MAT1100 LECTURE NOTES

11 FURTHER INTEGRAL CALCULUS

11.1 Further Integration formulae

Most of these important further integration formulae emanate from the fact that integration is the reverse process of differentiation.

1. $\int e^x dx = e^x + c$, since $\frac{d}{dx}(e^x) = e^x$.

2. $\int a^x dx = \dfrac{a^x}{\ln a} + c,\ a > 0,\ a \neq 1$, since $\frac{d}{dx}\left(\frac{1}{\ln a} a^x\right) = a^x$.

3. $\int \sin x dx = -\cos x + c$, since $\frac{d}{dx}(-\cos x) = \sin x$.

4. $\int \cos x dx = \sin x + c$, since $\frac{d}{dx}(\sin x) = \cos x$.

5. $\int \tan x dx = \ln|\sec x| + c$, since

$$\int \tan x\, dx = -\int \frac{-\sin x}{\cos x} dx = -\ln|\cos x|.$$

$$= \ln|(\cos x)^{-1}| = \ln\left|\frac{1}{\cos x}\right| = \ln|secx|.$$

6. $\int \cot x dx = \ln|\sin x| + c$, since

$$\int \cot x\, dx = \int \frac{\cos x}{\sin x} dx = \ln|\sin x|.$$

7. $\int \sec^2 x dx = \tan x + c$, since $\frac{d}{dx}(\tan x) = \sec^2 x$.

8. $\int \csc^2 x dx = -\cot x + c$, since $\frac{d}{dx}(\cot x) = -csc^2 x$.

9. $\int \sec x dx = \ln|\sec x + \tan x| + c$, since

$$\int \frac{\sec x + \tan x}{\sec x + \tan x} \sec x\, dx = \int \frac{sec^2 x + \sec x \tan x}{\sec x + \tan x} dx = \ln|\sec x + \tan x|.$$

10. $\int \csc x dx = \ln|\csc x - \cot x| + c$, since

$$\int \frac{\csc x - \cot x}{\csc x - \cot x} \csc x\, dx = \int \frac{csc^2 x - \csc x \cot x}{\csc x - \cot x} dx = \ln|\csc x - \cot x|.$$

11. $\int \sec x \tan x dx = \sec x + c$, since $\frac{d}{dx}(\sec x) = \sec x \tan x$.

12. $\int \csc x \cot x dx = -\csc x + c$, since $\frac{d}{dx}(-\csc x) = \csc x \cot x$

13. $\int \dfrac{1}{1+x^2} dx = \arctan x + c$, since $\frac{d}{dx}(\arctan x) = \frac{1}{x^2+1}$.

14. $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + c$, since $\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$.

11.2 Integration Methods

Apart from the above integration formulae there are other various methods used in the evaluation of indefinite integrals. These are

1. Integration by Substitution

To evaluate the integral $\int f(g(x))g'(x)dx$, it is often useful to substitute $g(x)$ with a new variable, say u, so that $u = g(x) and\ du = g'(x)dx$, and the integral becomes

$$\int f(g(x))g'(x)dx = \int f(u)du, \qquad (11.2.1)$$

which is easier to evaluate. After finding the right side of (11.2.1), replace *u* with $g(x)$ to obtain the results in terms of x.

Example 1 Find $\int \frac{2x}{(x^2+5)^{\frac{3}{2}}}dx$.

Solution: Here, note that $g(x) = x^2 + 5$, $f(g(x) = \frac{1}{(g(x))^{3/2}}$.

Thus we let, $u = x^2 + 5$ so that $du = 2xdx$.

$$\Rightarrow \int \frac{2x}{(x^2+5)^{\frac{3}{2}}}dx = \int \frac{1}{u^{\frac{3}{2}}}du = \int u^{-\frac{2}{3}}du = 3u^{\frac{1}{3}} + c = 3(x^2+5)^{\frac{1}{3}} + c.$$

Example 2 Evaluate the integral $\int \sin^5 x \cos x dx$.

