac{7}{2}\right) \cup \left[-\frac{5}{4}, \infty\right)$.

3.6 <u>Inequalities Involving Radicals functions</u>

To solve inequalities involving radicals we follow three steps below:

<u>Step</u> 1: Identify the values of the variable for which the radicand is nonnegative.

<u>Step</u> 2: Find critical points.

<u>Step</u> 3: Test values to check the solution or find intervals which contain values that satisfy the inequality.

<u>Example</u> 3.6.1 Solve each of the following inequalities:

1. $3+\sqrt{2x-7} \le 6$

<u>Solution</u>:

<u>Step</u> 1: The radicand must be nonnegative i.e. $2x-7 \ge 0 \quad \Rightarrow x \ge \frac{7}{2}$

<u>Step</u> 2: Critical points:

$$3+\sqrt{2x-7} = 6 \quad \Rightarrow \sqrt{2x-7} = 3 \quad \Rightarrow 2x-7 = 9 \quad \Rightarrow x = 8$$

Thus critical points are $x = \frac{7}{2}$ and $x = 8$.

Step 3: We test some values which satisfy the given inequality:

$x < \frac{7}{2}$	$\frac{7}{2} < x < 8$	$x > 8$
Take say $x = 0$: $3+\sqrt{2(0)-7} \le 6$ $3+\sqrt{-7} \le 6$. Not defined	Take say $x = 4$: $3+\sqrt{2(4)-7} \le 6$ $3+\sqrt{1} \le 6$ *i.e.* $4 < 6$, which is true	Take say $x = 9$: $3+\sqrt{2(8)-7} \le 6$ $3+\sqrt{11} \ne or < 6$. Not true

At critical points:

$$x = \frac{7}{2}, \quad 3+\sqrt{2(\tfrac{7}{2})-7} = 3+0 = 3 < 6 \text{, which is true.}$$

Thus $x = \frac{7}{2}$ is part of the solution set.

$$x = 8, \quad 3+\sqrt{2(8)-7} = 3+3 = 6 = 6 = RHS \text{, which is true.}$$

Thus $x = 8$ is also part of the solution set.

Therefore the solution set is $\left\{x \in \mathbf{R} : \frac{7}{2} \le x \le 8\right\} = \left[\frac{7}{2}, 8\right]$.

2. $\sqrt{2x+5} < \sqrt{9+x}$

<u>Solution</u>:

<u>Step</u> 1: The radicand for both $\sqrt{2x+5}$ and $\sqrt{9+x}$ must be nonnegative i.e.

$2x+5 \ge 0$ and $9+x \ge 0$

$$x \ge -\frac{5}{2} \quad \text{and } x \ge -9$$

Both $\sqrt{2x+5}$ and $\sqrt{9+x}$ are defined for values of $x \ge -\frac{5}{2}$.

<u>Step</u> 2: Critical points:

$$\sqrt{2x+5} = \sqrt{9+x} \quad \Rightarrow 2x+5 = 9+x \quad \Rightarrow x = 4$$

The critical points are $x = -\frac{5}{2}$ and $x = 4$.

Step 3: We test some values which satisfy the given inequality:

$x < -\frac{5}{2}$	$-\frac{5}{2} < x < 4$	$x > 4$
Not defined	Take say $x = 0$: $\sqrt{2(0)+5} \le \sqrt{9+0}$ $\sqrt{5} \le 3$, which is true	Take say $x = 5$: $\sqrt{2(5)+5} \le \sqrt{5+9}$ $\sqrt{15} \le \sqrt{14}$ Not true

At critical points:

$x = -\frac{5}{2}$, $\sqrt{2(-\frac{5}{2})+5} < \sqrt{9+(-\frac{5}{2})}$, $\Rightarrow 0 < \sqrt{\frac{13}{2}}$, which is true.

$x = 4$, $\sqrt{2(4)+5} < \sqrt{9+4}$ $\Rightarrow \sqrt{13} < \sqrt{13}$, which is not true. Thus $x = 4$ is not part of the solution set.

Therefore the solution set is $\left\{x \in \mathbf{R} : -\frac{5}{2} \le x < 4\right\} = [-\frac{5}{2}, 4)$.

3. $\sqrt{x+3}+\sqrt{x+7}>4$

<u>Solution</u>:

<u>Step</u> 1: The radicand for both $\sqrt{x+3}$ and $\sqrt{x+7}$ must be nonnegative i.e.

$x+3\geq 0$ and $x+7\geq 0$

$x\geq -3 \quad and \ x\geq -7$

Both $\sqrt{x+3}$ and $\sqrt{x+7}$ are defined for values of $x\geq -3$.

<u>Step</u> 2: Critical points: $x=-3$ is one the critical points.

To find other critical points we solve the equation

$\sqrt{x+3}+\sqrt{x+7}=4$

$\Rightarrow \left(\sqrt{x+3}\right)^2=\left(4-\sqrt{x+7}\right)^2$

$\Rightarrow x+3=16-8\sqrt{x+7}+x+7$

$\Rightarrow 8\sqrt{x+7}=20 \ or \ 2\sqrt{x+7}=5 \quad \Rightarrow 4(x+7)=25 \quad \Rightarrow x=-\frac{3}{4}$.

The critical points are $x=-3$ and $x=-\dfrac{3}{4}$.

Step 3: We test some values which satisfy the given inequality:

$x<-3$	$-3<x<-\dfrac{3}{4}$	$x>-\dfrac{3}{4}$
Not defined	Take say $x=-2$: $\sqrt{-2+3}+\sqrt{-2+7}>4$ $1+\sqrt{5}>4$, which is not true	Take say $x=$: $\sqrt{0+3}+\sqrt{0+7}>4$ $\sqrt{3}+\sqrt{7}>4$, which is true.

At critical points:

$x=-3,\quad \sqrt{-3+3}+\sqrt{-3+7}>4$, $\Rightarrow 0+2=2>4$, which is not true. Thus $x=-3$ is not part of the solution set.

$x=-\frac{3}{4}$, $\quad \sqrt{-\frac{3}{4}+3}+\sqrt{-\frac{3}{4}+7}<4 \quad \Rightarrow \frac{3}{2}+\frac{5}{2}=4<4$, which is not true. Thus $x=-\frac{3}{4}$ is not part of the solution set.

Therefore the solution set is $\{x \in \mathbf{R} : x > -\frac{3}{4}\} = (-\frac{3}{4}, \infty)$.

<u>TUTORIAL SHEET</u> 6

1. Solve each of the following inequalities:

(a) $-2 \le \dfrac{5-3x}{4} \le \dfrac{1}{2}$

(b) $2t^2 - 9t - 5 > 0$

(c) $(x-3)(3x+2)(x+4) < 0$

(d) $\dfrac{a-1}{a-5} \le 2$

(e) $\dfrac{1}{x-2} > \dfrac{1}{x+3}$

2. Solve each of the following inequalities:

(a) $|6x-11| < 2$ (b) $|2x+3| \ge 4$ (c) $\left|\dfrac{x+1}{x-2}\right| < 5$ (d) $\left|\dfrac{n+2}{n}\right| \ge 3$ (e) $|x+6| < |x-2|$

(f) $|k| > |2k-1|$

3. Solve each of the following inequalities:

(a) $10 - \sqrt{2x+7} \le 3$

(b) $3 \le \sqrt{2x+5} < 6$

(c) $\sqrt{2x+9} - \sqrt{9+x} > 0$

(d) $\sqrt{2} - \sqrt{x+6} \le -\sqrt{x}$

(e) $\sqrt{x-3} > \sqrt{x+4} - 1$.

MAT1100
LECTURE NOTES

5. PARTIAL FRACTIONS

You will recall, at secondary school, you were always asked to express a sum (or difference) of two or more fractions into a single fraction. In this course you will be required to resolve a single fraction into simpler fractions called **partial fractions**.

Theoretically, every proper fraction can be expressed as a sum of simpler fractions whose denominators are of the form $(ax+b)^n$ and $(ax^2+bx+c)^n$ where n is a positive integer.

5.1 <u>Proper Fractions</u>

1. Denominator with linear factors.

In general, an expression with two linear terms in the denominator can be split into partial fractions of the form $\dfrac{A}{ax+b}+\dfrac{B}{cx+d}$, where *A*, *B*, *a*, *b*, *c* and *d* are constants.

<u>Example</u> 5.1.1 Resolve $\dfrac{x-1}{(x+1)(x+3)}$ into partial fractions:

<u>Solution</u>: $$\frac{x-1}{(x-1)(x+3)}=\frac{A}{x+1}+\frac{B}{x+3}=\frac{A(x+3)+B(x+1)}{(x+1)(x+3)}$$

$$\Rightarrow \quad x-1\equiv A(x+3)+B(x+1),$$

which is an identity. This identity is satisfied by all real values *x*. Because of this we can substitute *x* by any real number and find appropriate values of the constant *A* and *B*.

In this case, for example, since the identity is satisfied by any real number, we can take, say $x=-1$, so that $-1-1=A(-1+3)+B(-1+1)$

$\Rightarrow -2=2A \quad \Rightarrow A=-1$.

Notice that $x=-1$ was strategically chosen so that the other constant B in the identity vanishes.

Also when we choose $x=-3$ the constant A in the identity vanishes and obtain $-4=-2B \quad \Rightarrow B=2$.

However, if one chooses different values of *x*, then simultaneous equations in terms of *A* and *B* which will have to be solved and obtain the required values of *A* and *B*.

For example,

when $x=0$, we have $-1=3A+B$ or $3A+B=-1$ (I)
and when $x=1$, we have $0=4A+2B$ or $2A+B=0$ (II). Solving these two equations simultaneously we obtain the same values of *A* and *B*.

Therefore,

$$\frac{x-1}{(x-1)(x+3)}=\frac{-1}{x+1}+\frac{2}{x+3}=\frac{2}{x+3}-\frac{1}{x+1}.$$

Sometimes the denominator may have more than two linear factors.

Example 5.1.2 Resolve $\frac{2x^2-12x-26}{(x+1)(x-2)(x+5)}$ into partial fractions:

Solution:

$$\frac{2x^2-12x-26}{(x+1)(x-2)(x+5)}=\frac{A}{x+1}+\frac{B}{x-2}+\frac{C}{x+5}$$

$$=\frac{A(x-2)(x+5)+B(x+1)(x+5)+C(x+1)(x-2)}{(x+1)(x-2)(x+5)}$$

$$\Rightarrow \quad 2x^2-12x-26\equiv A(x-2)(x+5)+B(x+1)(x+5)+C(x+1)(x-2)$$

When $x=-1$, we have $2+12-26\equiv-12A\Rightarrow A=1$

When $x=2$, we have $8-24-26=21B \quad \Rightarrow B=-2$

When $x=-5$, we have $50+60-26=28C \quad \Rightarrow C=3$

Therefore,

$$\frac{2x^2-12x-26}{(x+1)(x-2)(x+5)}=\frac{1}{x+1}-\frac{2}{x-2}+\frac{3}{x+5}.$$

2. Denominator with repeated linear factors

If a denominator has a repeated factor $(ax+b)^n$, then the repeated fractions can be split as $\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\frac{A_3}{(ax+b)^3}+\cdots+\frac{A_n}{(ax+b)^n}$.

Example: Resolve $\frac{x^2+10x+5}{(x+1)^2(x-1)}$ into partial fractions:

Solution: $$\frac{x^2+10x+5}{(x+1)^2(x-1)}=\frac{A_1}{x+1}+\frac{A_2}{(x+1)^2}+\frac{B}{x-1}$$

$$=A_1(x-1)(x+1)+A_2(x-1)+B(x+1)$$

$$\Rightarrow \quad x^2+10x+5\equiv A_1(x-1)(x+1)+A_2(x-1)+B(x+1).$$

When $x=1$, we have $1+10+5\equiv 2B\Rightarrow B=8$

When $x=-1$, we have $1-10+5=-2A_2 \quad \Rightarrow A_2=2$

Replacing these values of A_2 and B into the identity it becomes

$$-x^2-10x-5 \equiv A_1(x-1)(x+1)+2(x-1)+8(x+1)$$

Now, since this is an identity, we can use any value of x different from $x=1$ and $x=-1$, and obtain the correct value of A_1. Say, when $x=0$, we have $5=-A_1-2+8 \quad \Rightarrow A_1=1$.

Therefore,

$$\frac{x^2+10x+5}{(x+1)^2(x-1)}=\frac{1}{x+1}+\frac{2}{(x+1)^2}+\frac{8}{x-1}.$$

3. Denominator with an irreducible quadratic factor

If the denominator has an irreducible quadratic factor of the form ax^2+bx+c, i.e. where $b-4ac<0$, then the partial fraction decomposition will contain a term of the form

$$\frac{Ax+B}{ax^2+bx+c},$$

where A and B are constants.

If the denominator has a quadratic factor of the form ax^2+bx+c raised to the power k, where $b-4ac<0$, then the partial fraction decomposition will contain a term of the form

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_kx+B_k}{(ax^2+bx+c)^k},$$

where $A_1, A_2, \cdots, A_k, B_1, B_2, \cdots, B_k$ are constants.

Example 5.1.3 Resolve $\dfrac{4x^2+6x-10}{(x+3)(x^2+x+2)}$ into partial fractions.

Solution: The expression x^2+x+2 cannot be split into linear factors. Thus

$$\frac{4x^2+6x-10}{(x+3)(x^2+x+2)}=\frac{A}{x+3}+\frac{Bx+C}{x^2+x+2}=\frac{A(x^2+x+2)+(Bx+C)(x+3)}{(x+3)(x^2+x+2)}$$

$$\Rightarrow \qquad 4x^2+6x-10 \equiv A(x^2+x+2)+(Bx+C)(x+3) \qquad \text{(I)}$$

When $x=-3$, $4(3)^2+6(-3)-10=8A \ \Rightarrow 8=8A \ \Rightarrow A=1$.
Thus, identity (I) becomes

$$4x^2+6x-10 \equiv x^2+x+2+(Bx+C)(x+3) \qquad \text{(II)}$$

When $x=0$, $-10=2+3C \ \Rightarrow C=-4$.
$\Rightarrow$ identity (II) becomes

$$4x^2+6x-10 \equiv x^2+x+2+(Bx-4)(x+3) \qquad \text{(III)}$$

When $x=1$, $0=4+(B-4)(4) \ \Rightarrow B=3$.

Therefore,

$$\frac{4x^2+6x-10}{(x+3)(x^2+x+2)}=\frac{1}{x+3}+\frac{3x-4}{x^2+x+2}.$$

5.2 Improper Fractions

An improper fraction is of the form $\frac{q(x)}{f(x)}$ where $q(x)$ and $f(x)$ are polynomials in which the degree of $q(x)$ is greater than that of $f(x)$.

For example, $\frac{2x^4 - x^3 + 3x + 4}{x^2 - x - 2}$ is an improper fraction. An improper fraction can be resolved into partial fractions by first expressing it in the form

$\frac{q(x)}{f(x)} = h(x) + \frac{r(x)}{f(x)}$, where $h(x)$ and $r(x)$ are polynomials.

Example 5.1.4 Resolve $\frac{2x^4 - x^3 + 3x + 4}{x^2 - x - 2}$ into partial fractions.

Solution: We first need to rewrite the improper fraction as a mixed fraction as,

$$\frac{2x^4 - x^3 + 3x + 4}{x^2 - x - 2} = 2x^2 + x + 5 + \frac{10x + 14}{x^2 - x - 2}.$$

Then we resolve $\frac{10x + 14}{x^2 - x - 2}$ into partial fractions as before:

$$\frac{10x + 14}{x^2 - x - 2} = \frac{10x + 14}{(x-2)(x+1)} = \frac{A}{(x-2)} + \frac{B}{(x+1)} = \frac{A(x+1) + B(x-2)}{(x-2)(x+1)}$$

Thus, $10x + 14 \equiv A(x+1) + B(x-2)$

When $x = 2, \quad 34 = 3A \quad \Rightarrow A = \frac{34}{3}$

When $x = -1, \quad 4 = -3B \quad \Rightarrow B = -\frac{4}{3}$

$$\Rightarrow \frac{10x + 14}{x^2 - x - 2} = \frac{34}{3(x-2)} - \frac{4}{3(x+1)}.$$

Therefore,

$$\frac{2x^4 - x^3 + 3x + 4}{x^2 - x - 2} = 2x^2 + x + 5 + \frac{34}{3(x-2)} - \frac{4}{3(x+1)}.$$

TUTORIAL SHEET 8

Resolve the following fractions into partial fractions:

1. $\dfrac{11x-10}{(x-2)(x+1)}$

2. $\dfrac{x^2-18x+5}{(x-1)(x+2)(x-3)}$

3. $\dfrac{10x^2-73x+144}{x(x-4)^2}$

4. $\dfrac{5x^2+3x+6}{x(x^2-x+3)}$

5. $\dfrac{x^2+2}{x^2-1}$

6. $\dfrac{x^3+x^2+2}{(x^2+2)^2}$.

MAT1100 LECTURE NOTES

4 Differential Calculus

4.1 Limits of functions

Finite limits

Example 4.1.1 Let the function f be defined by

$$f(x) = x + 2.$$

What happens to the value of the function $f(x)$ as x gets closer to and closer to 2 from the left?

x	1	1.5	1.75	1.8	1.9	1.97	1.99	1.9999
$f(x) = x + 2$	3	3.5	3.75	3.8	3.9	3.97	3.99	3.9999

Clearly as x approaches 2 from the left, the value of $f(x)$ get closer to 4.

Here we say that $f(x)$ tends to 4 as x approaches 2 from the left and we write

$$f(x) \to 4 \text{ as } x \to 2^-.$$

In general, if $f(x) \to L$ as $x \to c^-$, then we say that L is the left limit of $f(x)$ as x approaches c from the left, and this is written as

$$\lim_{x \to c^-} f(x) = L.$$

This is the **left limit** of $f(x)$.

We also know that x can approach 2 through values of x greater than 2 i.e. x can approach 2 from the right, as shown in the table below.

x	3	2.5	2.3	2.2	2.1	2.05	2.001	2.0001
$f(x) = x + 2$	5	4.5	4.3	4.2	4.1	4.05	4.001	4.0001

Again, $f(x)$ gets closer to 4 as x approaches 2 through values greater than 2. Here we say that $f(x)$ tends to 4 as x approaches 2 from the right and write

$$f(x) \to 4 \text{ as } x \to 2^+.$$

Similarly, in general, if $f(x) \to L$ as $x \to c^+$ then we say that L is the right limit of $f(x)$ as x approaches c from the right, and this is written as

$$\lim_{x \to c^+} f(x) = L.$$

This is the **right limit** of $f(x)$.

If $\lim_{x\to c^-} f(x) = L = \lim_{x\to c^+} f(x)$ we say that the limit of $f(x)$ as approaches c exists and it is L. This is simply written as

$$\lim_{x\to c} f(x) = L.$$

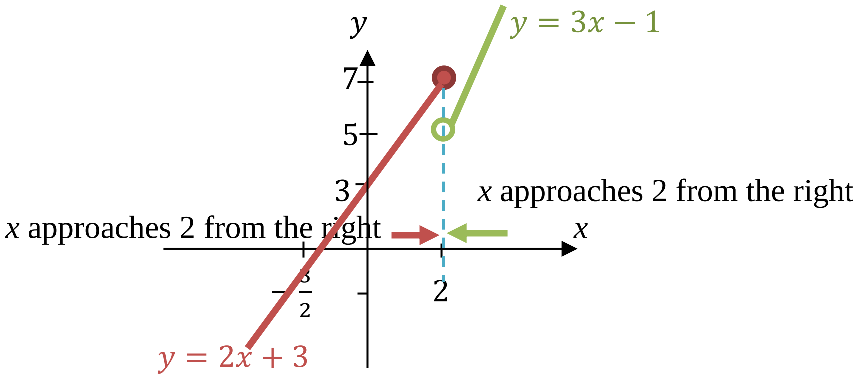
If $\lim_{x\to c^-} f(x) \neq \lim_{x\to c^+} f(x)$ then the limit of $f(x)$ as approaches c does not exists i.e. $\lim_{x\to c} f(x)$ does not exist.

Examples: Find the indicated limits. Hence state whether the limit exists or not.

1. $f(x) = \begin{cases} 2x+3, & x \leq 2 \\ 3x-1, & x > 2 \end{cases}$

   (a) $\lim_{x\to 2^+} f(x)$ (b) $\lim_{x\to 2^-} f(x)$

Solution:



(a) From the graph, we see that as x approaches 2 from the right, $f(x)$ tends to 5. Therefore,

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (3x-1) = 5.$$

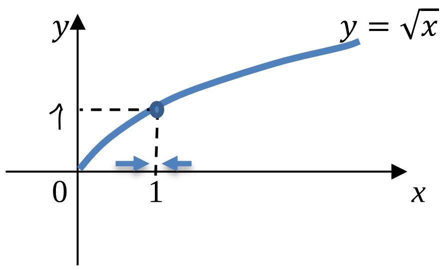
(b) From the graph, we see that as x approaches 2 from the left, $f(x)$ tends to 7. Therefore,

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (2x+3) = 7$$

Note that $\lim_{x\to 2} f(x)$ does not exist since the right limit is different from the left limit i.e. $\lim_{x\to 2^+} f(x) = 5 \neq 7 = \lim_{x\to 2^-} f(x)$.

2. $f(x) = \sqrt{x}$

   Solution:



(a) From the graph, we see that as x approaches 1 from the right, $f(x)$ tends to 1. Therefore,

$$\lim_{x\to 1^+} \sqrt{x} = 1.$$

(b) Also as x approaches 1 from the left, $f(x)$ tends to 1. Therefore,

$$\lim_{x\to 1^-} \sqrt{x} = 1.$$

Since, the left limit is equal to the right limit i.e.

$$\lim_{x\to 1^-} \sqrt{x} = 1 = \lim_{x\to 1^+} \sqrt{x},$$

$\lim_{x\to 1} \sqrt{x}$ exists and $\lim_{x\to 1} \sqrt{x} = 1$.

Theorem 4.1.1 If f and g are two functions such that

$$\lim_{x\to c} f(x) = L_1 \text{ and } \lim_{x\to c} g(x) = L_2,$$

where L_1 and L_2 are real number, then

1. $\lim_{x\to c}(f \pm g)(x) = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x) = L_1 \pm L_2$.

Example 4.1.2 $\lim_{x\to 1}(x^3 + 4x^2) = \lim_{x\to 1} x^3 + \lim_{x\to 1} 4x^2 = 1 + 4 = 5.$

2. $\lim_{x\to c}(f \times g)(x) = \lim_{x\to c} f(x) \times \lim_{x\to c} g(x) = L_1 \times .L_2$

Example 4.1.3 $\lim_{x\to -1}(2x+4)(x-1) = \lim_{x\to -1}(2x+4) \times \lim_{x\to -1}(x-1) = 2(-2) = -4$.

3. $\lim_{x\to c}\left(\dfrac{f}{g}\right)(x) = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \dfrac{L_1}{L_2}$, provided $L_2 \neq 0$.

Example 4.1.4 $\lim_{x\to 2} \dfrac{x^2}{2x-1} = \dfrac{\lim_{x\to 2} x^2}{\lim_{x\to 2}(2x-1)} = \dfrac{4}{3}$.

4. $\lim_{x\to c} \sqrt[p]{f(x)} = \sqrt[p]{L_1}$, provided $f(x) \geq 0$ when p is an even integer.

Example 4.1.5 $\lim_{x\to -7} \sqrt[3]{x-1} = \sqrt[3]{-8} = -2.$

Example 4.1.6 Evaluate the indicated limits

(a) $\lim_{x\to 2}(2x^3+4)$ (b) $\lim_{x\to 2}\sqrt{\dfrac{x^4-1}{x^3-1}}, x \neq 1$ (c) $\lim_{x\to 3} \dfrac{x^3-27}{x^2-x-6}, x \neq 3$.

Solutions:

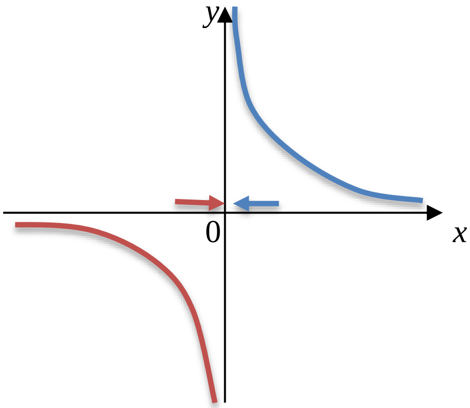
(a) $\lim_{x\to 2}(2x^3+4) = \lim_{x\to 2} 2x^3 + \lim_{x\to 2} 4 = 2(8) + 4 = 20.$

(b) $\lim_{x\to 2}\sqrt{\dfrac{x^4-2}{x^3-1}} = \lim_{x\to 2} \dfrac{\sqrt{x^4-2}}{\sqrt{x^3-1}} = \dfrac{\lim_{x\to 2}\sqrt{x^4-2}}{\lim_{x\to 2}\sqrt{x^3-1}} = \dfrac{\sqrt{14}}{\sqrt{7}} = \sqrt{\dfrac{14}{7}} = \sqrt{2}.$

(c) $\lim_{x\to3}\frac{x^3-27}{x^2-x-6}=\lim_{x\to3}\frac{(x-3)(x^2+3x+9)}{(x+2)(x-3)}=\lim_{x\to3}\frac{(x^2+3x+9)}{(x+2)}=\frac{\lim_{x\to3}(x^2+3x+9)}{\lim_{x\to3}(x+2)}=\frac{27}{5}$.

Infinite limits and limits at infinite

Consider the function $f(x)=\frac{1}{x}, x\neq 0$. The graph of $f(x)=\frac{1}{x}, x\neq 0$ is given below.

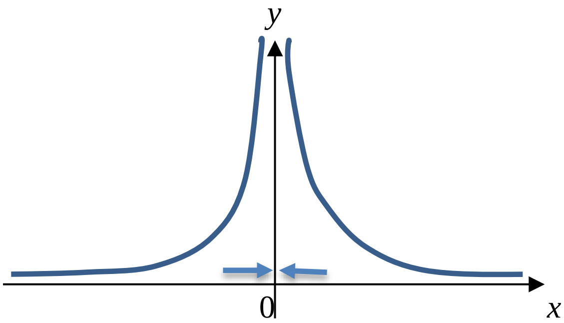


It can be seen from the graph that as $x\to 0^+, f(x)\to+\infty$. This means that $\lim_{x\to0^+}\frac{1}{x}=+\infty$. Similarly as $x\to0^-$, $f(x)\to-\infty$. This means that $\lim_{x\to0^-}\frac{1}{x}=-\infty$.

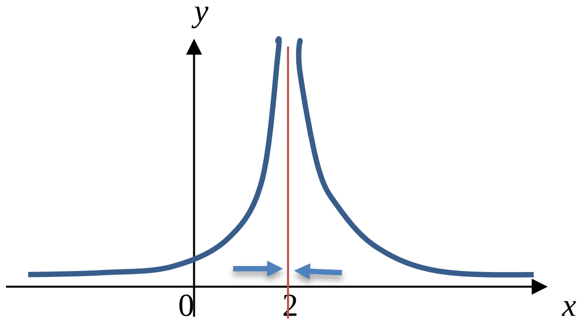
In this case, the function $f(x)=\frac{1}{x}, x\neq 0$ is said to have **infinite** limits as $x\to0$ either from the right or from the left.

However, note that since $\lim_{x\to0^+}\frac{1}{x}=+\infty\neq-\infty=\lim_{x\to0^-}\frac{1}{x}$, the limit $\lim_{x\to0}\frac{1}{x}$ does not exist.

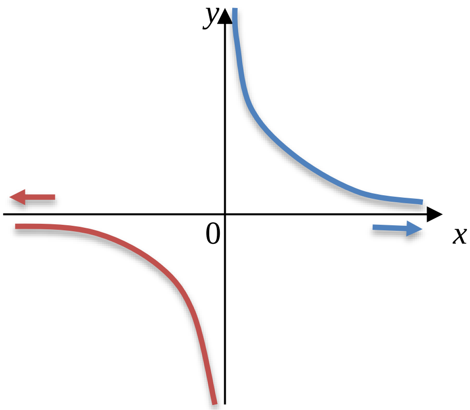
<u>Example</u> 4.1.7 1. $\lim_{x\to0}\frac{1}{x^2}=+\infty$ since, $\lim_{x\to0^-}x^2=+\infty=\lim_{x\to0^+}x^2$, as seen from the graph.



2. $\lim_{x\to2}\frac{1}{(x-2)^2}=+\infty$ since, $\lim_{x\to2^-}\frac{1}{(x-2)^2}=+\infty=\lim_{x\to2^+}\frac{1}{(x-2)^2}$, as seen from the graph.



Again from the graph below, as $x \to +\infty$, $f(x) \to 0$ implying that $\lim_{x\to+\infty} \frac{1}{x} = 0$.



Similarly, as $x \to -\infty$, $f(x) \to 0$ implying that $\lim_{x\to-\infty} \frac{1}{x} = 0$. These are **limits at infinity.**

Example: 4.1.7 Evaluate the indicated limits:

1. $\lim_{x\to+\infty} \frac{2x+5}{7x-3}, \quad x \neq \frac{3}{7}$ 2. $\lim_{x\to+\infty} \frac{\sqrt[3]{x^3+1}}{3x-5}, \quad x \neq \frac{5}{3}$.

Solutions:

1. $$\lim_{x\to+\infty} \frac{2x+7}{7x-3} = \lim_{x\to+\infty} \frac{x\left(2+7\left(\frac{1}{x}\right)\right)}{x\left(7-3\left(\frac{1}{x}\right)\right)} = \lim_{x\to+\infty} \frac{\left(2+7\left(\frac{1}{x}\right)\right)}{\left(7-3\left(\frac{1}{x}\right)\right)} = \frac{\lim_{x\to+\infty}\left(2+7\left(\frac{1}{x}\right)\right)}{\lim_{x\to+\infty}\left(7-3\left(\frac{1}{x}\right)\right)}$$

$$= \frac{2+7(0)}{7+3(0)} = \frac{2}{7}.$$

2. $$\lim_{x\to+\infty} \frac{\sqrt[3]{x^3+1}}{3x-5} = \lim_{x\to+\infty} \frac{x\left(\sqrt[3]{1+\frac{1}{x^3}}\right)}{x\left(3-5\left(\frac{1}{x}\right)\right)} = \lim_{x\to+\infty} \frac{\left(\sqrt[3]{1+\left(\frac{1}{x}\right)^3}\right)}{\left(3-5\left(\frac{1}{x}\right)\right)} = \frac{\lim_{x\to+\infty}\left(\sqrt[3]{1+\left(\frac{1}{x}\right)^3}\right)}{\lim_{x\to+\infty}\left(3-5\left(\frac{1}{x}\right)\right)}$$

$$= \frac{\sqrt[3]{1+(0)^3}}{(3-5(0))} = \frac{1}{3}.$$

4.2 Continuity of a Function

Definition: A function f is said to be **continuous** at a point $x = c$ if and only if

(a) $f(c)$ exists, i.e. f is defined at c,

(b) $\lim_{x\to c} f(x)$ exists i.e. $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x)$, and

(c) $\lim_{x\to c} f(x) = f(c)$.

When a function f is not continuous at c we say that f is **discontinuous** at c.

If a function is continuous at every point of interval I, it is said to be continuous on I.

Example 4.2.1 Let f be a function defined by

1. $f(x) = \begin{cases} 2, \text{ if } & x \geq 2 \\ -2, \text{ if } & x < 2 \end{cases}$. Is f continuous at $x = 2$?

2. $f(x) = \begin{cases} \frac{x^2+x-2}{x-1}, & \text{if } x \neq 1 \\ 3, & \text{if } x = 1 \end{cases}$, . Is f continuous at $x = 1$?

3. $f(x) = \dfrac{1}{x+1}$

(a) Is f continuous on $I_1 = \{x : 0 < x < 4\}$?

(b) Is f continuous on $I_2 = \{x : -3 < x < 4\}$?

Solutions:

1. By definition,

(a) $f(2) = 2 \Rightarrow f$ is defined at $x = 2$

(b) $\lim_{x\to 2^-} f(x) = -2$ and $\lim_{x\to 2^+} f(x) = 2$.

Since $\lim_{x\to 2^-} f(x) = -2 \neq 2 = \lim_{x\to 2^+} f(x)$, the limit $\lim_{x\to 2} f(x)$ does not exist.

Therefore, since one condition in the definition is not satisfied, the function

$f(x) = \begin{cases} 2, \text{ if } & x \geq 2 \\ -2, \text{ if } & x < 2 \end{cases}$ is not continuous at $x = 2$.

2. By definition,

(a) $f(1) = 3 \Rightarrow f$ is defined at $x = 1$

(b) $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{x^2+x-2}{x-1} = \lim_{x\to 1^-} \frac{(x+2)(x-1)}{(x-1)} = \lim_{x\to 1^-} (x+2) = 3$ and

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{x^2+x-2}{x-1} \lim_{x\to 1^+} \frac{(x+2)(x-1)}{(x-1)} = \lim_{x\to 1^+} (x+2) = 3.$

Since $\lim_{x\to 1^-} f(x) = 3 = \lim_{x\to 1^+} f(x)$, the limit $\lim_{x\to 1} f(x)$ exists, and

(c) $\lim_{x\to 1} f(x) = 3 = f(1)$.

Therefore, since all the conditions in the definition are satisfied, the function

$f(x)=\begin{cases}\dfrac{(x+2)(x-1)}{x-1}, & if \;\; x\neq 1\\ 3, & if \;\; x=1\end{cases}$ is f continuous at $x = 1$.

3. (a) Let c be any number in I_1. Then $f(c)=\dfrac{1}{c+1}$. This means that $f(x)$ is defined on I_1 and $\lim_{x\to c^-} f(x) = \frac{1}{c+1} = \lim_{x\to c^+} f(x) = \frac{1}{c+1}$ for every $c \in I_1$.⇒ $\lim_{x\to c}\dfrac{1}{x+1} = \frac{1}{c+1}$, and $\lim_{x\to c}\dfrac{1}{x+1} = f(c)$. Since all the conditions in the definition are satisfied for every $c \in I_1$, the function $f(x)=\dfrac{1}{x+1}$ is continuous on I_1.

(b) $f(x)=\dfrac{1}{x+1}$ is not defined at $x = -1$. Since the function is not continuous at a point $x = -1 \in I_2$, $f(x)$ is not continuous on I_2.

Properties of Continuous Functions

Suppose f and g are functions which are both continuous at c. Then

1. $f(x) \pm g(x)$ is continuous at c.
2. $f(x) \cdot g(x)$ is continuous at c.
3. $f(x)/g(x)$ is continuous at c, provided $g(c) \neq 0$.

4.3 The Derivative

Let the function f be defined by $y = f(x)$. Suppose x is increased by h. Then we shall denote the change in x by $\Delta x = h$, and then corresponding change in $y = f(x)$ will be $\Delta y = f(x + h) - f(x)$.

Thus the difference quotient is given by $\dfrac{\Delta y}{\Delta x} = \frac{f(x+h)-f(x)}{h}$.

Definition 4.3.1 Let the function f be defined by $y = f(x)$. Then from the first principle, the derivative of f with respect to x in an open interval in the domain of the function, is denoted by $\frac{dy}{dx} = f'(x)$ and is defined by

$$\frac{dy}{dx} = f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h},$$

provided the limit exists.

