

MAT 1100 LECTURE NOTES

10 Further Differential Calculus

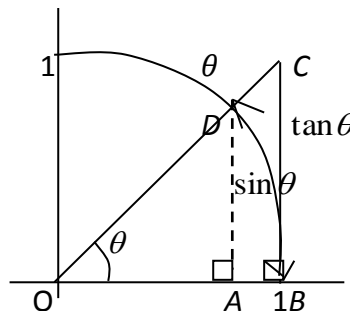
10.1 Some Important limits of Trigonometric functions

1. $\lim_{\theta \rightarrow 0} \left(\frac{\sin \theta}{\theta} \right) = 1.$

Proof: We consider an arc of a circle of radius 1, as shown in the diagram below.

Let the line OC make an angle of θ with the line OA .

The length of the arc of radius 1 with angle at centre θ is just θ . The length of the line $BC = \tan \theta$ and the length of the line $AD = \sin \theta$.



Clearly, from the diagram,

$$AD \leq \text{arc } BD \leq BC.$$

That is

$$\sin \theta \leq \theta \leq \tan \theta.$$

Dividing through by $\sin \theta$ we obtain

$$1 \leq \frac{\theta}{\sin \theta} \leq \frac{1}{\cos \theta},$$

and taking reciprocals, we have

$$\cos \theta \leq \frac{\sin \theta}{\theta} \leq 1.$$

But $\lim_{\theta \rightarrow 0} \cos \theta = 1$. This means that

$$\lim_{\theta \rightarrow 0} \cos \theta \leq \lim_{\theta \rightarrow 0} \left(\frac{\sin \theta}{\theta} \right) \leq \lim_{\theta \rightarrow 0} 1$$

implying that

$$1 \leq \lim_{\theta \rightarrow 0} \left(\frac{\sin \theta}{\theta} \right) \leq 1$$

Therefore,

$$\lim_{\theta \rightarrow 0} \left(\frac{\sin \theta}{\theta} \right) = 1$$

2. $\lim_{\theta \rightarrow 0} \left(\frac{\cos \theta - 1}{\theta} \right) = 0.$

Proof: $\frac{\cos \theta - 1}{\theta} = \frac{\cos \theta - 1}{\theta} \times \frac{\cos \theta + 1}{\cos \theta + 1} = \frac{\cos^2 \theta - 1}{\theta(\cos \theta + 1)} = \frac{-\sin^2 \theta}{\theta(\cos \theta + 1)} = (-1) \left(\frac{\sin \theta}{\theta} \right) \left(\frac{\sin \theta}{\cos \theta + 1} \right)$

But $\lim_{\theta \rightarrow 0} \left(\frac{\sin \theta}{\theta} \right) = 1$ and $\lim_{\theta \rightarrow 0} \left(\frac{\sin \theta}{\cos \theta + 1} \right) = 0.$

Therefore,

$$\lim_{\theta \rightarrow 0} \left(\frac{\cos \theta - 1}{\theta} \right) = (-1)(1)(0) = 0.$$

10.2 Derivative of Trigonometric functions

1. $\frac{d}{dx}(\sin x) = \cos x$

Proof: From first principle,

$$\begin{aligned} \frac{d}{dx}(\sin x) &= \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin x[\cos h - 1]}{h} + \lim_{h \rightarrow 0} \frac{\cos x \sin h}{h} \\ &= \sin x \lim_{h \rightarrow 0} \frac{[\cos h - 1]}{h} + \cos x \lim_{h \rightarrow 0} \frac{\sin h}{h} \\ &= (\sin x) \times 0 + (\cos x) \times 1 = \cos x \end{aligned}$$

2. $\frac{d}{dx}(\cos x) = -\sin x$

Proof: From first principle,

$$\begin{aligned} \frac{d}{dx}(\cos x) &= \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos x}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cos x[\cos h - 1]}{h} - \lim_{h \rightarrow 0} \frac{\sin x \sin h}{h} \\ &= \cos x \lim_{h \rightarrow 0} \frac{[\cos h - 1]}{h} - \sin x \lim_{h \rightarrow 0} \frac{\sin h}{h} \\ &= (\cos x) \times 0 - (\sin x) \times 1 = -\sin x \end{aligned}$$

3. $\frac{d}{dx}(\tan x) = \sec^2 x$

Proof:

$$\begin{aligned} \frac{d}{dx}(\tan x) &= \frac{d}{dx} \left(\frac{\sin x}{\cos x} \right) = \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x \end{aligned}$$

4. $\frac{d}{dx}(\cot x) = -\csc^2 x$

Proof:
$$\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) = \frac{\sin x(-\sin x) - \cos x(\cos x)}{\sin^2 x}$$

$$= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\csc^2 x$$

5.
$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

Proof:
$$\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{(\cos x) \times 0 - 1(-\sin x)}{\cos^2 x}$$

$$= \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x} \times \frac{\sin x}{\cos x} = \sec x \tan x$$

6.
$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

Proof:
$$\frac{d}{dx}(\csc x) = \frac{d}{dx}\left(\frac{1}{\sin x}\right) = \frac{(\sin x) \times 0 - 1(\cos x)}{\sin^2 x}$$

$$= \frac{-\cos x}{\sin^2 x} = \frac{-1}{\sin x} \times \frac{\cos x}{\sin x} = -\csc x \cot x$$

10.3 Derivatives of Exponential and logarithmic functions

1. The Derivative of $\log_b x$

Consider the expression

$$(1 + p)^{\frac{1}{p}}$$

Notice that for

$$p = 1 \quad (1 + 1)^1 = 2.0000$$

$$p = \frac{1}{2} \quad (1 + \frac{1}{2})^2 = 2.2500$$

$$p = \frac{1}{4} \quad (1 + \frac{1}{4})^4 = 2.4414$$

$$p = \frac{1}{100} \quad (1 + \frac{1}{100})^{100} = 2.7048$$

$$p = \frac{1}{1000} \quad (1 + \frac{1}{1000})^{1000} = 2.7181$$

etc.

This is suggesting that $\lim_{p \rightarrow 0} (1 + p)^{1/p}$ exists and it is an irrational number 2.7181 ... This is the number denoted by e . Thus,

$$\lim_{p \rightarrow 0} (1 + p)^{1/p} = e \approx 2.7181 \dots$$

We can now use this to prove, from first principle, that

$$\frac{d}{dx}(\log_b x) = \frac{1}{x} \log_b e$$

Proof:
$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{1}{h} [\log_b(x+h) - \log_b x] \\ &= \frac{1}{h} \left[\log_b \left(\frac{x+h}{x} \right) \right] \\ &= \frac{1}{h} \left[\log_b \left(1 + \frac{h}{x} \right) \right] \\ &= \frac{1}{x} \left[\frac{x}{h} \log_b \left(\frac{x+h}{x} \right) \right] \\ &= \frac{1}{x} \log_b \left(1 + \frac{h}{x} \right)^{x/h} \end{aligned}$$

Letting $p = h/x$ we note that as $h \rightarrow 0, p \rightarrow 0$. Therefore,

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \frac{1}{x} \lim_{p \rightarrow 0} \log_b (1+p)^{\frac{1}{p}} = \frac{1}{x} \log_b e$$

Example : Given that $y = \log_b(3x + 2x^4)$, find y' .