Solution: Here, note that $g(x) = \sin x$, $f(g(x)) = (\sin x)^5$, and

$g'(x) = \cos x\, dx$. Hence, letting $= \sin x$, $du = \cos dx$.

$$\Rightarrow \int \sin^5 x \cos x dx = \int u^5 du = \frac{1}{6}u^5 + c = \frac{1}{6}\sin^6 x + c.$$

Example 3 Evaluate each of the following the integrals:

(a) $\int x\sqrt{x+1}\,dx$.

Solution: Let $x+1 = u^2$ so that $x = u^2 - 1$. Then $dx = 2udu$.

Thus,

$$\int x\sqrt{x+1}\,dx = 2\int (u^2-1)u^2 du = 2\int (u^4 - u^2)du$$
$$= \frac{2}{5}u^5 - \frac{2}{3}u^3 + c = \frac{2}{5}\left(\sqrt{x+1}\right)^5 - \frac{2}{3}\left(\sqrt{x+1}\right)^3 + c$$

(b) $\int \frac{x}{\sqrt{3-x}}\,dx$.

Solution: Let $3-x=u^2$ so that $x=3-u^2$. Then $dx=-2udu$.
Thus,

$$\int \frac{x}{\sqrt{3-x}}dx = -2\int \frac{3-u^2}{u}du = -2\int\left(\frac{3}{u}-u\right)du$$

$$= -6\ln|u| + u^2 + c = -6\ln\sqrt{3-x} + (3-x) + c$$

(c) $\int (x+1)(x+3)^5\,dx$.

Solution: Let $x+3=u$ so that $x+1=u-2$. Then $dx=du$.
Thus,

$$\int (x+1)(x+3)^5 dx = \int (u-2)u^5 du = \int \left(u^6 - 2u^5\right)du$$

$$= \frac{1}{7}u^7 - \frac{2}{6}u^6 + c = \frac{1}{7}(x+3)^7 - \frac{1}{3}(x+3)^6 + c$$

2. Quick Integration by Inspection

We have two simple formulae which enable us to find integrals almost immediately.

(a) The first is

$$\int g'(x)[g(x)]^r dx = \frac{1}{r+1}[g(x)]^{r+1} + c,\ r \neq -1 \qquad (11.2.2)$$

This formula is justified by noting that

$$\frac{d}{dx}\left\{\frac{1}{r+1}[g(x)]^{r+1}\right\} = g'(x)[g(x)]^r .$$

<u>Example</u> 6.2.3 Find $\int 6x^2\left(2x^3+5\right)^3 dx$.

<u>Solution</u>: Here note that $g(x)=2x^3+5$ and $g'(x) = 6x^2$. Therefore

$$\int 6x^2\left(2x^3+5\right)^3 dx = \frac{1}{4}\left(2x^3+5\right)^4 + c .$$

(b) The second quick integration formula is

$$\int \frac{g'(x)}{g(x)}dx = \ln|g(x)| + c \ . \qquad (11.2.3)$$

This formula is justified by noting that

$$\frac{d}{dx}\{\ln|g(x)|\} = \frac{g'(x)}{g(x)}.$$

<u>Example</u>: 2.1 Find $\int \frac{1}{x\ln x} dx$.

<u>Solution</u>: Note that $g(x) = \ln x$ and $g'(x) = \frac{1}{x}$.

Therefore, $\int \frac{1}{x\ln x} dx = \ln(\ln x) + c$.

Note that both formulae emanate from the substitution method.