Example 4.3.1 Find the derivative of each of the following functions, using the first Principle:

1. $f(x) = x$
2. $f(x) = x^2$
3. $f(x) = x^3$
4. $f(x) = \frac{1}{x}, x \neq 0$
5. $f(x) = \sqrt{x}$

Solutions:

1. $f(x) = x \Rightarrow f(x+h) = x + h.$

Thus, $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{x+h-x}{h} = \lim_{h\to 0} \frac{h}{h} = \lim_{h\to 0} 1 = 1.$

2. $f(x) = x^2 \Rightarrow f(x+h) = (x+h)^2.$

Thus, $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2-x^2}{h} = \lim_{h\to 0} \frac{x^2+2xh+h^2-x^2}{h}$

$= \lim_{h\to 0} \frac{2xh+h^2}{h} = \lim_{h\to 0} \frac{h(2x+h)}{h} \lim_{h\to 0}(2x+h) = 2x.$

3. $f(x) = x^3 \Rightarrow f(x+h) = (x+h)^3.$

Thus, $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^3-x^3}{h} = \lim_{h\to 0} \frac{(x+h)(x+h)^2-x^3}{h}$

$= \lim_{h\to 0} \frac{(x+h)(x^2+2xh+h^2)-x^3}{h} = \lim_{h\to 0} \frac{x^3+3x^2h+3xh^2+h^3-x^3}{h}$

$= \lim_{h\to 0} \frac{3x^2h+3xh^2+h^3}{h} = \lim_{h\to 0} \frac{h(3x^2+3xh+h^2)}{h}$

$= \lim_{h\to 0}(3x^2 + 3xh + h^2) = 3x^2.$

4. $f(x) = \frac{1}{x} \Rightarrow f(x+h) = \frac{1}{x+h}.$

Thus, $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h}-\frac{1}{x}}{h} = \lim_{h\to 0} \frac{\frac{x-(x+h)}{x(x+h)}}{h} = \lim_{h\to 0} \frac{x-(x+h)}{hx(x+h)}$

$= \lim_{h\to 0} \frac{-h}{hx(x+h)} = \lim_{h\to 0} \frac{-1}{x^2+xh)} = -\frac{1}{x^2}.$

5. $f(x) = \sqrt{x} \Rightarrow f(x+h) = \sqrt{x+h}.$

Thus, $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}$

$= \lim_{h\to 0} \frac{(\sqrt{x+h}-\sqrt{x})}{h} \times \frac{(\sqrt{x+h}+\sqrt{x})}{(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0} \frac{(x+h-x)}{h(\sqrt{x+h}+\sqrt{x})}$

$$= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h}+\sqrt{x})} = \frac{1}{2\sqrt{x}}.$$

Rules of Differentiation

1. The derivative of any constant (number) is 0.

   Proof: Exercise.

Example 4.3.2 In general, if $f(x) = k$, where k is a constant, then $f'(x) = 0$.

2. If f is a function defined by $f(x) = x^n$, where $n \in \mathbb{Q}$, then

$$f'(x) = nx^{n-1}.$$

Proof: $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$

$$= \lim_{h\to 0} \frac{x^n + nx^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n - x^n}{h}$$

$$= \lim_{h\to 0} \frac{h\left(nx^{n-1} + \frac{n(n-1)}{2!}x^{n-2}h^3 + \cdots + nxh^{n-2} + h^{n-1}\right)}{h}$$

$$= nx^{n-1}.$$

Exercise 4.3.1 Verify the derivatives of the functions in Example 4.3.1 using the formula.

3. If f is a function defined by $f(x) = cx^n$, where $n \in \mathbb{Q}$, and c is a constant then

$$\boxed{f'(x) = ncx^{n-1}.}$$

   Proof: Exercise.

   Example 4.3.3: Find $f'(x)$ given that

   (a) $f(x) = -6x^{\frac{1}{3}}$ (b) $f(x) = 2x^{-4}$.

   Solutions:

   (a) $f'(x) = (-6)\left(\frac{1}{3}\right)x^{\frac{1}{3}-1} = -2x^{-\frac{2}{3}}$.

   (b) $f'(x) = 2(-4)x^{-4-1} = -8x^{-5}$.

4. If f is a function defined by $f(x) = u(x) \pm v(x)$, where u and v are differentiable functions of x, then the derivative of f is given by

$$\boxed{f'(x) = u'(x) \pm v'(x).}$$

   Proof: Exercise.

   Example 4.3.4 Find $f'(x)$ if $f(x) = 5x^2 + 4x^{-7}$.

   Solution: Let $u(x) = 5x^2$ and $v(x) = 4x^{-7}$. Then

$f'(x) = u'(x) + v'(x) = 5(2)x + 4(-7)x^{-8} = 10x - 28x^{-8}.$

5. The Product Rule: Let f be a function defined by $f(x) = u(x) \cdot v(x)$, where u and v are differentiable functions of x, then derivative of f is given by

$$\boxed{f'(x) = u'(x) \cdot v(x) + u(x) \cdot v(x).}$$

This is the **product rule** for differentiation.

Proof: $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h\to 0} \frac{u(x+h)v(x+h) - u(x)v(x)}{h}$$

$$= \lim_{h\to 0} \frac{u(x+h)v(x+h) - v(x)u(x+h) + v(x)u(x+h) - u(x)v(x)}{h}$$

$$= \lim_{h\to 0} \frac{v(x)[u(x+h) - u(x)] + u(x+h)[v(x+h) - v(x)]}{h}$$

$$= \lim_{h\to 0} \frac{v(x)[u(x+h) - u(x)]}{h} + \lim_{h\to 0} \frac{u(x+h)[v(x+h) - v(x)]}{h}$$

$$= \lim_{h\to 0} v(x) \cdot \lim_{h\to 0} \frac{u(x+h) - u(x)}{h} + \lim_{h\to 0} u(x+h) \cdot \lim_{h\to 0} \frac{v(x+h) - v(x)}{h}$$

$$= v(x)u'(x) + u(x)v'(x).$$

Example 4.3.5 Find $f'(x)$ if $f(x) = (3x^3 - 4x^5) \cdot (7x + 2x^2)$.

Solution: Let $u(x) = 3x^3 - 4x^5$ and $v(x) = 7x + 2x^2$. Then

$$f'(x) = v(x)u'(x) + u(x)v'(x)$$

$$= (7x + 2x^2)(9x^2 - 20x^4) + (3x^3 - 4x^5)(7 + 4x).$$

6. The Quotient Rule: Let f be a function defined by $f(x) = u(x)/v(x)$, where u and v are differentiable functions of x, then derivative of f is given by

$$\boxed{f'(x) = \frac{v(x)u'(x) - u(x)v'(x)}{[v(x)]^2}.}$$

This is the **quotient rule** for differentiation.

Proof: $f'(x) = \lim_{h\to\infty} \frac{1}{h}\left[\frac{u(x+h)}{v(x+h)} - \frac{u(x)}{v(x)}\right] = \lim_{h\to\infty} \frac{1}{h}\left[\frac{v(x)u(x+h) - u(x)v(x+h)}{v(x)v(x+h)}\right]$

$$= \lim_{h\to\infty} \frac{1}{h}\left[\frac{v(x)u(x+h) - u(x)v(x) + u(x)v(x) - u(x)v(x+h)}{v(x)v(x+h)}\right]$$

$$= \lim_{h\to\infty}\left[\frac{v(x)[u(x+h)-u(x)]}{h} - \frac{u(x)[v(x+h)-v(x)]}{h}\right] \times \frac{1}{v(x)v(x+h)}$$

$$= v(x)\left[\lim_{h\to\infty}\frac{u(x+h)-u(x)}{h} - u(x)\lim_{h\to\infty}\frac{v(x+h)-v(x)}{h}\right] \times \frac{1}{v(x)v(x+h)}$$

$$\therefore \qquad f'(x) = \frac{v(x)u'(x) - u(x)v'(x)}{[v(x)]^2}$$

Example 4.3.6 If $f(x) = \frac{(x^2+1)}{(2x-1}$, we let $u(x) = x^2 + 1$ and $v(x) = 2x - 1$

so that $u'(x) = 2x$ and $v'(x) = 2$. Then

$$f'(x) = \frac{v(x)u'(x) - u(x)v'(x)}{[v(x)]^2} = \frac{(2x-1)\times 2x - (x^2+1)\times 2}{(2x-1)^2}$$

$$= \frac{4x^2 - 2x - 2x^2 - 2}{(2x-1)^2} = \frac{2x^2 - 2x - 2}{(2x-1)^2} = \frac{2(x^2 - x - 1)}{(2x-1)^2}.$$

7. The Chain Rule

Suppose $h(x) = f[g(x)]$ in which $f(x)$ is differentiable at $g(x)$ and $g(x)$ is differentiable at *x*. Then $h(x)$ is also differentiable at *x* and

$$h'(x) = f'[g(x)] \times g'(x).$$

This is called the **chain rule** for differentiation and it is used to find the derivative of a function of another function (composite of two functions).

Note: If $= f[g(x)]$, we let $u = g(x)$ so that $y = f(u)$. Then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx},$$

where $\frac{dy}{du} = f'(u)$ and $\frac{du}{dx} = g'(x)$. Therefore

$$y' = f'(u) \times g'(x).$$

Example 4.3.7 Suppose $y = (3x^2 - 2x + 5)^{25}$. We let $u(x) = 3x^2 - 2x + 5$

so that $y = u^{25}$. Then $y' = 25u^{24}$ and $u'(x) = 6x - 2$.

Therefore,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 25u^{24} \times (6x - 2) = 25(3x^2 - 2x + 5)^{24}(6x - 2)$$

. $$= 25(6x - 2)(3x^2 - 2x + 5)^{24}.$$

8. <u>Differentiation of Implicit Functions</u>

All differentiation carried out so far involved equations of the form $y = f(x)$. But the equation $y + xy + y^2 = 2$, (say) do exist but cannot easily be expressed in the form $y = f(x)$. In such a case we say that $y = f(x)$ is implied by the equation $y + xy + y^2 = 2$. Such functions are called implicit functions.

Now, if $y = f(x)$, equations such as $y + xy + y^2 = 2$ may be written as

$$f(x) + xf(x) + [f(x)]^2 = 2.$$

This equation can be differentiated term by term and have

$$f'(x) + [f(x) + xf'(x)] + 2[f(x)]f'(x) = 0.$$

Since $\frac{dy}{dx} = f'(x)$ we can rewrite the new equation as

$$\frac{dy}{dx} + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0.$$

Making $\frac{dy}{dx}$ the subject of the formula we have

$$\frac{dy}{dx} = \frac{-y}{1 + x + 2y}.$$

Thus, we say that we have differentiated implicitly.

<u>Example</u> 4.3.8 Find $\frac{dy}{dx}$ given that $x^2 - xy^2 + y^3 = 5$.

<u>Solution</u>: We differentiate term by term

$$\frac{d}{dx}(x^2) - \frac{d}{dx}(xy^2) + \frac{d}{dx}(y^3) = \frac{d}{dx}(5)$$

and obtain

$$2x - \left[y^2 + 2xy\frac{dy}{dx}\right] + 3y^2\frac{dy}{dx} = 0$$

$$[3y^2 - 2xy]\frac{dy}{dx} = y^2 - 2x.$$

Therefore, making $\frac{dy}{dx}$ the subject of the formula we obtain

$$\frac{dy}{dx} = \frac{y^2 - 2x}{3y^2 - 2xy}.$$

<u>TUTORIAL SHEET</u> 7

1. Sketch the graph of the following functions defined and intuitively determine the indicated limits:

(a) $f(x) = 3x^2$: (i) $\lim_{x\to 0^+} f(x)$ (ii) $\lim_{x\to 0^-} f(x)$

(b) $f(x) = 2x^2 + 4$: (i) $\lim_{x\to 2^+} f(x)$ (ii) $\lim_{x\to 3^-} f(x)$

(c) $f(x) = \begin{cases} 2x+3, & x \le 2 \\ 3x-1, & x > 2 \end{cases}$ (i) $\lim_{x\to 1^+} f(x)$ (ii) $\lim_{x\to 1^-} f(x)$. Does the limit $\lim_{x\to 1} f(x)$ exist?

(d) $f(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & x \ne 2 \\ 5, & x = 2 \end{cases}$ (i) $\lim_{x\to 2^+} f(x)$ (ii) $\lim_{x\to 2^-} f(x)$

Does the limit $\lim_{x\to 2} f(x)$ exist?

2. Evaluate each of the following limits:

(a) $\lim_{x\to -5} \dfrac{x^2}{2x-3}$ (b) $\lim_{x\to 2} \dfrac{x^3-8}{x-2}$ (c) $\lim_{x\to -2} \dfrac{x^3+8}{x+2}$ (d) $\lim_{x\to 2} \sqrt{\dfrac{x^2-4}{x^2-3x+2}}$

(e) $\lim_{x\to 0}\left(\dfrac{1}{x} - \dfrac{1}{x^2}\right)$

3. Evaluate each of the following limits:

(a) $\lim_{x\to +\infty} \dfrac{5x-1}{-x+4}$ (b) $\lim_{x\to +\infty} \dfrac{x^2+3x+7}{3x^2-2x-1}$ (c) $\lim_{x\to -\infty} \dfrac{\sqrt{x^2+6}}{x+6}$ (d) $\lim_{x\to 2} \sqrt{\dfrac{x^2-4}{x^2-3x+2}}$

4. Sketch the graph of each of the following functions defined and determine if each is continuous at the given point *c*:

(a) $f(x) = \sqrt{3x-1}$ at $c = \frac{1}{3}$ (b) $f(x) = \begin{cases} |x-2|, & x \ne 2 \\ 1, & x = 2 \end{cases}$ at $c = 2$

(c) $f(x) = \begin{cases} 2x+1, & x \le 1 \\ -x+4, & x > 1 \end{cases}$ at $c = 1$ (d) $f(x) = \begin{cases} x+2, & x \le -1 \\ x^2, & -1 < x < 1 \\ 3x+1, & x \ge 1 \end{cases}$ at $c = -1, 1$.

5. A function *f* is defined on an interval *I*. Determine whether *f* is continuous on *I*.

(a) $f(x) = \dfrac{1}{x^2+1}$, $I = \{x : -1 < x < 5\}$ (b) $f(x) = \dfrac{x}{x-1}$, $I = \{x : -3 < x < 1\}$

(c) $f(x) = \dfrac{x^2+3x-1}{2x^2+3}$, $I = \{x : -2 < x < 2\}$

(d) $f(x)=\begin{cases} \dfrac{x^2+x-2}{x+2}, & x\neq -2 \\ -3 & x=-2 \end{cases}$, $I=\{x: -3<x<3\}$.

6. Define $f(c)$ so that the function is continuous:

(a) $f(x)=\dfrac{(x-1)(x+2)}{x-1}$, $x\neq 1, c=1$ (b) $f(x)=\dfrac{x^2-3x+2}{x^2-4}$, $x\neq \pm 2, c=2$

(c) $f(x)=\dfrac{x^2-1}{x+1}$, $x\neq -1, c=-1$ (d) $f(x)=\dfrac{x^3+8}{x+2}$, $x\neq -2, c=-2$.

7. Find the derivative of each of the following functions from first principle:

(a) $f(x)=\sqrt{x+1}$ (b) $f(x)=3x^2+4x-8$ (c) $f(x)=\dfrac{1}{\sqrt{1-x}}$ (d) $f(x)=\dfrac{1}{x-3}$.

8. Differentiate with respect to x:

(a) x^4-9x^3+6 (b) $x^3\sqrt{x-2}$ (c) $\dfrac{x^2-7x+4}{x^3+2}$ (d) $\left(3x^3-x^2\right)^6$ (e) $\dfrac{(x^2+3)^2}{x^{3/2}}$.

9. Use implicit differentiation to find $\frac{dy}{dx}$ of each of the following functions:

(a) $x^2+y^2+8x-2y-8=0$ (b) $2x^2+2y^2-3x+2y+1=0$

(c) $x^3+xy-x^2y^2=7$.

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LECTURE NOTES

6 Integral Calculus I

6.1 The Indefinite Integral

6.1.1 Integration as the reverse process of differentiation

Recall that when a function $f(x)$ is differentiated $f'(x)$ is obtained. In indefinite integral when a function $f'(x)$ is given, we find a function which was differentiated to give $f'(x)$.

The symbol for integration is $\int$. For example, for $f'(x) = nx^{n-1}$, to find $f(x)$ we write $\int (nx^{n-1})dx = f(x)$. In general, if $\frac{dy}{dx} = f'(x)$ then $\int f'(x)dx = y$.

Now, the process of finding the function which was differentiated is called **integration**, and clearly it is the reverse operation of differentiation.

<u>Example</u> 6.1.1 Find what was differentiated to give us $2x$.

Here. clearly, the majority of people will say it was x^2. Others would say it was $x^2 + 1$, May be you may say $x^2 - 29$. All these are correct answers. Thus, we can receive an infinite number of correct answers. Even $x^2 + c$, where c is a constant, is also a correct answer. $f(x) = x^2 + c$ is a general answer for the integral of $2x$. Therefore,

$$\int 2x dx = x^2 + c,$$

and c is called the ***constant of integration***.

Similarly,

$$\boxed{\int f'(x)dx = f(x) + c.}$$

<u>Definition</u> 6.1.1 Suppose that $\dfrac{dy}{dx} = f(x), \quad a < x < b$. Then a function $F(x) = \int f(x)dx$ is called the **indefinite integral** of $f(x)$.

<u>Note:</u> If $F(x)$ is an integral of $f(x)$ with respect to x then $F(x)+c$ where c is any constant is also such an integral. Therefore all indefinite integrals of $f(x)$ are contained in the formula $y = F(x)+c$. This is the more reason why the indefinite integral of $f(x)$ is written in the form

$$\boxed{\int f(x)dx = F(x)+c.}$$

6.1.2 Fundamental integration formulae

A number of fundamental integration formulae below follow immediately from the standard differentiation formulae, while other may be checked by differentiation. These formulae are important in the evaluation of some integrals.

1. If k is any constant then

$$\int kdx = kx + c.$$

2. $\int x^n dx = \frac{1}{n+1}x^{n+1} + c, n \neq -1.$

   Example: $\int x^{14}\, dx = \frac{1}{15}x^{15} + c$

3. $\int \frac{1}{x}dx = \ln|x| + c$, since $\frac{d}{dx}(\ln x) = \frac{1}{x}$. (This will be proved when we do Further Differential Calculus.)

4. $\int \frac{1}{x+k}dx = \ln|x + k| + c$. (This follows from 3.)

5. $\int kf(x)dx = k\int f(x)dx$, where k is any constant.

   Example: $\int 4x^3\, dx = 4\int x^3\, dx = 4\left(\frac{1}{4}\right)x^4 + c = x^4 + c.$

6. $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$

   Examples: (a) $\int (x^2 - 5x^3)\, dx = \int x^2 dx - 5\int x^3 dx$

$$= \frac{1}{3}x^3 - \frac{5}{4}x^4 + c.$$

   (b) $\int (3x + x^2 - 6)\, dx = 3\int xdx + \int x^2\, dx - 6\int dx$

$$= \frac{3}{2}x^2 + \frac{1}{3}x^3 - 6x + c$$

Miscellaneous Examples

Evaluate each of the following indefinite integrals:

1. $\int \sqrt[3]{x^2}dx.$

   Solution: $\int \sqrt[3]{x^2}dx = \int (x^2)^{\frac{1}{3}}dx = \int x^{\frac{2}{3}}dx = \frac{3}{5}x^{\frac{5}{3}} + c$

2. $\int (3 - 2x)^2 dx.$

   Solution: $\int (3 - 2x)^2 dx = \int (9 - 12x + 4x^2)dx$

$$= 9x - 6x^2 + \frac{4}{3}x^3 + c.$$

3. $\int \frac{x^2-4}{x-2}dx.$

   Solution: $\int \frac{x^2-4}{x-2}dx = \int \frac{(x-2)(x+2)}{x-2}dx = \int (x + 2)dx = \frac{1}{2}x^2 + 2x + c.$

4. $\int \frac{x^3-2}{\sqrt{x}} dx.$

Solution: $\int \frac{x^3-2}{\sqrt{x}} dx = \int \frac{x^3-2}{x^{\frac{1}{2}}} dx = \int x^{-\frac{1}{2}}(x^3-2)dx$

$$= \int \left(x^{\frac{5}{2}} - 2x^{-\frac{1}{2}}\right) dx$$

$$= \int x^{\frac{5}{2}}dx - 2\int x^{-\frac{1}{2}}dx$$

$$= \frac{2}{7}x^{\frac{7}{2}} - 2\left(\frac{2}{1}x^{\frac{1}{2}}\right) = \frac{2}{7}x^{\frac{7}{2}} - 4x^{\frac{1}{2}} + c.$$

5. $\int \frac{x-4}{\sqrt{x}+2} dx.$

Solution: $\int \frac{x-4}{\sqrt{x}+2} dx = \int \frac{(\sqrt{x}+2)(\sqrt{x}-2)}{\sqrt{x}+2} dx = \int(\sqrt{x}-2)dx$

$$= \int x^{\frac{1}{2}}dx - 2\int dx$$

$$= \frac{2}{3}x^{\frac{3}{2}} - 2x + c.$$

6. $\int \frac{x^2+3x-4}{x-2} dx$

Solution: We can rewrite the integrand as a mixed fraction by dividing $x-2$ into into x^2+3x-4:

$$\begin{array}{r l}
 & x+5 \\
x-2 \,\big) & x^2+3x-4 \\
 & -(x^2-2x) \\
 & \quad 5x-4 \\
 & \quad -(5x-10) \\
 & \qquad 6
\end{array}$$

$\Rightarrow \qquad \frac{x^2+3x-4}{x-2} = x+5+\frac{6}{x-2}$

$\therefore \quad \int \frac{x^2+3x-4}{x-2} dx = \int \left(x+5+\frac{6}{x-2}\right) dx$

$$= \int x\,dx + \int 5\,dx + \int \frac{6}{x-2} dx$$

$$= \int x\,dx + 5\int dx + 6\int \frac{1}{x-2} dx$$

$$= \frac{1}{2}x^2 + 5x + 6\ln|x-2| + c$$

6.2 The Definite Integral

Integration can be used to find the area under the graph of a function on an interval $[a,b]$. It can also be used to find volumes, central points and so many other useful things.

Definition 6.2.1 If $\int f(x)dx = F(x)+c$, and f is a continuous function on the interval $[a,b]$ then the definite integral of $f(x)$ from $x=a$ to $x=b$ is written as $\int_a^b f(x)\,dx$ and is given by

$$\boxed{\int_a^b f(x)\,dx = [F(x)+c]_a^b = F(b)-F(a)\ .}$$

The numbers a and b are called the **lower** and **upper limits** of integration, respectively.

Example: Evaluate the following definite integrals:

1. $\int_2^5 (2x+3)\,dx$.

Solution: $\int_2^5 (2x+3)\,dx = [x^2+3x]_2^5 = (5^2+3(5))-(2^2+3(2)) = 30$.

2. $\int_0^1 (3x^2+2)^2\,xdx$.

Solution: $\int_0^1 (3x^2+2)^2 dx = \int_0^1 (9x^4+12x^2+4)dx$

$$= (\tfrac{9}{5}x^5+4x^3+4x)_0^1$$

$$= \left(\tfrac{9}{5}(1)^5+4(1)^3+4(1)\right)-\left(\tfrac{9}{5}(0)^5+4(0)^3+4(0)\right)$$

$$= \tfrac{9}{5}+4+4 = \tfrac{49}{5}.$$

6.2.1 Properties of Definite Integrals

If $f(x)$ and $g(x)$ are continuous functions on the interval of integration $[a,b]$, then

1. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$

Example: $\int_1^3 3x^2\,dx = 3\int_1^3 x^2 dx = 3\left(\tfrac{1}{3}\right)(x^3)_1^3 = (x^3)_1^3 = (3)^3-(1)^3$

$$= 27-1 = 26.$$

2. $\int_a^b [f(x)\pm g(x)]dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

Example: $\int_2^5 (2x+3)\,dx = \int_2^5 2xdx + 3\int_2^5 dx = (x^2)_2^5 + 3(x)_2^5$

$$= (5)^2-(2)^2+3[5-2]$$
$$= 25-4+3(3)$$
$$= 30.$$

3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where $a < c < b$.

Example: $\int_1^3 3x^2\,dx = (x^3)_1^3 = (3)^3 - (1)^3 = 27 - 1 = 26.$

$$\int_1^2 3x^2 dx + \int_2^3 3x^2 dx = (x^3)_1^2 + (x^3)_2^3$$

$$= [(2)^3 - (1)^3] + [(3)^3 - (2)^3]$$

$$= [8 - 1] + [27 - 8]$$

$$= 7 + 19 = 26 = \int_1^3 3x^2\,dx.$$

4. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$.

Example: $\int_1^3 3x^2\,dx = 26.$

$$\int_3^1 3x^2\,dx = (x^3)_3^1 = (1)^3 - (3)^3 = 1 - 27 = -26$$

$$= -\int_1^3 3x^2\,dx.$$

5. $\int_a^a f(x)\,dx = 0$.

Example: $\int_4^4 \frac{1}{x}\,dx = (\ln x)_4^4 = \ln 4 - \ln 4 = 0$

<u>TUTORIAL SHEET</u> 9

1. Integrate the following functions with respect to *x*:

(a) $\frac{4}{x^3} - \frac{1}{x^2} - x^2$ (b) $6\sqrt{x} - 3x^3 + x^{-2} + 2$

2. Evaluate the following integrals:

(a) $\int (9x^5 + 1)dx$ (b) $\int \frac{1}{\sqrt[3]{x}} dx$ (c) $\int \left(7x^3 - \frac{4}{x^2}\right) dx$ (d) $\int 2x(3x^3 - 5x^2 + 1)dx$

(e) $\int \frac{5x+4}{\sqrt[3]{x}} dx$ (f) $\int (2t + 1)(t^2 + t)dt$ (g) $\int u(3 - 4u^2)^2\, du$

(h) $\int (3x^2 - 2x)^2(x^2 - 1)dx$ (i) $\int x^2\sqrt{x}dx$ (j) $\int \sqrt{\frac{x+3}{x^3}} dx.$

3. Find the definite integrals:

(a) $\int_2^5 (4 - 3x)dx$ (b) $\int_{-1}^0 (t^{1/t} - t^{2/3})dt$ (c) $\int_{-1}^1 (3y + 4)^2 dy$

(d) $\int_1^4 \left(\sqrt{t} + 4\right)dt$ (e) $\int_0^3 (v - 2)^2 dv$ (f) $\int_0^2 \frac{(3x-4)^2}{3x+1} dx.$

MAT1100

<u>LECTURE NOTES</u>

7. BINOMIAL EXPANSIONS

7.1 Binomial Expansion for $(\boldsymbol{a}+\boldsymbol{b})^{\boldsymbol{n}}$ for $\boldsymbol{n} \in \mathbb{Z}^{+}$

A binomial is the sum (or difference) of two terms. For example, $a+b,\ 3x+5y,$ p^2-2q are all binomials.

It is sometimes necessary to expand a power of a binomial. For example,

$$(a+b)^3 = (a+b)(a+b)^2 = (a+b)(a^2+2ab+b^2) = a^3+3a^2b+3ab^2+b^3$$

Multiplication of higher powers of binomials is a tedious process. A quicker way is by the use of the Pascal’s Triangle which we shall develop in the next section.

7.1.1 Pascal’s Triangle

Consider the following expansion:

$$(a+b)^1 = a+b$$

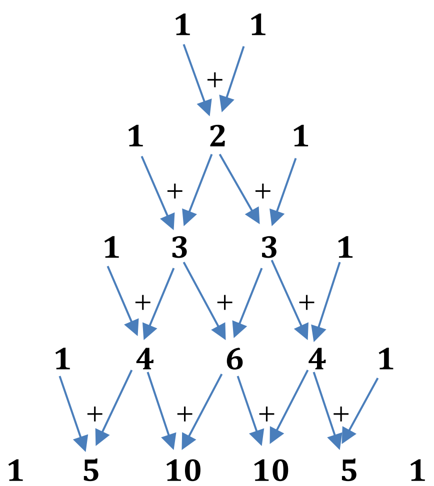
$$(a+b)^2 = a^2+2ab+b^2$$

$$(a+b)^3 = (a+b)(a+b)^2 = a^3+3a^2b+3ab^2+b^3$$

$$(a+b)^4 = (a+b)(a+b)^3 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$$

$$(a+b)^5 = (a+b)(a+b)^4 = a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$$

From the expansions, writing the coefficients only, in a triangular fashion we have



etc.

This is the **Pascal’s triangle**.

From the triangle, one can easily write down the coefficients of the terms in the expansion of $(a+b)^6$. These are 1, 6, 15, 20, 15, 6 and 1. Therefore, using these coefficients we have

$$(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

Notice that in the expansion of $(a+b)^n$, the powers of a which start with n, thereafter at each stage it reduces by 1 and ends with 0 while those of b start with 0 and increases by 1 at each stage and ends with n.

Example 7.2.2 Use Pascal's Triangle to expand:

(a) $(2x+y)^5$

Solution: In the expansion of $(a+b)^5$, the coefficients of the terms are

1 5 10 10 5 1.

Taking $a = 2x$ and $b = y$, we have

$$\begin{aligned}(2x+y)^5 &= (2x)^5 + 5(2x)^4y + 10(2x)^3y^2 + 10(2x)^2y^3 + 5(2x)y^4 + y^5\\ &= (2^5)x^5 + 5(2^4)x^4y + 10(2^3)x^3y^2 + 10(2^2)x^2y^3 + 5(2x)y^4 + y^5\\ &= 32x^5 + 80x^4y + 80x^3y^2 + 40x^2y^3 + 10xy^4 + y^5.\end{aligned}$$

(b) $(x^2-y)^4$

Solution: $(x^2-y)^4 = (x^2+(-y))^4$. In the expansion of $(a+b)^4$, the coefficients of the terms are $1, 4, 6, 4, 1$. Taking $a = x^2$ and $b = (-y)$,

$$\begin{aligned}(x^2-y)^4 &= (x^2)^4 + 4(x^2)^3(-y) + 6(x^2)^2(-y)^2 + 4(x^2)(-y)^3 + (-y)^4\\ &= x^8 - 4x^6y + 6x^4y^2 - 4x^2y^3 + y^4\end{aligned}$$

(c) $\left(\sqrt{2} - \frac{1}{\sqrt{2}}\right)^6$

Solution: $\left(\sqrt{2} - \frac{1}{\sqrt{2}}\right)^6 = \left(\sqrt{2} + \left(-\frac{1}{\sqrt{2}}\right)\right)^6$. Now in the expansion of $(a+b)^6$ the coefficients of the terms are 1, 6, 15, 20, 15, 6 and 1. Taking $a = \sqrt{2}$ and $b = \left(-\frac{1}{\sqrt{2}}\right)$,

$$\begin{aligned}\left(\sqrt{2} - \frac{1}{\sqrt{2}}\right)^6 &= \left(\sqrt{2}\right)^6 + 6\left(\sqrt{2}\right)^5\left(-\frac{1}{\sqrt{2}}\right) + 15\left(\sqrt{2}\right)^4\left(-\frac{1}{\sqrt{2}}\right)^2 + 20\left(\sqrt{2}\right)^3\left(-\frac{1}{\sqrt{2}}\right)^3\\ &\quad + 15\left(\sqrt{2}\right)^2\left(-\frac{1}{\sqrt{2}}\right)^4 + 6\left(\sqrt{2}\right)\left(-\frac{1}{\sqrt{2}}\right)^5 + \left(-\frac{1}{\sqrt{2}}\right)^6\\ &= 8 - 24 + 30 - 20 + \frac{15}{2} - \frac{3}{2} + \frac{1}{8} = \frac{1}{8}.\end{aligned}$$

7.2 Factorials

The product

$$1 \times 2 \times 3 \times 4 \times 5 \times 6 \times \cdots \times 98 \times 99 \times 100$$

is a very large number, and is difficult to multiply out without using a calculator, and is cumbersome to write, even when left in factor form as above. So this clumsy product is denoted by 100! and is read '100 factorial'.

In general, $n!$ represents the number $1 \times 2 \times 3 \times \cdots \times (n-2) \times (n-1) \times n$ i.e.

$$n! = 1 \times 2 \times 3 \times \cdots \times (n-2) \times (n-1) \times n$$

or
$$n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1,$$

since multiplication is commutative.

Note that $n!$ can also be written as

$$\begin{aligned} n! &= n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1, \text{ or} \\ &= n \times (n-1) \times (n-2) \times (n-3)!, \text{ or} \\ &= n \times (n-1) \times (n-2)!, \text{ or} \\ &= n \times (n-1)! \\ &\text{etc.} \end{aligned}$$

Clearly, $1! = 1$. Now, it should be noted that 0! is defined as 1 i.e. $0! = 1$.

Example 7.1.1 (a) Evaluate (i) 10! (ii) $\frac{12!}{8!}$

(b) Write in factorial form:

(i) $96 \times 97 \times 98 \times 99 \times 100$

(ii) $\frac{9\times8\times7\times6}{4\times3\times2}$.

7.2.2 Binomial coefficients

The binomial coefficients are positive integers that occur as coefficients in the binomial expansion. For $n \geq r \geq 0$ the binomial coefficient is denoted by $\binom{n}{r}$ and is defined by

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}$$

Example 7.2.3 Evaluate the binomial coefficients:

(a) $\binom{10}{7}$ (b) $\binom{100}{0}$ (c) $\binom{16}{1}$

Solutions:

(a) $\binom{10}{7} = \frac{10!}{(10-7)!\times7!} = \frac{10!}{3!\times7!} = \frac{10\times9\times8\times7!}{3!\times7!} = \frac{10\times9\times8}{3\times2\times1} = 120.$

(b) $\binom{100}{0} = \frac{100!}{100!\times0!} = \frac{100!}{100!\times1} = 1.$

(c) $\binom{16}{1} = \frac{16!}{(16-1)!\times1!} = \frac{16\times15!}{15!\times1} = 16.$

7.2.3 Binomial theorem

The Binomial theorem for positive integers $n \geq r$ is given by

$$(a+b)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1}b + \binom{n}{2} a^{n-2}b^2 + \cdots$$
$$+ \binom{n}{r-1} a^{n-r+1}b^{r-1} + \cdots + \binom{n}{n-1} ab^{n-1} + \binom{n}{n} b^n.$$

NOTE:

1. In the expansion of $(a+b)^n$the powers of a start with n and at each stage it reduce by 1 i.e. in the second term the power of is $n-1$, in the third it will be $n-2$, in the fourth term it will be $n-3$ etc. and in the last term its power will be 0. While the power of b will start with 0 and start increasing by 1 and in the last term its power will be n.

2. The r^{th} term for $r \leq n$ in the expansions of the binomial $(a+b)^n$, where $n \in \mathbb{Z}^+$, is given by $\binom{n}{r-1} a^{n-r+1}b^{r-1}$.

3. In the expansion $\binom{n}{0} = \binom{n}{n} = 1$ and $\binom{n}{1} = \binom{n}{n-1} = n$.

4. In the case where $a = 1$ and $b = x$, the binomial expansion becomes

$$(1+x)^n = \binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 + \cdots + \binom{n}{r-1} x^{r-1} + \cdots$$
$$+ \binom{n}{n-1} x^{n-1} + \binom{n}{n} x^n.$$

This can also be written as

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots + \frac{n(n-1)\cdots(n-r)}{(r-1)!}x^{r-1} + \cdots$$
$$+nx^{n-1} + x^n.$$

Example 4.2.4 Use the Binomial theorem to expand:

(a) $(2x+3y)^5$ (b) $(1-\frac{1}{2}x)^4$

Solutions:

(a) $(a+b)^5 = \binom{5}{0} a^5 + \binom{5}{1} a^4b + \binom{5}{2} a^3b^2 + \binom{5}{3} a^2b^3 + \binom{5}{4} ab^4 + \binom{5}{5} b^5.$

In this expansion we can make $a = 2x$ and $b = 3y$ so that

$$(2x+3y)^5 = \binom{5}{0} (2x)^5 + \binom{5}{1} (2x)^4(3y) + \binom{5}{2} (2x)^3(3y)^2 + \binom{5}{3} (2x)^2(3y)^3$$
$$+ \binom{5}{4} (2x)(3y)^4 + \binom{5}{5} (3y)^5$$
$$= 32x^5 + 5(3)(16)x^4y + 10(8)(9)x^3y^2 + 10(4)(27)x^2y^3$$
$$+5(2)(81)xy^4 + 143y^5$$
$$= 32x^5 + 240x^4y + 720x^3y^2 + 1080x^2y^3 + 810xy^4 + 143y^5.$$

(b) $(1+a)^4 = \binom{4}{0} + \binom{4}{1} a^2 + \binom{4}{2} a^3 + \binom{4}{3} a^3 + \binom{4}{4} a^4.$

In this expansion we can make $a = \left(-\frac{x}{2}\right)$ so that

$$(1-\tfrac{1}{2}x)^4 = \binom{4}{0} + \binom{4}{1}\left(-\frac{x}{2}\right) + \binom{4}{2}\left(-\frac{x}{2}\right)^2 + \binom{4}{3}\left(-\frac{x}{2}\right)^3 + \binom{4}{4}\left(-\frac{x}{2}\right)^4$$

$$= 1 + 4\left(-\frac{1}{2}\right)x + 6\left(-\frac{1}{2}\right)^2 x^2 + 4\left(-\frac{1}{2}\right)^3 x^3 + \left(-\frac{1}{2}\right)^4 x^4$$

$$= 1 - 2x + 6\left(\frac{1}{4}\right)x^2 - 4\left(\frac{1}{8}\right)x^3 + \left(\frac{1}{16}\right)x^4$$

$$= 1 - 2x + \frac{3}{2}x^2 - \frac{1}{2}x^3 + \frac{1}{16}x^4.$$

Example 4.2.5 Find the term in x^3 in the expansion of $(3+x)^5$.