Solution: Let $t = 3x + 2x^4$. Then $y = \log_b t$. Therefore by chain rule

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dt} \times \frac{dt}{dx} = \frac{1}{t} \log_b e \times (3 + 8x^3) \\ &= \frac{3+8x^3}{3x+2x^4} \log_b e. \end{aligned}$$

2. The Derivative of $\log_e x$ or $\ln x$

$$\frac{d}{dx}(\log_e x) = \frac{1}{x} \log_e e = \frac{1}{x}$$

3. The Derivative of e^x

$$\frac{d}{dx}(e^x) = e^x$$

Proof: Let $y = e^x$. Then $x = \ln y$.

Differentiating with respect to y we have $\frac{dx}{dy} = \frac{1}{y}$.

Now,

$$\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{1/y} = y = e^x.$$

Example : Find $f'(x)$ given that $f(x) = e^{2x} + e^{x \ln x}$.

Solution: Let $u = e^{2x}$ and $v = e^{x \ln x}$. Then

$$\begin{aligned} \frac{du}{dx} &= e^{2x} \times \frac{d}{dx}(2x) = 2e^{2x} \\ \frac{dv}{dx} &= e^{x \ln x} \times \frac{d}{dx}(x \ln x) = e^{x \ln x} \times (\ln x + x \cdot \frac{1}{x}) \end{aligned}$$

$$= (\ln x + 1)e^{x \ln x}$$

$$f'(x) = 2e^{2x} + (\ln x + 1)e^{x \ln x}.$$

4. The Derivative of b^x

$$\frac{d}{dx}(b^x) = b^x \ln b$$

Proof: $y = b^x \Rightarrow x = \log_b y$. Changing the base from b to e we have

$$x = \frac{\log_e y}{\log_e b} = \frac{1}{\log_e b} \log_e y.$$

Differentiating x with respect to y we obtain

$$\frac{dx}{dy} = \frac{1}{\log_e b} \times \frac{1}{y} = \frac{1}{y \log_e b} = \frac{1}{y \ln b} = \frac{1}{b^x \ln b}$$

Therefore

$$\frac{dy}{dx} = b^x \ln b.$$

Example: Find $f'(x)$ given that $f(x) = 3^{x-1} \cdot 2^x$.

Solution: $f(x) = 3^{x-1} \cdot 2^x = \frac{1}{3}(3^x) \times 2^x$

$$= 2^x \left(\frac{1}{3}(3^x \ln 3) \right) + \frac{1}{3} 3^x (2^x \ln 2)$$

$$= \frac{1}{3} 2^x 3^x (\ln 3 + \ln 2)$$

$$= 2^x 3^{x-1} \ln 6$$

5. Derivatives of Inverse trigonometric functions

Derivatives of $y = \sin^{-1}x$, $y = \cos^{-1}x$ and $y = \tan^{-1}x$

1. $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$, provided $|x| < 1$.

Proof: $y = \sin^{-1}x$. Then $x = \sin y$.

$$\text{Thus } \frac{dx}{dy} = \cos y \Rightarrow \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}.$$

2. $\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$, provided $|x| < 1$.

Proof: $y = \cos^{-1}x$. Then $x = \cos y$.

$$\text{Thus } \frac{dx}{dy} = -\sin y \Rightarrow \frac{dy}{dx} = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1-\cos^2 y}} = -\frac{1}{\sqrt{1-x^2}}.$$

3. $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$, $\forall x \in \mathbb{R}$

Proof: $y = \tan^{-1}x$. Then $x = \tan y$.

$$\text{Thus } \frac{dx}{dy} = \sec^2 y \Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}.$$

Example: Calculate the derivate of the given function.

(a) $\sin^{-1}\sqrt{x}$ (b) $\cos^{-1}(1 - 2x)$ (c) $\tan^{-1}(x^3 + 1)$

Solution: (a) We let $y = \sin^{-1}\sqrt{x}$ and $u = \sqrt{x}$. Then $y = \sin^{-1}u$.

Thus, by the chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$

But $\frac{dy}{du} = \frac{1}{\sqrt{1-u^2}}$ and $\frac{du}{dx} = \frac{1}{2\sqrt{x}}$.

This means that $\frac{dy}{dx} = \frac{1}{\sqrt{1-u^2}} \times \frac{1}{2\sqrt{x}}$.

Therefore

$$\frac{d}{dx} \sin^{-1}\sqrt{x} = \frac{1}{2\sqrt{x}\sqrt{1-x}} = \frac{1}{2\sqrt{x-x^2}}.$$

(b) We let $y = \cos^{-1}(1 - 2x)$ and $u = 1 - 2x$. Then $y = \cos^{-1}u$.

Thus, by the chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$

But $\frac{dy}{du} = -\frac{1}{\sqrt{1-u^2}}$ and $\frac{du}{dx} = -2$.

This means that $\frac{dy}{dx} = -\frac{1}{\sqrt{1-u^2}} \times (-2)$.

Therefore

$$\frac{d}{dx} \cos^{-1}(1 - 2x) = -\frac{(-2)}{\sqrt{1-(1-2x)^2}} = \frac{2}{\sqrt{4x-4x^2}} = \frac{1}{\sqrt{x-x^2}}.$$

(c) We let $y = \tan^{-1}(x^3 + 1)$ and $u = (x^3 + 1)$. Then $y = \tan^{-1}u$.

Thus, by the chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$

But $\frac{dy}{du} = \frac{1}{1+u^2}$ and $\frac{du}{dx} = 3x^2$.

This means that $\frac{dy}{dx} = \frac{1}{1+u^2} \times 3x^2$.