3. Integration by Parts

Suppose f and g are differentiable functions of x. Then by the rule of the derivative of products we have

$$(f.g)' = f.g' + f'.g \tag{11.2.4}$$

Integrating both sides of equation (11.2.4) with respect to x yields

$$\int \frac{d}{dx}[f(x).g(x)]\,dx = \int [f(x).g'(x) + f'(x).g(x)]dx \tag{11.2.5}$$

i.e.

$$f(x).g(x) = \int [f(x).g'(x)]dx + \int [f'(x).g(x)]dx \tag{11.2.6}$$

Thus, from (3.3) we have the **integration by parts** formula which is written as

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx. \tag{11.2.7}$$

The formula is easily remembered when the following substitutions are made:

$$u = f(x) \text{ and } dv = g'(x)dx \Rightarrow du = f'(x)dx \text{ and } v = \int g'(x)dx.$$

These substitutions transforms (3.4) to

$$\boxed{\int udv = uv - \int vdu} \tag{11.2.8}$$

<u>Example</u>: 3.1 Find $\int x^2 e^x dx$.

<u>Solution</u>: Let $u = x^2$ and $dv = e^x dx$. $\Rightarrow du = 2x$ and $v = \int e^x dx = e^x$.

Thus, $\int x^2 e^x dx = uv - \int vdu = x^2 e^x - 2\int xe^x dx$.

To evaluate $\int xe^x dx$ we use integration by parts again.

Thus, we let $u = x$ and $dv = e^x dx$. $\Rightarrow du = dx$ and $v = e^x$.

$\Rightarrow \int xe^x dx = uv - \int vdu = xe^x - \int e^x\, dx = xe^x - e^x + c_1$

Therefore,

$$\int x^2 e^x\, dx = x^2 e^x - 2(xe^x - e^x + c_1)$$

$$= x^2 e^x - 2(xe^x - e^x) + c$$

$$= x^2 e^x - 2e^x(x - 1) + c$$

Now to determine which part of the integral is u and the other dv in the application of formula (3.5) the following two rules may be used:

(i) The part selected as dv must be easily integrated.

(ii) $\int vdu$ must not be more complicated than $\int udv$.

However, the following is the list of common integrals with suggestions for the choices of u and dv:

1. For integrals of the form

$$\int x^n e^x dx,\ \int x^n \sin x\, dx,\ \int x^n \cos x\, dx,$$

let $u = x^n$ and let $dv = e^x dx, dv = \sin x dx\ or\ dv = \cos x\, dx$.

2. For integral of the form

$$\int x^n \ln x\, dx,\ \int x^n \arcsin x\, dx,\ \int x^n \arccos x\, dx,$$

let $u = \ln x, \arcsin x\ or\ \arccos x$ and $dv = x^n dx$.

<u>Example </u>3.2 Find $\int x^2 \ln x dx$.

<u>Solution</u>: Since the derivative of $\ln x$ is a power of x, let $u = \ln x$ and $dv = x^2 dx$ so that $du = \frac{1}{x} dx$ and $v = \frac{1}{3} x^3$.

Applying equation (3.5) gives

$$\int x^2 \ln x\, dx = \tfrac{1}{3} x^3 \ln x - \tfrac{1}{3} \int x^3 . \tfrac{1}{x} dx = \tfrac{1}{3} x^3 \ln x - \tfrac{1}{3} \int x^2\, dx$$

$$= \frac{1}{3}x^3 \ln x - \frac{1}{3}.\frac{1}{3}x^3 + c$$

$$= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + c$$

Example 3.3 Find $\int \arcsin x \, dx$.

Since it is easier to differentiate $\arcsin x$ than integrating it,

let $u = \arcsin x$ and $dv = dx$ so that $du = \frac{1}{\sqrt{1-x^2}}dx$ and $v = x$.