Solutions: Sometimes you don't need to expand the whole expression to find just one term. You just need to use the rth term. For example, tet the term in x^3 be the rth term. Then the rth term is

$$\binom{5}{r-1} 3^{5-(r-1)} x^{r-1}.$$

Thus, $r - 1 = 3 \Rightarrow r = 4 \Rightarrow \binom{5}{3} 3^2 x^3 = \frac{5!}{2!\times 3!} \times 9x^3 = 90x^3.$

Example 4.2.6 The coefficient of x^4 in the expansion of $(2+ax)^6$ is 60.
Find possible values of the constant a.

Solutions: Let the coefficient of x^4be the rth term. The rth term is

$$\binom{6}{r-1} 2^{6-(r-1)}(ax)^{r-1} = \binom{6}{r-1} 2^{7-r}(a^{r-1})x^{r-1}.$$

$\Rightarrow r - 1 = 4 \Rightarrow r = 5.$

Therefore, the coefficient of x^4 is

$$\binom{6}{5-1} 2^{7-5}(a)^{5-1} = \binom{6}{4} \times 2^2 \times a^4 = \frac{6!}{2!\times 4!} \times 2^2 \times a^4 = \frac{6\times 5\times 4!}{2\times 4!} \times 4 \times a^4$$

$$= 60a^4 = 60$$

$\Rightarrow a^4 = 1 \Rightarrow a = 1.$

Example 4.2.6 Find the 5th term in the expansion of $(2+3x)^8$.

Solutions: The rth (5th) term is

$$\binom{8}{5-1} 2^{8-(5-1)}(3x)^{5-1} = \binom{8}{4} 2^{8-(5-1)}(3)^{5-1}x^{5-1} = \frac{8!}{4!\times 4!} \times 2^4 \times 3^4 x^4$$

$$= \frac{8\times 7\times 6\times 5}{4\times 3\times 2\times 1} \times 16 \times 81x^4$$

$$= 90720x^4$$

7.3 Binomial Expansion of $(1+x)^n$ for $n \in \mathbb{Q}$

The Binomial theorem can be extended to the expansion of $(1+x)^n$ where $n \in \mathbb{Q}$ provided that $|x| < 1$. The expansion is given by

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

provided $|x| < 1$.

It should be noted that for rational n the expansion is only valid for $(1+x)^n$ and it is **not** for $(a+x)^n$, where $a \neq 1$.

In the expansion of $(a+x)^n$ for $\boldsymbol{n} \in \mathbb{Q}$ where $a \neq 1$ one must first rewrite $(a+x)^n$ as $a^n\left(1+\frac{x}{a}\right)^n$ in which case

$$(a+x)^n = a^n\left(1 + n\left(\frac{x}{a}\right) + \frac{n(n-1)}{2!}\left(\frac{x}{a}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{x}{a}\right)^3 + \cdots\right),$$

and the expansion is valid for $\left|\frac{x}{a}\right| < 1$ i.e. $|x| < 2$ or $-a < x < a$.

Example 4.2.7 Expand each of the following binomials in ascending powers of x up to and including the term containing x^3 stating the set of values of x for which each expansion is valid.

(a) $(1-2x)^{\frac{1}{3}}$ (b) $(2+x)^{-2}$

Solution: (a) $(1-2x)^{\frac{1}{3}} = 1 + \frac{1}{3}(-2x) + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}(-2x)^2 + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}(-2x) + \cdots$

$$= 1 - \frac{2}{3}x - \frac{4}{9}x^2 - \frac{40}{81}x^3 + \cdots,\ |-2x| < 1\ i.e.\ 2|x| < 1.$$

Therefore the range of values of x for which the expansion is valid is $|x| < \frac{1}{2}$ or $-\frac{1}{2} < x < \frac{1}{2}$.

(b) $(2+x)^{-2} = \left[2\left(1+\frac{x}{2}\right)\right]^{-2} = 2^{-2}\left(1+\frac{x}{2}\right)^{-2} \frac{1}{4}\left(1+\frac{x}{2}\right)^{-2}$

$$= \frac{1}{4}\left(1 + (-2)\left(\frac{x}{2}\right) + \frac{-2(-2-1)}{2!}\left(\frac{x}{2}\right)^2 + \frac{-2(-2-1)(-2-2)}{3!}\left(\frac{x}{2}\right)^3 + \cdots\right)$$

$$= \frac{1}{4}\left(1 - x + \frac{3}{4}x^2 - \frac{1}{2}x^3 + \cdots\right),\ \left|\frac{x}{2}\right| < 1\ i.e.\ |x| < 2 \text{ or } -2 < x < 2.$$

Therefore the range of values of x for which the expansion is valid is $|x| < 2$ or $-2 < x < 2$.

Example 4.2.8 By substituting 0.02 for x in $(1-x)^{\frac{1}{2}}$ and its expansion, find $\sqrt{2}$ correct to five decimal places.

Solution: $(1-x)^{\frac{1}{2}} = 1 + \left(\frac{1}{2}\right)(-x) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(-x)^2$

$$+ \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(-x)^3 + \cdots$$

$$= 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 + \cdots, \ |x| < 1$$

Since, $-1 < 0.02 < 1$, we can replace x by 0.02 in the expansion. i.e.

$$(1 - 0.02)^{\frac{1}{2}} = 1 - \frac{1}{2}(0.02) - \frac{1}{8}(0.02)^2 - \frac{1}{16}(0.02)^3 + \cdots$$

$$(0.98)^{\frac{1}{2}} = 1 - 0.01 - 0.00005 - 0.0000005 + \cdots \approx 0.989945$$

$$\Rightarrow \ \sqrt{\frac{98}{100}} \approx 0.989945$$

$$\Rightarrow \ \frac{7\sqrt{2}}{10} \approx 0.989945$$

$$\Rightarrow \ \sqrt{2} \approx 1.414207$$

$\therefore \quad \sqrt{2} \approx 1.41421$ correct to 5 decimal places.

Example 4.2.9 Expand $\frac{5}{(1+3x)(1-2x)}$ as a series of ascending powers of x giving:

(a) the first four terms

(b) the range of values of x for which the expansion is valid.

Solution: (a) Resolving $\frac{5}{(1+3x)(1-2x)}$ into partial fractions we have

$$\frac{5}{(1+3x)(1-2x)} = \frac{3}{1+3x} + \frac{2}{1-2x} = 3(1+3x) + 2(1-2x)$$

$$3(1+3x)^{-1} = 3\left[1 + (-1)(3x) + \frac{-1(-1-1)}{2!}(3x)^2 + \frac{-1(-1-1)(-1-2)}{3!}(3x)^2 + \cdots\right]$$

$$= 3[1 - 3x + 9x^2 - 27x^3 + - \cdots]$$

$$= 3 - 9x + 27x^2 - 81x^3 + - \cdots, \ |3x| < 1 \text{ or } |x| < \frac{1}{3} \text{ or } -\frac{1}{3} < x < \frac{1}{3}.$$

$$2(1-2x)^{-1} = 2\left[1 + (-1)(-2x) + \frac{-1(-1-1)}{2!}(-2x)^2 + \frac{-1(-1-1)(-1-2)}{3!}(-2x)^2 + \cdots\right]$$

$$= 2[1 + 2x + 4x^2 + 8x^3 + \cdots]$$

$$= 1 + 4x + 8x^2 + 16x^3 + - \cdots, \ |-2x| < 1 \text{ or } |x| < \frac{1}{2} \text{ or } -\frac{1}{2} < x < \frac{1}{2}.$$

(b) The range of values of x for which the expansion is valid must satisfy both the expansion for $(1+3x)^{-1}$ and $(1-2x)^{-1}$. Thus we are looking for the intersection of the two ranges $-\frac{1}{3} < x < \frac{1}{3}$ and $-\frac{1}{2} < x < \frac{1}{2}$. Therefore the required range is

$-\frac{1}{3} < x < \frac{1}{3}$.

TUTORIAL SHEET 10

1. Compute explicitly the given quantity:

(a) $\frac{5!}{3!0!}$ (b) $\binom{12}{7}$ (c) $\binom{12}{\binom{5}{3}}$.

2. If k and r are integers such that $0 \le r \le k$, prove that

$$\binom{k}{r} = \binom{k+1}{r} - \binom{k}{r-1}.$$

3. Show that $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$.

4. Find the first three terms in the expansion of each of the following:

(a) $(2-x)^5$ (b) $(1-4y)(1-5y)^8$ (c) $(1+2x)^6$ (d) $(x-2y^2)^4$.

5. Given that $(1+2y)^7 = 1 + Ax + Bx^2 + Cx^3 + \cdots$, A, B find and C.

6. Write the term indicated in the expansion of each of the following expansions:

(a) fourth term of $(1+2y)^7$; (b) fifth term of $\left(x^2 - \frac{1}{x}\right)^8$:

7. Find the term independent of x in the expression of $\left(x^2 - \frac{3}{x}\right)^n$ when
(a) $n = 3$ (b) $n = 12$.

8. The first three terms in the expansion of $(1+kx)^n$ are 1, $14x$ and $84x^2$ respectively. Find the values of the constants k and n and the coefficients of and terms.

9. Expand the following functions as a series of ascending powers of x up to and including the term in x^3. In each case give the range of values of x for which the expansion is valid:

(a) $(1-2x)^{\frac{1}{2}}$ (b) $(4+x)^{-1}$ (c) $\frac{2x+1}{(x-1)(x+1)}$ (d) $\frac{1}{\sqrt{x+1}}$.

10. By substituting 0.08 for x in $(1+x)^{\frac{1}{2}}$ and its expansion find $\sqrt{3}$ correct to four significant figures.

11. Use a suitable binomial expansion to find $\frac{1}{\sqrt{0.99}}$ correct to five decimal places.

12. Find the expansion of $(1+y)^6$.

(a) By writing $y = x + x^2$, find the first four terms, in the expansion of

$(1+x+x^2)^6$.

(b) By putting $x = 0.01$ in your first four terms, find an approximation for $(1.0101)^6$.

8. PRINCIPLE OF MATHEMATICAL INDUCTION

Let P_n be a statement in terms of n, where n is a positive integer. To prove that P_n is true using the principle of mathematical induction, usually we follow four steps:

Step 1: Prove that P_1 is true i.e. the statement is true for $n = 1$;

Step 2: Assume that P_k is true i.e. assume that the statement is true for $n = k$ where $k > 1$.

Step 3: Using the assumption in Step 2 show the general statement P_{k+1} is true i.e. general statement is true for $n = k + 1$.

Step 4: Conclude that the general statement is then true for all positive integers, n.

Example 8.1: Prove, by mathematical induction, that all integers $n \in \mathbb{Z}^+$.

(a) $P_n: 1 + 2 + 3 + 4 + 5 + \cdots + n = \frac{1}{2}n(n + 1)$

Proof: Step 1: Is P_1 true for $n = 1$?

Left hand side (LHS) = 1;

Right hand side (RHS) = $\frac{1}{2}(1)(1 + 1) = \frac{1}{2}(2) = 1 = LHS$

$\Rightarrow$ it is true for $n = 1$ i.e. P_1 is true.

Step 2: Assume that it is true for $n = k$ i.e. P_k is true.

i.e. $\quad 1 + 2 + 3 + 4 + 5 + \cdots + k = \frac{1}{2}k(k + 1)$.

Step 3: Prove that P_{k+1} is true i.e. the general statement is true for $n = k + 1$ also.

i.e. by using the assumption prove that

$$1 + 2 + 3 + 4 + 5 + \cdots + k + (k + 1) = \frac{1}{2}(k + 1)(k + 2)$$

Pf: $\quad LHS = \underbrace{1 + 2 + 3 + 4 + 5 + \cdots + k}_{\text{(by assumption)}} + (k + 1)$

$$= \frac{1}{2}k(k + 1) + (k + 1)$$

$$= (k + 1)\left(\frac{1}{2}k + 1\right) = (k + 1)\left(\frac{1}{2}k + \frac{2}{2}\right) = \frac{1}{2}(k + 1)(k + 2) = RHS$$

$\Rightarrow$ it true for $n = k + 1$ also.

Step 4: Since it is true for $n = 1, n = k$ and $n = k + 1$, we conclude that it is true for all $n \in \mathbb{N}$.

Note: The truth of P_k implies the truth of P_{k+1}.

(b) $P_n: 1 + 3 + 5 + 7 + 9 + \cdots + (2n - 1) = n^2$.

Proof: Step 1: Is it true for $n = 1$?

$LHS = 1$; $\quad RHS = 1^2 = 1 = LHS$

$\Rightarrow$ it is true for $n = 1$.

Step 2: Assume that it is true for $n = k$.

i.e. $1 + 2 + 3 + 4 + 5 + \cdots + (2k - 1) = k^2$.

Step 3: Prove, by using the assumption, that it is true for

$n = k + 1$ also. i.e. prove that

$1 + 2 + 3 + 4 + 5 + \cdots + (2k - 1) + (2k + 1) = (k + 1)^2$

Pf: $LHS = \underbrace{1 + 2 + 3 + 4 + 5 + \cdots + (2k - 1)} + (2k + 1)$

$= \quad k^2 \quad + (2k + 1)$

$= k^2 + 2k + 1 = (k + 1)^2 = RHS$

$\Rightarrow$ it true for $n = k + 1$ also.

Step 4: Since it is true for $n = 1, n = k$ and $n = k + 1$, we conclude that the statement P_n is true for all $n \in \mathbb{N}$.

(c) $P_n: 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + \cdots + n^2 = \frac{1}{6}n(n + 1)(2n + 1)$

Proof: Step 1: Is it true for $n = 1$?

$LHS = 1^2 = 1; RHS = \frac{1}{6}(1)(1 + 1)(2(1) + 1) = \frac{1}{6}(1)(2)(3) = 1 = LHS$

$\Rightarrow$ it is true for $n = 1$.

Step 2: Assume that it is true for $n = k$.

i.e. $1^2 + 2^2 + 3^2 + 4^2 + 5^2 + \cdots + k^2 = \frac{1}{6}k(k + 1)(2k + 1)$

Step 3: Prove, by using the assumption, that it is true for $n = k + 1$ also.

i.e. prove that

$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + \cdots + k^2 + (k + 1)^2 = \frac{1}{6}(k + 1)(k + 2)(2k + 3)$

Pf: $LHS = \underbrace{1^2 + 2^2 + 3^2 + 4^2 + 5^2 + \cdots + k^2} + (k + 1)^2$

$= \quad \frac{1}{6}k(k + 1)(2k + 1) \quad + (k + 1)^2$

$= \frac{1}{6}(k + 1)(k(2k + 1) + 6(k + 1) = \frac{1}{6}(k + 1)(2k^2 + 7k + 6)$

$= \frac{1}{6}(k + 1)(k + 2)(2k + 3) = RHS$

$\Rightarrow$ it true for $n = k + 1$ also.

Step 4: Since it is true for $n = 1, n = k$ and $n = k + 1$, we conclude that it is true for all $n \in \mathbb{N}$.

(d) $P_n: 1^3 + 2^3 + 3^3 + 4^3 + 5^3 + \cdots + n^3 = \frac{1}{4}n^2(n + 1)^2$

Proof: Step 1: Is it true for $n = 1$?

$LHS = 1^3 = 1; RHS = \frac{1}{4}(1^2)(1+1)^2 = \frac{1}{4}(1)(4) = 1 = LHS$

$\Rightarrow$ it is true for $n = 1$.

Step 2: Assume that it is true for $n = k$.

i.e. $1^3 + 2^3 + 3^3 + 4^3 + 5^3 + \cdots + k^3 = \frac{1}{4}k^2(k+1)^2$

Step 3: Prove, by using the assumption, that it is true for $n = k + 1$ also.

i.e. prove that

$$1^3 + 2^3 + 3^3 + 4^3 + 5^3 + \cdots + k^3 + (k+1)^3 = \frac{1}{4}(k+1)^2(k+2)^2$$

Pf: LHS

$$= \underbrace{1^3 + 2^3 + 3^3 + 4^3 + 5^3 + \cdots + k^3}_{} + (k+1)^3$$

$$= \frac{1}{4}k^2(k+1)^2 + (k+1)^3$$

$$= \frac{1}{4}(k+1)^2\,(k^2 + 4k + 4)$$

$$= \frac{1}{4}(k+1)^2(k+2)^2 = RHS.$$

$\Rightarrow$ it true for $n = k + 1$ also.

Step 4: Since it is true for $n = 1, n = k$ and $n = k + 1$, we conclude that it is true for all $n \in \mathbb{N}$.

Example 8.2 Prove, by mathematical induction, that

(a) $P_n: n^3 - 7n + 9$ is divisible by 3 for all integers $n \in \mathbb{Z}^+$.

Proof: Step 1: Is it true for $n = 1$?

$1^3 - 7(1) + 9 = 3$, which is divisible by 3.

Thus, it is true for $n = 1$.

Step 2: Assume that it is true for $n = k$.

i.e. $k^3 - 7k + 9$ is divisible by 3 i.e. $\frac{k^3-7k+9}{3} = p$, or $k^3 - 7k + 9 = 3p$ where $p \in \mathbb{N}$.

Step 3: Prove, by using the assumption, that it is true for $n = k + 1$ also.

i.e. prove that

$(k+1)^3 - 7(k+1) + 9$ is divisible by 3

Pf: $(k+1)^3 - 7(k+1) + 9 = k^3 + 3k^2 + 3k + 1 - 7k - 7 + 9$

$$= \underbrace{(k^3 - 7k + 9)}_{3p} + 3k^2 + 3k - 6$$

$$= 3p + 3k^2 + 3k - 6$$

$$= 3(p + k^2 + k - 3),$$

which is a multiple of 3, since $p, k \in \mathbb{N}$.

Thus, it true for $n = k + 1$ i.e. P_n is divisible by 3 also.

Step 4: Since it is true for $n = 1, n = k$ and $n = k + 1$, we conclude that it is true for all $n \in \mathbb{N}$.

(b) $9^n - 1$ is divisible by 8 for $n \in \mathbb{N}$.

Proof: Step 1: Is it true for $n = 1$?

$9^1 - 1 = 8$, which is divisible by 8.

Thus, it is true for $n = 1$.

Step 2: Assume that it is true for $n = k$ i.e. $9^k - 1$ is divisible by 8

$9^k - 1 = 8p$, where $p \in \mathbb{N} \Rightarrow 9^k = 8p + 1$, where $p \in \mathbb{N}$.

Step 3: Prove, by using the assumption, that it is true for $n = k + 1$ also.

i.e. prove that

$9^{k+1} - 1$ is divisible by 8

Pf:
$$
\begin{aligned}
9^{k+1} - 1 &= 9(9^k) - 1 \\
&= 9(8p + 1) - 1 \\
&= 72p + 8 \\
&= 8(9p + 1)
\end{aligned}
$$

which is divisible by 8, since $9p + 1 \in \mathbb{N}$. Thus, it true for $n = k + 1$ also.

Step 4: Since it is true for $n = 1, n = k$ and $n = k + 1$, we conclude that it is true for all $n \in \mathbb{N}$.

Example 8.3 Prove each of the following in equalities hold by mathematical induction:

(a) P_n: $4^{n-1} > n^2$ for $n \geq 3$.

Proof: Step 1: Is it true for $n = 3$?

$LHS = 4^{3-1} = 4^2 = 16.$

$RHS = 3^2 = 9 < LHS$

Thus, it is true for $n = 3$.

Step 2: Assume that it is true for $n = k$.

i.e. $4^{k-1} > k^2$ for $k \geq 3$.

Step 3: Prove, by using the assumption, that it is true for $n = k + 1$ also.

i.e. prove that

$4^{(k+1)-1} > (k + 1)^2$ for $k \geq 3$

Pf: $LHS = 4^{(k-1)+1} = 4(4^{k-1}) > 4k^2$, by assumption, since $4^{k-1} > k^2$

for $k \geq 3$.

i.e. $4^{(k-1)+1} > 4k^2 = 2k^2 + 2k^2 > 2k^2 + 2k$, since $2k^2 > 2k$ for $k \geq 3$.

and $k^2 > k$ for $k \geq 3$

i.e. $4^{(k-1)+1} > 2k^2 + 2k = k^2 + k^2 + 2k > k^2 + 2k + 1$, since $k^2 > 1$ for $k \geq 3$.

Thus, $4^{(k+1)-1} > (k+1)^2 = RHS$, for $k \geq 3$, since $k^2 + 2k + 1 = (k+1)^2$.

Hence, it true for $n = k + 1$ also.

Step 4: Since it is true for $n = 3, n = k$ and $n = k + 1$, we conclude that it is true for all $n \in \mathbb{N}$.

(b) $P_n: (2n)! > 2^n (n!)^2, n \geq 2$.

Proof: Step 1: Is it true for $n = 2$?

$$LHS = \big(2(2)\big)! = 4! = 24.$$

$$RHS = 2^2(2!)^2 = 8 < 24 = LHS$$

Thus, it is true for $n = 2$.

Step 2: Assume that it is true for $n = k$.

i.e. $(2k)! > 2^k (k!)^2, k \geq 2$.

Step 3: Prove, by using the assumption, that it is true for $n = k + 1$ also.

i.e. prove that

$(2(k+1))! > 2^{k+1}((k+1)!)^2, k \geq 2$

Pf: $LHS = (2(k+1))! = (2k+2)! = (2k+2)(2k+1)(2k)!$,

$> 2(k+1)(2k+1)2^k(k!)^2$, by assumption, since $(2k)! > 2^k(k!)^2$ for $k \geq 2$.

$(2(k+1))! > 2(k+1)(k+1)2^k(k!)^2$, since $2k + 1 > k + 1$ for $k \geq 2$.

$= 2^{k+1}(k+1)k!\,(k+1)k! = 2^{k+1}[(k+1)!]^2$.

Hence, it true for $n = k + 1$ also.

Step 4: Since it is true for $n = 2$, for $n = k$ and $n = k + 1$, we conclude that it is true for all $n \in \mathbb{N}$.

<u>TUTORIAL SHEET</u> 11

1. Prove, using the principle of mathematical induction, that for positive integers n

(a) $1\cdot 2+2\cdot 3+3\cdot 4+\cdots+n(n+1)=\dfrac{n(n+1)(n+2)}{3}$

(b) $4^1+4^2+4^3+\cdots+4^n=\dfrac{4}{3}\left(4^n-1\right)$

(c) $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\dfrac{1}{3\cdot 4}+\cdots+\dfrac{1}{n(n+1)}=\dfrac{n}{n+1}$

(d) $1+2^1+2^2+2^3+\cdots+2^n=2^{n+1}-1.$

2. Use the principle of mathematical induction to prove the following statements for all $n\in\mathbf{Z}^+$:

(a) 8^n-1 is divisible by 7

(b) $3^{2n-1}+1$ is divisible by 4.

3. Use mathematical induction to prove that each of the following statements are true for all $n\in\mathbf{Z}^+$:

(a) $3^n\geq 2n+1$

(b) $4^n\geq 4n$

(c) $n^2\geq n$

(d) $2^n\geq n+1$

MAT 1100 LECTURE NOTES

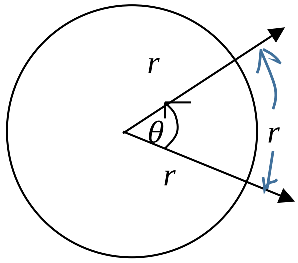
9. TRANSCEDENTAL FUNCTIONS.

9.1 TRIGONOMETRIC FUNCTIONS

9.1.1 The Radian measure

The **radian** is another basic unit of angle measure that is used extensively in subsequent mathematics courses and in various mathematical applications in the physical sciences.

One radian is the measure of the central angle of a circle in which the sides of the angle intercept an arc equal in length to the radius of the circle.



From the diagram, θ is an angle of measure 1 radian. Thus, since the circumference of a circle is given by $C = 2\pi r$, and each arc of length r determines an angle of 1 radian, there are $\frac{2\pi r}{r} = 2\pi$ radians in one complete revolution. This means that

$$2\pi \; radians = 360^o$$

or, equivalently,

$$\pi \; radians = 180^o$$

The two angle units of measure are related as follows:

$$1 \; radian = \frac{180}{\pi} \; degrees \quad \text{and } 1 \; degree = \frac{\pi}{180} \; radians$$

Exercise: Change (a) 150^o to radians

(b) $\frac{3\pi}{4}$ radians to degrees

Exercise: Complete the following table:

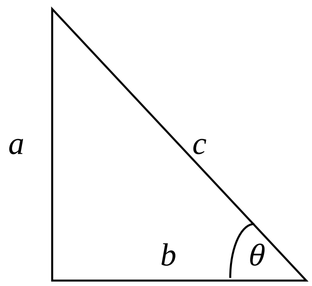
DEGREES	RADIANS
30^o	
	$\frac{\pi}{4}$
60^o	
	$\frac{\pi}{2}$
180^o	
	$\frac{3\pi}{2}$
360^o	

9.1.2 <u>Trigonometric Ratios</u>

These are ratios of a right angled triangle.

<u>Primary trigonometric Ratios</u>

For the right angled triangle



1. Sine ratio is defined by

$$\sin\theta = \frac{\text{oppositeside}}{\text{hypotenuse}} = \frac{a}{c}$$

2. Cosine ratio is defined by

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{b}{c}$$

3. Tangent ratio is defined by

$$\tan\theta = \frac{\text{oppositeside}}{\text{adjcent side}} = \frac{a}{b}$$

<u>Note</u>: $\tan\theta = \frac{\sin\theta}{\cos\theta}$ since $\frac{\sin\theta}{\cos\theta} = \frac{a/c}{b/c} = \frac{a}{b} = \tan\theta$

<u>Secondary trigonometric Ratios</u>

These are the trigonometric ratios of a right angled triangle which are the reciprocals of the primary ratios.

1. Secant ratio is defined as

$$\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent side}} = \frac{c}{b}$$

<u>Note</u>: $\sec\theta = \frac{1}{\cos\theta}$ since $\frac{1}{\cos\theta} = \frac{1}{b/c} = \frac{c}{b} = \sec\theta$

2. Cosecant ratio is defined as

$$\csc\theta = \frac{\text{hypotenuse}}{\text{adjacent side}} = \frac{c}{a}$$

<u>Note</u>: $\csc\theta = \frac{1}{\sin\theta}$ since $\frac{1}{\sin\theta} = \frac{1}{a/c} = \frac{c}{a} = \csc\theta$

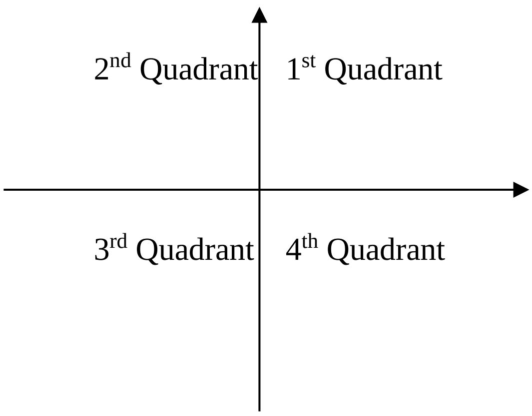
3. Cotangent ratio is defined as

$$\cot\theta = \frac{\text{adjacent side}}{\text{oppositeside}} = \frac{b}{a}$$

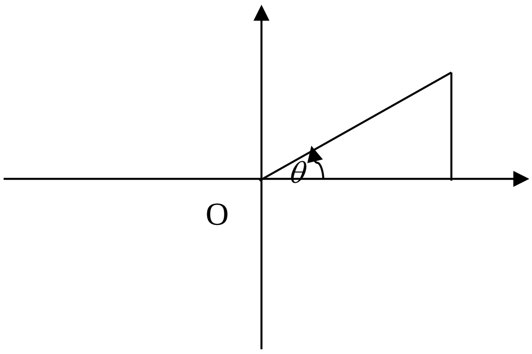
<u>Note</u>: $\cot\theta = \frac{1}{\tan\theta}$ since $\frac{1}{\tan\theta} = \frac{1}{a/b} = \frac{b}{a} = \csc\theta$

<u>Standard Trigonometric Ratios for Some angles</u>

The Cartesian plane has Four Quadrants

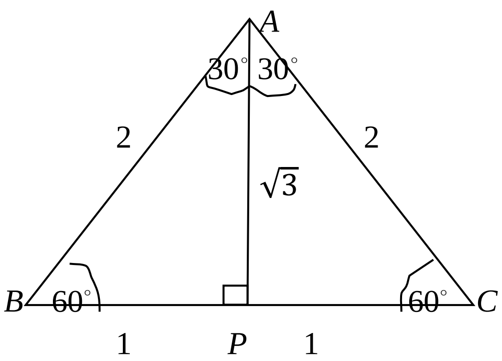


1. Standard ratios of angles ($\theta = 30^\circ$, 45°, 60°) the in the first quadrant



(a) $\sin 30^\circ$, $\cos 30^\circ$, $\tan 30^\circ$, $\sin 60^\circ$, $\cos 60^\circ$, $\tan 60^\circ$

To obtain these ratios we consider an equilateral triangle *ABC* where each side is 2 units and each angle is 60°. A perpendicular from *A* to *BC* bisects *BC* and angle *A*, as shown in the diagram.



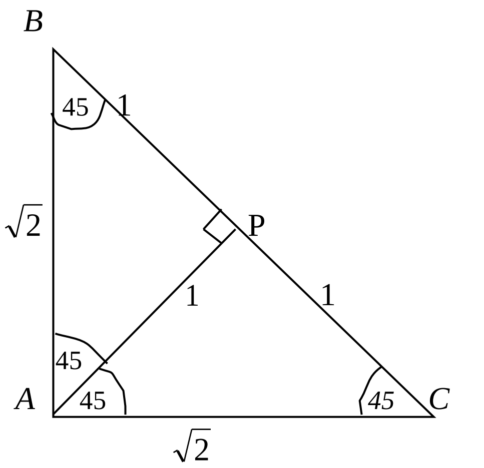
By Pythagoras theorem, $AP = \sqrt{3}$. Clearly from the triangle,

$$\sin 30^\circ = \frac{1}{2},\ \cos 30^\circ = \frac{\sqrt{3}}{2}, \tan 30^\circ = \frac{1}{\sqrt{3}};$$

$$\sin 60^\circ = \frac{1}{\sqrt{3}},\ \cos 60^\circ = \frac{1}{2}, \tan 60^\circ = \sqrt{3}$$

(b) $\sin 45^\circ$, $\cos 45^\circ$, $\tan 45^\circ$

To obtain these ratios we construct a right angled isosceles triangle *ABC* whose hypotenuse is of length 2 units and angle *A* is 90°. A perpendicular from *A* to *BC* bisects *BC* and angle *A*, as shown in the diagram.

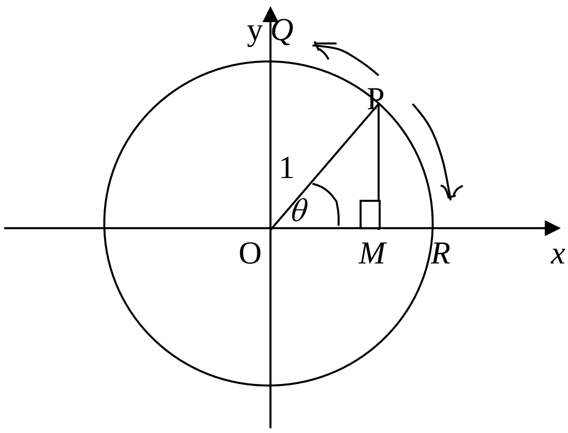


By Pythagoras theorem, $AB = \sqrt{2}$. It follows that

$$\sin 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2},\ \cos 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2},\ \tan 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

(c) $\sin 0^\circ$, $\cos 0^\circ$, $\tan 0^\circ$

Take the Cartesian plane and consider a unit circle with centre at the origin. Let P be a point on the circle and PM be perpendicular to the x – axis as shown in the diagram.



From the right angled triangle OMP,

$$\sin\theta = \frac{PM}{OP},\ \cos\theta = \frac{OM}{OP},\ \tan\theta = \frac{PM}{OP}.$$

Now, as P approaches R as it moves on the circle, both θ and PM approach 0 whereas OM approaches 1. Thus as θ becomes 0°

$$\sin 0^\circ = \frac{0}{1} = 0,\ \cos 0^\circ = \frac{1}{1} = 1,\ \tan 0^\circ = \frac{0}{1} = 0.$$

i.e. $\sin 0^\circ = 0$, $\cos 0^\circ = 1$, $\tan 0^\circ = 0$.

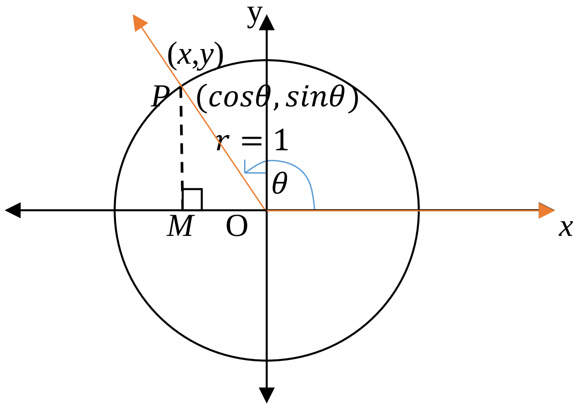
Similarly, as P approaches Q as it moves on the circle, θ approaches 90° whereas PM approaches 1. Thus, as θ becomes 90°

$$\sin 0^\circ = \tfrac{1}{1} = 1,\ \cos 90^\circ = \frac{0}{1} = 0,\ \tan 0^\circ = \frac{1}{0} \text{ which is not defined}.$$

Therefore,

$$\sin 90^\circ = 1,\ \cos 90^\circ = 0,\ \tan 0^\circ \ \textit{undefined}.$$

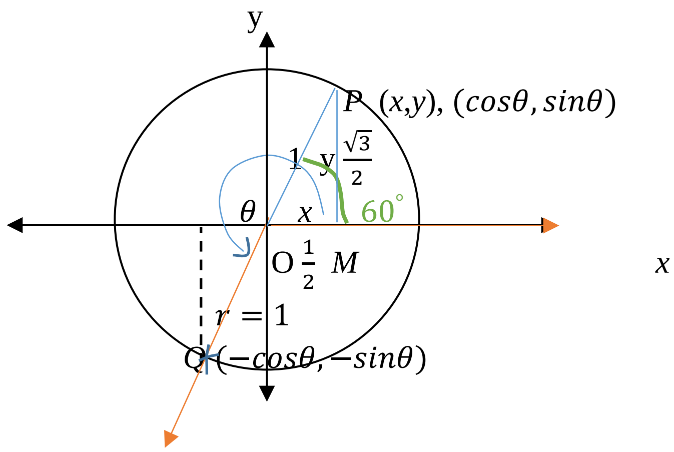
We now consider a unit circle on the Cartesian plane with centre at the origin.



Choosing $r = 1$, for any angle we obtain

$$sin\theta = \tfrac{y}{1} = y,\ cos\theta = \tfrac{x}{1} = x \text{ and } \quad tan\theta = \tfrac{y}{x} = \tfrac{sin\theta}{cos\theta}, \text{ provided } cos\theta \neq 0.$$

<u>Example</u> Find $\sin\theta$, $\cos\theta$, $\tan\theta$ for $\theta = 240^o$.