Therefore

$$\frac{d}{dx} \tan^{-1}(x^3 + 1) = \frac{3x^2}{1+(x^3+1)^2} = \frac{3x^2}{x^6+2x^3+2}$$

Derivatives of $y = \sec^{-1}x$, $y = \operatorname{cosec}^{-1}x$ and $y = \cot^{-1}x$

4. (a) $\frac{d}{dx} \sec^{-1}x = \frac{1}{x\sqrt{x^2-1}}$, $x > 1$

(b) $\frac{d}{dx} \sec^{-1}x = \frac{-1}{x\sqrt{x^2-1}}$, $x < -1$

In general,

(c) $\frac{d}{dx} \sec^{-1}x = \frac{-1}{|x|\sqrt{x^2-1}}$, $x < -1$ or $x > 1$

5. (a) $\frac{d}{dx} \operatorname{cosec}^{-1}x = \frac{-1}{x\sqrt{x^2-1}}$, $x > 1$

(b) $\frac{d}{dx} \operatorname{cosec}^{-1}x = \frac{1}{x\sqrt{x^2-1}}$, $x < -1$

In general,

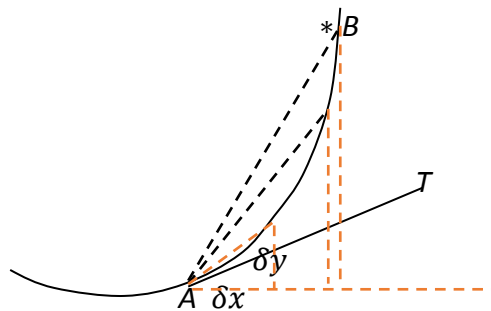
(c) $\frac{d}{dx} \operatorname{cosec}^{-1}x = \frac{-1}{|x|\sqrt{x^2-1}}$, $x < -1$ or $x > 1$

6. $\frac{d}{dx} \cot^{-1}x = \frac{-1}{1+x^2}$, $\forall x \in \mathbb{R}$

Exercise: Prove 4 to 6 the same way as in 1 to 3.

2.5.1 Gradient function

The gradient of a curve at any point on the curve is defined as the gradient of the tangent to the curve at that point and measures the rate of increase of y with respect to x .



The better approximation to the gradient of the tangent AT is obtained when B moves along the curve as it approaches A i.e. as $B \rightarrow A$ on the curve

gradient of chord $AB \rightarrow$ gradient of tangent AT

i.e. *gradient of $AT = \lim_{B \rightarrow A} \{\text{gradient of chord } AB\}$*

$$\begin{aligned} &= \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} \\ &= \frac{dy}{dx} \end{aligned}$$

Thus, if the curve is defined by $y = f(x)$, then $\frac{dy}{dx} = f'(x)$ gives the gradient of the curve at any point on the curve. Thus

$$\frac{dy}{dx} = f'(x)$$

is called the gradient function. Therefore, the gradient function, at the point where it is defined, gives the gradient of the curve at the point.

Example: Find the gradient of the curve $y = 2x(x^2 - 6)$ at the point $x = 2$.

Solution: The gradient function is

$$\frac{dy}{dx} = 6x^2 - 12.$$

Therefore the gradient of the curve at the point $x = 2$ is

$$\frac{dy}{dx} \Big|_{x=2} = 6(2)^2 - 12 = 12.$$

Note: The gradient of the curve at any given point gives the gradient of the tangent to the curve at that point.

2.5.2 Equations of Tangents and Normal lines to a curve

Example Find the equation of the tangent and of the normal to the curve $y = x^2 + 5x - 2$ at the point where the curve cuts the line $x = 4$.

Solution: Gradient function $\frac{dy}{dx} = 2x + 5$.

At $x = 4$, $\frac{dy}{dx} = 2(4) + 5 = 13 \Rightarrow$ *gradient of the tangent* = 13.

The gradient of the normal to the curve is $-\frac{1}{13}$.

When $x = 4$, $y = 4^2 + 5(4) - 2 = 34$. Thus the curve cuts the line $x = 4$ at the point (4,34).

Therefore, the equation of the tangent is

$$y - 34 = 13(x - 4)$$

or

$$13x - y - 18 = 0.$$

The equation of the normal to the curve is

$$y - 34 = -\frac{1}{13}(x - 4)$$

or

$$x + 13y - 446 = 0.$$

2.5.3 Increasing and Decreasing Functions

A function f is said to be increasing over an interval if for any two points x_1 and x_2 of the interval such that $x_1 < x_2$, we have $f(x_1) < f(x_2)$.

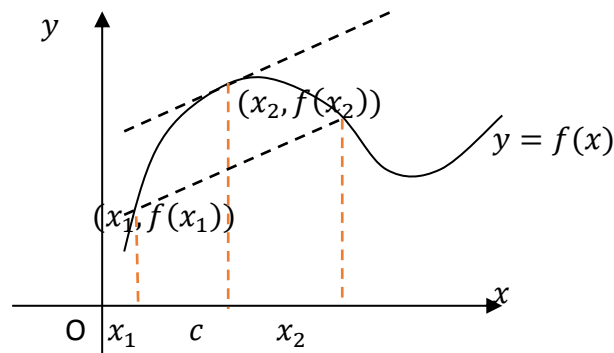
Similarly f is decreasing over the interval if $x_1 < x_2$ implies

$$f(x_1) < f(x_2).$$

The following theorem helps to determine whether a function is increasing or decreasing:

Theorem 2.1.1 If a function f is differentiable, then it is *increasing* over the interval on which $f'(x) > 0$ and it is *decreasing* over any interval on which $f'(x) < 0$.

Proof: Let $f'(c) > 0$. We show that f is increasing on the interval $[x_1, x_2]$



Now, if a function f is continuous on a closed interval $[x_1, x_2]$ and is differentiable in the open interval (x_1, x_2) , then there exist a real number $c \in (x_1, x_2)$ such that the gradient of the tangent to the curve at $x = c$ is equal to that of the line through the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ i.e.

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

From (2.4.1) we have

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

Since $x_1 < x_2$, $x_2 - x_1 > 0$. From the hypothesis, $f'(c) > 0$. Thus the right-hand side of (2.4.2) is positive. This implies that

$$f(x_2) - f(x_1) > 0$$

$$\Rightarrow f(x_2) > f(x_1).$$

Therefore by definition f is increasing.

For a decreasing function over an interval $[x_1, x_2]$, the proof is similar.

Example: Find the intervals in which the function f defined by

$$f(x) = x^3 - 6x^2 + 9x - 7.$$

is (a) increasing

(b) decreasing.

Solution: $f(x) = x^3 - 6x^2 + 9x - 7$. Differentiating, we get

$$f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3)$$

or $f'(x) = 3(x - 1)(x - 3)$.

Now, $f'(x)$ is zero only for the values $x = 1$ and $x = 3$. The two points where the derivative is zero divides the x -axis into three intervals:

$$-\infty < x < 1, 1 < x < 3 \text{ and } 3 < x < \infty.$$

If $x < 1$, then $x < 3 \Rightarrow x - 1 < 0$ and $x - 3 < 0$. Hence $(x - 1)(x - 3) > 0$.

Thus,

$$f'(x) = 3(x - 1)(x - 3) > 0.$$

and f is increasing.

Similarly, if $x > 3$, then $x > 1 \Rightarrow x - 1 > 0$ and $x - 3 > 0$, implying that $(x - 1)(x - 3) > 0$. Thus,

$$f'(x) = 3(x - 1)(x - 3) > 0.$$

and f is increasing.