Thus,

$\int x \arcsin x \, dx = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} dx.$

To evaluate $\int \frac{x}{\sqrt{1-x^2}} dx$ use the substitution method. Let $t = 1 - x^2$

so that $dt = -2xdx$ or $xdx = -\frac{1}{2}dt$. Thus,

$\int \frac{x}{\sqrt{1-x^2}} dx = -\frac{1}{2}\int \frac{1}{\sqrt{t}} dt = -\frac{1}{2}\int t^{-\frac{1}{2}} dt = -\frac{1}{2}\left(\frac{2}{1}\right) t^{\frac{1}{2}} = -t^{\frac{1}{2}} = -\sqrt{1-x^2}.$

Therefore,

$$\int x \arcsin x \, dx = x \arcsin x + \sqrt{1-x^2} + c.$$

3. For integrals of the form

$$\int e^x \sin x \, dx \; or \int e^x \cos x \, dx,$$

let u to be either e^x or $\sin x$ ($\cos x$). If you take $u = e^x$ then

$dv = \sin x dx$ (or $\cos x \, dx$). If you take $u = \sin x$ or $\cos x$ then

$dv = e^x dx$. Upon integration you will obtain the same result.

Example 3.4 Find $\int e^x \cos x dx$.

Solution: Since there is no clue as to which function to take as u , arbitrarily choose $u = e^x$ and $dv = \cos x dx$. Then $du = e^x dx$ and $v = \sin x dx$. Applying (3.5) yields

$$\int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx \, . \qquad (11.2.9)$$

$\int e^x \sin x dx$ can be evaluated the same way. Let $u = e^x$ and

$dv = \sin x dx \;\; \Rightarrow du = e^x du$ and $v = -\cos x$. Thus

$$\int e^x \sin x dx = -e^x \cos x + \int e^x \cos x dx$$

Therefore, replacing the integral of $\int e^x \sin x\, dx$ in (3.6) we have

$$\int e^x \cos x dx = e^x \sin x - [-e^x \cos x + \int e^x \cos x dx]$$

$$\int e^x \cos x\, dx = e^x \sin x + e^x \cos x - \int e^x \cos x\, dx$$

$$\Rightarrow \qquad 2\int e^x \cos x dx = e^x \sin x + e^x \cos x$$

$$\Rightarrow \qquad \int e^x \cos x dx = \tfrac{1}{2}[e^x \sin x + e^x \cos x] + c$$

4. Integration by Partial Fractions

Example: 4.1 Evaluate $\int \frac{x^4 - x^3 - x - 1}{x^2(x-1)} dx$.

Solution: $\frac{x^4 - x^3 - x - 1}{x^2(x-1)} = x - \frac{x+1}{x^2(x-1)}$

$$\frac{x+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} = \frac{Ax(x-1) + B(x-1) + Cx^2}{x^2(x-1)}$$

$$x + 1 \equiv Ax(x-1) + B(x-1) + Cx^2$$

When $x = 0, \quad 1 = -B \quad \Rightarrow B = -1$

$$\Rightarrow \quad x + 1 \equiv Ax(x-1) - (x-1) + Cx^2$$

When $x = 1, \quad 2 = C \quad \Rightarrow C = 2$

$$\Rightarrow \quad x + 1 \equiv Ax(x-1) - (x-1) + 2x^2$$

When $x = -1, \quad 0 = 2A + 2 + 2 \quad \Rightarrow A = -2$

$$\frac{x+1}{x^2(x-1)} = \frac{-2}{x} - \frac{1}{x^2} + \frac{2}{x-1}$$

$$\Rightarrow \quad \frac{x^4 - x^3 - x - 1}{x^2(x-1)} = x - \left(\frac{-2}{x} - \frac{1}{x^2} + \frac{2}{x-1}\right) = x + \frac{2}{x} + \frac{1}{x^2} - \frac{2}{x-1}$$

$$\therefore \quad \int \frac{x^4 - x^3 - x - 1}{x^2(x-1)} dx = \int \left(x + \frac{2}{x} + x^{-2} - \frac{2}{x-1}\right) dx$$

$$= \frac{1}{2}x^2 + 2\ln|x| - \frac{1}{x} - \ln|x-1| + c$$

Example: 4.2 Evaluate the integral $\int \frac{3x+2}{x^3+x}dx$

Solution: We resolve $\frac{3x+2}{x^3+x}$ into partial fractions.

$$\frac{3x+2}{x^3+x} = \frac{3x+2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1)+x(Bx+C)}{x(x^2+1)}$$