$\angle POM = 60^\circ$ implying that $\sin 60^\circ = \dfrac{PM}{OP} = \dfrac{y}{1} = y \Rightarrow y = \dfrac{\sqrt{3}}{2}$

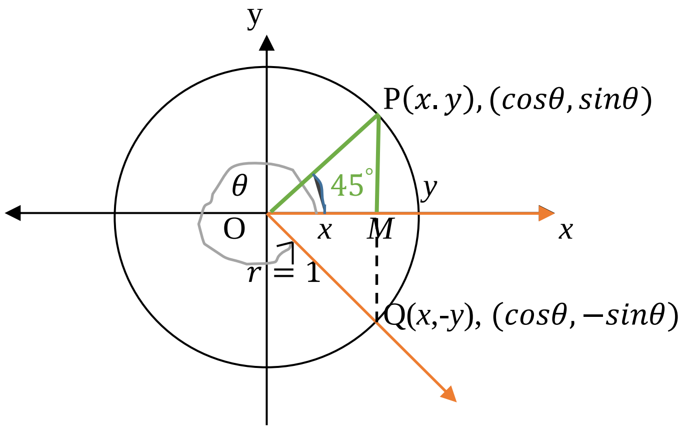
and $\cos 60^\circ = \dfrac{OM}{OP} = \dfrac{x}{1} = x \Rightarrow x = \dfrac{1}{2}$. Thus point P has coordinates $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

Thus, point Q has coordinates $\left(-\dfrac{1}{2}, -\dfrac{\sqrt{3}}{2}\right)$

Therefore,

$$\sin 240^\circ = -\frac{\sqrt{3}}{2}, \quad \cos 240^\circ = -\frac{1}{2}, \quad \tan 240^\circ = \frac{\sin 240^\circ}{\cos 240^\circ} = \frac{-\sqrt{3}/2}{-1/2} = \sqrt{3}$$

Example Find $\sin\theta$, $\cos\theta$, $\tan\theta$ for $\theta = 315^o$.



$\angle POM = 45^\circ$ implying that $\sin 45^\circ = \frac{PM}{OP} = \frac{y}{1} = y \Rightarrow y = \frac{\sqrt{2}}{2}$

and $\cos 45^\circ = \frac{OM}{OP} = \frac{x}{1} = x \Rightarrow x = \frac{\sqrt{2}}{2}$. Thus point P has coordinates $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

Thus, point Q has coordinates $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.

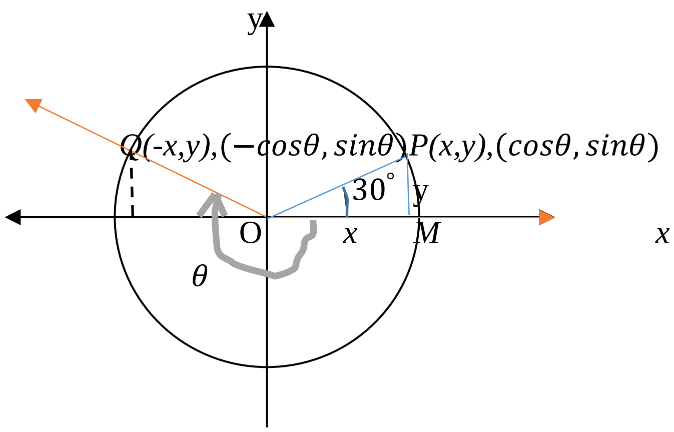
Therefore,

$$\sin 315^\circ = -\frac{\sqrt{2}}{2}, \quad \cos 315^\circ = \frac{\sqrt{2}}{2}, \quad \tan 315^\circ = \frac{\sin 45^\circ}{\cos 45^\circ} = \frac{-\sqrt{2}/2}{\sqrt{2}/2} = -1$$

Example Find $\sin\theta$, $\cos\theta$, $\tan\theta$ for $\theta = -210^o$.

$\angle POM = 30^\circ$ implying that $\sin 30^\circ = \frac{PM}{OP} = \frac{y}{1} = y \Rightarrow y = \frac{1}{2}$

and $\cos 30^\circ = \frac{OM}{OP} = \frac{x}{1} = x \Rightarrow x = \frac{\sqrt{3}}{2}$. Thus, point P has coordinates $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.



Thus, point Q has coordinates $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

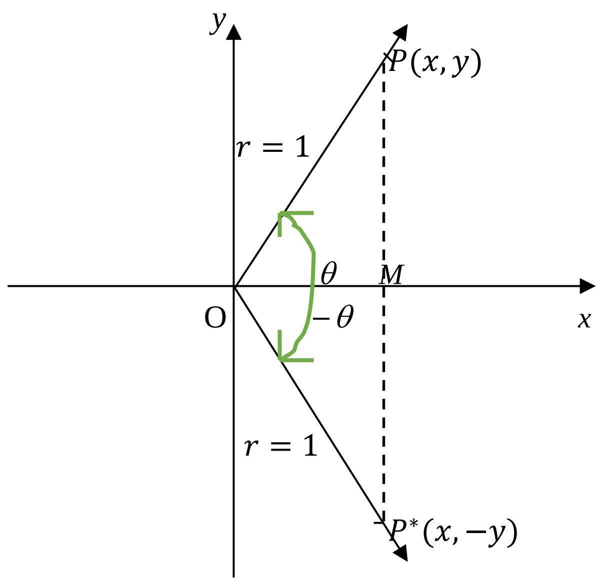
Therefore,

$$\sin(-210^\circ) = \frac{1}{2}, \quad \cos(-210^\circ) = -\frac{\sqrt{3}}{2}, \quad \tan(-210^\circ) = \frac{\sin(-210^\circ)}{\cos(-210)^\circ} = \frac{1/2}{-\sqrt{3}/2} = -\frac{1}{\sqrt{3}}$$

<u>Exercise</u>: Complete the following table:

θ	θ *in Radians*	$sin\theta$	$cos\theta$	$tan\theta$
0°				
90°				
120°				
180°				
-135°				
-30°				
270°				
-300°				
360°				
510°				

In the diagram,



if point $P(x, y)$ lies in the first quadrant then

$$sin\theta = y,\ cos\theta = x,\ \ tan\theta = y/x.$$

The image P^* of P under a reflection in the x – axis has coordinates $(x, -y)$. Thus

$$\text{Sin}(-\theta) = -y = -sin\theta, \cos(-\theta) = x = cos\theta,$$

$$tan(-\theta) = -\left(\frac{y}{x}\right) = -tan\theta.$$

Therefore,

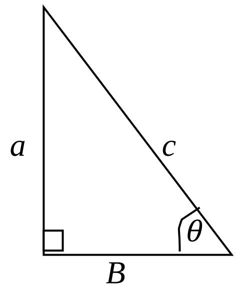
$$\sin(-\theta) = -sin\theta;\ \cos(-\theta) = cos\theta,\ \ tan(-\theta) = -tan\theta.$$

For example,

$$\text{Sin}(-30^o) = -sin30^o = -\frac{1}{2};\ \cos(-30^o) = cos30^o = \frac{\sqrt{3}}{2},$$
$$tan(-30^o) = -tan30^o = -\frac{1}{\sqrt{3}}.$$

9.1.3 Trigonometric Identities

We consider a right angled triangle.



By Pythagorus theorem,

$$a^2 + b^2 = c^2.$$

$$sin\theta = \frac{a}{c};\ cos\theta = \frac{b}{c};\ tan\theta = \frac{a}{b}$$

$$sin^2\theta + cos^2\theta = \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = \frac{a^2}{c^2} + \frac{b^2}{c^2} = \frac{a^2+b^2}{c^2} = \frac{c^2}{c^2} = 1$$

i.e. 1. $$\boxed{sin^2\theta + cos^2\theta = 1}$$

This is the basic trigonometric identity from which the other trigonometric identity are derived.

For example,

2. $\boxed{sin^2\theta = 1 - cos^2\theta}$

3. $\boxed{cos^2\theta = 1 - sin^2\theta}$

When we divide 1 by $cos^2\theta$,

$$\frac{sin^2\theta}{cos^2\theta} + \frac{cos^2\theta}{cos^2\theta} = \frac{1}{cos^2\theta} \Rightarrow$$

4. $\boxed{tan^2\theta + 1 = sec^2\theta}$

When we divide 1 by $sin^2\theta$,

$$\frac{sin^2\theta}{sin^2\theta} + \frac{cos^2\theta}{sin^2\theta} = \frac{1}{sin^2\theta} \Rightarrow$$

5. $\boxed{1 + cot^2\theta = csc^2\theta}$

<u>Examples</u>: Verify each of the following identity:

1. $\frac{\sin\theta}{\csc\theta} + \frac{\cos\theta}{\sec\theta} = 1$

<u>Proof</u>: $LHS = \frac{\sin\theta}{\csc\theta} + \frac{\cos\theta}{\sec\theta} = \frac{\sin\theta}{\frac{1}{\sin\theta}} + \frac{\cos\theta}{\frac{1}{\cos\theta}} = \frac{(\sin\theta)^2}{1} + \frac{(\cos\theta)^2}{1} = \sin^2\theta + \cos^2\theta = 1$

$= RHS.$

2. $\cos x + \cos x \tan^2 x = \sec x$

<u>Proof</u>: $LHS = \cos x + \cos x \tan^2 x = \cos x\,(1 + \tan^2 x) = \cos x \sec^2 x$

$= \cos x \left(\frac{1}{\cos^2 x}\right) = \frac{1}{\cos x} = \sec x = RHS.$

3. $\frac{1}{\sec^2\theta} = (1+\sin\theta)(1-\sin\theta)$

<u>Proof</u>: $RHS = (1+\sin\theta)(1-\sin\theta) = 1 - \sin^2\theta = \cos^2\theta = \frac{1}{\frac{1}{\cos^2 x}} = \frac{1}{\sec^2\theta} = LHS.$

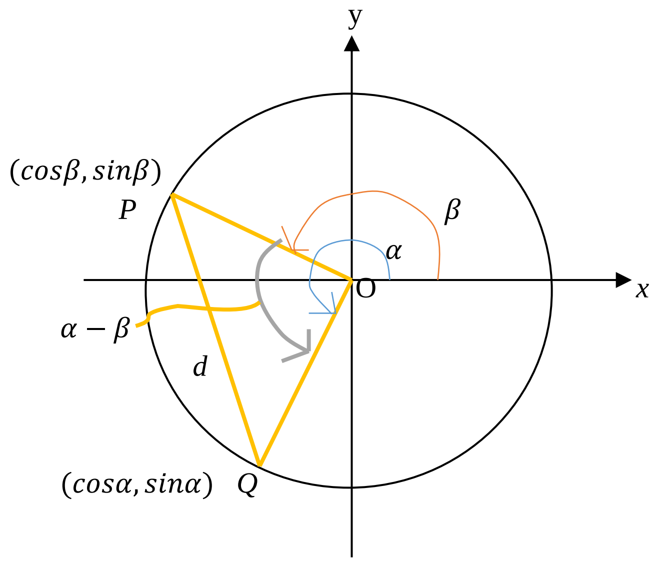
4. $\frac{\cos x + \tan x}{\sin x \cos x} = \csc x + \sec^2 x.$

Proof: $LHS = \frac{\cos x + \tan x}{\sin x \cos x} = \frac{\cos x + \frac{\sin x}{\cos x}}{\sin x \cos x} = \frac{\frac{\cos^2 x + \sin x}{\cos x}}{\sin x \cos x} = \frac{\cos^2 x + \sin x}{\sin x \cos^2 x} = \frac{1}{\sin x} + \frac{1}{\cos^2 x}$

$= \csc x + \sec^2 x = RHS.$

<u>Example</u> 1. If $\sin\theta = \frac{3}{5}$ and $\cos\theta < 0$, find the values of the other trigonometric functions.

<u>Sum and Difference Formulas</u>



We consider a unit circle and two angle α and β and the angle representing $\alpha - \beta$. In the triangle OPQ we let P have coordinates $(cos\beta, sin\beta)$, point Q have coordinates $(cos\alpha, sin\alpha)$ and let $PQ = d$. Using the distance formula

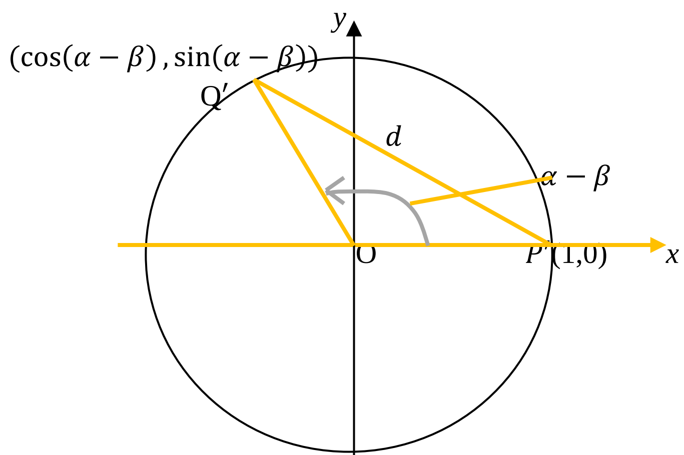
$$d = \sqrt{(cos\alpha - cos\beta)^2 + (sin\alpha - sin\beta)^2}$$

$$d^2 = (cos\alpha - cos\beta)^2 + (sin\alpha - sin\beta)^2$$

$$=cos^2\alpha - 2cos\alpha cos\beta + cos^2\beta + sin^2\alpha - 2sin\alpha sin\beta + sin^2\beta$$

$$=(sin^2\alpha + cos^2\alpha) - 2(cos\alpha cos\beta + sin\alpha sin\beta) + (sin^2\beta + cos^2\beta)$$

i.e. $d^2 = 2 - 2cos\alpha cos\beta - 2sin\alpha sin\beta$ (I)



The terminal side of angle $\alpha - \beta$ cut the unit circle at the point Q'$(\cos(\alpha - \beta), \sin(\alpha - \beta))$. Let the distance $P'Q' = D$. Then

$$D = \sqrt{[\cos(\alpha - \beta) - 1]^2 + [\sin(\alpha - \beta) - 0]^2}$$

Thus,

$$D^2 = 2 - 2\cos(\alpha - \beta) \qquad \text{(II)}$$

The triangles OPQ and $OP'Q'$ are congruent, thus $D = d$. This means that

$$2 - 2cos\alpha cos\beta - 2sin\alpha sin\beta = 2 - 2\cos(\alpha - \beta)$$

Therefore

6. $$\boxed{\cos(\alpha-\beta) = cos\alpha cos\beta + sin\alpha sin\beta}$$

If in the formula for $\cos(\alpha-\beta)$ we replace β with $(-\beta)$ we obtain

7. $$\boxed{\cos(\alpha+\beta) = cos\alpha cos\beta - sin\alpha sin\beta}$$

<u>Example</u>: Find an exact value for $cos\,\frac{5\pi}{12}$.

<u>Solution</u>: Let $\alpha = \pi/4$ and $\pi/6$ so that $\frac{\pi}{4}+\frac{\pi}{6}=\frac{3\pi+2\pi}{12}=\frac{5\pi}{12}$. Then

$$\cos\left(\frac{5\pi}{12}\right) = cos\left(\frac{\pi}{4}\right)cos\left(\frac{\pi}{6}\right) - sin\left(\frac{\pi}{4}\right)sin\left(\frac{\pi}{6}\right)$$
$$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{6}}{4}-\frac{\sqrt{2}}{4} = \frac{\sqrt{6}-\sqrt{2}}{4}.$$

The difference formula for cosine can generate other identities involving a 90^o angle.

8. $\cos(90^0-\alpha) = cos90^o cos\alpha + sin90^o sin\alpha = 0 + 1\cdot sin\alpha = sin\alpha$

i.e.

$$\boxed{\cos(90^0-\alpha) = sin\alpha}$$

If we substitute α with $90^0-\alpha$ in 8 we obtain

$\cos(90^0-(90^o-\alpha)) = sin(90^o-\alpha)$

i.e.

9. $$\boxed{\cos\alpha = sin(90^o-\alpha)}$$

10. $\tan(90^o-\alpha) = \frac{sin(90^o-\alpha)}{cos(90^o-\alpha)} = \frac{cos\alpha}{sin\alpha} = cot\alpha$

i.e.

$$\boxed{\tan(90^0-\alpha) = cot\alpha}$$

To develop the formula for $\sin(\alpha+\beta)$ we use identity 8 and replace α with $\alpha+\beta$. Thus

$$\text{Sin}(\alpha+\beta) = \cos\big(90^0-(\alpha+\beta)\big)$$
$$= cos[(90^0-\alpha)-\beta]$$
$$= \cos(90^o-\alpha)\,cos\beta + \sin(90^o-\alpha)sin\beta$$
$$= \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

11. $$\boxed{\text{Sin}(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta}$$

For $\text{Sin}(\alpha-\beta)$ we replace β with $(-\beta)$ in 11 so that

$$\text{Sin}(\alpha + (-\beta)) = \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta)$$

Thus,

12. $$\boxed{\text{Sin}(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta}$$

The formula for $\tan(\alpha + \beta)$ follows directly from the sine and cosine relationships.

$$\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}$$

Dividing both the numerator and the denominator yields

$$\tan(\alpha+\beta) = \frac{\frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta} + \frac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\frac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta} - \frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} = \frac{\frac{\sin\alpha}{\cos\alpha} + \frac{\sin\beta}{\cos\beta}}{1 - \frac{\sin\alpha}{\cos\alpha}\frac{\sin\beta}{\cos\beta}}$$

13 $$\boxed{\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}}$$

To obtain the formula for $\tan(\alpha - \beta)$ we replace β with $(-\beta)$ in 13 to get

14. $$\boxed{\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}}$$

<u>Example</u>: Find the exact value of $tan\ 195^o$.
<u>Solution</u>: Let $\alpha = 135^o$ and $\beta = 60^o$. then

$$\tan(135^\circ + 60^\circ) = \frac{\tan 135^\circ + \tan 60^\circ}{1 - \tan 135^\circ \tan 60^\circ}$$

$$= \frac{-1+\sqrt{3}}{1-(-1)(\sqrt{3})}$$

$$= \frac{-1+\sqrt{3}}{1+\sqrt{3}} \times \frac{1-\sqrt{3}}{1-\sqrt{3}}$$

$$= \frac{-1+\sqrt{3}+\sqrt{3}-3}{1^2-(\sqrt{3})^2} = \frac{-4+2\sqrt{3}}{1-3} = \frac{-4+2\sqrt{3}}{-2}$$

$$= \frac{-4}{-2} + \frac{2\sqrt{3}}{-2} = 2 - \sqrt{3}.$$

When we add the identities (7)

$$\cos(\alpha+\beta) = cos\alpha cos\beta - sin\alpha sin\beta$$

and (6)

$$\cos(\alpha-\beta) = cos\alpha cos\beta + sin\alpha sin\beta$$

we have

$$\cos(\alpha+\beta) + \cos(\alpha-\beta) = 2\cos\alpha\cos\beta.$$

From this we get the product identity

15. $$cos\alpha cos\beta = \frac{1}{2}[cos(\alpha+\beta) + cos(\alpha-\beta)]$$

When we subtract identity (6) from identity (7) we get the product identity

16. $$sin\alpha sin\beta = \frac{1}{2}[cos(\alpha-\beta) - cos(\alpha+\beta)]$$

Similarly adding (11)

$$\sin(\alpha+\beta) = sin\alpha cos\beta + cos\alpha sin\beta$$

and (12)

$$\sin(\alpha-\beta) = sin\alpha cos\beta - cos\alpha sin\beta$$

we obtain the product identities

17. $$sin\alpha cos\beta = \frac{1}{2}[sin(\alpha+\beta) + sin(\alpha-\beta)]$$

Subtracting the identities we get the product identity.

18. $$cos\alpha sin\beta = \frac{1}{2}[sin(\alpha+\beta) - sin(\alpha-\beta)]$$

When we let $A = \alpha+\beta$ and $B = \alpha-\beta$, it follows $\alpha = \frac{A+B}{2}$ and $\beta = \frac{A-B}{2}$. Replacing these in 15 we obtain the identity

19. $$cosA + cosB = 2cos\frac{A+B}{2}cos\frac{A-B}{2}$$

Replacing them in 16 we have the identity

20. $$cosB - cosA = 2cos\frac{A+B}{2}sin\frac{A-B}{2}$$

Replacing the same in 17 and 18 we obtain the identities

21. $$sinA + sinB = 2sin\frac{A+B}{2}cos\frac{A-B}{2}$$

and

22. $$sinA - sinB = 2cos\frac{A+B}{2}sin\frac{A-B}{2}$$

Multiple angle Formulas

In the formula for $\sin(\alpha + \beta)$, if we let $\alpha = \beta$ we have

$$\sin(\alpha + \alpha) = \sin\alpha\cos\alpha + \cos\alpha\, sin\alpha$$

i.e.

23. $$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

Note that $sin2\alpha \neq 2sin\alpha$. For example, if $\alpha = 30^0$,

$sin2(30^0) = sin60^0 = \frac{\sqrt{3}}{2}$ and $2sin30^0 = 2 \times \frac{1}{2} = 1 \neq \frac{\sqrt{3}}{2} = sin2(30^0)$.

Putting α in place for β in the formula for $\cos(\alpha + \beta)$ yields

$$\cos(\alpha + \alpha) = cos\alpha cos\alpha - sin\alpha sin\alpha$$

i.e.

24. $$\cos 2\alpha = cos^2\alpha - sin^2\alpha$$

From 24 we get

$$\cos 2\alpha = cos^2\alpha - (1 - cos^2\alpha)$$

i.e.

25. $$\cos 2\alpha = 2cos^2\alpha - 1$$

Again from 24 we obtain

$$\cos 2\alpha = (1 - sin^2\alpha) - sin^2\alpha$$

i.e.

26. $$\cos 2\alpha = 1 - 2sin^2\alpha$$

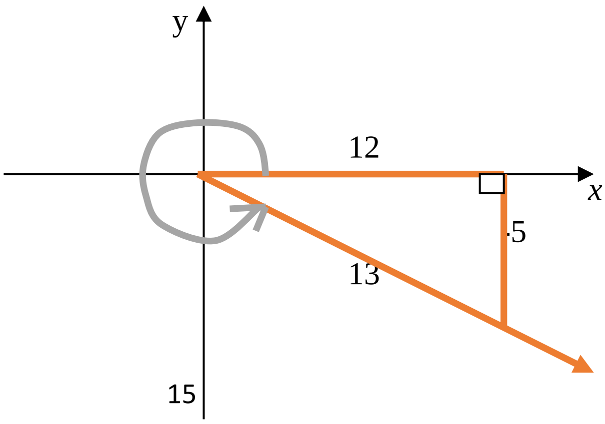
Substituting $\alpha = \beta$ in the formula for $\tan(\alpha + \beta)$ yields

$$\tan(\alpha + \alpha) = \frac{tan\alpha + tan\alpha}{1 - tan\alpha tan\alpha} = \frac{2tan\alpha}{1 - tan^2\alpha}$$

i.e.

27. $$\tan 2\alpha = \frac{2tan\alpha}{1 - tan^2\alpha}$$

Example: Find $sin2\theta, cos2\theta,$ and $tan2\theta$ if $sin\theta = -\frac{5}{13}$ and is a fourth quadrant angle.



Therefore, $cos\theta = \frac{12}{13}$ and $tan\theta = -\frac{5}{12}$. Now using the identities for double angles we get

(i) $sin2\theta = 2sin\theta cos\theta = 2 \times \left(-\frac{5}{13}\right) \times \left(\frac{12}{13}\right) = -\frac{120}{169}$

(ii) $cos2\theta = cos^2\theta - sin^2\theta = \left(\frac{12}{13}\right)^2 - \left(-\frac{5}{13}\right)^2 = \frac{144}{169} - \frac{25}{169} = \frac{119}{169}$

(iii) $\tan 2\theta = \frac{2tan\theta}{1-tan^2\theta} = \frac{2\left(-\frac{5}{12}\right)}{1-\left(-\frac{5}{12}\right)^2} = \frac{-\frac{10}{12}}{1-\frac{25}{144}} = -\frac{5}{6} \times \frac{144}{119} = -\frac{120}{119}$.

Half angle Formulas

Half-angle formulas for $\sin(\alpha/2)$ and $\cos(\alpha/2)$ are a direct consequence of the identities for $cos2\alpha$.

$$cos2\alpha = 1 - 2sin^2\alpha$$

$$\Rightarrow \quad sin^2\alpha = \frac{1}{2}(1 - cos2\alpha)$$

$$\Rightarrow \quad sin\alpha = \pm\sqrt{\frac{1}{2}(1 - cos2\alpha)}$$

Replacing α with $\alpha/2$ yields

28. $$\sin \alpha/2 = \pm\sqrt{\frac{1}{2}(1 - cos\alpha)}$$

Using the formula $\cos 2\alpha = 2cos^2\alpha - 1$ we can obtain the half-angle formula for $\cos\left(\frac{\alpha}{2}\right)$ *as follows*:

$$\cos 2\alpha = 2cos^2\alpha - 1$$

$$\Rightarrow \quad cos^2\alpha = \frac{1}{2}(1 + cos2\alpha)$$

$$\Rightarrow \quad cos\alpha = \pm\sqrt{\frac{1}{2}(1 + cos2\alpha)}$$

Replacing α with $\alpha/2$ yields

29. $$\cos \alpha/2 = \pm\sqrt{\frac{1}{2}(1 + cos\alpha)}$$

In the formulas for $\sin(\alpha/2)$ and $\cos(\alpha/2)$, the choice of the plus or minus sign is determined by the quadrant in which $\alpha/2$ lies.

For example if $\alpha/2$ is in the first and second quadrants then

$$\sin \alpha/2 = \sqrt{\frac{1}{2}(1 - cos\alpha)},$$

If it is in the third and the fourth quadrants, then

$$\sin \alpha/2 = -\sqrt{\frac{1}{2}(1 - cos\alpha)}.$$

To obtain the formula for $\tan(\frac{\alpha}{2})$ we proceed as follows:

$$\tan\left(\frac{\alpha}{2}\right) = \frac{\sin\alpha/2}{\cos\alpha/2} = \pm\left(\sqrt{\frac{1}{2}(1-cos\alpha)}\right)/\pm\left(\sqrt{\frac{1}{2}(1+cos\alpha)}\right)$$

i.e.

30. $$\boxed{\tan\ \alpha/2 = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}}$$

Or

$$\tan\ \alpha/2 = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}\times\frac{1-\cos\alpha}{1-\cos\alpha}} = \pm\sqrt{\frac{(1-cos\alpha)^2}{1-cos^2\alpha}} = \pm\sqrt{\frac{(1-cos\alpha)^2}{sin^2\alpha}}$$

$$= \pm\frac{1-cos\alpha}{sin\alpha}$$

i.e.

31. $$\boxed{\tan\ \alpha/2 = \frac{1-cos\alpha}{sin\alpha}}$$

We no longer need the $\pm$ sign because $1 - cos\alpha$ is never negative, and sin α and tan $\alpha/2$ the same sign.

Now,

$$\frac{1-cos\alpha}{sin\alpha} = \frac{1-cos\alpha}{sin\alpha}\times\frac{1+cos\alpha}{1+cos\alpha} = \frac{1-cos^2\alpha}{sin\alpha(1-cos\alpha)} = \frac{sin^2\alpha}{sin\alpha(1+cos\alpha)} = \frac{sin\alpha}{1-cos\alpha}$$

i.e.

32. $$\boxed{\tan\frac{\alpha}{2} = \frac{sin\alpha}{1-cos\alpha}}$$

Example: Find an exact value of tan 67.5^0.

Solution: $\tan\ 67.5^0 = \tan\frac{1}{2}(135)^0 = \frac{1-cos135^0}{\sin 135^0} = \frac{1-(-\sqrt{2}/2)}{\sqrt{2}/2}$

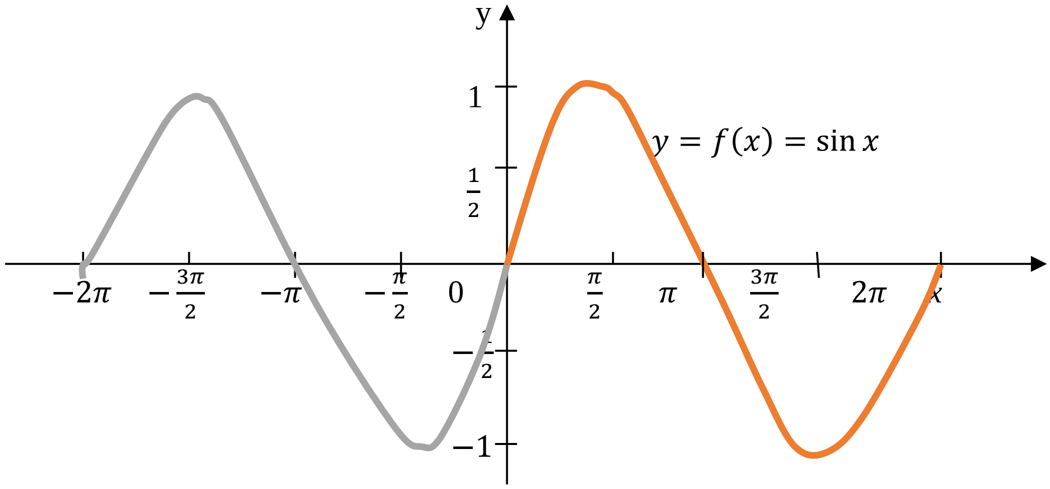
$$= \frac{2+\sqrt{2}}{\sqrt{2}} = \sqrt{2}+2$$

9.1.4 Graphing Trigonometric Functions

We look at the graphs of $y = sinx\ and\ y = cosx$.

1. The sine curve

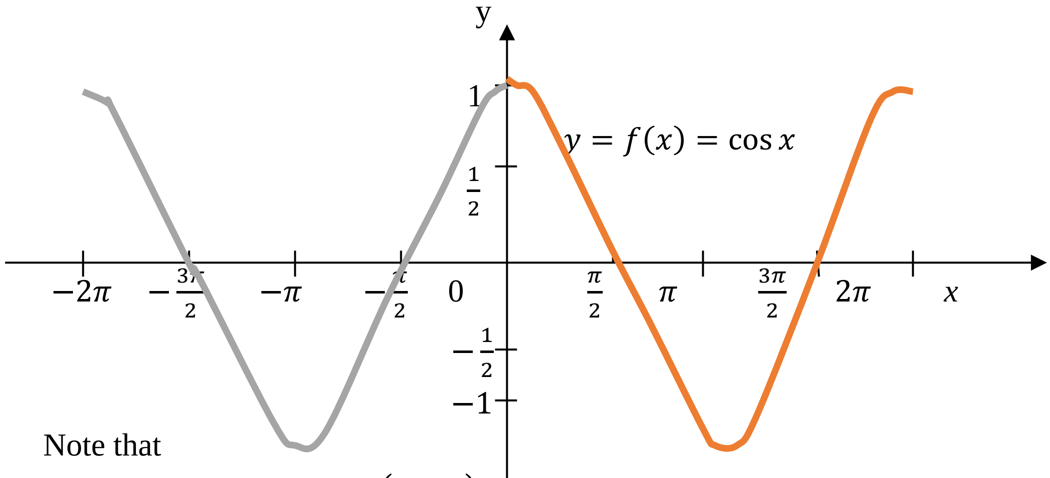
x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$f(x)$=sin$x$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{-\sqrt{3}}{2}$	-1	$\frac{-\sqrt{3}}{2}$	$-\frac{1}{2}$	0



Note that the graph of sine function repeats itself after 2π radians. Thus it is said to have a period of 2π.

2. The cosine curve

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
f(*x*)=sin*x*	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$\frac{-1}{2}$	$\frac{-\sqrt{3}}{2}$	-1	$\frac{-\sqrt{3}}{2}$	$\frac{-1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1



Note that

$$\sin\left(x + \frac{\pi}{2}\right) = \cos x$$

Similarly, the cosine function has a period of 2π.

<u>Period Amplitude and Phase Shift</u>

Period

A function *f* is called **periodic** if there exists a positive real number *p* such that

$$f(x + p) = f(x),$$

for all *x* in the domain of *f*. The smallest value of *p* is called the **period** of the function.

From the graphs of the sine and cosine functions it is evident that 2π is the smallest value for which $\sin(x + 2\pi) = \sin x$ and $\cos(x + 2\pi) = \cos x$. Therefore both the sine and the cosine functions are periodic with period 2π. Now, we consider the graph of a function of the form $f(x) = \sin bx$, where $b > 0$. One cycle of the graph is completed as bx increases form 0 to 2π. When $bx = 0, x = 0$ and $bx = 2\pi,\ \ x = \frac{2\pi}{b}$. The same holds for $f(x) = \cos bx$. Therefore we can state that the period of $f(x) = \sin bx$ and also of $f(x) = \cos bx$, where $b > 0$, is $\frac{2\pi}{b}$. If $b < 0$, then we can first apply the appropriate property, $\sin(-x) = -\sin x$ or $\cos(-x) = \cos x$ and find the period accordingly.

Amplitude

The amplitude of the graph of $f(x) = a\sin x$ or $f(x) = a\cos x$ is $|a|$ and it is the maximum functional value attained by $y = f(x)$.

For example, the amplitude of $f(x) = -\frac{1}{2}\cos x$ is $\left|-\frac{1}{2}\right| = \frac{1}{2}$ and that of $f(x) = 3\sin 2x$ is $|3| = 3$.

Phase Shift

You will note that the graph of $f(x) = \sin(x - \frac{\pi}{2})$ is the basic sine curve shifted $\frac{\pi}{2}$ units to the right. Likewise, the graph of $f(x) = \cos(x + \pi)$ is the basic cosine curve shifted π units to the left. Each number $\frac{\pi}{2}$ and π in this case represents the amount of shift to the right and to the left, called to the **phase shift** of the graph.
In general, the phase shift of $f(x) = \sin(x - c)$ or $f(x) = \cos(x - c)$ is $|c|$. **If c is positive, the phase shift is to the right and if c is negative, the phase shift is to the left.**

Example: Find the period, amplitude, and phase shift of $f(x) = 3\sin(2x + \frac{\pi}{2})$ and sketch the curve in the interval
$0 \le x \le \pi$.

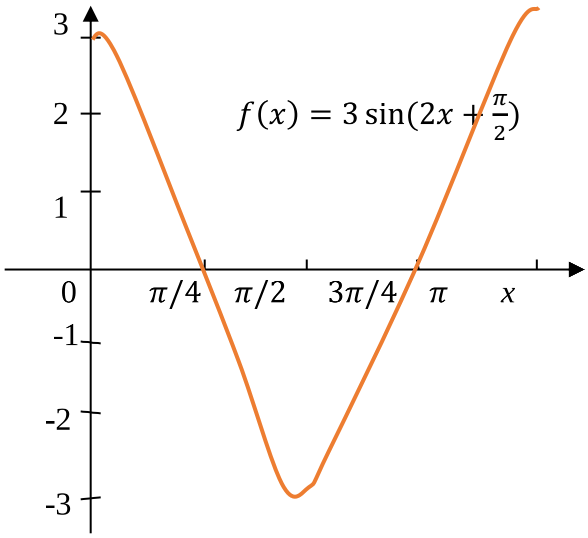
Solution: We rewrite the function in the following form for easy determination of the required quantities:

$$f(x) = 3\sin 2(x - (-\frac{\pi}{4}))$$

amplitude is $|3| = 3$

period is $\frac{2\pi}{2} = \pi$

phase shift is $\left|-\frac{\pi}{4}\right| = \frac{\pi}{4}$ *units to the left.*

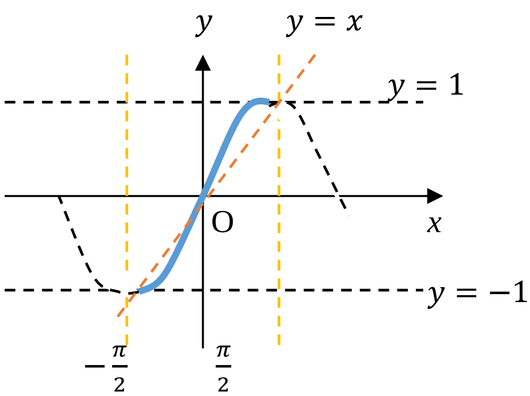


9.1.5 Inverses of trigonometric functions

Inverse functions of $sinx$, $cosx$ and $tanx$

Recall that a function *f* has an inverse function if and only if it is one-to-one and onto.