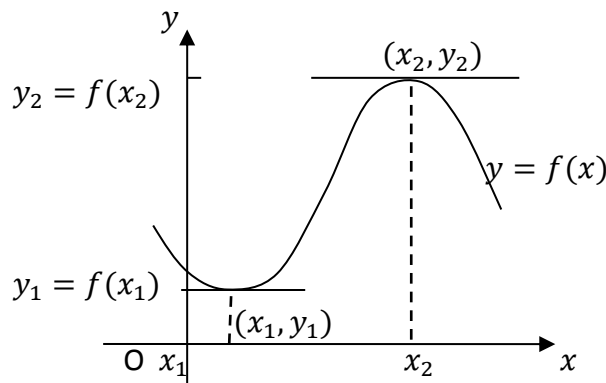
If $1 < x < 3$, then $x - 1 > 0$ and $x - 3 < 0$ and therefore

$(x - 1)(x - 3) < 0 \Rightarrow f'(x) = 3(x - 1)(x - 3) < 0$. Hence, in the interval $1 < x < 3$ f is decreasing.

Therefore, f is increasing in the intervals $-\infty < x < 1$ and $3 < x < \infty$, and is decreasing in the interval $1 < x < 3$.

2.5.4 Stationary points

A stationary value of a function $f(x)$ is any value of $f(x)$ at which its rate of change with respect to x is zero. i.e. stationary values of $f(x)$ occur when $\frac{d}{dx}(f(x)) = 0$.



In the diagram the function has stationary values at $f(x_1)$ and $f(x_2)$.

The values of x at which the function f attains its stationary values are

called **critical values** or **critical points**.

The points (x_1, y_1) and (x_2, y_2) are called **stationary points**.

At critical values, the gradient of the curve is zero. Thus, at critical values $\frac{d}{dx}(f(x)) = 0$.

Example Find the critical values of the function

$$f(x) = 2x^3 - 3x^2 - 36x + 5.$$

Hence, state the stationary value of the function.

Solution: Differentiating w.r.t. x we get

$$f'(x) = 6x^2 - 6x - 36.$$

Thus, the critical values satisfy the equation

$$f'(x) = 6x^2 - 6x - 36 = 0$$

or $x^2 - x - 6 = 0$

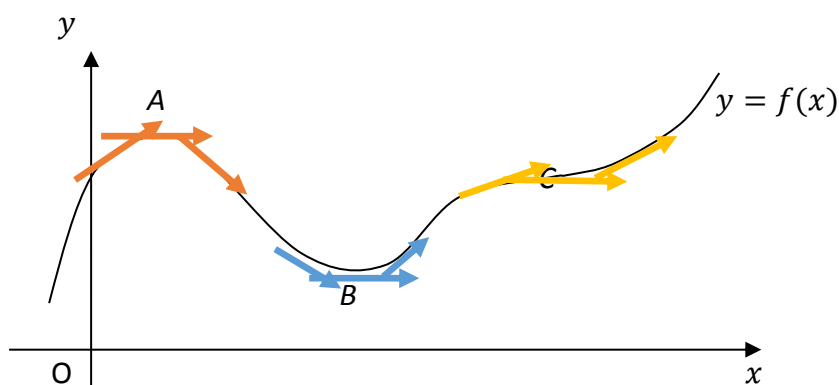
or $(x + 2)(x - 3) = 0$

Therefore, the critical values are $x = -2$ and $x = 3$.

Hence, the stationary values of the function are $f(-2) = 49$ and

$$f(3) = -76.$$

A curve may have several stationary points which are also called **turning points** of the curve. These occur in different categories:



Nature of Stationary Values or Points

Method 1 (First Derivative test)

Let a curve be defined by $y = f(x)$. When moving through a stationary point if

- (a) $\frac{dy}{dx} > 0$ followed by $\frac{dy}{dx} = 0$ and then $\frac{dy}{dx} < 0$, then the stationary point is maximum;

(b) $\frac{dy}{dx} < 0$ followed by $\frac{dy}{dx} = 0$ and then $\frac{dy}{dx} > 0$, then the stationary point is minimum;

(c) $\frac{dy}{dx} > 0$ followed by $\frac{dy}{dx} = 0$ and then $\frac{dy}{dx} > 0$ or $\frac{dy}{dx} < 0$ followed by $\frac{dy}{dx} = 0$ and then $\frac{dy}{dx} < 0$, then the turning point is point of inflexion.

	Maximum	Minimum	Inflexion
Sign of $\frac{dy}{dx}$ when moving through a stationary point			

Example: Find the relative maxima and the relative minima for the function

$$f(x) = (x^2 - 4)^2.$$

Solution: At critical values: $f'(x) = 2(x^2 - 4) \times 2x = 0$.

i.e.

$$x(x + 2)(x - 2) = 0$$

Thus, the critical values are $x = 0, x = -2$ and $x = 2$.

Next, we look at the signs of the values of $f'(x)$ in the neighbourhood of these critical values.

For the critical value $x = -2$, we shall consider the signs of the values $f'(-\frac{5}{2})$ and $f'(-\frac{3}{2})$.

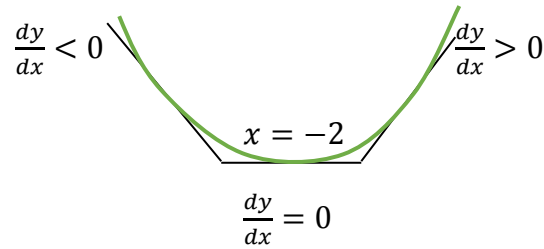
$$\text{At } x = -\frac{5}{2}, f'(-\frac{5}{2}) = 2 \left[\left(-\frac{5}{2}\right)^2 - 4 \right] \times 2 \left(-\frac{5}{2}\right) = -\frac{45}{2} < 0.$$

$$\text{At } x = -2, f'(-2) = 0.$$

$$\text{At } x = -\frac{3}{2}, f'(-\frac{3}{2}) = 2 \left[\left(-\frac{3}{2}\right)^2 - 4 \right] \times 2 \left(-\frac{3}{2}\right) = \frac{27}{2} > 0.$$

Moving along the curve from $-\frac{5}{2}$ through $x = -2$ to $-\frac{3}{2}$ the gradient of the curve is - followed by 0 and then +. This means that the function has a relative minimum at $x = -2$ and this is $f(-2) = ((-2)^2 - 4)^2 = 0$

The shape of the curve in the neighbourhood of $x = -2$ is as follows:



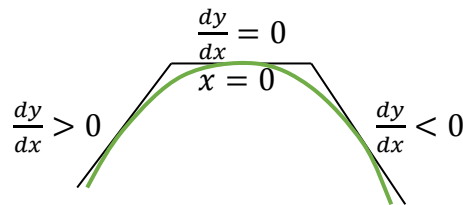
For the critical value $x = 0$, we shall consider the signs of the values $f'(-\frac{1}{2})$ and $f'(\frac{1}{2})$.

$$\text{At } x = -\frac{1}{2}, f'(-\frac{1}{2}) = 2 \left[\left(-\frac{1}{2}\right)^2 - 4 \right] \times 2 \left(-\frac{1}{2}\right) = \frac{15}{2} > 0.$$

$$\text{At } x = 0, f'(0) = 0.$$

$$\text{At } x = \frac{1}{2}, f'(\frac{1}{2}) = 2 \left[\left(\frac{1}{2}\right)^2 - 4 \right] \times 2 \left(\frac{1}{2}\right) = -\frac{15}{2} < 0.$$

Moving along the curve from $-\frac{1}{2}$ through $x = 0$ to $\frac{1}{2}$ the gradient of the curve is positive, followed by 0 and then negative. This means that the function has a relative maximum at $x = 0$ and the relative maximum is $f(0) = (0^2 - 4)^2 = 16$.