$\Rightarrow \qquad 3x+2 \equiv A(x^2+1)+x(Bx+C)$

When $x=0, \quad 2=A$

$\Rightarrow \qquad 3x+2 \equiv 2(x^2+1)+x(Bx+C)$

When $x=1, \quad 5=4+B+C \quad \Rightarrow B+C=1$

When $x=-1 \quad -1=4+B-C \quad \Rightarrow B-C=-5$

Thus, $B=-2$ and $C=3$

$$\frac{3x+2}{x^3+x} == \frac{2}{x} + \frac{3-2x}{x^2+1}$$

Therefore,

$$\int \frac{3x+2}{x^3+x}dx = \int\left(\frac{2}{x}+\frac{3-2x}{x^2+1}\right)dx = 2\int \frac{1}{x}dx + 3\int \frac{1}{x^2+1}dx - \int \frac{2x}{x^2+1}dx$$
$$= 2\ln|x| + 3\tan^{-1} x - \ln\left(x^2+1\right) + c$$

5. Trigonometric Integrals

Some trigonometric integrals are evaluated using some of the following trigonometric identities:

1. $\sin^2\theta + \cos^2\theta = 1$
2. $1+\tan^2\theta = \sec^2\theta$
3. $1+\cot^2\theta = \csc^2\theta$
4. $\sin^2\theta = \frac{1}{2}\left(1-\cos 2\theta\right)$
5. $\cos^2\theta = \frac{1}{2}\left(1+\cos 2\theta\right)$
6. $\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$
7. $\sin\alpha\cos\beta = \frac{1}{2}\left(\sin(\alpha-\beta)+\sin(\alpha+\beta)\right)$
8. $\sin\alpha\sin\beta = \frac{1}{2}\left(\cos(\alpha-\beta)-\cos(\alpha+\beta)\right)$
9. $\cos\alpha\cos\beta = \frac{1}{2}\left(\cos(\alpha-\beta)+\cos(\alpha+\beta)\right)$

Below are two substitution rules which are used in finding the indicated integrals:

(i) For the integral of the form

$$\int \sin^m x \cos^n x dx,$$

if *m* is odd, substitute $u = \cos x$. If *n* is odd, substitute $u = \sin x$.

Example 5.1: Find $\int \sin^3 x \cos^{\frac{3}{4}} x dx$.

Solution: Here $m = 3$, which is odd. Then we let

$u = \cos x$ so that $du = -\sin x dx$.

Now, we need to rewrite as

$$\begin{aligned}
\int \sin^3 x \cos^{\frac{3}{4}} x dx &= \int \sin^2 x \sin x \cos^{\frac{3}{4}} x dx \\
&= \int (1 - \cos^2 x) \cos^{\frac{3}{4}} x \sin x dx \\
&= -\int \left(\cos^{\frac{3}{4}} - \cos^{\frac{7}{4}} x \right) (-\sin x) dx \\
&= -\int \left(u^{\frac{3}{4}} - u^{\frac{7}{4}} x \right) du \\
&= -\left(\frac{4}{7} u^{\frac{7}{4}} - \frac{4}{11} u^{\frac{11}{4}} \right) + c \\
&= -\left(\frac{4}{7} \cos^{\frac{7}{4}} x - \frac{4}{11} \cos^{\frac{11}{4}} x \right) + c
\end{aligned}$$

(ii) For the integral of the form

$$\int \tan^m x \sec^n x dx,$$

If *m* is odd, substitute $u = \sec x$, and if *n* is even, substitute $u = \tan x$.