The sine function $y = sinx$ is not one-to-one unless its domain is restricted to the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$



In this restricted domain the sine function has an inverse function. Therefore, the inverse function of $y = sinx$ is the function that assigns to each number $x \in [-1,1]$ the unique number $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ such that $x = siny$.
The inverse of $y = sinx$ is written

$$\boldsymbol{y = sin^{-1}x \text{ or } y = arcsinx} \qquad (1.7.1)$$

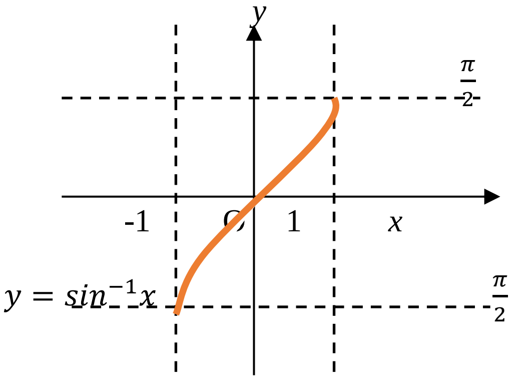
Thus,

$$\boxed{\boldsymbol{y = sin^{-1}x \iff x = siny}}$$

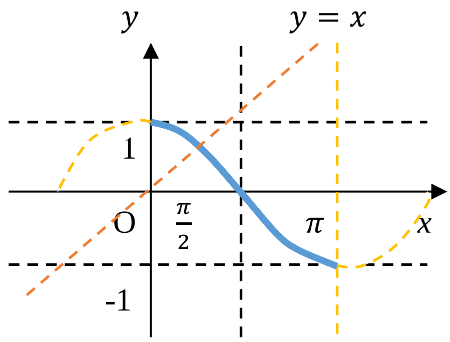
It should be noted that $y = sin^{-1}x \neq \frac{1}{sinx}$.

$\frac{1}{sinx}$ is the reciprocal of $\sin x$ and **not** the inverse of $\sin x$.

The graph of $y = sin^{-1}x$ which is reflection of $y = sinx$ in the line $y = x$ is shown below.



Similarly the cosine function $y = cosx$ has an inverse function only when its domain is restricted, say to $[0, \pi]$. In this case its range is $[-1,1]$. Thus for every $y \in [-1,1]$ there is a unique $x \in [0, \pi]$ such that $x = cosy$.



Therefore, $y = cosx$ has an inverse function, which is written as

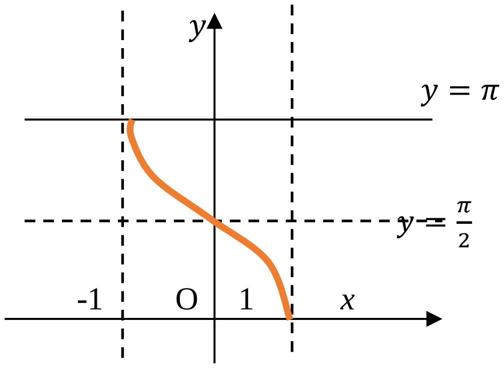
$$\boldsymbol{y = cos^{-1}x \text{ or } y = arccosx} \qquad (1.7.2)$$

Thus,

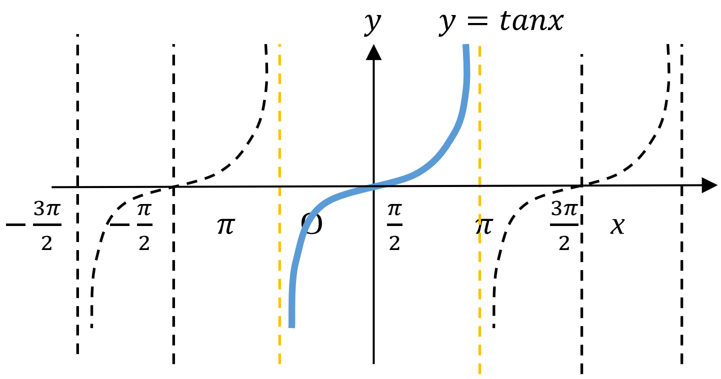
$$\boxed{y = cos^{-1}x \iff x = cosy}$$

Similarly, $= cos^{-1}x \neq \frac{1}{\cos x}$.

The graph of $y = cos^{-1}x$ which is reflection of $y = cosx$ in the line is shown below.



For the inverse function of $y = tanx$. We first consider the graph of $y = tanx$.



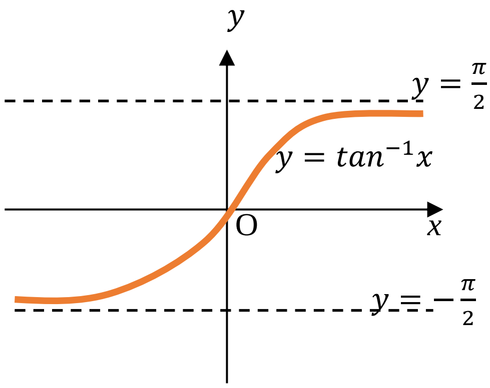
The function $y = tanx$ can take on values of y in the interval $(-\infty, \infty)$. To get a unique x for given y, we restrict x to the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. Thus the function $tanx$: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \rightarrow (-\infty, \infty)$ is one-to-one and has an inverse function written

$$\boldsymbol{y = tan^{-1}x \text{ or } y = arctanx} \qquad (1.7.3)$$

and

$$\boxed{y = \tan^{-1}x \quad \Leftrightarrow x = \tan y}$$

The domain of $y = tan^{-1}x$ is $(-\infty, \infty)$ and its range is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.
The graph of $y = tan^{-1}x$ is shown below.



The other three trigonometric functions $f(x) = \sec x$, $f(x) = \csc x$ and

$f(x) = \cot x$ with restricted domains also have respectively the inverse functions $f^{-1}(x) = \sec^{-1} x$, $f^{-1}(x) = \csc^{-1} x$, and $f^{-1}(x) = \cot^{-1} x$. The graphs of these inverse functions can be constructed the same way.

9.1.6 Trigonometric Equations

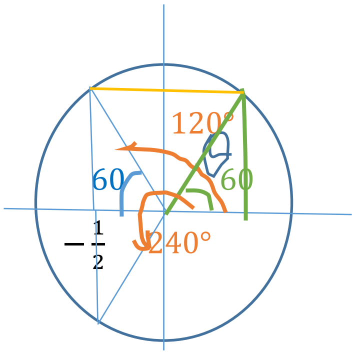
Examples:

1. Solve for θ, if $0 \leq \theta < 360^o$:

(a) $2\cos\theta + 1 = 0$

(b) $\cos 2\theta - \cos\theta = 0$

Solutions: (a) $2\cos\theta + 1 = 0 \Rightarrow 2\cos\theta = -1$



$\Rightarrow \cos\theta = -\frac{1}{2} \Rightarrow \theta = \cos^{-1}\left(-\frac{1}{2}\right) = 180° - 60° = 120°$

Or $\theta = \cos^{-1}\left(-\frac{1}{2}\right) = 180° + 60° = 240°$.

(b) $\cos 2\theta - \cos\theta = 0 \Rightarrow 2\cos^2\theta - 1 - \cos\theta = 0$

$\Rightarrow 2\cos^2\theta - \cos\theta - 1 = 0$

$\Rightarrow (2\cos\theta + 1)(\cos\theta - 1) = 0$

$\Rightarrow 2\cos\theta + 1 = 0$ or $\cos\theta - 1 = 0$

$\Rightarrow \cos\theta = -\frac{1}{2}$ or $\cos\theta = 1$

$\Rightarrow \theta = \cos^{-1}\left(-\frac{1}{2}\right)$ or $\theta = \cos^{-1}(1)$

$\Rightarrow \theta = \cos^{-1}\left(-\frac{1}{2}\right) = 120°, 240°$ or $\theta = \cos^{-1}(1) = 0°, 360°$.

$\therefore \boldsymbol{\theta = 0°, 120°, 240°, 360°}$

2. Solve for x, if $0 \le x < 2\pi$:

(a) $2sin^2x + sinx - 1 = 0$

(b) $sec^2x + tan^2x = 3$

(c) $sinx + cosx = \sqrt{2}$

(d) $\sin 2x - 2\cos x + \sin x - 1 = 0$.

Solutions: (a) $2sin^2x + sinx - 1 = 0$

$(2\sin x - 1)(\sin x + 1) = 0$

$\Rightarrow 2\sin x - 1 = 0$ or $\sin x + 1 = 0$

$\Rightarrow \sin x = \frac{1}{2}$ or $\sin x = -1$

The sine is positive in the 1st and 2nd quadrants. Thus

$x = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}, \frac{5\pi}{6}$

And we know that $\sin x = -1 \Rightarrow x = \frac{3\pi}{2}$.

$\therefore \boldsymbol{x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}}$

(b) $sec^2 x + tan^2 x = 3$

$$(1 + tan^2 x) + tan^2 x = 3$$

$$\Rightarrow \quad 2tan^2 x + 1 = 3 \Rightarrow \quad 2tan^2 x = 2$$

$$\Rightarrow \quad tan^2 x = 1 \Rightarrow \tan x = \pm 1$$

Now we know that the tangent is positive in the 1st and 3rd quadrants. Thus,

$$x = \tan^{-1}(1) = \frac{\pi}{4}, \frac{5\pi}{4}.$$

The tangent is negative in the 2nd and 4th quadrants. Thus,

$$x = \tan^{-1}(1) = \frac{3\pi}{4}, \frac{7\pi}{4}.$$

$$\therefore \quad \boldsymbol{x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}}.$$

(c) $sinx + cosx = \sqrt{2}$

Squaring both sides we have

$$(sinx + cosx)^2 = \left(\sqrt{2}\right)^2$$

$$\Rightarrow \quad \sin^2 x + 2\sin x \cos x + \cos^2 x = 2$$

$$\Rightarrow \quad \sin^2 x + 2\sin x \cos x + (1 - \sin^2 x) = 2$$

$$\Rightarrow \quad 2\sin x \cos x = 1 \Rightarrow \sin 2x = 1 \Rightarrow 2x = \frac{\pi}{2} \; or \; \frac{5\pi}{2} \text{ or } \frac{7\pi}{2} \text{ etc.}$$

Since $0 \le x < 2\pi$,

$$x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}.$$

But since we changed the question by squaring, we need to test these angles in the give equation.

For $x = \frac{\pi}{4}$, $LHS = \sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2} = RHS$

$\Rightarrow \frac{\pi}{4}$ is part of the solution.

For $x = \frac{5\pi}{4}$, $LHS = \sin\left(\frac{5\pi}{4}\right) + \cos\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = -\sqrt{2} \neq \sqrt{2} = RHS$

$\Rightarrow \frac{5\pi}{4}$ is not part of the solution.

For $x = \frac{7\pi}{4}$, $LHS = \sin\left(\frac{7\pi}{4}\right) + \cos\left(\frac{7\pi}{4}\right) = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = 0 \neq \sqrt{2} = RHS$

$\Rightarrow \frac{7\pi}{4}$ is not part of the solution.

Therefore the only angle which satisfy the given equation is $\boldsymbol{\frac{\pi}{4}}$.

(d) $\sin 2x - 2\cos x + \sin x - 1 = 0$

$\Rightarrow (2\sin x \cos x) - 2\cos x + \sin x - 1 = 0$

$\Rightarrow 2\cos x\,(\sin x - 1) + (\sin x - 1) = 0$

$\Rightarrow (\sin x - 1)(2\cos x + 1) = 0$

$\Rightarrow \sin x - 1 = 0$ and $2\cos x + 1 = 0$

$\Rightarrow \sin x = 1$ and $\cos x = -\frac{1}{2}$

$\Rightarrow x = \sin^{-1}(1) = \frac{\pi}{2}$ and $x = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}, \frac{5\pi}{3}$.

$\therefore \quad \boldsymbol{x = \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{3}}$.

General Solution of Trigonometric Equations

The general solution is an expression which represents all angles which satisfy the given trigonometric equation. Thus, the general solution is an infinite set of angles which satisfy the equation.

To find the general solution of a trigonometric equation , you may need to use the following:

(i) The circular function graph,

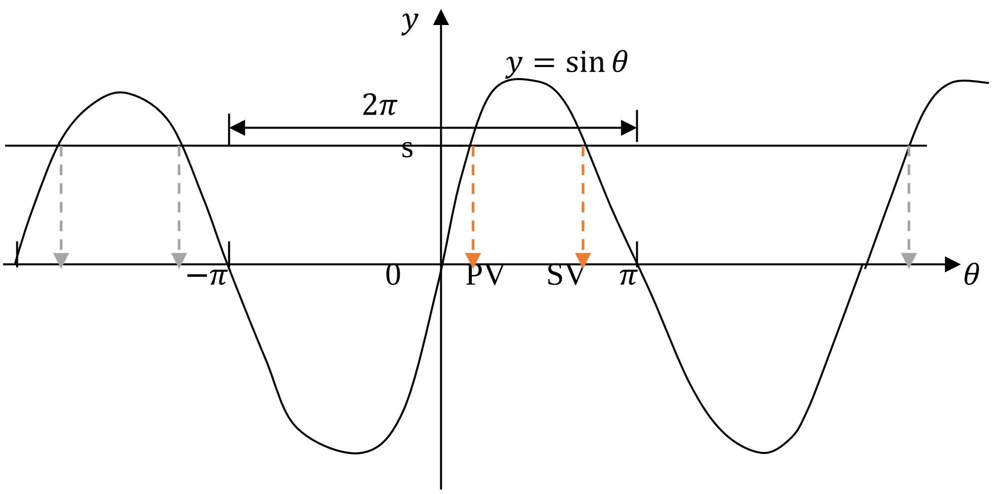
(ii) The period of each circular function,

(iii) The principal solution, and except when the tangent ratio is involved, the secondary solution.

Consider the equation

$$\sin\theta = s$$

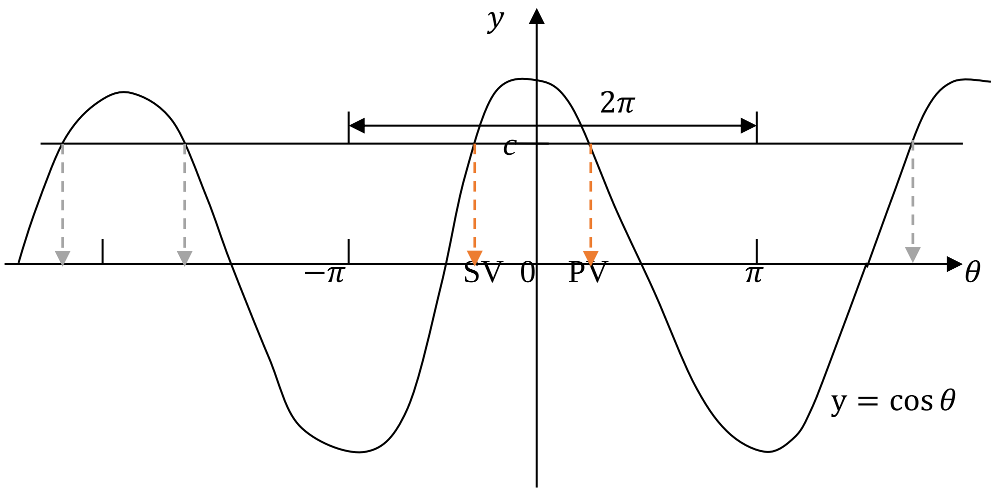
where $|s| \leq 1$. The period, 2π, of the sine function is covered in the interval $[-\pi, \pi]$, which includes both the Principle Value (PV) and the Secondary Value (SV) of θ. So, by adding or subtracting any multiple of 2π to either the PV or the SV, we get another angle with the same sine ratio.



Therefore the general solution of the equation $\sin\theta = s$, where $|s| \leq 1$ is

$$\theta = \begin{cases} PV + 2n\pi \\ SV + 2n\pi \end{cases} \text{ or } \begin{cases} PV + 360n^{\circ} \\ SV + 360n^{\circ} \end{cases} \text{ where } n \in \mathbb{Z}.$$

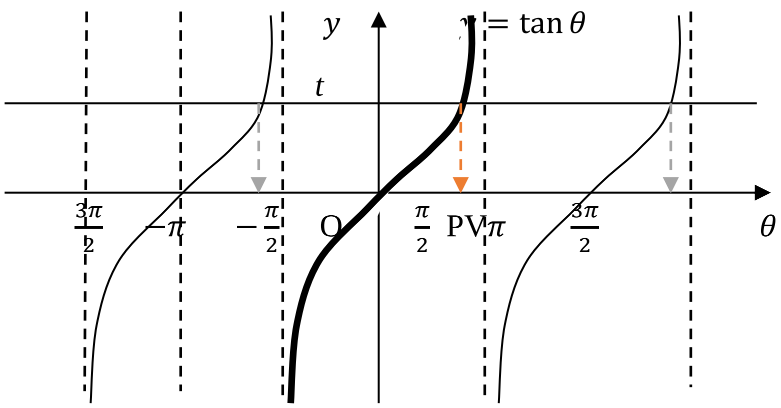
A similar situation arise when we consider the equation $\cos\theta = c$ because both the PV and the SV of the cosine function occur within one period, 2π.



Thus, by adding or subtracting any multiple of 2π to either the PV or the SV we get the general solution of the equation $\cos\theta = c$. However, you note that for the cosine, the PV and the SV are equal in value but opposite sign i.e. $PV = -PV$. Therefore, the general solution for the equation $\cos\theta = c$ is given by

$$\theta = \{\pm PV + 2n\pi \text{ or } \{\pm PV + 360n^{\circ} \text{ where } n \in \mathbb{Z}.$$

When we consider the equation $\tan\theta = t$, we note that only the PV is included in the period $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. The other angles with the same tangent ratio are obtained by combining multiples of π, the period.



Therefore the general solution of the equation $\tan\theta = t$ is given by

$$\theta = \{PV + \pi \text{ or } \{PV + 180n^{\circ} \text{ where } n \in \mathbb{Z}.$$

<u>Examples</u> Find the general solution of each of the following trigonometric equations:

1. (a) $\tan\theta = 1$ (b) $\tan\theta = -\sqrt{3}$
2. (a) $\cos\theta = \frac{1}{\sqrt{2}}$ (b) $\cos\theta = -\frac{1}{2}$
3. $\sin\theta = \frac{1}{2}$
4. $\cos 2\theta = \frac{\sqrt{3}}{2}$.

<u>Solutions</u>: 1. (a) $\tan\theta = 1 \Rightarrow PV = \frac{\pi}{4}$ or $45°$.

Therefore, the general solution is

$$\theta = \frac{\pi}{4} + \pi n, n \in \mathbb{Z} \text{ or } \theta = 45° + 180° n, n \in \mathbb{Z}.$$

(b) $\tan\theta = -\sqrt{3}\ PV = -\frac{\pi}{3}$ or $-60°$.

Therefore, the general solution is

$$\theta = -\frac{\pi}{3} + \pi n, n \in \mathbb{Z} \text{ or } \theta = -60° + 180° n, n \in \mathbb{Z}.$$

2. (a) $\cos\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) \Rightarrow PV = \frac{\pi}{4}$ or $45°$.

Therefore, the general solution is

$$\theta = \frac{\pi}{4} + 2\pi n, n \in \mathbb{Z} \text{ or } \theta = 45° + 360° n, n \in \mathbb{Z}.$$

(b) $\cos\theta = -\frac{1}{2} \Rightarrow PV = \pm\frac{2\pi}{3}$ or $\pm 120°$

Therefore, the general solution is

$$\theta = \left\{\pm\frac{2\pi}{3} + 2n\pi \text{ or } \{\pm 120° + 360n^{\circ} \text{ where } n \in \mathbb{Z}.\right.$$

3. $\sin\theta = \frac{1}{2} \Rightarrow PV = \frac{\pi}{6}$ or $30°$ and $SV = \frac{5\pi}{6}\ or\ 150°$.

Therefore, the general solution is

$$\theta = \begin{cases} \frac{\pi}{6} + 2n\pi \\ \frac{5\pi}{6} + 2n\pi \end{cases} \text{ or } \begin{cases} 30° + 360n^{\circ} \\ 150° + 360n^{\circ} \end{cases} \text{ where } n \in \mathbb{Z}.$$

4. $\cos 2\theta = \frac{\sqrt{3}}{2} \Rightarrow 2\theta = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \pm\frac{\pi}{6} \Rightarrow \theta = \pm\frac{\pi}{12} \Rightarrow PV = \pm\frac{\pi}{12}$

or $15°$.

Therefore, the general solution is

$$\theta = \left\{\pm\frac{\pi}{12} + 2n\pi \text{ or } \{\pm 15° + 360n^{\circ} \text{ where } n \in \mathbb{Z}.\right.$$

<u>TUTORIAL SHEET</u> 12

1. Find the quadrant that contains the terminal side of θ if the given conditions are true:

   (a) $\sin\theta < 0$ and $\cos\theta > 0$ (b) $\sin\theta < 0$ and $\cot\theta > 0$

   (c) $\sec\theta < 0$ and $\tan\theta > 0$ (d) $\csc\theta > 0$ and $\cot\theta < 0$.

2. Without using a calculator, change each of the following to radians:

   (a) 420° (b) 570° (c) -45°.

3. Without using a calculator, change each of the following to degrees:

   (a) $\frac{7\pi}{6}$ (b) $-\frac{4\pi}{3}$ (c) $\frac{17\pi}{4}$.

4. Find exact values. Do not use a calculator:

   (a) $\sin 120^\circ$ (b) $\cos 150^\circ$ (c) $\tan 300^\circ$ (d) $\csc(-135^\circ)$ (e) $\sec 420^\circ$

   (f) $\sin\frac{2\pi}{3}$ (g) $\tan\frac{4\pi}{3}$ (h) $\cos\left(-\frac{5\pi}{3}\right)$ (i) $\cot\left(-\frac{13\pi}{3}\right)$ (j) $\sec\left(-\frac{7\pi}{6}\right)$.

5. If $\tan\alpha = -\frac{4}{3}$ and $\sin\beta = -\frac{3}{5}$, where α is a second quadrant angle and β is a fourth quadrant angle, find $\sin(\alpha-\beta)$ and $\tan(\alpha+\beta)$.

6. Find the exact values without using a table or a calculator:

   (a) $\sin 15^\circ$ (b) $\tan 15^\circ$ (c) $\tan 75^\circ$ (d) $\cos 345^\circ$ (e) $\cos\frac{\pi}{12}$

   (f) $\sin\frac{7\pi}{12}$ (g) $\tan\frac{11\pi}{12}$.

7. Verify each of the following identities:

   (a) $\sin(\alpha+90^\circ) = \cos\alpha$ (b) $\cos(\alpha+90^\circ) = -\sin\alpha$ (c) $\sin(\alpha+\pi) = -\sin\alpha$

   (d) $\cos(\alpha-\pi) = -\cos\alpha$ (e) $\tan(\alpha+\frac{\pi}{4}) = \frac{1+\tan\alpha}{1-\tan\alpha}$ (f) $\tan(\alpha-\frac{\pi}{4}) = \frac{\tan\alpha-1}{\tan\alpha+1}$.

8. Graph each of the following functions in the indicated intervals:

   (a) $f(x) = \sin\left(x+\frac{\pi}{2}\right), \quad -\frac{\pi}{2} \le x \le \frac{3\pi}{2}$ (b) $f(x) = \cos\left(x-\frac{\pi}{2}\right), \quad \frac{\pi}{2} \le x \le \frac{5\pi}{2}$

   (c) $f(x) = 2+\sin x, \quad -\pi \le x \le 2\pi$ (d) $f(x) = -2+\sin\left(x-\frac{\pi}{2}\right), \quad \frac{\pi}{2} \le x \le \frac{5\pi}{2}$.

9. Find the period, amplitude, and phase shift of the given function and draw the graph of the function:

   (a) $f(x) = 2\sin\left(x-\frac{\pi}{3}\right)$ (b) $f(x) = 3\cos 2\left(x+\frac{\pi}{2}\right)$ (c) $f(x) = -\frac{1}{2}\cos\left(2x-\frac{\pi}{2}\right)$

(d) $f(x) = -2\tan\left(x - \frac{\pi}{2}\right)$.

10. Solve each of the following equations for θ, if $0^\circ \leq \theta \leq 360^\circ$. Do not use a calculator or a table:

(a) $2\sin\theta + \sqrt{2} = 0$ (b) $\sin^2\theta - 1 = 0$ (c) $2\cos^2\theta = \cos\theta$

(d) $2\cos^3\theta = \cos\theta$ (e) $2\sin^2\theta - \cos\theta - 1 = 0$ (f) $\tan\theta = \cot\theta$.

11. Solve each of the following equations for θ, if $0 \leq x \leq 2\pi$. Do not use a calculator or a table:

(a) $2\tan x \sec x - \tan x = 0$ (b) $2\cos^2\theta + 3\cos x + 1 = 0$

(c) $\sec^2 x - \sec x - 2 = 0$ (d) $\sin x = 1 - \cos x$

(e) $\sin x \cos x - \cos x + \sin x - 1 = 0$ (f) $\tan x + 1 = \sec x$.

12. Solve each of the following equation for θ, where $0 \leq \theta < 2\pi$. Do not use a calculator or table.

(a) $\cos\theta = \sin 2\theta$ (b) $\cos 2\theta - 3\sin\theta - 2 = 0$ (c) $\tan 2\theta + \sec 2\theta = 1$

(d) $2 - \sin^2\theta = 2\cos^2\frac{\theta}{2}$ (e) $\sin\frac{\theta}{2} + \cos\theta = 1$.

9.2 Exponential functions

Exponential functions occur naturally in real life. Scientists can model the growth of an organism in any given culture by using exponential functions.

Exponential functions are ones of the form

$$y = b^x \text{ or } f(x) = b^x$$

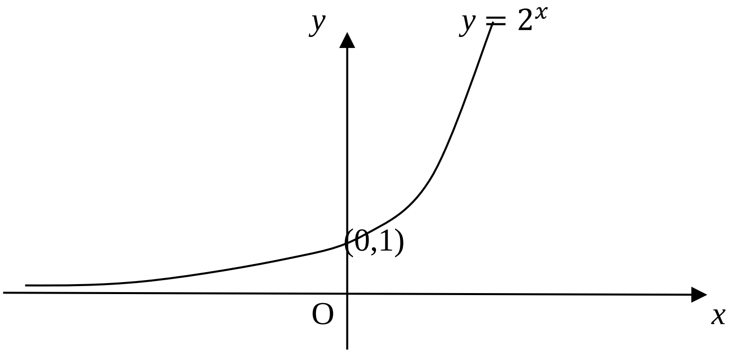
Graphs of exponential functions $y = b^x$ all pass through the point $(0,1)$ because $b^0 = 1$.

Example 9.2.1: Sketch the graph of $f(x) = 2^x$ for the domain $x \in \mathbf{R}$. Hence, state its range.

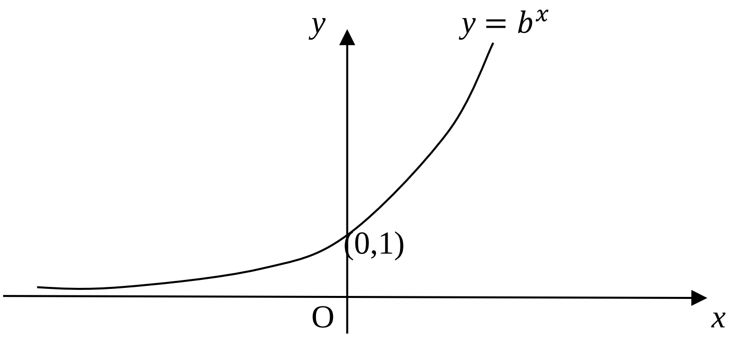
Solution We have a table of values.

x	-2	-1	0	1	2	3	4	5	6
y	0.25	0.5	1	2	4	8	16	32	64

Plotting these points on the Cartesian plane yields the following graph:

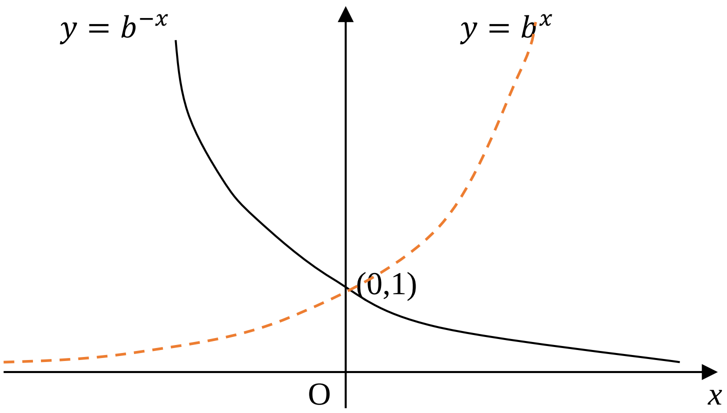


In general the graph of an exponential function $y = b^x$, for all $b > 1$ and $x \in \mathbf{R}$, is of the form



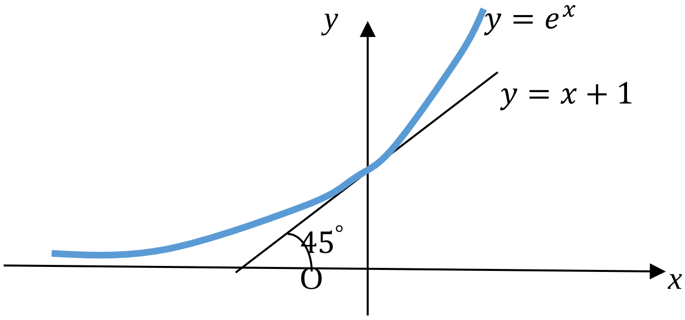
Clearly, the graph of $y = 1^x$ for all $x \in \mathbf{R}$ is a straight line $y = 1$.
The graph of $y = b^{-x}$ for all $b > 1$, $x \in \mathbf{R}$ is of the form

y



Note that the graph of $f(x) = b^{-x}$ is the reflection in the *y* – axis of the graph of $f(x) = b^{x}$.

The function $y = e^{x}$ is called **the exponential function** as opposed to an exponential function $y = b^{x}$. The exponential function $y = e^{x}$ is special in the sense that the tangent to the curve at the point (0,1) makes an angle of 45° with the *x* – axis i.e. the line $y = x + 1$ is tangent to the curve at the point (0,1).



Example 9.2.2 The price of a used car can be represented by the formula

$$P = 16000e^{\frac{-t}{10}},$$

where *P* is the rice in K and *t* is the age in years from new.
Calculate:

(a) The new price

(b) The value at 5 years old

(c) What does the model suggests about the eventual value of the car.

Use this to sketch the graph of *P* against *t*.

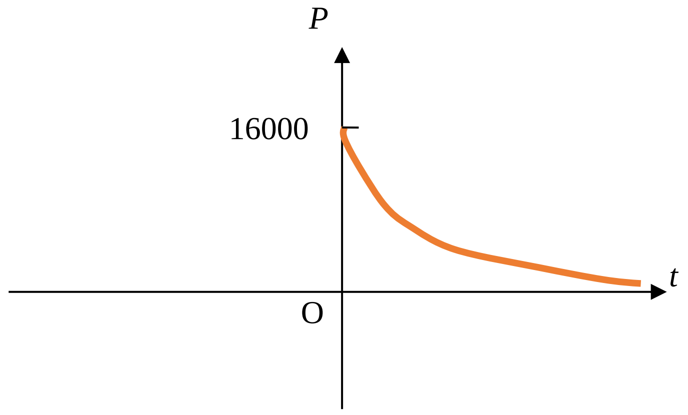
Solution: (a) Substituting *t* = 0 yields $P = 16000e^{\frac{-0}{10}} = 16000 \times 1 = 16000$
Thus, the new price is K16000.

(b) Substituting t = 5 yields $P = 16000e^{\frac{-5}{10}} = 16000 \times e^{-\frac{1}{2}} = 9704.49$.

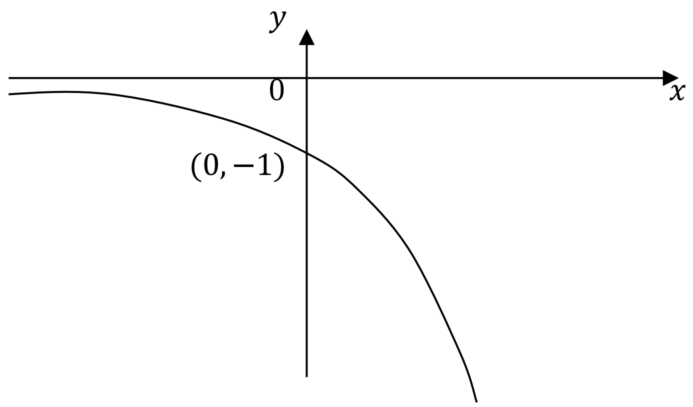
The price after 5 years K9704.49

(c) As $t \to \infty, \to 0$. Therefore $P \to 16000 \times 0 = 0$.

The eventual value is zero.

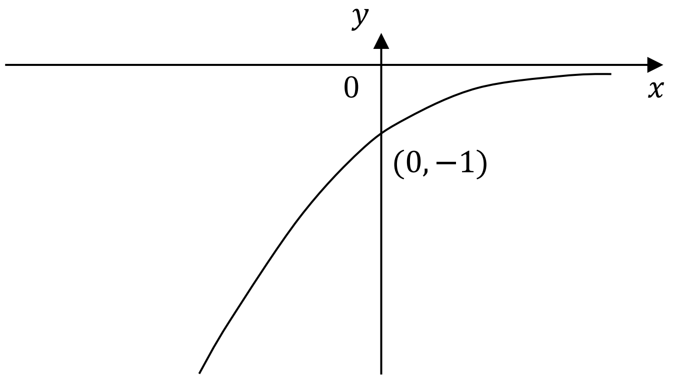


The graph of $y = -b^x$, for $b > 1$ is as follows:



Note that the graph of $f(x) = -b^x$ is the reflection the graph of $f(x) = b^x$ in the x – axis.

The graph of $y = -b^{-x}$, for $b > 1$ is as follows:



Similarly, the graph of $f(x) = -b^{-x}$ is the reflection the graph of $f(x) = b^{-x}$ in the x – axis.

Example 9.2.3 The number of infected people with a disease varies according to the formula

$$N = 300 - 100e^{-0.5t},$$

where N is the number of people infected with the diseases and t is the time in years after detection.

(a) How many people were first diagnosed with the disease?

(b) What is the long term prediction of how this disease will spread?

(c) Graph N against t.

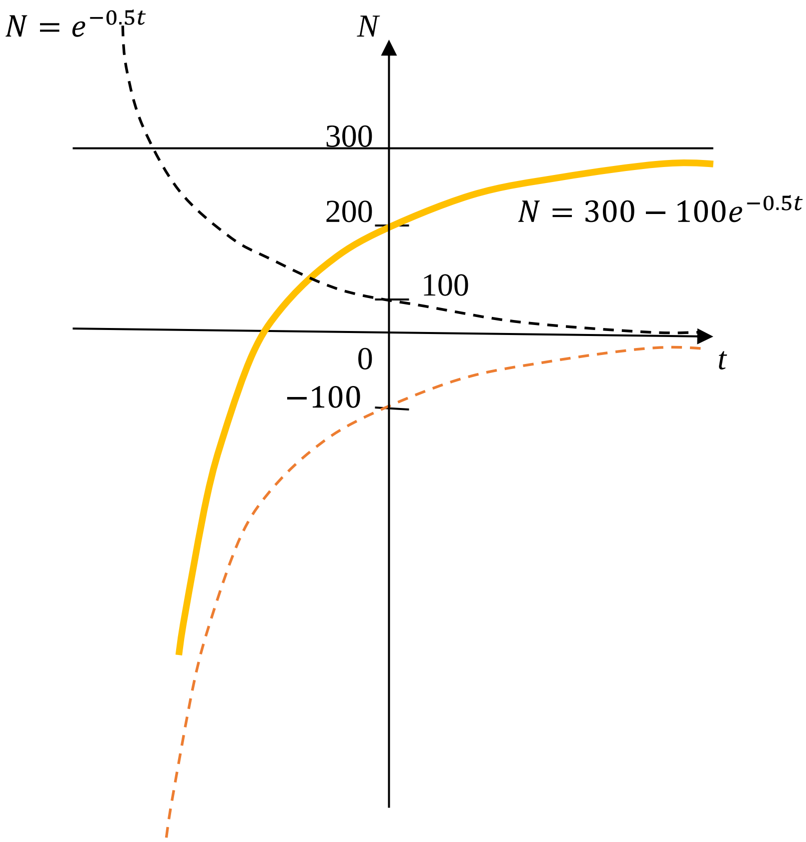
Solution: (a) Substituting $t = 0$ yields $N = 300 - 100e^{-0.5(0)} = 200$

Thus, the number of people first diagnosed with the disease is 200.