For the critical value $x = 2$, we shall consider the signs of the values

$$f'(\frac{3}{2}) \text{ and } f'(\frac{5}{2}).$$

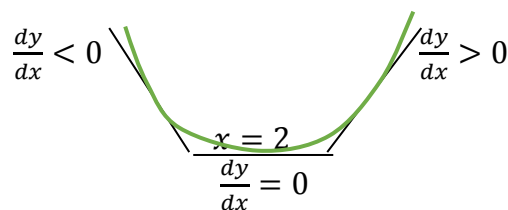
$$\text{At } x = \frac{3}{2}, f'(\frac{3}{2}) = 2 \left[\left(\frac{3}{2}\right)^2 - 4 \right] \times 2 \left(\frac{3}{2}\right) = -\frac{27}{2} < 0.$$

$$\text{At } x = 2, f'(2) = 0.$$

$$\text{At } x = \frac{5}{2}, f'(\frac{5}{2}) = 2 \left[\left(\frac{5}{2}\right)^2 - 4 \right] \times 2 \left(\frac{5}{2}\right) = \frac{45}{2} > 0.$$

Moving along the curve from $\frac{3}{2}$ through $x = 2$ to $\frac{5}{2}$ the gradient of the curve is negative, followed by 0 and then positive. This means that the function has a relative minimum at $x = 2$ and this is $f(2) = ((2)^2 - 4)^2 = 0$

The shape of the curve in the neighbourhood of $x = 2$ is as follows:



Note that the points $(-2,0)$ and $(2,0)$ are minimum turning points and the point $(0,16)$ is the maximum turning point for given curve.

Method 2 (The Second Derivative test)

Let the function be defined by $y = f(x)$. Then $\frac{dy}{dx} = f'(x)$ is the first derivative of y w.r.t. x , $\frac{d^2y}{dx^2} = f''(x)$ is the second derivative of y w.r.t. x , and $\frac{d^3y}{dx^3} = f'''(x)$ is the third derivative of y w.r.t. x . In general, $\frac{d^n y}{dx^n} = f^{(n)}(x)$ is the n th derivative of y w.r.t. x .

Example Find the first, second and the third derivatives of the function $y = x^2 + \frac{23}{x}$, with respect to x .

Solution: The first derivative of y w.r.t. x is $\frac{dy}{dx} = 2x - \frac{23}{x^2}$, the second derivative of y w.r.t. x is $\frac{d^2y}{dx^2} = 2 + \frac{46}{x^3}$ and the third derivative of y w.r.t. x is $\frac{d^3y}{dx^3} = -\frac{138}{x^4}$.

For the curve $y = f(x)$ we can also determine the nature of the stationary value (or the turning point) at critical values by finding the value of $\frac{d^2y}{dx^2} = f''(x)$, and where necessary, the value of $\frac{d^3y}{dx^3} = f'''(x)$.

This is because the second derivative, $\frac{d^2y}{dx^2} = f''(x)$, measures the change in gradient.

1. If $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$, the critical value yields a minimum stationary value of the function and hence it yield a minimum turning point.
2. If $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$, the critical value yields a maximum stationary value of the function and hence it yields a maximum turning point.
3. If $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$, the critical value yields a minimum or maximum turning point or point of inflexion.

In this case we need to use method 1 to determine the nature of the stationary value.

Example: Determine the nature of the stationary values for the curve $y = x^4 - 9$.

Solution: $\frac{dy}{dx} = 4x^3 = 0 \Rightarrow x = 0$ is the only critical value.

At this critical value, $\frac{d^2y}{dx^2} = 0$.

Moving through $-\frac{1}{2}$, 0 and then $\frac{1}{2}$ we have:

$$\text{At } x = -\frac{1}{2}, \frac{dy}{dx} = 4\left(-\frac{1}{2}\right)^3 = -\frac{1}{2} < 0;$$

$$\text{At } x = 0, \frac{dy}{dx} = 4(0)^3 = 0;$$

$$\text{At } x = \frac{1}{2}, \frac{dy}{dx} = 4\left(\frac{1}{2}\right)^3 = \frac{1}{2} > 0.$$

Therefore, the curve has a minimum stationary value at $x = 0$, and this $y = 0^4 - 9 = -9$.

Note that though at the point $x = 0$, $\frac{d^2y}{dx^2} = 0$, and in this case the critical value does not yield a point of inflexion but a minimum turning point.

4. If $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$, but $\frac{d^3y}{dx^3} \neq 0$, the critical value yields a point of inflexion.

Example: Determine the nature of the turning point for the curve $y = x^3 - 3x^2 + 3x$.

Solution: $\frac{dy}{dx} = 3x^2 - 6x + 3 = 0 \Rightarrow 3(x - 1)^2 = 0 \Rightarrow x = 1$. The curve only has one critical value.

$$\frac{d^2y}{dx^2} = 6x - 6, \text{ and at } x = 1, \frac{d^2y}{dx^2} = 6(1) - 6 = 0 \text{ and } \frac{d^3y}{dx^3} = 6 \neq 0.$$

Therefore, at $x = 1$, the curve has a point of inflexion and this point of inflexion is (1,1).

To confirm that indeed this is a point of inflexion, we use method 1.

We find the values of $\frac{dy}{dx}$ at $x = -\frac{1}{2}$, $x = 1$ and $x = \frac{3}{2}$.

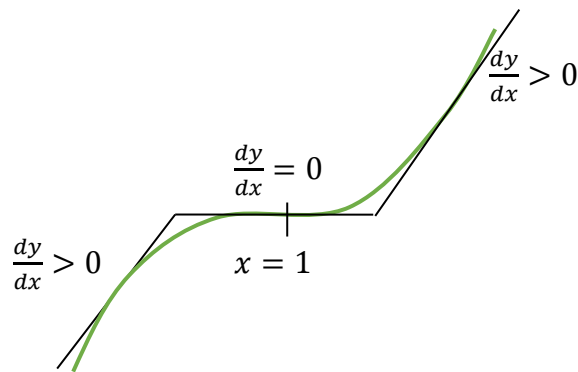
$$\text{At } x = \frac{1}{2}, \frac{dy}{dx} = \frac{3}{4} - 3 + 3 = \frac{3}{4} > 0;$$

$$\text{At } x = 1, \frac{dy}{dx} = 3 - 6 + 3 = 0;$$

$$\text{At } x = \frac{3}{2}, \frac{dy}{dx} = \frac{27}{4} - 9 + 3 = \frac{3}{4} > 0.$$

Moving along the curve from $\frac{1}{2}$ through $x = 1$ to $\frac{3}{2}$ the gradient of the curve is + followed by 0 and then +. This means that the curve has a point of inflexion at $x = 1$.

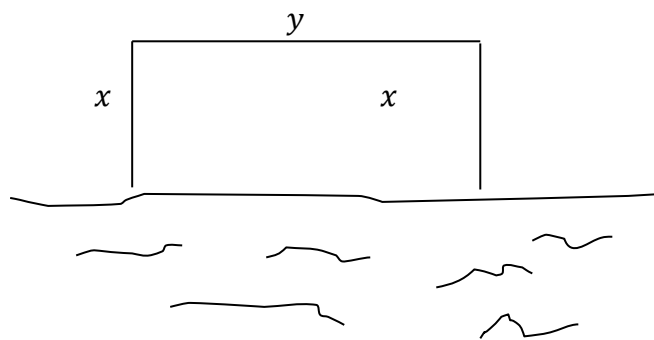
Note that the shape of the curve in the neighbourhood of $x = 1$ is as shown in the diagram below:



2.4.5 Applications of Maxima and Minima

Example: A farmer having 120m of fencing wishes to contain a cow in a rectangular plot of land along the bank of a river. What should the dimensions of the rectangle be to provide the cow with maximum grazing ground? (Assume that no fencing is needed along the river, which is flowing along a straight edge.)

Solution:



Notice that

$$2x + y = 120$$

$$\Rightarrow y = 120 - 2x \quad (1)$$

Area of the rectangle is

$$A = xy \quad (2)$$

Replacing (1) into (2) yields

$$A = x(120 - 2x) = 120x - 2x^2$$

$$\text{or} \quad A = 120x - 2x^2$$

To maximise the area A , we need to differentiate A w.r.t. x and find the critical values.

$$\frac{dA}{dx} = 120 - 4x = 0$$

$$\Rightarrow x = 30.$$

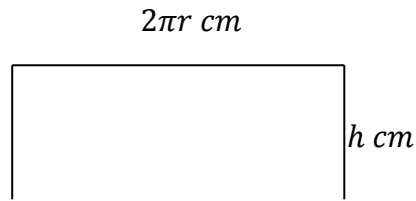
Applying the second derivative test, we have

$$\frac{d^2A}{dx^2} = -4 < 0$$

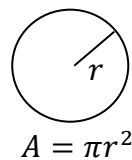
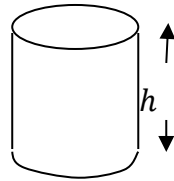
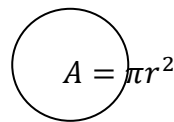
This means that $x = 30$ yields the maximum area and hence, the largest grazing ground. Therefore the dimension of the rectangle are $30m \times 60m$.

Example: A soup manufacturing company wishes to pack 25 cm^3 of mushroom soup in a can in the form of a right circular cylinder. Find the dimensions of the can if the surface area is to be minimum.

Solution: Let the dimensions of the rectangle be $2\pi r \times h$.



Curved surface area of a cylinder is $2\pi rh$.



The Volume of the cylinder is

$$V = \pi r^2 h = 25$$

i.e.
$$h = \frac{25}{\pi r^2}.$$

Surface area of the metal used is

$$A = 2\pi rh + 2\pi r^2$$

i.e.
$$A = 2\pi r \left(\frac{25}{\pi r^2} \right) + 2\pi r^2$$

or
$$A = \frac{50}{r} + 2\pi r^2.$$

Thus, to minimize A, we need to find the critical values, by first differentiating A w.r.t. r and equating the derivative to zero.

$$\frac{dA}{dr} = -\frac{50}{r^2} + 4\pi r = 0$$

$$\Rightarrow r^3 = \frac{25}{2\pi}$$

$$\Rightarrow r = \sqrt[3]{\frac{25}{2\pi}}$$

The only critical value is $r = \sqrt[3]{\frac{25}{2\pi}}$.

Applying the second derivative we find

$$\frac{d^2A}{dr^2} = \frac{100}{r^3} + 4\pi.$$

$$\text{At } r = \sqrt[3]{\frac{25}{2\pi}}, \quad \frac{d^2A}{dr^2} = \frac{100}{\left(\sqrt[3]{\frac{25}{2\pi}}\right)^3} + 4\pi = 8\pi + 4\pi = 12\pi > 0.$$

Therefore, A is minimum when $r = \sqrt[3]{\frac{25}{2\pi}}$. The corresponding value of h is $h = \frac{25}{\pi \left(\sqrt[3]{\frac{25}{2\pi}}\right)^2} = \frac{25}{\pi} \left(\frac{2\pi}{25}\right)^{\frac{2}{3}}$.

2.5.6 Curve Sketching and Graphs of Rational Functions

1. Graphs of Polynomial Functions

Example: Sketch the graph of $y = x^4 - 4x^3 + 4x^2 + 1$.

Solution: $\frac{dy}{dx} = 4x^3 - 12x^2 + 8x = 4x(x^2 - 3x + 2) = 4x(x - 1)(x - 2)$.

The critical points are $x = 0, x = 1$ and $x = 2$.

$$\frac{d^2y}{dx^2} = 12x^2 - 24x + 8 = 4(3x^2 - 6x + 2).$$

At $x = 0, \frac{d^2y}{dx^2} = 8 > 0 \Rightarrow x = 0$ yields a minimum turning point, which is (0,1).

At $x = 1, \frac{d^2y}{dx^2} = -4 < 0 \Rightarrow x = 1$ yields a maximum turning point, which is (1,2).

At $x = 2, \frac{d^2y}{dx^2} = 8 > 0 \Rightarrow x = 2$ yields a minimum turning point, which is (2,1).