(b) As $t \to \infty, e^{-0.5t} \to 0$. Therefore $N \to 300$.

The long term prediction of how long the disease will spread is 300.

(c)



9.2.1 Basic Properties of an Exponential function

An exponential function $f(x) = b^x$ has the following properties:

(a) $f(x_1 + x_2) = b^{x_1+x_2} = b^{x_1} \cdot b^{x_2} = f(x_1) \cdot f(x_2)$

(b) $f(x_1 - x_2) = b^{x_1-x_2} = b^{x_1}/b^{x_2} = f(x_1)/f(x_2)$

(c) $f(x_1) < f(x_2)$ if and only if $x_1 < x_2$

i.e. $b^{x_1} < b^{x_2}$ if and only if $x_1 < x_2$

(d) $f(x) \to +\infty$ as $x \to +\infty$ i.e. $b^x \to +\infty$ as $x \to +\infty$

(e) $f(x) = b^x \to 0$ as $x \to -\infty$ or as $x \to +\infty$

9.3 Logarithmic functions

Let b> 0 $(b \neq 1)$. Then the function $f(x) = b^x$ has domain $\{x|x \in \boldsymbol{R}\}$ and range $\{y|y > 0\}$. It is self evident that *f* is one-to-one. Therefore it has an inverse function f^{-1} with domain $\{x|x > 0\}$ and range $\{y|y \in \boldsymbol{R}\}$. This inverse function is denoted by

$$f^{-1}(x) = log_b x,$$

Read as logarithm to base *b* of *x*. The function f^{-1} is called the **logarithmic function**. The graph of f^{-1} is a reflection of the graph of *f* through the line $y = x$.

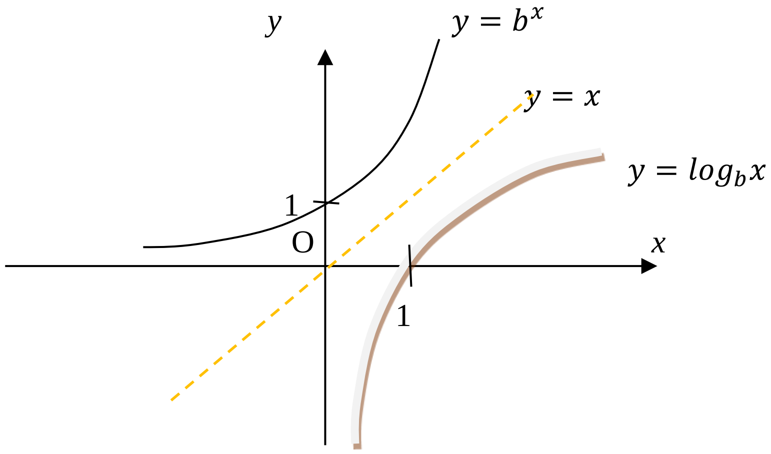
Note that if f^{-1} is the inverse $y = log_b x$ of the function $y = b^x (\Rightarrow x = b^y)$ then

$$x = b^{log_b x} = b^y,$$

implying that

$$y = log_b x \text{ if and only if } x = b^y$$

The graphs of $y = b^x$ and $f^{-1}(x) = log_b x$ are shown in the figure below.



The graph of the logarithmic function $y = log_b x$ passes through the point (1,0). This follows from the fact that the graph of an exponential function $y = b^x$ passes through the point (0,1).

Important points about the graph of $y = log_b x$:

- as $x \to 0^+, y \to -\infty$
- $log_b x$ does not exist for negative values of *x*
- when $x = 1, y = 0$
- as $x \to +\infty, y \to +\infty$ (slowly).

Since the range of $y = b^x$ is $y > 0$, the domain of $y = log_b x$ is $x > 0$.

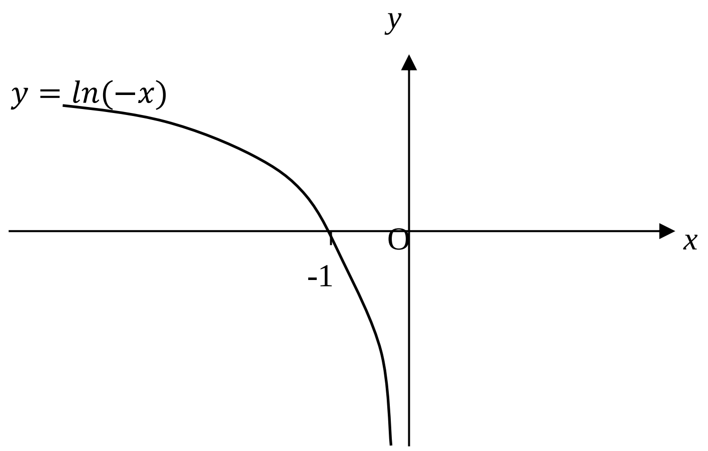
Thus the logarithmic function is not defined for negative values of x.

The inverse of $y = e^x$ is $y = log_e x$ and is called the **natural logarithmic function**. Often it is written as $y = lnx$.

Example 9.3.1 Sketch the graphs of

(a) $y = \ln(-x)$ (b) $y = \ln(3 - x)$ (c) $y = 2 + \ln(3x)$

Graph (a)



Note that $y = \ln(-x)$ is only defined for $x < 0$ and passes through the Point (-1,0).

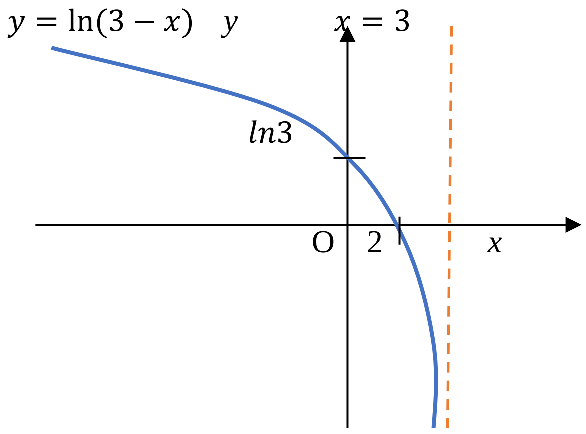
(b) For $y = \ln(3 - x)$,

when $x \to 3, y \to -\infty$

y does not exist for values of $x > 3$

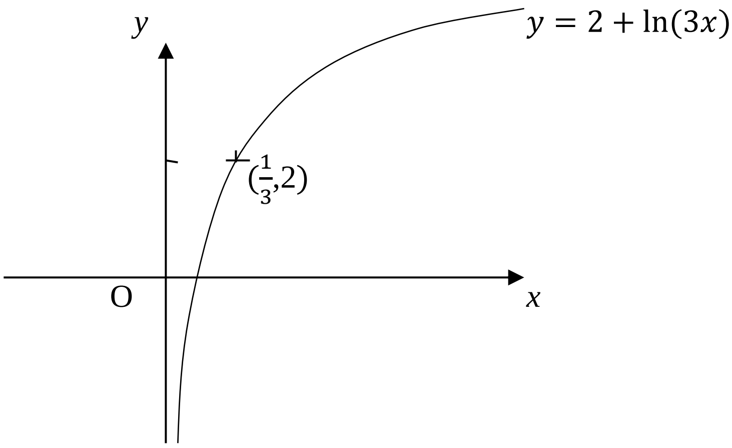
when $x = 2, \ln(3 - 2) = ln1 = 0$

as $x \to \infty$, $y \to \infty$ (slowly)



(c) For $y = 2 + \ln(3x)$, as $x \to 0, y \to -\infty$, and when $x = \frac{1}{3}, y = 2 + ln1 = 2$.

Now as $x \to \infty, y \to \infty$ (slowly).



9.3.1 Basic Properties of Logarithms

Let b, u and v be positive numbers and $b > 1$. Then

(a) $log_b xy = log_b x + log_b y$

(b) Proof

Let $u = log_b x$ and $v = log_b y$. Then $x = b^u$ and $y = b^v$.

Thus

$$x.y = b^u.b^v = b^{u+v} \text{ if and only if } log_b x.y = u + v$$

Therefore

$$log_b xy = log_b x + log_b y.$$

(c) $log_b \frac{x}{y} = log_b x - log_b y$

Proof: Let $u = log_b x$ and $v = log_b y$. Then $x = b^u$ and $y = b^v$.

Thus

$$\frac{x}{y} = \frac{b^u}{b^v} = b^{u-v} \text{ if and only if } log_b \frac{x}{y} = u - v.$$

Therefore

$$log_b \frac{x}{y} = log_b x - log_b y.$$

(d) $log_b 1 = 0$

Proof: Let $u = log_b 1$. Then $b^u = 1$, implying that $u = 0$.

Therefore,

$$log_b 1 = 0.$$

(e) $log_b b = 1$

Proof: Let $u = log_b b$. Then $b^u = b$, implying that $u = 1$.

Therefore,

$$log_b b = 1.$$

(f) $log_b x^n = n log_b x$

Proof: Let $u = log_b x$. Then $b^u = x$. Thus $x^n = (b^u)^n = b^{nu}$.

Implying that

$$nu = log_b x^n$$

Therefore,

$$nlog_b x = log_b x^n.$$

The following formula is used to change the base of the logarithm to another:

(g) $log_b x = \frac{log_a x}{log_a b}$

Proof: Let $N = log_b x$. Then $x = b^N$, implying that

$$log_a x = log_a b^N = Nlog_a b$$

$$N = \frac{log_a x}{log_a b}$$

Therefore,

$$log_b x = \frac{log_a x}{log_a b}.$$

The special case of (g) is obtained when $x = a$.

(h) $log_b a = \frac{1}{log_a b}$

Clearly, by (g) and (e), we have

$$log_b a = \frac{log_a a}{log_a b} = \frac{1}{log_a b}.$$

Example 9.3.2 Simplify:

(a) $\frac{1}{2}log25 - 2log3 + 2log6$ (b) $\log(x + 1) - \log(x^2 - 1)$

Solution: (a) $\frac{1}{2}log25 - 2log3 + 2log6 = log25^{\frac{1}{2}} - log3^2 + log6^2$

$$= log5 - log9 + log36$$

$$= log\frac{5\times36}{9}$$

$$= log20$$

(b) $\log(x + 1) - \log(x^2 - 1) = log\frac{(x+1)}{(x^2-1)} = log\frac{(x+1)}{(x+1)(x-1)} = log\frac{1}{x-1} = -\log(x - 1).$

9.3.2 Exponential and Logarithmic Equations

Example 9.3.3 Solve the equations:

(a) $4^{2x+1} = 3$ (b) $2(2^{2x}) - 5(2^x) + 2 = 0$ (c) $log_2 x + log_x 2 = 2$.

Solution (a) $4^{2x+1} = 3$

$\Rightarrow \quad 4(4^{2x}) = 3$

$\Rightarrow \quad 4^{2x} = \frac{3}{4}$

$\Rightarrow \quad ln4^{2x} = ln\frac{3}{4}$

$\Rightarrow \quad 2xln4 = ln\frac{3}{4}$

$\Rightarrow \quad 2x = ln\frac{3}{4}/ln4$

$\therefore \quad x = ln\frac{3}{4}/2ln4$

(b) $2(2^{2x}) - 5(2^x) + 2 = 0$

$2(2^x)^2 - 5(2^x) + 2 = 0$

Letting $u = 2^x$ we have

$2u^2 - 5u + 2 = 0$

$(2u - 1)(u - 2) = 0$

$u = \frac{1}{2}$ or $u = 2$

$\Rightarrow 2^x = \frac{1}{2}$ or $2^x = 2$

Taking logarithms base 2 on both sides yields

$log_2 2^x = log_2\left(\frac{1}{2}\right) = log_2 1 - log_2 2 = -1$

$\Rightarrow \ xlog_2 2 = -1$

$\therefore \quad x = -1$ or $log_2 2^x = log_2 2 \ \Rightarrow xlog_2 2 = log_2 2 \quad \Rightarrow x = 1$

(c) $log_2 x + log_x 2 = 2$

Changing the logarithmic base from *x* to 2 we have

$log_2 x + \frac{log_2 2}{log_2 x} = 2$

or $log_2 x + \frac{1}{log_2 x} = 2$

Multiplying through by $log_2 x$ yields

$(log_2 x)^2 + 1 = 2log_2 x$

Letting $u = log_2 x$ we have the equation

$u^2 - 2u + 1 = 0$

or $(u-1)^2 = 0$

$\Rightarrow u = 1 \;\; \Rightarrow \;\; log_2 x = 1$

$\therefore \quad x = 2.$

Example 9.3.4 Solve the simultaneous equations

$$2logy = log2 + logx \text{ and } 2^y = 4^x$$

Solution Let $2logy = log2 + logx$ be (I) and $2^y = 4^x$ (II)

Using (II) we have

$$2^y = 4^x \;\; \Rightarrow \;\; 2^y = (2^2)^x = 2^{2x} \;\Rightarrow y = 2x$$

Substituting this in (I) we have

$$2log2x = log2 + logx$$

$\Rightarrow \quad \log(2x)^2 = \log 2x$

$\Rightarrow \quad (2x)^2 = 2x$

$\Rightarrow \quad (2x)^2 - 2x = 0$ or $2x(2x-1) = 0$

$\Rightarrow \quad 2x = 0$ or $2x = 1$

$\Rightarrow \quad x = 0$ or $x = 1/2$

But the equation will not be defined at $x = 0$, since log0 is not defined.

$\therefore \quad x = \frac{1}{2} \;\; \Rightarrow \;\; y = 2\left(\frac{1}{2}\right) = 1.$

<u>TUTORIAL SHEET</u> 13

1. Solve each of the following equations.

   (a) $3^x = 27$ (b) $(\frac{1}{2})^x = \frac{1}{16}$ (c) $27^{4x} = 9^{x+1}$ (d) $\left(\frac{1}{8}\right)^{-2t} = 2^{t+3}$.

2. Sketch the graph of each of the following functions.

   (a) $f(x) = 3^x$ (b) $f(x) = \left(\frac{1}{4}\right)^x$ (c) $f(x) = 2^{x-1}$ (d) $f(x) = 2^{-|x|}$

3. Suppose that in a certain bacteria culture, the equation $Q(t) = 1000e^{0.4t}$ expresses the number of bacteria present as a function of time *t*, where *t* is expressed in hours. How many bacteria are present at the end of 2 hours? 3 hours? % hours?

4. Suppose that a certain substance has a half-life of 20 years. If there are presently 2500 milligrams of the substance, then the equation $Q(t) = 2500(2)^{-t/20}$ yields the amount remaining after t years. How much remains after 40 years? 50% years?

5. Write each of the following in algorithmic form. For example, $2^4 = 16$ becomes $\log_2 16 = 4$.

   (a) $10^1 = 10$ (b) $\left(\frac{2}{3}\right)^{-3} = \frac{27}{8}$ (c) $10^{-2} = 0.01$ (d) $10^5 = 100{,}000$.

6. Write each of the following in exponential form.
   For example, $\log_2 8 = 3$ becomes $2^3 = 8$.
   (a) $\log_2 64 = 6$ (b) $\log_{10} 0.00001 = -5$ (c) $\log_2\left(\frac{1}{16}\right) = -4$.

7. Solve each of the following equations:

   (a) $2^{2x} + 3(2^x) - 4 = 0$ (b) $3^{2x+1} - 26\left(3^x\right) - 9 = 0$ (c) $4^x - 6(2^x) - 16 = 0$.

8. Solve each of the following equations:

   (a) $\log_5 x = 2$ (b) $\log_4 m = \frac{3}{2}$ (c) $\log_b 3 = \frac{1}{2}$.

9. Express the following as a simple logarithm. For example
   $3\log_b\left(\frac{x^3}{y^2}\right) = \log_b x^9 - \log_b y^6$.

   (a) $2\log_b x + 4\log_b y - 3\log_b z$ (b) $2\log_b x + \frac{1}{2}\log_b(x-1) - 4\log_b(2x+5)$
   (c) $\frac{1}{2}\log_b x - 3\log_b x + 4\log_b y$.

10. Solve each of the following equations.
    (a) $\log_{10} 5 + \log_{10} x = 1$ (b) $\log_{10} x + \log_{10}(x-3) = 1$
    (c) $\log_{10}(x-4) + \log_{10}(x-1) = 1$ (d) $\log_{10}(x+2) - \log_{10} x = 1$.

11. Solve the simultaneous equations.
    (a) $2\lg y = \lg 2 + \lg x$; $2^y = 4^x$ (b) $\log_3 x = y = \log_9(2x-1)$.

12. Solve the simultaneous equation

$\log(x+y)=0; \quad 2\log x=\log(y+1)$.

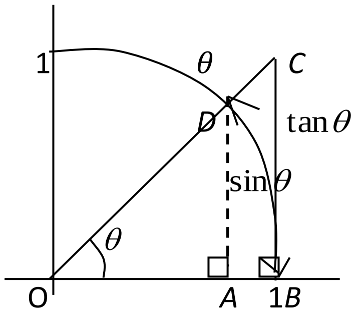
MAT 1100 LECTURE NOTES

10 Further Differential Calculus

10.1 Some Important limits of Trigonometric functions

1. $\lim_{\theta\to 0}\left(\frac{sin\theta}{\theta}\right) = 1.$

   Proof: We consider an arc of a circle of radius 1, as shown in the diagram below. Let the line OC make an angle of θ with the line OA.
   The length of the arc of radius 1 with angle at centre θ is just θ. The length of the line $BC = \tan\theta$ and the length of the line $AD = \sin\theta$.



   Clearly, from the diagram,
   $$AD \le arc\ BD \le BC.$$
   That is
   $$sin\theta \le \theta \le \tan\theta.$$
   Dividing through by $\sin\theta$ we obtain
   $$1 \le \frac{\theta}{\sin\theta} \le \frac{1}{\cos\theta},$$
   and taking reciprocals, we have
   $$\cos\theta \le \frac{\sin\theta}{\theta} \le 1.$$
   But $\lim_{\theta\to 0}\cos\theta = 1$. This means that
   $$\lim_{\theta\to 0}\cos\theta \le \lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right) \le \lim_{\theta\to 0} 1$$
   implying that
   $$1 \le \lim_{\theta\to 0}\left(\frac{sin\theta}{\theta}\right) \le 1$$
   Therefore,
   $$\lim_{\theta\to 0}\left(\frac{sin\theta}{\theta}\right) = 1$$

2. $\lim_{\theta\to 0}\left(\frac{\cos\theta - 1}{\theta}\right) = 0.$

   Proof: $\frac{\cos\theta-1}{\theta} = \frac{\cos\theta-1}{\theta}\times\frac{\cos\theta+1}{\cos\theta+1} = \frac{cos^2\theta-1}{\theta(\cos\theta+1)} = \frac{-sin^2\theta}{\theta(\cos\theta+1)} = (-1)\left(\frac{\sin\theta}{\theta}\right)\left(\frac{\sin\theta}{\cos\theta+1}\right)$

   But $\lim_{\theta\to 0}\left(\frac{sin\theta}{\theta}\right) = 1$ and $\lim_{\theta\to 0}\left(\frac{sin\theta}{\cos\theta+1}\right) = 0.$

Therefore,

$$\lim_{\theta\to 0}\left(\frac{\cos\theta-1}{\theta}\right)=(-1)(1)(0)=0.$$

10.2 Derivative of Trigonometric functions

1. $$\boxed{\frac{d}{dx}(\sin x)=\cos x}$$

Proof: From first principle,

$$\frac{d}{dx}(\sin x)=\lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$
$$=\lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h}$$
$$=\lim_{h\to 0}\frac{\sin x[\cos h-1]}{h}+\lim_{h\to 0}\frac{\cos x\sin h}{h}$$
$$=\sin x\lim_{h\to 0}\frac{[\cos h-1]}{h}+\cos x\lim_{h\to 0}\frac{\sin h}{h}$$
$$=(\sin x)\times 0+(\cos x)\times 1=\cos x$$

2. $$\boxed{\frac{d}{dx}(\cos x)=-\sin x}$$

Proof: From first principle,

$$\frac{d}{dx}(\cos x)=\lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}$$
$$=\lim_{h\to 0}\frac{\cos x\cos h-\sin x\sin h-\cos x}{h}$$
$$=\lim_{h\to 0}\frac{\cos x[\cos h-1]}{h}-\lim_{h\to 0}\frac{\sin x\sin h}{h}$$
$$=\cos x\lim_{h\to 0}\frac{[\cos h-1]}{h}-\sin x\lim_{h\to 0}\frac{\sin h}{h}$$
$$=(\cos x)\times 0-(\sin x)\times 1=-\sin x$$

3. $$\boxed{\frac{d}{dx}(\tan x)=\sec^2 x}$$

Proof: $$\frac{d}{dx}(\tan x)=\frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)=\frac{\cos x\cos x-\sin x(-\sin x)}{\cos^2 x}$$

$$=\frac{\cos^2 x+\sin^2 x}{\cos^2 x}=\frac{1}{\cos^2 x}=\sec^2 x$$

4. $$\frac{d}{dx}(\cot x)=-\csc^2 x$$

Proof: $$\frac{d}{dx}(\cot x)=\frac{d}{dx}\left(\frac{\cos x}{\sin x}\right)=\frac{\sin x(-\sin x)-\cos x(\cos x)}{\sin^2 x}$$

$$=\frac{-(\sin^2 x+\cos^2 x)}{\sin^2 x}=-\frac{1}{\sin^2 x}=-\csc^2 x$$

5. $$\boxed{\frac{d}{dx}(\sec x)=\sec x\tan x}$$

Proof: $$\frac{d}{dx}(\sec x)=\frac{d}{dx}\left(\frac{1}{\cos x}\right)=\frac{(\cos x)\times 0-1(-\sin x)}{\cos^2 x}$$

$$=\frac{\sin x}{\cos^2 x}=\frac{1}{\cos x}\times\frac{\sin x}{\cos x}=\sec x\tan x$$

6. $$\boxed{\frac{d}{dx}(\csc x)=-\csc x\cot x}$$

Proof: $$\frac{d}{dx}(\csc x)=\frac{d}{dx}\left(\frac{1}{\sin x}\right)=\frac{(\sin x)\times 0-1(\cos x)}{\sin^2 x}$$

$$=\frac{-\cos x}{\sin^2 x}=\frac{-1}{\sin x}\times\frac{\cos x}{\sin x}=-\csc x\cot x$$

10.3 Derivatives of Exponential and logarithmic functions

1. The Derivative of $log_b x$

Consider the expression

$$(1+p)^{\frac{1}{p}}$$

Notice that for

$p=1$	$(1+1)^1=2.0000$
$p=\frac{1}{2}$	$(1+\frac{1}{2})^2=2.2500$
$p=\frac{1}{4}$	$(1+\frac{1}{4})^4=2.4414$
$p=\frac{1}{100}$	$(1+\frac{1}{100})^{100}=2.7048$
$p=\frac{1}{1000}$	$(1+\frac{1}{1000})^{1000}=2.7181$

etc.

This is suggesting that $lim_{p\to 0}(1+p)^{1/p}$ exists and it is an irrational number 2.7181 ⋯. This is the number denoted by *e*. Thus,

$$\boxed{lim_{p\to 0}(1+p)^{1/p}=e\approx 2.7181\cdots}.$$

We can now use this to prove, from first principle, that

$$\frac{d}{dx}(\log_b x) = \frac{1}{x}\log_b e$$

Proof: $$\frac{f(x+h)-f(x)}{h} = \frac{1}{h}[log_b(x+h) - log_b x]$$

$$= \frac{1}{h}\left[log_b\left(\frac{x+h}{x}\right)\right]$$

$$= \frac{1}{h}\left[log_b\left(1+\frac{h}{x}\right)\right]$$

$$= \frac{1}{x}\left[\frac{x}{h}log_b\left(\frac{x+h}{x}\right)\right]$$

$$= \frac{1}{x}log_b\left(1+\frac{h}{x}\right)^{x/h}$$

Letting $p = h/x$ we note that as $h \to 0, p \to 0$. Therefore,

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \frac{1}{x}\lim_{p\to 0}\log_b(1+p)^{\frac{1}{p}} = \frac{1}{x}\log_b e$$

Example : Given that $y = log_b(3x+2x^4)$, find y'.

Solution: Let $t = 3x+2x^4$. Then $y = log_b t$. Therefore by chain rule

$$\frac{dy}{dx} = \frac{dy}{dt}\times\frac{dt}{dx} = \frac{1}{t}log_b e \times (3+8x^3)$$

$$= \frac{3+8x^3}{3x+2x^4}log_b e.$$

2. The Derivative of $log_e x$ or lnx

$$\frac{d}{dx}(log_e x) = \frac{1}{x}log_e e = \frac{1}{x}$$

3. The Derivative of e^x

$$\frac{d}{dx}(e^x) = e^x$$

Proof: Let $y = e^x$. Then $x = lny$.

Differentiating with respect to y we have $\frac{dx}{dy} = \frac{1}{y}$.

Now,

$$\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{1/y} = y = e^x.$$

Example : Find $f'(x)$ given that $f(x) = e^{2x} + e^{xlnx}$.

Solution: Let $u = e^{2x}$ and $v = e^{xlnx}$. Then

$$\frac{du}{dx} = e^{2x}\times\frac{d}{dx}(2x) = 2e^{2x}$$

$$\frac{dv}{dx} = e^{x\ln x}\times\frac{d}{dx}(xlnx) = e^{x\ln x}\times(\ln x + x.\frac{1}{x})$$

$= (lnx + 1)e^{xln\,x}$

$f'(x) = 2e^{2x} + (\ln x + 1)e^{x\ln x}$.

4. The Derivative of b^x

$$\boxed{\frac{d}{dx}(b^x) = b^x \ln b}$$

Proof: $y = b^x \quad \Rightarrow \quad x = log_b y$. Changing the base from *b* to *e* we have

$$x = \frac{log_e y}{log_e b} = \frac{1}{log_e b} log_e y.$$

Differentiating *x* with respect to *y* we obtain

$$\frac{dx}{dy} = \frac{1}{log_e b} \times \frac{1}{y} = \frac{1}{ylog_e b} = \frac{1}{ylnb} = \frac{1}{b^x lnb}$$

Therefore

$$\frac{dy}{dx} = b^x lnb.$$

Example: Find $f'(x)$ given that $(x) = 3^{x-1}.2^x$.

Solution: $f(x) = 3^{x-1}.2^x = \frac{1}{3}(3^x) \times 2^x$

$$= 2^x\left(\frac{1}{3}(3^x ln3)\right) + \frac{1}{3}3^x(2^x ln2)$$

$$= \frac{1}{3}2^x 3^x(ln3 + ln2)$$

$$= 2^x 3^{x-1} ln6$$

5. Derivatives of Inverse trigonometric functions

Derivatives of $y = sin^{-1}x$, $y = cos^{-1}x$ and $y = tan^{-1}x$

1. $\boxed{\frac{d}{dx}(sin^{-1}x) = \frac{1}{\sqrt{1-x^2}},}$ provided $|x| < 1$.

   Proof: $y = sin^{-1}x$. Then $x = siny$.

   Thus $\frac{dx}{dy} = cosy \quad \Rightarrow \frac{dy}{dx} = \frac{1}{cosy} = \frac{1}{\sqrt{1-sin^2y}} = \frac{1}{\sqrt{1-x^2}}.$

2. $\boxed{\frac{d}{dx}(cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}},}$ provided $|x| < 1$.

   Proof: $y = cos^{-1}x$. Then $x = cosy$.

   Thus $\frac{dx}{dy} = -siny \quad \Rightarrow \frac{dy}{dx} = -\frac{1}{siny} = -\frac{1}{\sqrt{1-cos^2y}} = -\frac{1}{\sqrt{1-x^2}}.$

3. $\boxed{\frac{d}{dx}(tan^{-1}x) = \frac{1}{1+x^2}, \forall x \in \mathbb{R}}$

   Proof: $y = tan^{-1}x$. Then $x = tany$.

Thus $\frac{dx}{dy} = sec^2 y \;\; \Rightarrow \; \frac{dy}{dx} = \frac{1}{sec^2 y} = \frac{1}{1+tan^2 y} = \frac{1}{1+x^2}$.

<u>Example</u>: Calculate the derivate of the given function.

(a) $sin^{-1}\sqrt{x}$ (b) $cos^{-1}(1-2x)$ (c) $tan^{-1}(x^3+1)$

<u>Solution</u>: (a) We let $y = sin^{-1}\sqrt{x}$ and $u = \sqrt{x}$. Then $y = sin^{-1}u$.

Thus, by the chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$

But $\frac{dy}{du} = \frac{1}{\sqrt{1-u^2}}$ and $\frac{du}{dx} = \frac{1}{2\sqrt{x}}$.

This means that $\frac{dy}{dx} = \frac{1}{\sqrt{1-u^2}} \times \frac{1}{2\sqrt{x}}$.

Therefore

$$\frac{d}{dx} sin^{-1}\sqrt{x} = \frac{1}{2\sqrt{x}\sqrt{1-x}} = \frac{1}{2\sqrt{x-x^2}}.$$

(b) We let $y = cos^{-1}(1-2x)$ and $u = 1-2x$. Then $y = cos^{-1}u$.

Thus, by the chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$

But $\frac{dy}{du} = -\frac{1}{\sqrt{1-u^2}}$ and $\frac{du}{dx} = -2$.

This means that $\frac{dy}{dx} = -\frac{1}{\sqrt{1-u^2}} \times (-2)$.

Therefore

$$\frac{d}{dx} cos^{-1}(1-2x) = -\frac{(-2)}{\sqrt{1-(1-2x)^2}} = \frac{2}{\sqrt{4x-4x^2}} = \frac{1}{\sqrt{x-x^2}}.$$

(c) We let $y = tan^{-1}(x^3+1)$ and $= (x^3+1)$. Then $y = tan^{-1}u$.

Thus, by the chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$

But $\frac{dy}{du} = \frac{1}{1+u^2}$ and $\frac{du}{dx} = 3x^2$.

This means that $\frac{dy}{dx} = \frac{1}{1+u^2} \times 3x^2$.

Therefore

$$\frac{d}{dx}tan^{-1}(x^3+1) = \frac{3x^2}{1+(x^3+1)^2} = \frac{3x^2}{x^6+2x^3+2}.$$

Derivatives of $y = sec^{-1}x$, $y = cosec^{-1}x$ and $y = cot^{-1}x$

4. (a) $\frac{d}{dx}sec^{-1}x = \frac{1}{x\sqrt{x^2-1}},\ x > 1$

(b) $\frac{d}{dx}sec^{-1}x = \frac{-1}{x\sqrt{x^2-1}},\ x < -1$

In general,

(c) $\frac{d}{dx}sec^{-1}x = \frac{-1}{|x|\sqrt{x^2-1}},\ x < -1\ or\ x > 1$

5. (a) $\frac{d}{dx}cosec^{-1}x = \frac{-1}{x\sqrt{x^2-1}},\ x > 1$

(b) $\frac{d}{dx}cosec^{-1}x = \frac{1}{x\sqrt{x^2-1}},\ x < -1$

In general,

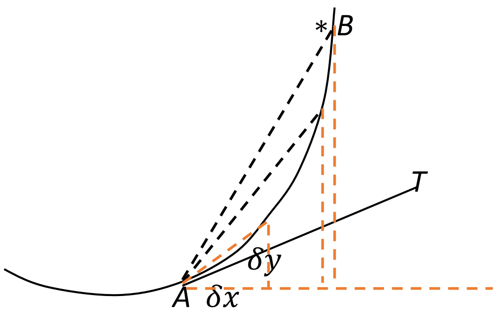
(c) $\frac{d}{dx}cosec^{-1}x = \frac{-1}{|x|\sqrt{x^2-1}},\ x < -1\ or\ x > 1$

6. $\frac{d}{dx}cot^{-1}x = \frac{-1}{1+x^2},\ \forall x \in \mathbb{R}$

Exercise: Prove 4 to 6 the same way as in 1 to 3.

2.5.1 Gradient function

The gradient of a curve at any point on the curve is defined as the gradient of the tangent to the curve at that point and measures the rate of increase of *y* with respect to *x*.



The better approximation to the gradient of the tangent AT is obtained when *B* moves along the curve as it approaches *A* i.e. as $B \to A$ on the curve

$$gradient\ of\ chord\ AB \to gradient\ of\ tangent\ AT$$

i.e. $$grandient\ of\ AT = lim_{B\to A}\{gradient\ of\ chord\ AB\}$$

$$= lim_{\delta x \to 0}\frac{\delta y}{\delta x}$$

$$= \frac{dy}{dx}$$

Thus, if the curve is define by $y = f(x)$, then $\frac{dy}{dx} = f'(x)$ gives the gradient of the curve at any point on the curve. Thus

$$\frac{dy}{dx} = f'(x)$$

is called the gradient function. Therefore, the gradient function, at the point where it is defined, gives the gradient of the curve at the point.

Example: Find the gradient of the curve $y = 2x(x^2 - 6)$ at the point $x = 2$.

Solution: The gradient function is

$$\frac{dy}{dx} = 6x^2 - 12.$$

Therefore the gradient of the curve at the point $x = 2$ is

$$\frac{dy}{dx}|_{x=2} = 6(2)^2 - 12 = 12.$$

Note: The gradient of the curve at any given point gives the gradient of the tangent to the curve at that point.

2.5.2 Equations of Tangents and Normal lines to a curve

Example Find the equation of the tangent and of the normal to the curve $y = x^2 + 5x - 2$ at the point where the curve cuts the line $x = 4$.

Solution: Gradient function $\frac{dy}{dx} = 2x + 5$.

At $x = 4$, $\frac{dy}{dx} = 2(4) + 5 = 13 \Rightarrow gradient\ of\ the\ tangent = 13$.

The gradient of the normal to the curve is $-\frac{1}{13}$.

When $x = 4$, $y = 4^2 + 5(4) - 2 = 34$. Thus the curve cuts the line $x = 4$ at the point $(4,34)$.

Therefore, the equation of the tangent is

$$y - 34 = 13(x - 4)$$

or

$$13x - y - 18 = 0.$$

The equation of the normal to the curve is

$$y - 34 = -\frac{1}{13}(x - 4)$$

or

$$x + 13y - 446 = 0.$$

2.5.3 Increasing and Decreasing Functions

A function *f* is said to be increasing over an interval if for any two points x_1 and x_2 of the interval such that $x_1 < x_2$, we have $f(x_1) < f(x_2)$.

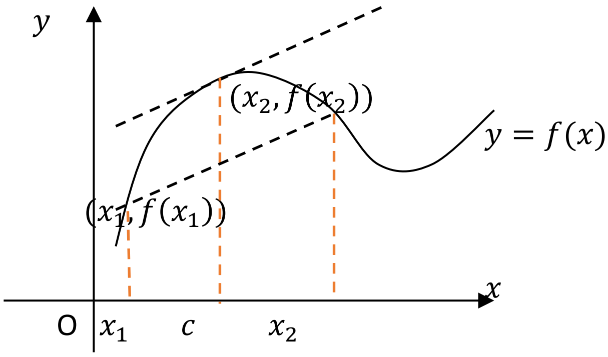
Similarly *f* is decreasing over the interval if $x_1 < x_2$ implies

$f(x_1) < f(x_2)$.

The following theorem helps to determine whether a function is increasing or decreasing:

Theorem 2.1.1 If a function *f* is differentiable, then it is *increasing* over the interval on which $f'(x) > 0$ and it is *decreasing* over any interval on which $f'(x) < 0$.