To find points of inflexions we set the second derivative to zero and find the critical points which yields these points. Thus

$$\frac{d^2y}{dx^2} = 0 \text{ when } 3x^2 - 6x + 2 = 0 \text{ i.e. when}$$

$$x = \frac{6 \pm \sqrt{36-24}}{6} = \frac{6 \pm \sqrt{12}}{6} = 1 \pm \frac{1}{\sqrt{3}}$$

$$\frac{d^3y}{dx^3} = 24x - 24.$$

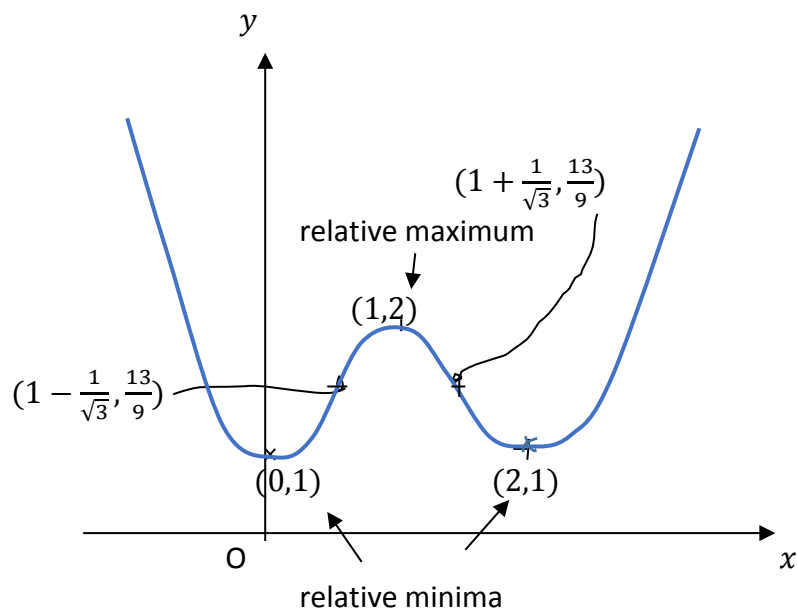
$$\text{At } x = 1 + \frac{1}{\sqrt{3}}, \frac{d^3y}{dx^3} = 24 \left(1 + \frac{1}{\sqrt{3}}\right) - 24 = 24 + \frac{24}{\sqrt{3}} - 24 = \frac{24}{\sqrt{3}} \neq 0$$

$\Rightarrow x = 1 + \frac{1}{\sqrt{3}}$ yields a point of inflexion, which is $(1 + \frac{1}{\sqrt{3}}, \frac{13}{9})$.

$$\text{At } x = 1 - \frac{1}{\sqrt{3}}, \frac{d^3y}{dx^3} = 24 \left(1 - \frac{1}{\sqrt{3}}\right) - 24 = 24 - \frac{24}{\sqrt{3}} - 24 = -\frac{24}{\sqrt{3}} \neq 0$$

$\Rightarrow x = 1 - \frac{1}{\sqrt{3}}$ yields the other point of inflexion, which is $(1 - \frac{1}{\sqrt{3}}, \frac{13}{9})$.

Finally, plot these points and sketch the curve which passes through the points.

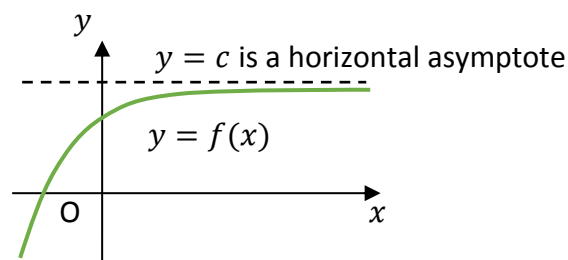


2. Graphs of Rational Functions

Curve sketching by looking at what happens either as $x \rightarrow \pm\infty$ or when $f(x) \rightarrow \pm\infty$ as approaches a finite value.

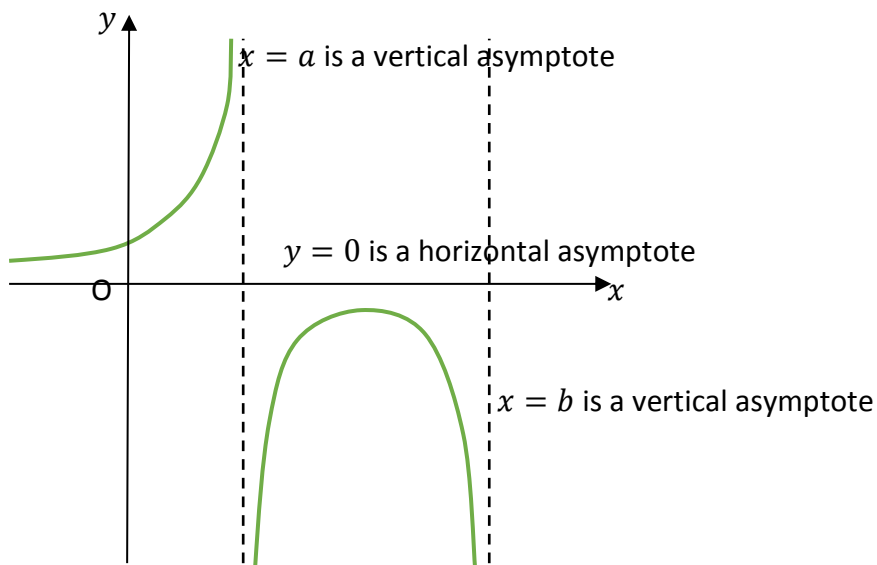
Definition (Horizontal Asymptote)

Let c be a real number. Suppose $f(x) \rightarrow c$ as $x \rightarrow \infty$ or as $x \rightarrow -\infty$. Then the horizontal line $y = c$ is called a **horizontal asymptote** of the graph of f .



Definition (Vertical Asymptote)

Let a be a real number. Suppose $f(x) \rightarrow \pm\infty$ as $x \rightarrow a^+$ or as $x \rightarrow a^-$. Then the vertical line $x = a$ is called a **vertical asymptote** of the graph of f .



Example: Sketch the graph of $y = \frac{x}{1+x}$.

Solution: Note that the function is not defined at $x = -1$.

Now we observe the following:

If $x < -1$, then $y = \frac{x}{1+x} > 0$, (since the numerator is negative and the denominator is negative.)

If $-1 < x < 0$, then $y = \frac{x}{1+x} < 0$.

If $x = 0$, then $y = \frac{x}{1+x} = 0$

If $x > 0$, then $0 < y = \frac{x}{1+x} < 1$.

Furthermore,

$\lim_{x \rightarrow -1^+} \frac{x}{1+x} = -\infty$ (since the numerator is negative and the denominator is positive) and

$\lim_{x \rightarrow -1^-} \frac{x}{1+x} = \infty$ (since the numerator and the denominator are both negative).

Thus, the line $x = -1$ is a vertical asymptote.