Proof: Let $f'(c) > 0$. We show that f is increasing on the interval $[x_1, x_2]$



Now, if a function *f* is continuous on a closed interval $[x_1, x_2]$ and is differentiable in the open interval (x_1, x_2), then there exist a real number $c \in (x_1, x_2)$ such that the gradient of the tangent to the curve at $x = c$ is equal to that of the line through the points

$(x_1, f(x_1))$ and $(x_2, f(x_2))$ i.e.

$$f'(x) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

From (2.4.1) we have

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

Since $x_1 < x_2$, $x_2 - x_1 > 0$. From the hypothesis, $f'(c) > 0$. Thus

the right-hand side of (2.4.2) is positive. This implies that

$$f(x_2) - f(x_1) > 0$$

$$\Rightarrow \qquad f(x_2) > f(x_1).$$

Therefore by definition *f* is increasing.

For a decreasing function over an interval $[x_1, x_2]$, the proof is similar.

Example: Find the intervals in which the function *f* defined by

$$f(x) = x^3 - 6x^2 + 9x - 7.$$

is (a) increasing

(b) decreasing.

Solution: $f(x) = x^3 - 6x^2 + 9x - 7$. Differentiating, we get

$$f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3)$$

or $$f'(x) = 3(x-1)(x-3).$$

Now, $f'(x)$ is zero only for the values $x = 1$ and $x = 3$. The two points where the derivative is zero divides the *x* - axis into three intervals:

$$-\infty < x < 1,\ 1 < x < 3 \text{ and } 3 < x < \infty.$$

If $x < 1$, then $x < 3 \Rightarrow x - 1 < 0$ and $x - 3 < 0$. Hence $(x-1)(x-3) > 0$.

Thus,

$$f'(x) = 3(x-1)(x-3) > 0.$$

and *f* is increasing.

Similarly, if $x > 3$, then $x > 1 \Rightarrow x - 1 > 0$ and $x - 3 > 0$, implying that $(x-1)(x-3) > 0$. Thus,

$$f'(x) = 3(x-1)(x-3) > 0.$$

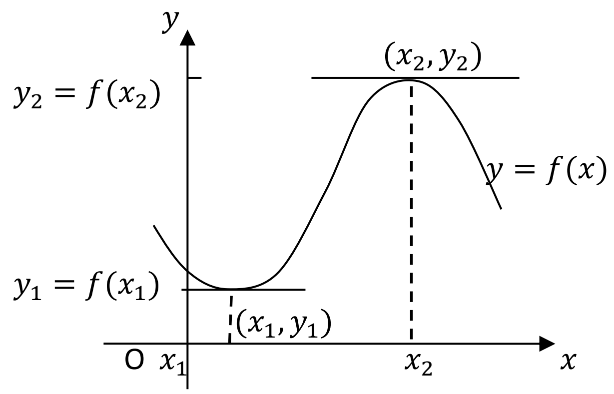
and *f* is increasing.

If $1 < x < 3$, then $x - 1 > 0$ and $x - 3 < 0$ and therefore $(x-1)(x-3) < 0 \Rightarrow f'(x) = 3(x-1)(x-3) < 0$. Hence, in the interval $1 < x < 3$ *f* is decreasing.

Therefore, *f* is increasing in the intervals $-\infty < x < 1$ and $3 < x < \infty$, and is decreasing in the interval $1 < x < 3$.

2.5.4 Stationary points

A stationary value of a function $f(x)$ is any value of $f(x)$ at which its rate of change with respect to *x* is zero. i.e. stationary values of $f(x)$ occur when $\frac{d}{dx}(f(x)) = 0$.



In the diagram the function has stationary values at $f(x_1)$ and $f(x_2)$.

The values of x at which the function *f* attains its stationary values are called **critical values** or **critical points**.

The points (x_1, y_1) and (x_2, y_2) are called **stationary points**.

At critical values, the gradient of the curve is zero. Thus, at critical values $\frac{d}{dx}(f(x)) = 0$.

<u>Example</u> Find the critical values of the function

$$f(x) = 2x^3 - 3x^2 - 36x + 5.$$

Hence, state the stationary value of the function.

<u>Solution</u>: Differentiating w.r.t. *x* we get

$$f'(x) = 6x^2 - 6x - 36.$$

Thus, the critical values satisfy the equation

$$f'(x) = 6x^2 - 6x - 36 = 0$$

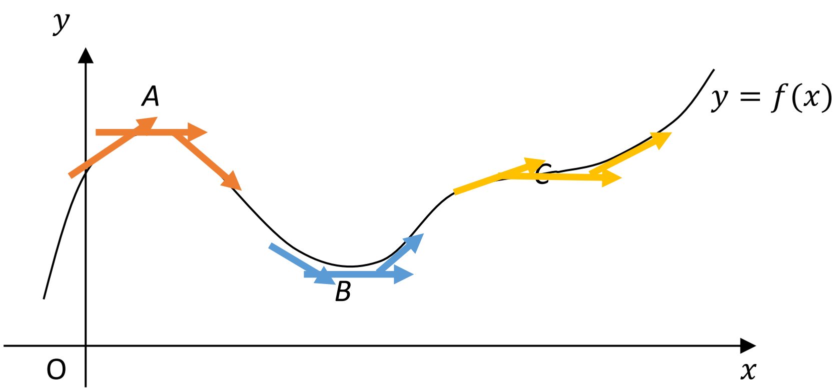
or $$x^2 - x - 6 = 0$$

or $$(x + 2)(x - 3) = 0$$

Therefore, the critical values are $x = -2$ and $x = 3$.

Hence, the stationary values of the function are $f(-2) = 49$ and $f(3) = -76$.

A curve may have several stationary points which are also called **turning points** of the curve. These occur in different categories:



<u>Nature of Stationary Values or Points</u>

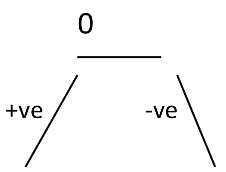
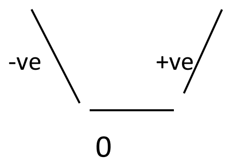
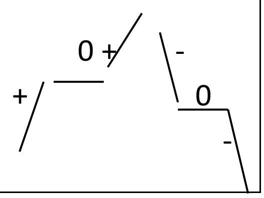
<u>**Method 1**</u> (First Derivative test)

Let a curve be defined by $y = f(x)$. When moving through a stationary point if

(a) $\frac{dy}{dx} > 0$ followed by $\frac{dy}{dx} = 0$ and then $\frac{dy}{dx} < 0$, then the stationary point is maximum;

(b) $\frac{dy}{dx} < 0$ followed by $\frac{dy}{dx} = 0$ and then $\frac{dy}{dx} > 0$, then the stationary point is minimum;

(c) $\frac{dy}{dx} > 0$ followed by $\frac{dy}{dx} = 0$ and then $\frac{dy}{dx} > 0$ or $\frac{dy}{dx} < 0$ followed by $\frac{dy}{dx} = 0$ and then $\frac{dy}{dx} < 0$, then the turning point is point of inflexion.

	Maximum	Minimum	Inflexion
Sign of $\frac{dy}{dx}$ when moving through a stationary point			

Example: Find the relative maxima and the relative minima for the function

$$f(x) = (x^2 - 4)^2.$$

Solution: At critical values: $f'(x) = 2(x^2 - 4) \times 2x = 0$.
i.e.

$$x(x + 2)(x - 2) = 0$$

Thus, the critical values are $x = 0, x = -2$ and $x = 2$.

Next, we look at the signs of the values of $f'(x)$ in the neighbourhood of these critical values.

For the critical value $x = -2$, we shall consider the signs of the values $f(-\frac{5}{2})$ and $f(-\frac{3}{2})$.

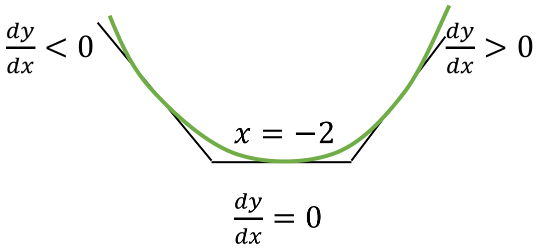
At $x = -\frac{5}{2}$, $f'\left(-\frac{5}{2}\right) = 2\left[\left(-\frac{5}{2}\right)^2 - 4\right] \times 2\left(-\frac{5}{2}\right) = -\frac{45}{2} < 0$.

At $x = -2$, $f'(-2) = 0$.

At $x = -\frac{3}{2}$, $f'(-\frac{3}{2}) = 2\left[\left(-\frac{3}{2}\right)^2 - 4\right] \times 2\left(-\frac{3}{2}\right) = \frac{27}{2} > 0$.

Moving along the curve from $-\frac{5}{2}$ through $x = -2$ to $-\frac{5}{2}$ the gradient of the curve is - followed by 0 and then + . This means that the function has a relative minimum at $x = -2$ and this is $f(-2) = ((-2)^2 - 4)^2 = 0$

The shape of the curve in the neighbourhood of $x = -2$ is as follows:



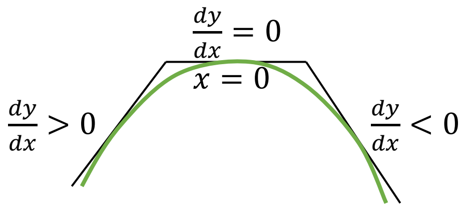
For the critical value $x = 0$, we shall consider the signs of the values $f'(-\frac{1}{2})$ and $f'(\frac{1}{2})$.

At $x = -\frac{1}{2}$, $f'\left(-\frac{1}{2}\right) = 2\left[\left(-\frac{1}{2}\right)^2 - 4\right] \times 2\left(-\frac{1}{2}\right) = \frac{15}{2} > 0.$

At $x = 0$, $f'(0) = 0.$

At $x = -\frac{1}{2}$, $f'\left(\frac{1}{2}\right) = 2\left[\left(\frac{1}{2}\right)^2 - 4\right] \times 2\left(\frac{1}{2}\right) = -\frac{15}{2} < 0.$

Moving along the curve from $-\frac{1}{2}$ through $x = 0$ to $\frac{1}{2}$ the gradient of the curve is positive, followed by 0 and then negative. This means that the function has a relative maximum at $x = 0$ and the relative maximum is $f(0) = (0^2 - 4)^2 = 16$.



For the critical value $x = 2$, we shall consider the signs of the values

$f'(\frac{3}{2})$ and $f'(\frac{5}{2})$.

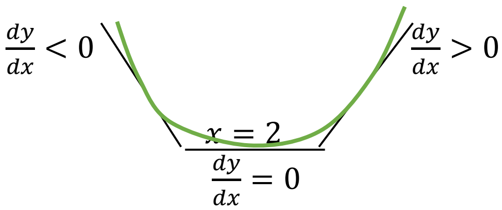
At $x = \frac{3}{2}$, $f'\left(\frac{3}{2}\right) = 2\left[\left(\frac{3}{2}\right)^2 - 4\right] \times 2\left(\frac{3}{2}\right) = -\frac{27}{2} < 0.$

At $x = 2$, $f'(2) = 0.$

At $= \frac{5}{2}$, $f'(\frac{5}{2}) = 2\left[\left(\frac{5}{2}\right)^2 - 4\right] \times 2\left(\frac{5}{2}\right) = \frac{45}{2} > 0.$

Moving along the curve from $\frac{3}{2}$ through $x = 2$ to $\frac{5}{2}$ the gradient of the curve is negative, followed by 0 and then positive. This means that the function has a relative minimum at $x = 2$ and this is $f(2) = ((2)^2 - 4)^2 = 0$

The shape of the curve in the neighbourhood of $x = 2$ is as follows:



Note that the points $(-2,0)$ and $(2,0)$ are minimum turning points and the point $(0,16)$ is the maximum turning point for given curve.

Method 2 (The Second Derivative test)

Let the function be defined by $y = f(x)$. Then $\frac{dy}{dx} = f'(x)$ is the first derivative of y w.r.t. x, $\frac{d^2y}{dx^2} = f''(x)$ is the second derivative of y w.r.t. x, and $\frac{d^3y}{dx^3} = f'''(x)$ is the third derivative of y w.r.t. x. In general, $\frac{d^ny}{dx^n} = f^{(n)}(x)$ is the *nth* derivative of y w.r.t. x.

Example Find the first, second and the third derivatives of the function $y = x^2 + \frac{23}{x}$, with respect to x.

Solution: The first derivative of y w.r.t. x is $\frac{dy}{dx} = 2x - \frac{23}{x^2}$, the second derivative of y w.r.t. x is $\frac{d^2y}{dx^2} = 2 + \frac{46}{x^3}$ and the third derivative of y w.r.t. x is $\frac{d^3y}{dx^3} = -\frac{158}{x^4}$.

For the curve $y = f(x)$ we can also determine the nature of the stationary value (or the turning point) at critical values by finding the value of $\frac{d^2y}{dx^2} = f''(x)$, and where necessary, the value of $\frac{d^3y}{dx^3} = f'''(x)$.
This is because the second derivative, $\frac{d^2y}{dx^2} = f''(x)$, measures the change in gradient.

1. If $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$, the critical value yields a minimum stationary value of the function and hence it yield a minimum turning point.
2. If $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$, the critical value yields a maximum stationary value of the function and hence it yields a maximum turning point.
3. If $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$, the critical value yields a minimum or maximum turning point or point of inflexion.

   In this case we need to use method 1 to determine the nature of the stationary value.

   Example: Determine the nature of the stationary values for the curve $y = x^4 - 9$.

   Solution: $\frac{dy}{dx} = 4x^3 = 0 \Rightarrow x = 0$ is the only critical value.

At this critical value, $\frac{d^2y}{dx^2} = 0$.
Moving through $-\frac{1}{2}$, 0 and then $\frac{1}{2}$ we have:

At $x = -\frac{1}{2}, \frac{dy}{dx} = 4(-\frac{1}{2})^3 = -\frac{1}{2} < 0$;

At $x = 0, \frac{dy}{dx} = 4(0)^3 = 0$;

At $x = \frac{1}{2}, \frac{dy}{dx} = 4(\frac{1}{2})^3 = \frac{1}{2} > 0$.

Therefore, the curve has a minimum stationary value at $x = 0$, and this $y = 0^4 - 9 = -9$.
Note that though at the point $x = 0$, $\frac{d^2y}{dx^2} = 0$, and in this case the critical value does not yield a point of inflexion but a minimum turning point.

4. If $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$, but $\frac{d^3y}{dx^3} \neq 0$, the critical value yields a point of inflexion.

<u>Example</u>: Determine the nature of the turning point for the curve
$y = x^3 - 3x^2 + 3x$.

<u>Solution</u>: $\frac{dy}{dx} = 3x^2 - 6x + 3 = 0 \Rightarrow 3(x-1)^2 = 0 \Rightarrow x = 1$. The curve only has one critical value.

$\frac{d^2y}{dx^2} = 6x - 6$, and at $x = 1, \frac{d^2y}{dx^2} = 6(1) - 6 = 0$ and $\frac{d^3y}{dx^3} = 6 \neq 0$.

Therefore, at $x = 1$, the curve has a point of inflexion and this point of inflexion is $(1,1)$.

To confirm that indeed this is a point of inflexion, we use method 1.

We find the values of $\frac{dy}{dx}$ at $x = -\frac{1}{2}, x = 1$ and $x = \frac{3}{2}$.

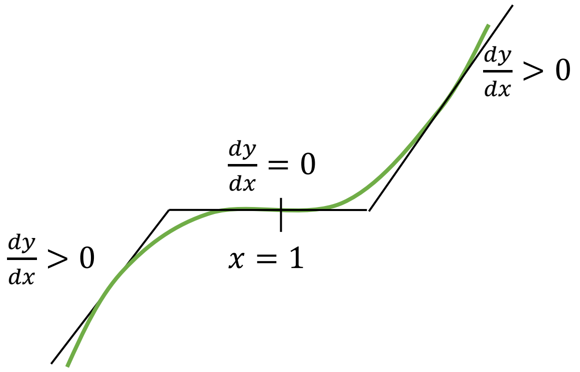
At $x = \frac{1}{2}, \frac{dy}{dx} = \frac{3}{4} - 3 + 3 = \frac{3}{4} > 0$;

At $x = 1, \frac{dy}{dx} = 3 - 6 + 3 = 0$;

At $x = \frac{3}{2}, \frac{dy}{dx} = \frac{27}{4} - 9 + 3 = \frac{3}{4} > 0$.

Moving along the curve from $\frac{1}{2}$ through $x = 1$ to $\frac{3}{2}$ the gradient of the curve is + followed by 0 and then + . This means that the curve has a point of inflexion at $x = 1$.

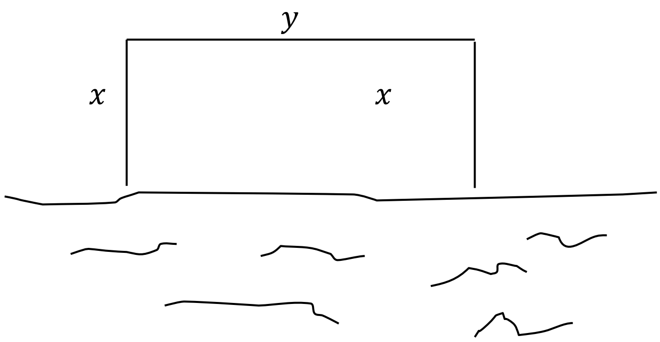
Note that the shape of the curve in the neighbourhood of $x = 1$ is as shown
In the diagram below:



2.4.5 Applications of Maxima and Minima

Example: A farmer having 120m of fencing wishes to contain a cow in a rectangular plot of land along the bank of a river. What should the dimensions of the rectangle be to provide the cow with maximum grazing ground? (Assume that no fencing is needed along the river, which is flowing along a straight edge.)

Solution:



Notice that

$$2x + y = 120$$

$$\Rightarrow \quad y = 120 - 2x \qquad (1)$$

Area of the rectangle is

$$A = xy \qquad (2)$$

Replacing (1) into (2) yields

$$A = x(120 - 2x) = 120x - 2x^2$$

or $\quad A = 120x - 2x^2$

To maximise the area A, we need to differentiate A w.r.t. x and find the critical values.

$$\frac{dA}{dx} = 120 - 4x = 0$$

$$\Rightarrow \quad x = 30.$$

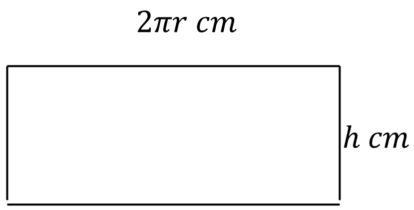
Applying the second derivative test, we have

$$\frac{d^2A}{dx^2} = -4 < 0$$

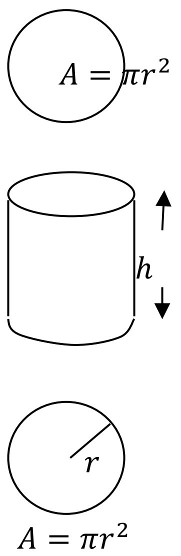
This means that $x = 30$ yields the maximum area and hence, the largest grazing ground. Therefore the dimension of the rectangle are $30m \times 60m$.

Example: A soup manufacturing company wishes to pack 25 cm^3 of mushroom soup in a can in the form of a right circular cylinder. Find the dimensions of the can if the surface area is to be minimum.

Solution: Let the dimensions of the rectangle be $22\pi r \ \times h$.



Curved surface area of a cylinder is $2\pi rh$.



The Volume of the cylinder is

$$V = \pi r^2 h = 25$$

i.e. $$h = \frac{25}{\pi r^2}.$$

Surface area of the metal used is

$$A = 2\pi rh + 2\pi r^2$$

i.e. $$A = 2\pi r\left(\frac{25}{\pi r^2}\right) + 2\pi r^2$$

or $$A = \frac{50}{r} + 2\pi r^2.$$

Thus, to minimize A, we need to find the critical values, by first differentiating A w.r.t. r and equating the derivative to zero.

$$\frac{dA}{dr} = -\frac{50}{r^2} + 4\pi r = 0$$

$\Rightarrow$ $$r^3 = \frac{25}{2\pi}$$

$\Rightarrow$ $$r = \sqrt[3]{\frac{25}{2\pi}}.$$

The only critical value is $r = \sqrt[3]{\frac{25}{2\pi}}$.

Applying the second derivative we find

$$\frac{d^2A}{dr^2} = \frac{100}{r^3} + 4\pi.$$

At $r = \sqrt[3]{\frac{25}{2\pi}}$,

$$\frac{d^2A}{dr^2} = \frac{100}{\left(\sqrt[3]{\frac{25}{2\pi}}\right)^3} + 4\pi = 8\pi + 4\pi = 12\pi > 0.$$

Therefore, *A* is minimum when $r = \sqrt[3]{\frac{25}{2\pi}}$. The corresponding value of *h* is $h = \frac{25}{\pi\left(\sqrt[3]{\frac{25}{2\pi}}\right)^2} = \frac{25}{\pi}\left(\frac{2\pi}{25}\right)^{\frac{2}{3}}$.

2.5.6 Curve Sketching and Graphs of Rational Functions

1. Graphs of Polynomial Functions

Example: Sketch the graph of $y = x^4 - 4x^3 + 4x^2 + 1$.

Solution: $\frac{dy}{dx} = 4x^3 - 12x^2 + 8x = 4x(x^2 - 3x + 2) = 4x(x-1)(x-2)$.

The critical points are $x = 0, x = 1$ and $x = 2$.

$\frac{d^2y}{dx^2} = 12x^2 - 24x + 8 = 4(3x^2 - 6x + 2)$.

At $x = 0, \frac{d^2y}{dx^2} = 8 > 0 \Rightarrow x = 0$ yields a minimum turning point, which is (0,1).

At $x = 1, \frac{d^2y}{dx^2} = -4 < 0 \Rightarrow x = 1$ yields a maximum turning point, which is (1,2).

At $x = 2, \frac{d^2y}{dx^2} = 8 > 0 \Rightarrow x = 2$ yields a minimum turning point, which is (2,1).

To find points of inflexions we set the second derivative to zero and find the critical points which yields these points. Thus

$\frac{d^2y}{dx^2} = 0$ when $3x^2 - 6x + 2 = 0$ i.e. when

$x = \frac{6 \pm \sqrt{36-24}}{6} = \frac{6 \pm \sqrt{12}}{6} = 1 \pm \frac{1}{\sqrt{3}}.$

$\frac{d^3y}{dx^3} = 24x - 24.$

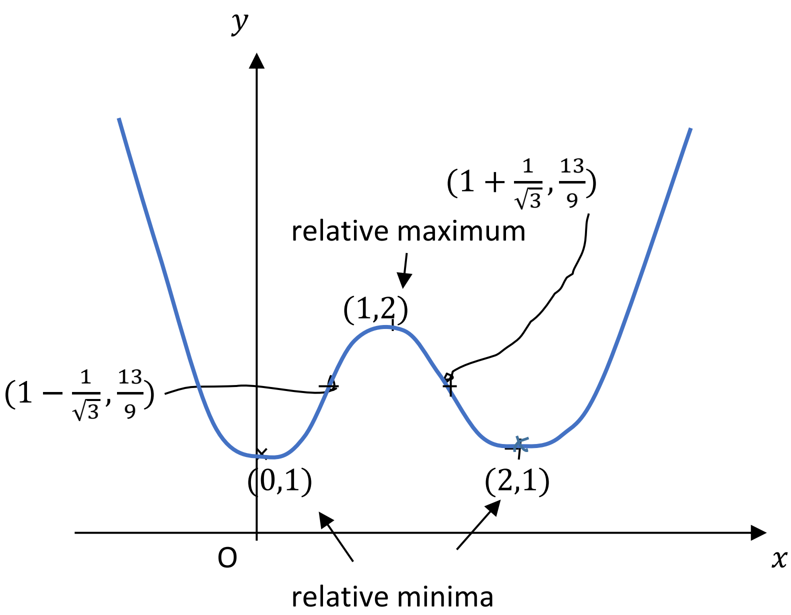
At $x = 1 + \frac{1}{\sqrt{3}}, \frac{d^3y}{dx^3} = 24\left(1 + \frac{1}{\sqrt{3}}\right) - 24 = 24 + \frac{24}{\sqrt{3}} - 24 = \frac{24}{\sqrt{3}} \neq 0$

$\Rightarrow x = 1 + \frac{1}{\sqrt{3}}$ yields a point of inflexion, which is $(1 + \frac{1}{\sqrt{3}}, \frac{13}{9})$.

At $x = 1 - \frac{1}{\sqrt{3}}, \frac{d^3y}{dx^3} = 24\left(1 - \frac{1}{\sqrt{3}}\right) - 24 = 24 - \frac{24}{\sqrt{3}} - 24 = -\frac{24}{\sqrt{3}} \neq 0$

$\Rightarrow x = 1 - \frac{1}{\sqrt{3}}$ yields the other point of inflexion, which is $(1 - \frac{1}{\sqrt{3}}, \frac{13}{9})$.

Finally, plot these points and sketch the curve which passes through the points.

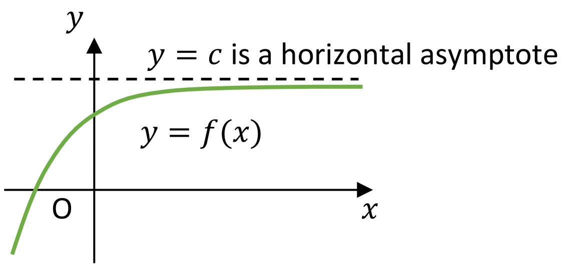


2. Graphs of Rational Functions

Curve sketching by looking at what happens either as $x \to \pm\infty$ or when $f(x) \to \pm\infty$ as approaches a finite value.

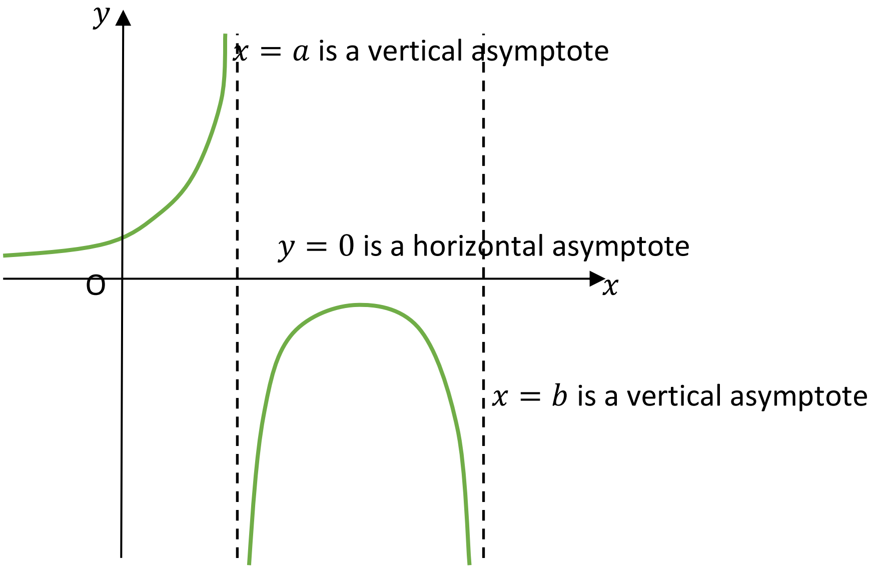
Definition (Horizontal Asymptote)

Let c be a real number. Suppose $f(x) \to c$ as $x \to \infty$ or as $x \to -\infty$. Then the horizontal line $y = c$ is called a **horizontal asymptote** of the graph of f.



Definition (Vertical Asymptote)

Let a be a real number. Suppose $f(x) \to \pm\infty$ as $x \to a^+$ or as $x \to a^-$.
Then the vertical line $x = a$ is called a **vertical asymptote** of the graph of f.



Example: Sketch the graph of $y = \frac{x}{1+x}$.

Solution: Note that the function is not defined at $x = -1$.

Now we observe the following:

If $x < -1$, then $y = \frac{x}{1+x} > 0$, (since the numerator is negative and the denominator is negative.)

If $-1 < x < 0$, then $y = \frac{x}{1+x} < 0$.

If $x = 0$, then $y = \frac{x}{1+x} = 0$

If $x > 0$, then $0 < y = \frac{x}{1+x} < 1$.

Furthermore,

$$lim_{x \to -1^+} \frac{x}{1+x} = -\infty$$ (since the numerator is negative and the

denominator is positive) and

$$lim_{x \to 1^-} \frac{x}{1+x} = \infty$$ (since the numerator and the denominator are both

negative).

Thus, the line $x = -1$ is a vertical asymptote.
Also,

$$lim_{x \to \infty} \frac{x}{1+x} = lim_{x \to \infty} \frac{1}{\left(\frac{1}{x}\right)+1} = 1.$$

and

$$lim_{x\to-\infty}\frac{x}{1+x} = lim_{x\to\infty}\frac{1}{\left(\frac{1}{x}\right)+1} = 1$$

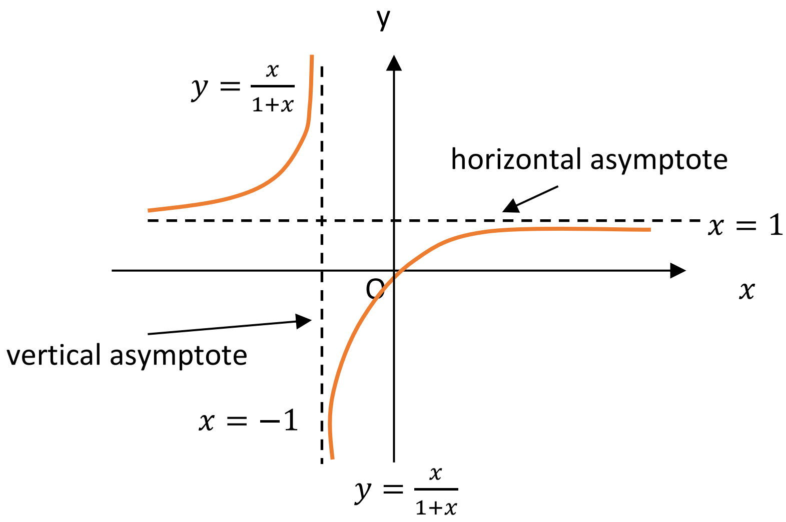
Therefore, as $x \to \pm\infty, y \to 1$. Thus, the line $y = 1$ is a horizontal asymptote for the function $y = \frac{x}{1+x}$.

Now $\frac{dy}{dx} = \frac{(1+x)(1)-x(1)}{(1+x)^2} = \frac{1}{(1+x)^2}$.

Clearly, $\frac{dy}{dx} > 0\ \forall x \in \mathbb{R}, x \neq -1$ ⇒the function is always increasing (except at $x = -1$) where it is not defined.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(1+x)^{-2} = -\frac{-2}{(1+x)^3} \neq 0\ \forall x \in \mathbb{R} \text{ except } x = -1.$$

This means that the curve has no critical points.



TUTORIAL SHEET 14

1. Find the derivative of each of the following functions from first principle:
   (a) $\sin 2x$ (b) $\cos 3x$.

2. Differentiate with respect to *x*:

   (a) $\dfrac{\sin x+\cos x}{\sin x-\cos x}$ (b) $\dfrac{\sin(3x+1)}{\cos(2x+3)}$ (c) $\tan^4(7x^2+3x+9)$

   (d) $\sin^2(2x)\times\cos^3(5x)$.

3. Differentiate each of the following functions:

   (a) $f(x)=e^{3-4x}$ (b) $f(x)=e^{\sin x+\cos 2x}$ (c) $f(t)=e^t\sin^2(\cos t)$

   (d) $f(u)=\cos(e^{3u})$ (e) $f(x)=6^{5x}$ (f) $f(x)=2^{x^2}5^{x-1}$

4. For each of the following implicit functions, find $\dfrac{dy}{dx}$:

   (a) $y+xy+y^2=2$ (b) $x^2+xy^2+y^3=2$ (c) $x=ye^x$ (d) $\dfrac{1}{x^2}+\dfrac{1}{y^2}=\dfrac{1}{4}$

   (e) $\sin x\cos y=2$ (f) $xe^y=x+1$.

5. Differentiate each of the following functions:

   (a) $f(x)=\log_e(5x+1)$ (b) $f(x)=\ln(x^2\sin x)$ (c) $f(t)=\log_3(x^2+e^x)$

   (d) $f(u)=e^{x\ln x}$ (e) $f(x)=\log_2[\log_5(\sqrt{x^2+1})]$.

6. Use the rules of logarithms to simplify the expression and find $f'(x)$:

   (a) $f(x)=\ln\dfrac{(x+1)^{16}(2x^2+x)^8}{\sqrt{x^2+4}}$ (b) $f(x)=\ln\dfrac{(e^{2x}+6)^7\sqrt{x+4}}{\left(e^{-x}+e^x\right)^5}$.

7. For each of the given functions, find $\dfrac{d^2y}{dx^2}$, $\dfrac{d^3y}{dx^3}$ and $\dfrac{d^4y}{dx^4}$:

   (a) $y=2x^4-3x^2$ (b) $\sin x+\cos y==7$.

8. Find the indicated value:

   (a) $f(x)=\sqrt{4-x}$, $f''(-5)$ (b) $f(x)=\left(x^3-2x\right)^3$, $f''(1)$

   (c) $f(t)=\sqrt{2t+3}$, $f'''(\tfrac{1}{2})$.

9. If $x^3y=2$, find the value of $\dfrac{d^2y}{dx^2}$ when $x=1$.

10. Let *f* be a function defined by $f(x)=2x-x^2$.

    (a) Find the equation of the tangent and normal line to the graph of *f* at the point

$\left(\frac{1}{2}, \frac{3}{4}\right)$.

(b) At what point on the curve is the tangent line parallel to the x – axis?

11. Find the point at which the slope of the tangent line to the curve $y = 5 + 3x - x^3$ is 3.

12. Find the equation of the tangent line to the curve $2y^2 - 3x^2 = 6$ at point $(2,3)$.

13. Find where f is increasing and where it is decreasing, and sketch the graph of f:

(a) $f(x) = x^3 - 3x^2 - 9x + 15$ (b) $f(x) = 4x^4 + x^2$.

14. Suppose the rate R of an autocatalytic reaction is given by

$$R = kx(4 - x), \quad 0 \leq x \leq 4$$

where k is a positive constant and x is the amount of substance produced. Find the interval where the rate R is increasing and where it is decreasing. Sketch the graph.

15. Find the critical values and determine the relative maxima and the relative minima. Find the absolute extrema if any exist.

(a) $f(x) = x^3 + 3x^2 - 12x + 7, \quad -\infty < x < \infty$

(b) $f(x) = 2x^4 - x^2, \quad -1 \leq x \leq 1$.

16. If the selling price x is related to the profit P by the equation

$$P = 5000x - 125x^2$$

(a) For what range of values of x is the profit increasing?

(b) For what range of values of x is the profit decreasing?

(c) Determine the value of x that would yield maximum profit.

17. Find the turning points on $y = 3x^4 + 4x^3 - 12x^2$. Give a rough sketch of the curve.

18. An open rectangular box is made from a square sheet of cardboard by removing a square from each corner and joining the cut edges. If the cardboard is of edge $0.5\ m$, find the maximum volume of the box.

19. The concentration of hydrogen ion in a solution is given by $X = H + \frac{10^{-5}}{H}$. For what value of H is the concentration a minimum?

20. Sketch the graph of each given function showing clearly any asymptotes, turning points, points of intersection with the coordinate axes and the behavior of the curve when a and /or y are very large:

(a) $y = \dfrac{1}{1-x}$ (b) $y = \dfrac{x-2}{x-4}$ (c) $y = \dfrac{x}{(1-x)^2}$ (d) $y = \dfrac{x-2}{(x+1)(x-1)}$.

11 FURTHER INTEGRAL CALCULUS

11.1 Further Integration formulae

Most of these important further integration formulae emanate from the fact that integration is the reverse process of differentiation.

1. $\int e^x dx = e^x + c$, since $\frac{d}{dx}(e^x) = e^x$.

2. $\int a^x dx = \frac{a^x}{\ln a} + c,\ a > 0,\ a \neq 1$, since $\frac{d}{dx}\left(\frac{1}{\ln a} a^x\right) = a^x$.

3. $\int \sin x dx = -\cos x + c$, since $\frac{d}{dx}(-\cos x) = \sin x$.

4. $\int \cos x dx = \sin x + c$, since $\frac{d}{dx}(\sin x) = \cos x$.

5. $\int \tan x dx = \ln|\sec x| + c$, since

$$\int \tan x\, dx = -\int \frac{-\sin x}{\cos x} dx = -\ln|\cos x|.$$

$$= \ln|(\cos x)^{-1}| = \ln\left|\frac{1}{\cos x}\right| = \ln|secx|.$$

6. $\int \cot x dx = \ln|\sin x| + c$, since

$$\int \cot x\, dx = \int \frac{\cos x}{\sin x} dx = \ln|\sin x|.$$

7. $\int \sec^2 x dx = \tan x + c$, since $\frac{d}{dx}(\tan x) = \sec^2 x$.

8. $\int \csc^2 x dx = -\cot x + c$, since $\frac{d}{dx}(\cot x) = -csc^2\, x$.

9. $\int \sec x dx = \ln|\sec x + \tan x| + c$, since

$$\int \frac{\sec x+\tan x}{\sec x+\tan x} \sec x\, dx = \int \frac{sec^2 x+\sec x \tan x}{\sec x+\tan x} dx = \ln|\sec x + \tan x|.$$

10. $\int \csc x dx = \ln|\csc x - \cot x| + c$, since

$$\int \frac{\csc x-\cot x}{\csc x-\cot x} \csc x\, dx = \int \frac{csc^2 x-\csc x \cot x}{\csc x-\cot x} dx = \ln|\csc x - \cot x|.$$

11. $\int \sec x \tan x dx = \sec x + c$, since $\frac{d}{dx}(\sec x) = \sec x \tan x$.

12. $\int \csc x \cot x dx = -\csc x + c$, since $\frac{d}{dx}(-\csc x) = \csc x \cot x$

13. $\int \frac{1}{1+x^2} dx = \arctan x + c$, since $\frac{d}{dx}(\arctan x) = \frac{1}{x^2+1}$.

14. $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + c$, since $\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$.

11.2 Integration Methods

Apart from the above integration formulae there are other various methods used in the evaluation of indefinite integrals. These are

1. Integration by Substitution

To evaluate the integral $\int f(g(x))g'(x)dx$, it is often useful to substitute $g(x)$ with a new variable, say u, so that $u = g(x) and\ du = g'(x)dx$, and the integral becomes

$$\int f(g(x))g'(x)dx = \int f(u)du, \tag{11.2.1}$$

which is easier to evaluate. After finding the right side of (11.2.1), replace *u* with $g(x)$ to obtain the results in terms of x.

Example 1 Find $\int \frac{2x}{(x^2+5)^{\frac{3}{2}}}dx$.

Solution: Here, note that $g(x) = x^2 + 5$, $f(g(x) = \frac{1}{(g(x))^{3/2}}$.

Thus we let, $u = x^2 + 5$ so that $du = 2xdx$.

$$\Rightarrow \int \frac{2x}{(x^2+5)^{\frac{3}{2}}}dx = \int \frac{1}{u^{\frac{3}{2}}}du = \int u^{-\frac{2}{3}}du = 3u^{\frac{1}{3}} + c = 3(x^2+5)^{\frac{1}{3}} + c.$$

Example 2 Evaluate the integral $\int \sin^5 x \cos x dx$.

Solution: Here, note that $g(x) = \sin x$, $f(g(x)) = (\sin x)^5$, and

$g'(x) = \cos x\, dx$. Hence, letting $= \sin x$, $du = \cos dx$.

$$\Rightarrow \int \sin^5 x \cos x dx = \int u^5 du = \frac{1}{6}u^5 + c = \frac{1}{6}\sin^6 x + c.$$

Example 3 Evaluate each of the following the integrals:

(a) $\int x\sqrt{x+1}\,dx$.

Solution: Let $x+1 = u^2$ so that $x = u^2 - 1$. Then $dx = 2udu$.

Thus,

$$\int x\sqrt{x+1}\,dx = 2\int (u^2-1)u^2 du = 2\int (u^4 - u^2)du$$
$$= \frac{2}{5}u^5 - \frac{2}{3}u^3 + c = \frac{2}{5}\left(\sqrt{x+1}\right)^5 - \frac{2}{3}\left(\sqrt{x+1}\right)^3 + c$$

(b) $\int \frac{x}{\sqrt{3-x}} dx$.

Solution: Let $3-x=u^2$ so that $x=3-u^2$. Then $dx=-2udu$.
Thus,

$$\int \frac{x}{\sqrt{3-x}}dx = -2\int \frac{3-u^2}{u}du = -2\int\left(\frac{3}{u}-u\right)du$$

$$= -6\ln|u| + u^2 + c = -6\ln\sqrt{3-x} + (3-x) + c$$

(c) $\int (x+1)(x+3)^5 \, dx$.

Solution: Let $x+3=u$ so that $x+1=u-2$. Then $dx=du$.
Thus,

$$\int (x+1)(x+3)^5 dx = \int (u-2)u^5 du = \int \left(u^6 - 2u^5\right)du$$

$$= \frac{1}{7}u^7 - \frac{2}{6}u^6 + c = \frac{1}{7}(x+3)^7 - \frac{1}{3}(x+3)^6 + c$$

2. Quick Integration by Inspection

We have two simple formulae which enable us to find integrals almost immediately.

(a) The first is

$$\int g'(x)[g(x)]^r dx = \frac{1}{r+1}[g(x)]^{r+1} + c,\ r \neq -1 \qquad (11.2.2)$$

This formula is justified by noting that

$$\frac{d}{dx}\left\{\frac{1}{r+1}[g(x)]^{r+1}\right\} = g'(x)[g(x)]^r .$$

Example 6.2.3 Find $\int 6x^2\left(2x^3+5\right)^3 dx$.

Solution: Here note that $g(x)=2x^3+5$ and $g'(x) = 6x^2$. Therefore

$$\int 6x^2\left(2x^3+5\right)^3 dx = \frac{1}{4}\left(2x^3+5\right)^4 + c .$$

(b) The second quick integration formula is

$$\int \frac{g'(x)}{g(x)}dx = \ln|g(x)| + c \ . \qquad (11.2.3)$$

This formula is justified by noting that

$$\frac{d}{dx}\{\ln|g(x)|\} = \frac{g'(x)}{g(x)}.$$

Example: 2.1 Find $\int \frac{1}{x\ln x}dx$.

Solution: Note that $g(x) = \ln x$ and $g'(x) = \frac{1}{x}$.

Therefore, $\int \frac{1}{x\ln x}dx = \ln(\ln x) + c$.

Note that both formulae emanate from the substitution method.

3. Integration by Parts

Suppose f and g are differentiable functions of x. Then by the rule of the derivative of products we have

$$(f.g)' = f.g' + f'.g \qquad (11.2.4)$$

Integrating both sides of equation (11.2.4) with respect to x yields

$$\int \frac{d}{dx}[f(x).g(x)]\,dx = \int [f(x).g'(x) + f'(x).g(x)]dx \qquad (11.2.5)$$

i.e.

$$f(x).g(x) = \int [f(x).g'(x)]dx + \int [f'(x).g(x)]dx \qquad (11.2.6)$$

Thus, from (3.3) we have the **integration by parts** formula which is written as

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx. \qquad (11.2.7)$$

The formula is easily remembered when the following substitutions are made:

$u = f(x)$ and $dv = g'(x)dx \Rightarrow du = f'(x)dx$ and $v = \int g'(x)dx$.

These substitutions transforms (3.4) to

$$\boxed{\int udv = uv - \int vdu} \qquad (11.2.8)$$

Example: 3.1 Find $\int x^2e^x dx$.

Solution: Let $u = x^2$ and $dv = e^x dx$. $\Rightarrow du = 2x$ and $v = \int e^x dx = e^x$.

Thus, $\int x^2e^x dx = uv - \int vdu = x^2e^x - 2\int xe^x dx$.

To evaluate $\int xe^x dx$ we use integration by parts again.

Thus, we let $u = x$ and $dv = e^x dx$. $\Rightarrow du = dx$ and $v = e^x$.

$\Rightarrow \int xe^x dx = uv - \int vdu = xe^x - \int e^x\, dx = xe^x - e^x + c_1$

Therefore,

$$\int x^2 e^x\, dx = x^2 e^x - 2(xe^x - e^x + c_1)$$

$$= x^2 e^x - 2(xe^x - e^x) + c$$

$$= x^2 e^x - 2e^x(x-1) + c$$

Now to determine which part of the integral is u and the other dv in the application of formula (3.5) the following two rules may be used:

(i) The part selected as dv must be easily integrated.

(ii) $\int vdu$ must not be more complicated than $\int udv$.

However, the following is the list of common integrals with suggestions for the choices of u and dv:

1. For integrals of the form

$$\int x^n e^x dx,\ \int x^n \sin x\, dx,\ \int x^n \cos x\, dx,$$

let $u = x^n$ and let $dv = e^x dx, dv = \sin x dx\ or\ dv = \cos x\, dx.$

2. For integral of the form

$$\int x^n \ln x\, dx,\ \int x^n \arcsin x\, dx,\ \int x^n \arccos x\, dx,$$

let $u = \ln x, \arcsin x\ or \arccos x$ and $dv = x^n dx.$

Example 3.2 Find $\int x^2 \ln x dx$.

Solution: Since the derivative of $\ln x$ is a power of x , let $u = \ln x$ and $dv = x^2 dx$ so that $du = \frac{1}{x} dx$ and $v = \frac{1}{3}x^3$.

Applying equation (3.5) gives

$$\int x^2 \ln x\, dx = \tfrac{1}{3}x^3 \ln x - \tfrac{1}{3}\int x^3 . \tfrac{1}{x} dx = \tfrac{1}{3}x^3 \ln x - \tfrac{1}{3}\int x^2\, dx$$

$$= \frac{1}{3}x^3 \ln x - \frac{1}{3}.\frac{1}{3}x^3 + c$$

$$= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + c$$

Example 3.3 Find $\int \arcsin x \, dx$.

Since it is easier to differentiate $\arcsin x$ than integrating it,

let $u = \arcsin x$ and $dv = dx$ so that $du = \frac{1}{\sqrt{1-x^2}}dx$ and $v = x$.

Thus,

$\int x \arcsin x \, dx = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} dx.$

To evaluate $\int \frac{x}{\sqrt{1-x^2}} dx$ use the substitution method. Let $t = 1 - x^2$

so that $dt = -2xdx$ or $xdx = -\frac{1}{2}dt$. Thus,

$\int \frac{x}{\sqrt{1-x^2}} dx = -\frac{1}{2}\int \frac{1}{\sqrt{t}} dt = -\frac{1}{2}\int t^{-\frac{1}{2}} dt = -\frac{1}{2}\left(\frac{2}{1}\right) t^{\frac{1}{2}} = -t^{\frac{1}{2}} = -\sqrt{1-x^2}.$

Therefore,

$$\int x \arcsin x \, dx = x \arcsin x + \sqrt{1 - x^2} + c.$$

3. For integrals of the form

$$\int e^x \sin x \, dx \; or \int e^x \cos x \, dx,$$

let *u* to be either e^x or $\sin x$ ($\cos x$). If you take $u = e^x$ then

$dv = \sin x dx$ (or $\cos x \, dx$). If you take $u = \sin x$ or $\cos x$ then

$dv = e^x dx$. Upon integration you will obtain the same result.

Example 3.4 Find $\int e^x \cos x dx$.

Solution: Since there is no clue as to which function to take as u, arbitrarily choose $u = e^x$ and $dv = \cos x dx$. Then $du = e^x dx$ and $v = \sin x dx$. Applying (3.5) yields

$$\int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx. \qquad (11.2.9)$$

$\int e^x \sin x dx$ can be evaluated the same way. Let $u = e^x$ and

$dv = \sin x dx \;\; \Rightarrow du = e^x du$ and $v = -\cos x$. Thus

$$\int e^x \sin x dx = -e^x \cos x + \int e^x \cos x dx$$

Therefore, replacing the integral of $\int e^x \sin x\, dx$ in (3.6) we have

$$\int e^x \cos x dx = e^x \sin x - [-e^x \cos x + \int e^x \cos x dx]$$

$$\int e^x \cos x\, dx = e^x \sin x + e^x \cos x - \int e^x \cos x\, dx$$

$$\Rightarrow \quad 2\int e^x \cos x dx = e^x \sin x + e^x \cos x$$

$$\Rightarrow \quad \int e^x \cos x dx = \tfrac{1}{2}[e^x \sin x + e^x \cos x] + c$$

4. Integration by Partial Fractions

Example: 4.1 Evaluate $\int \frac{x^4 - x^3 - x - 1}{x^2(x-1)} dx$.

Solution: $\frac{x^4 - x^3 - x - 1}{x^2(x-1)} = x - \frac{x+1}{x^2(x-1)}$

$$\frac{x+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} = \frac{Ax(x-1) + B(x-1) + Cx^2}{x^2(x-1)}$$

$$x + 1 \equiv Ax(x-1) + B(x-1) + Cx^2$$

When $x = 0, \quad 1 = -B \quad \Rightarrow B = -1$

$$\Rightarrow \quad x + 1 \equiv Ax(x-1) - (x-1) + Cx^2$$

When $x = 1, \quad 2 = C \quad \Rightarrow C = 2$

$$\Rightarrow \quad x + 1 \equiv Ax(x-1) - (x-1) + 2x^2$$

When $x = -1, \quad 0 = 2A + 2 + 2 \quad \Rightarrow A = -2$

$$\frac{x+1}{x^2(x-1)} = \frac{-2}{x} - \frac{1}{x^2} + \frac{2}{x-1}$$

$$\Rightarrow \quad \frac{x^4 - x^3 - x - 1}{x^2(x-1)} = x - \left(\frac{-2}{x} - \frac{1}{x^2} + \frac{2}{x-1}\right) = x + \frac{2}{x} + \frac{1}{x^2} - \frac{2}{x-1}$$

$$\therefore \quad \int \frac{x^4 - x^3 - x - 1}{x^2(x-1)} dx = \int \left(x + \frac{2}{x} + x^{-2} - \frac{2}{x-1} \right) dx$$

$$= \frac{1}{2}x^2 + 2\ln|x| - \frac{1}{x} - \ln|x-1| + c$$

Example: 4.2 Evaluate the integral $\int \frac{3x+2}{x^3+x}dx$

Solution: We resolve $\frac{3x+2}{x^3+x}$ into partial fractions.

$$\frac{3x+2}{x^3+x} = \frac{3x+2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1)+x(Bx+C)}{x(x^2+1)}$$

$\Rightarrow \quad 3x+2 \equiv A(x^2+1)+x(Bx+C)$

When $x=0, \quad 2=A$

$\Rightarrow \quad 3x+2 \equiv 2(x^2+1)+x(Bx+C)$

When $x=1, \quad 5=4+B+C \quad \Rightarrow B+C=1$

When $x=-1 \quad -1=4+B-C \quad \Rightarrow B-C=-5$

Thus, $B=-2$ and $C=3$

$$\frac{3x+2}{x^3+x} == \frac{2}{x} + \frac{3-2x}{x^2+1}$$

Therefore,

$$\int \frac{3x+2}{x^3+x}dx = \int\left(\frac{2}{x}+\frac{3-2x}{x^2+1}\right)dx = 2\int\frac{1}{x}dx + 3\int\frac{1}{x^2+1}dx - \int\frac{2x}{x^2+1}dx$$
$$= 2\ln|x| + 3\tan^{-1}x - \ln\left(x^2+1\right) + c$$

5. Trigonometric Integrals

Some trigonometric integrals are evaluated using some of the following trigonometric identities:

1. $\sin^2\theta + \cos^2\theta = 1$
2. $1+\tan^2\theta = \sec^2\theta$
3. $1+\cot^2\theta = \csc^2\theta$
4. $\sin^2\theta = \frac{1}{2}\left(1-\cos 2\theta\right)$
5. $\cos^2\theta = \frac{1}{2}\left(1+\cos 2\theta\right)$
6. $\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$
7. $\sin\alpha\cos\beta = \frac{1}{2}\left(\sin(\alpha-\beta)+\sin(\alpha+\beta)\right)$
8. $\sin\alpha\sin\beta = \frac{1}{2}\left(\cos(\alpha-\beta)-\cos(\alpha+\beta)\right)$
9. $\cos\alpha\cos\beta = \frac{1}{2}\left(\cos(\alpha-\beta)+\cos(\alpha+\beta)\right)$

Below are two substitution rules which are used in finding the indicated integrals:

(i) For the integral of the form

$$\int \sin^m x \cos^n x dx,$$

if *m* is odd, substitute $u = \cos x$. If *n* is odd, substitute $u = \sin x$.

Example 5.1: Find $\int \sin^3 x \cos^{\frac{3}{4}} x dx$.

Solution: Here $m = 3$, which is odd. Then we let

$u = \cos x$ so that $du = -\sin x dx$.

Now, we need to rewrite as

$$\int \sin^3 x \cos^{\frac{3}{4}} x dx = \int \sin^2 x \sin x \cos^{\frac{3}{4}} x dx$$

$$= \int (1 - \cos^2 x) \cos^{\frac{3}{4}} x \sin x dx$$

$$= -\int \left(\cos^{\frac{3}{4}} - \cos^{\frac{7}{4}} x \right) (-\sin x) dx$$

$$= -\int \left(u^{\frac{3}{4}} - u^{\frac{7}{4}} x \right) du$$

$$= -\left(\frac{4}{7} u^{\frac{7}{4}} - \frac{4}{11} u^{\frac{11}{4}} \right) + c$$

$$= -\left(\frac{4}{7} \cos^{\frac{7}{4}} x - \frac{4}{11} \cos^{\frac{11}{4}} x \right) + c$$

(ii) For the integral of the form

$$\int \tan^m x \sec^n x dx,$$

If *m* is odd, substitute $u = \sec x$, and if *n* is even, substitute $u = \tan x$.

Example 5.2 Find $\int \tan^6 x \sec^4 x dx$

Solution: Here $n = 4$ and is even. So we let $u = \tan x$ so that

$du = \sec^2 x dx$. Thus, rewriting the integral we have

$$\int \tan^6 x \sec^4 x dx = \int \tan^6 x \sec^2 x \sec^2 x dx$$

$$= \int \tan^6 x (1 + \tan^2 x) \sec^2 x dx$$

$$= \int u^6 (1 + u^2) du = \int (u^6 + u^8) du$$

$$= \frac{1}{7} u^7 + \frac{1}{9} u^9 + c$$

$$= \frac{1}{7} \tan^7 x + \frac{1}{9} \tan^9 x + c$$

More examples: Evaluate each of the following examples:

1. $\int \sin^2 x \cos^2 x dx$.

<u>Solution</u>: We rewrite the integral in terms of the identities

$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$ and $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

and obtain

$$\int \sin^2 x \cos^2 x dx = \int \tfrac{1}{2}(1-\cos 2x)\cdot \tfrac{1}{2}(1+\cos 2x)dx$$
$$= \tfrac{1}{4}\int (1-\cos^2 2x)dx$$
$$= \tfrac{1}{4}\int \left(1-\tfrac{1}{2}(1+\cos 2(2x))\right)dx$$
$$= \tfrac{1}{4}\int \left(\tfrac{1}{2}-\tfrac{1}{2}\cos 4x\right)dx = \tfrac{1}{8}\int (1-\cos 4x)dx$$
$$= \tfrac{1}{8}\left(x-\tfrac{1}{4}\sin 4x\right)+c$$

2. $\int 2\sin 4x \cos 3x dx$

<u>Solution</u>: We rewrite the integral in terms of the identities

$\sin\alpha\cos\beta = \frac{1}{2}\left(\sin(\alpha-\beta)+\sin(\alpha+\beta)\right)$.

Thus

$$2\sin 4x\cos 3x = \sin(4x-3x)+\sin(4x+3x)$$
$$= \sin x + \sin 7x$$

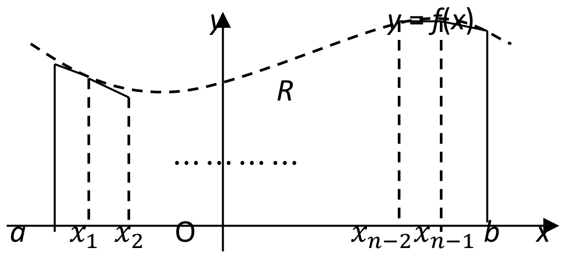
Therefore,

$$\int 2\sin 4x\cos 3x dx = \int (\sin x + \sin 7x)dx$$
$$= -\cos x - \tfrac{1}{7}\cos 7x + c$$

11.3 Application of definite integrals

1 Area under the curve

(a) Let $[a,b]$ be an interval on which a given function $f(x) > 0\ \forall x \in [a,b]$ and is continuous on the interval. Let $x_0, x_1, \cdots, x_{n-1}, x_n$ be points on the interval such that $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ and let $\Delta x = x_n - x_{n-1}$.



Thus, as $\Delta x \to 0$ and $n \to \infty$ the area between the curve $y = f(x)$ and the *x* - axis from $x = a$ to $x = b$, is given by

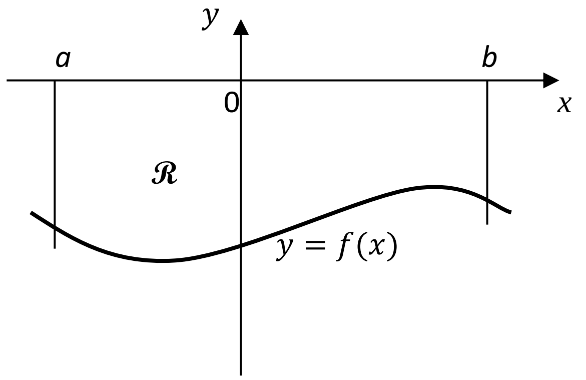
$$A \approx \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k)\Delta x = \int_a^b f(x)dx .$$

i.e.

$$A = \int_a^b f(x)dx \qquad (11.3.1)$$

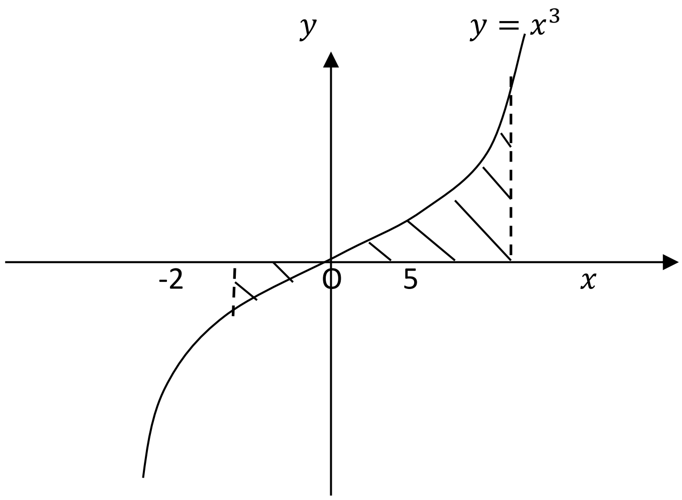
(b)) If $f(x)$ is continuous and $f(x) < 0$ on the interval $[a,b]$, then the area *A* of the region *R* between the curve and the *x* – axis is given by

$$A = -\int_a^b f(x)dx = \int_b^a f(x)dx \qquad (11.3.2)$$



Example 11.3.1 Find the area bounded by the curve $y = x^3$ and the *x* – axis from $x = -2$ to $x = 5$

Solution



$$Area = -\int_{-2}^{0} x^3\, dx + \int_0^5 x^3\, dx$$

$$= -\frac{1}{4}(x^4)_{-2}^{0} + \frac{1}{4}(x^4)_0^5$$

$$= \frac{1}{4}((-2)^4 - 0^4) + \frac{1}{4}(5^4 - 0^4)$$

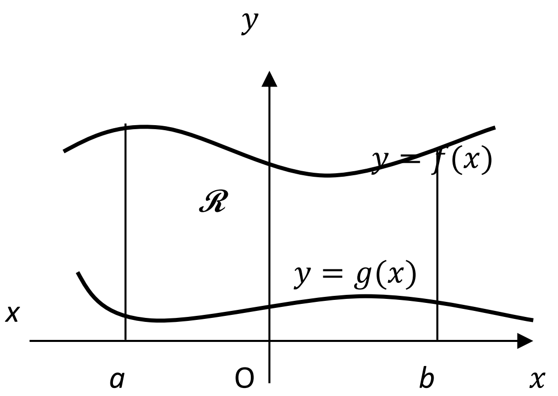
$$= \frac{1}{4}(16 + 625) = \frac{641}{4} \text{ sq. Unit}$$

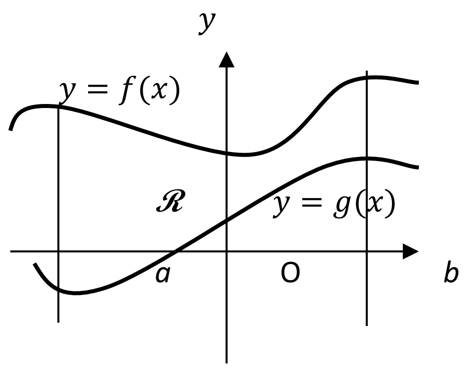
2 Area between two Curves

If we assume that $f(x)$ and $g(x)$ are continuous functions such that $0 \le g(x) \le f(x)$ for $a \le x \le b$, then the area *A* of the region $\mathscr{R}$ between the curves $y = f(x)$ and $y = g(x)$ from $x = a$ and $x = b$, is given by

$$A = \int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b [f(x) - g(x)]dx \qquad (11.3.3)$$

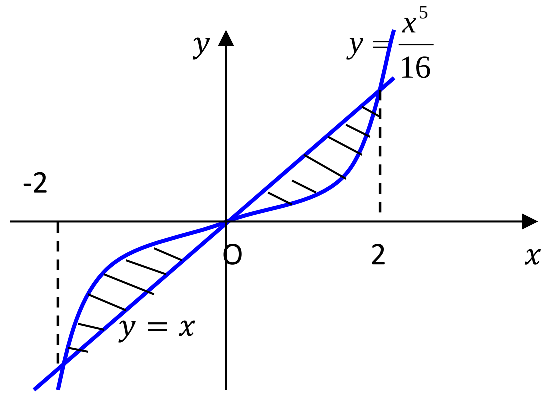
The formula (11.3.3) holds when one or both curve $y = f(x)$ and $y = g(x)$ lie partially or completely below the *x* – axis.





Example 11.3.2 Find the area of the region bounded by the curves $y = x$ and $y = \frac{x^5}{16}$.

Solution



Solution: First we must find the points of intersection of the two curves by solving the two equations simultaneously.

$$x = \frac{x^5}{16}.$$

$$x - \frac{x^5}{16} = 0$$

$$x\left(1 - \frac{x^4}{16}\right) = 0$$

$$x\left(1 - \frac{x^2}{4}\right)\left(1 + \frac{x^2}{4}\right) = 0$$

$$x\left(1 - \frac{x}{2}\right)\left(1 + \frac{x}{2}\right)\left(1 + \frac{x^2}{4}\right) = 0$$

$\Rightarrow \qquad x = 0,\ x = 2,\ x = -2$

Therefore, the area bounded by the two curves is

$$A = \int_{-2}^{0}\left(\frac{x^5}{16} - x\right)d + \int_{0}^{2}\left(x - \frac{x^5}{16}\right)dx$$

$$= \left(\frac{x^6}{96} - \frac{x^2}{2}\right)_{-2}^{0} + \left(\frac{x^2}{2} - \frac{x^6}{96}\right)_{0}^{2}$$

$$= 0 - \left(\frac{(-2)^6}{96} - \frac{(-2)^2}{2}\right) + \left(\frac{2^2}{2} - \frac{2^6}{96}\right) - 0$$

$$= -\frac{64}{96} + 2 + 2 - \frac{64}{96} = \frac{8}{3} \text{ square units.}$$

3 More Applications of Integration

Example 11.3.3. The rate of increase of a population *P* of microorganisms at time *t* is given by $\frac{dP}{dt} = kP$, where *k* is a positive constant. Given that at $t = 0$ the population was of size 8, and at $t = 1$ the population is 56, find the size of the population at time $t = 2$.

Solution: Here we are asked to find *P*. This can be found by using the fact that integration is the reverse operation of differentiation.

Thus, to find *P* we first separate the variables of $\frac{dP}{dt} = kP$ and integrate both sides.

i.e. $\int \frac{1}{P} dP = k \int dt$

$$\Rightarrow \qquad ln|P| = kt + c. \qquad (11.3.4)$$

Next we need to use the given conditions to find the values of *k* and *c*.

For $t = 0, P = 8 \Rightarrow ln8 = k(0) + c \Rightarrow c = ln8.$

This means that (11.3.4) becomes

$$ln|P| = kt + ln8. \tag{11.3.5}$$

Also, replacing $t = 1 \; and \; P = 56 \; in \; (11.3.5) \; we \; have$

$\Rightarrow ln56 = k(1) + ln8 \Rightarrow k = ln56 - ln8 = ln\frac{56}{8} = ln7.$

Hence, (11.3.5) becomes

$$ln|P| = tln7 + ln8.$$

or $$ln|P| = ln8(7^t).$$

or $$P = 8 \times 7^t$$

Therefore, at time $t = 2$,

$$P = 8 \times 7^2 = 392$$

Example 11.3.4 The mass *M* at time *t* of the leaves of a certain plant caries according to the differential equation

$$\frac{dM}{dt} = M - M^2.$$

(a) Given that $t = 0, M = 0.5$, find an expression for *M* in terms of *t*.
(b) Find a value for *M* when $t = ln2$.

(c) Explain what happens to the value of *M* as *t* increases.

Solution: (a) First we must separate the variables in the given differential equation and integrate both sides:

$$\int \frac{1}{M-M^2} dM = \int dt$$

$$\int \frac{1}{M(1-M)} dM = \int dt$$

$$\int \left(\frac{1}{M} + \frac{1}{1-M}\right) dM = \int dt$$

$$ln|M| - ln|1 - M| = t + c$$

$$ln\left|\frac{M}{1-M}\right| = t + c \tag{11.3.6}$$

When $t = 0 \; and \; M = 0.5$, we have

$$ln1 = 0 + c \Rightarrow c = 0$$

Therefore, (11.3.6) becomes

$$ln\left|\frac{M}{1-M}\right| = t$$

or $\frac{M}{1-M} = e^t$

$$M = \frac{e^t}{1+e^t} \;\; or \; M = 1 - \frac{1}{1+e^t} \qquad (11.3.7)$$

(b) When $t = ln2, M = \frac{e^{ln2}}{1+e^{ln2}} = \frac{2}{3}$.

(c) As *t* increases, *M* approaches 1.

<u>Example</u> 11.3.5 The rate of depreciation $\frac{dV}{dt}$ of a machine is inversely proportional to the square of $t + 1$, where *V* is the value of the machine *t* years after it has been purchased. If the initial value of the machine was worth \$500,000, and its value decreased by \$100,000 in the first year, estimate its value after 4 years.

<u>Solution</u>: $\frac{dV}{dt} \alpha \frac{1}{(t+1)^2} \;\; \Rightarrow \frac{dV}{dt} = \frac{k}{(t+1)^2}$

Separating the variables in the differential equation and integrating both sides we have

$$\int dV = k \int \frac{1}{(t+1)^2} dt$$

$$\Rightarrow \qquad V = \frac{-k}{t+1} + c \qquad (11.3.8)$$

Note that when $t = 0, V = 500{,}000$.

Thus, substituting these in (11.3.8) we have

$$500{,}000 = -k + c$$

$$\Rightarrow \qquad c = 500{,}000 + k \qquad (11.3.9)$$

Also, when $t = 1, \;V = 400{,}000$. Substituting in (11.3.8) we have

$$400{,}000 = \frac{-k}{2} + c$$

$$\Rightarrow \qquad 400{,}000 = \frac{-k}{2} + (500{,}000 + k)$$

$$\Rightarrow \qquad k = -200{,}000$$

$$\Rightarrow \qquad c = 300{,}000$$

Hence, (11.3.8) becomes

$$V = \frac{-200{,}000}{t+1} + 300{,}000 \qquad (11.3.10)$$

Therefore, using ((11.3.10)) we find *V* after *t* = 4 years:

$$V = \frac{-200{,}000}{4+1} + 300{,}000 = \$250{,}000.$$

TUTORIAL SHEET 15

1. Integrate the following functions with respect to *x*:

(a) $(2+7x)^3$ (b) $3\sqrt{2-5x}$ (c) $\dfrac{1}{(4x+5)^3}$ (d) $4\sqrt{x}+\sqrt{4x+1}-4(1-3x)^3$.

2. Evaluate the following integrals:

(a) $\int x(x^2+3)^3\,dx$ (b) $\int x(x^2+3)^3\,dx$ (c) $\int \dfrac{2x}{\sqrt{x^2+1}}dx$ (d) $\int \dfrac{\cos x}{1+\sin x}dx$

(e) $\int \dfrac{2x+3}{(x^2+3x+4)^3}dx$ (g) $\int \dfrac{x^2}{1+x^3}dx$ (h) $\int \cos^3 x dx$ ·

3. Evaluate the following integrals:

(a) $\int xe^{-x}dx$ (b) $\int x^3 \ln 3x dx$ (c) $\int (1+x)e^x dx$ (d) $\int x^2 \sin x dx$

(e) $\int \tan^{-1} x dx$ ·

4. Use partial fractions to find the following integrals:

(a) $\int \dfrac{x}{x+1}dx$ (b) $\int \dfrac{x^2-2}{x^2-1}dx$ (c) $\int \dfrac{3}{(x+1)(x+2)}dx$ (d) $\int \dfrac{x^2+x+5}{x(x+1)^2}dx$

(e) $\int \dfrac{12x}{(2-x)(3-x)(4-x)}dx$ (f) $\int \dfrac{x^2+2x+4}{(2x-1)(x^2-1)}dx$

6. Evaluate the following definite integrals:

(a) $\int_2^3 (x^2+2x-1)dx$ (b) $\int_1^2 xe^{x^2}dx$ (c) $\int_{\pi/6}^{\pi/3} cos2xdx$

(d) $\int_0^{\pi/6} sin^2 xdx$ (e) $\int_1^2 \frac{x}{x^2+1}dx$ (f) $\int_0^1 x^2(x^3+1)^4 dx$.

7. Find the areas bounded by the specified lines and curves in the following questions:

(a) The *x*- and *y*-axes, the line $x = 3$ and the curve $y = x^2 + 1$.

(b) The *x*-axis, the lines $x = 1$ and $x = 4$ and the curve $xy = 2$.

(c) The *x*-axis, and the curve $y = 1 - x^2$.

(d) The *y*-axis, and the curve $x = 9 - y^2$.

(e) The curve $y = x(x-1)(x-2)$, and the *x*-axis.

8. Compute the area of the region for which $x \in [-1,2]$ and which is bounded by the graphs of the function *f* and g where

$f(x) = -\frac{1}{4}x^2 + 1$ and $g(x) = \frac{1}{2}x^2 - 3$.

9. The Goldegg Chicken Farm produces $E(t)$ dozen eggs in *t* days. Its production rate (in dozen/day) is given by

$$E'(t) = 150 + \frac{2}{5}t,$$

find

(a) the number of eggs produced in t days assuming $E(0) = 0$

(b) the number of eggs produced in one year (365 days)

(c) the average daily production.

10. Consider an (idealized) experiment in which a colony of live bacteria is introduced to a limited food supply. Suppose the rate of change in the number of *N* of live bacteria with respect to time is given by

$$N'(t) = 6000t^2 - 75t^4.$$

Find the size $N(t)$ of the population of the bacteria at time t if initially 1000 bacteria were introduced to the food supply.

11. An object on the ground is projected vertically with initial velocity of 32 m/sec. If the acceleration $a(t) = -10.6$ m/sec^2, find

(a) the velocity function,

(b) the distance at time t,

(c) the height the object will attain,

(d) the height of the object in 5 seconds.