Also,

$$\lim_{x \rightarrow \infty} \frac{x}{1+x} = \lim_{x \rightarrow \infty} \frac{1}{\left(\frac{1}{x}\right)+1} = 1.$$

and

$$\lim_{x \rightarrow -\infty} \frac{x}{1+x} = \lim_{x \rightarrow \infty} \frac{1}{\left(\frac{1}{x}\right)+1} = 1$$

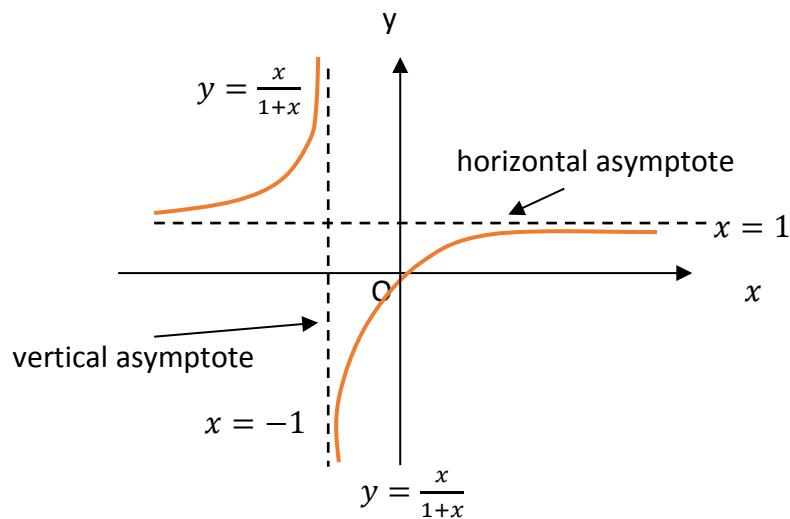
Therefore, as $x \rightarrow \pm\infty, y \rightarrow 1$. Thus, the line $y = 1$ is a horizontal asymptote for the function $y = \frac{x}{1+x}$.

$$\text{Now } \frac{dy}{dx} = \frac{(1+x)(1)-x(1)}{(1+x)^2} = \frac{1}{(1+x)^2}.$$

Clearly, $\frac{dy}{dx} > 0 \forall x \in \mathbb{R}, x \neq -1 \Rightarrow$ the function is always increasing (except at $x = -1$) where it is not defined.

$$\frac{d^2y}{dx^2} = \frac{d}{dx} (1+x)^{-2} = -\frac{-2}{(1+x)^3} \neq 0 \forall x \in \mathbb{R} \text{ except } x = -1.$$

This means that the curve has no critical points.



TUTORIAL SHEET 14

- Find the derivative of each of the following functions from first principle:
(a) $\sin 2x$ (b) $\cos 3x$.
- Differentiate with respect to x :
(a) $\frac{\sin x + \cos x}{\sin x - \cos x}$ (b) $\frac{\sin(3x+1)}{\cos(2x+3)}$ (c) $\tan^4(7x^2 + 3x + 9)$
(d) $\sin^2(2x) \times \cos^3(5x)$.
- Differentiate each of the following functions:
(a) $f(x) = e^{3-4x}$ (b) $f(x) = e^{\sin x + \cos 2x}$ (c) $f(t) = e^t \sin^2(\cos t)$
(d) $f(u) = \cos(e^{3u})$ (e) $f(x) = 6^{5x}$ (f) $f(x) = 2^{x^2} 5^{x-1}$
- For each of the following implicit functions, find $\frac{dy}{dx}$:
(a) $y + xy + y^2 = 2$ (b) $x^2 + xy^2 + y^3 = 2$ (c) $x = ye^x$ (d) $\frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{4}$
(e) $\sin x \cos y = 2$ (f) $xe^y = x + 1$.
- Differentiate each of the following functions:
(a) $f(x) = \log_e(5x + 1)$ (b) $f(x) = \ln(x^2 \sin x)$ (c) $f(t) = \log_3(x^2 + e^x)$
(d) $f(u) = e^{x \ln x}$ (e) $f(x) = \log_2[\log_5(\sqrt{x^2 + 1})]$.
- Use the rules of logarithms to simplify the expression and find $f'(x)$:
(a) $f(x) = \ln \frac{(x+1)^{16}(2x^2+x)^8}{\sqrt{x^2+4}}$ (b) $f(x) = \ln \frac{(e^{2x} + 6)^7 \sqrt{x+4}}{(e^{-x} + e^x)^5}$.
- For each of the given functions, find $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$ and $\frac{d^4y}{dx^4}$:
(a) $y = 2x^4 - 3x^2$ (b) $\sin x + \cos y = 7$.
- Find the indicated value:
(a) $f(x) = \sqrt{4-x}$, $f''(-5)$ (b) $f(x) = (x^3 - 2x)^3$, $f''(1)$
(c) $f(t) = \sqrt{2t+3}$, $f'''(\frac{1}{2})$.
- If $x^3y = 2$, find the value of $\frac{d^2y}{dx^2}$ when $x = 1$.
- Let f be a function defined by $f(x) = 2x - x^2$.
(a) Find the equation of the tangent and normal line to the graph of f at the point

$$\left(\frac{1}{2}, \frac{3}{4}\right).$$

(b) At what point on the curve is the tangent line parallel to the x – axis?

11. Find the point at which the slope of the tangent line to the curve $y = 5 + 3x - x^3$ is 3.
12. Find the equation of the tangent line to the curve $2y^2 - 3x^2 = 6$ at point (2,3).
13. Find where f is increasing and where it is decreasing, and sketch the graph of f :
- (a) $f(x) = x^3 - 3x^2 - 9x + 15$ (b) $f(x) = 4x^4 + x^2$.

14. Suppose the rate R of an autocatalytic reaction is given by

$$R = kx(4 - x), \quad 0 \leq x \leq 4$$

where k is a positive constant and x is the amount of substance produced. Find the interval where the rate R is increasing and where it is decreasing. Sketch the graph.

15. Find the critical values and determine the relative maxima and the relative minima. Find the absolute extrema if any exist.

(a) $f(x) = x^3 + 3x^2 - 12x + 7, \quad -\infty < x < \infty$

(b) $f(x) = 2x^4 - x^2, \quad -1 \leq x \leq 1$.

16. If the selling price x is related to the profit P by the equation

$$P = 5000x - 125x^2$$

- (a) For what range of values of x is the profit increasing?
- (b) For what range of values of x is the profit decreasing?
- (c) Determine the value of x that would yield maximum profit.

17. Find the turning points on $y = 3x^4 + 4x^3 - 12x^2$. Give a rough sketch of the curve.

18. An open rectangular box is made from a square sheet of cardboard by removing a square from each corner and joining the cut edges. If the cardboard is of edge $0.5 m$, find the maximum volume of the box.

19. The concentration of hydrogen ion in a solution is given by $X = H + \frac{10^{-5}}{H}$. For what value of H is the concentration a minimum?

20. Sketch the graph of each given function showing clearly any asymptotes, turning points, points of intersection with the coordinate axes and the behavior of the curve when x and /or y are very large:

(a) $y = \frac{1}{1-x}$ (b) $y = \frac{x-2}{x-4}$ (c) $y = \frac{x}{(1-x)^2}$ (d) $y = \frac{x-2}{(x+1)(x-1)}$.