Example 5.2 Find $\int \tan^6 x \sec^4 x dx$

Solution: Here $n = 4$ and is even. So we let $u = \tan x$ so that

$du = \sec^2 x dx$. Thus, rewriting the integral we have

$$\begin{aligned}
\int \tan^6 x \sec^4 x dx &= \int \tan^6 x \sec^2 x \sec^2 x dx \\
&= \int \tan^6 x (1 + \tan^2 x) \sec^2 x dx \\
&= \int u^6 (1 + u^2) du = \int (u^6 + u^8) du \\
&= \frac{1}{7} u^7 + \frac{1}{9} u^9 + c \\
&= \frac{1}{7} \tan^7 x + \frac{1}{9} \tan^9 x + c
\end{aligned}$$

More examples: Evaluate each of the following examples:

1. $\int \sin^2 x \cos^2 x dx$.

<u>Solution</u>: We rewrite the integral in terms of the identities

$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$ and $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

and obtain

$$\int \sin^2 x\cos^2 x dx = \int \tfrac{1}{2}(1-\cos 2x)\cdot\tfrac{1}{2}(1+\cos 2x)dx$$
$$= \tfrac{1}{4}\int (1-\cos^2 2x)dx$$
$$= \tfrac{1}{4}\int \left(1-\tfrac{1}{2}(1+\cos 2(2x))\right)dx$$
$$= \tfrac{1}{4}\int \left(\tfrac{1}{2}-\tfrac{1}{2}\cos 4x\right)dx = \tfrac{1}{8}\int (1-\cos 4x)dx$$
$$= \tfrac{1}{8}\left(x-\tfrac{1}{4}\sin 4x\right)+c$$

2. $\int 2\sin 4x\cos 3x dx$

<u>Solution</u>: We rewrite the integral in terms of the identities

$\sin\alpha\cos\beta = \frac{1}{2}\left(\sin(\alpha-\beta)+\sin(\alpha+\beta)\right)$.

Thus

$$2\sin 4x\cos 3x = \sin(4x-3x)+\sin(4x+3x)$$
$$= \sin x + \sin 7x$$

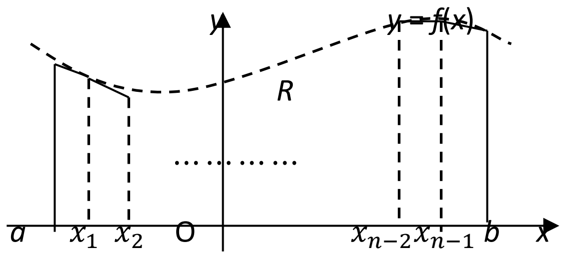
Therefore,

$$\int 2\sin 4x\cos 3x dx = \int (\sin x + \sin 7x)dx$$
$$= -\cos x - \tfrac{1}{7}\cos 7x + c$$

11.3 Application of definite integrals

1 Area under the curve

(a) Let $[a,b]$ be an interval on which a given function $f(x) > 0\ \forall x \in [a,b]$ and is continuous on the interval. Let $x_0, x_1, \cdots, x_{n-1}, x_n$ be points on the interval such that $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ and let $\Delta x = x_n - x_{n-1}$.



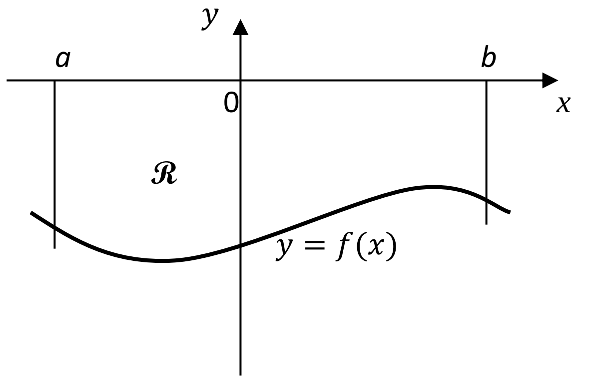
Thus, as $\Delta x \to 0$ and $n \to \infty$ the area between the curve $y = f(x)$ and the *x* - axis from $x = a$ to $x = b$, is given by

$$A \approx \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k)\Delta x = \int_a^b f(x)dx .$$

i.e. $$A = \int_a^b f(x)dx \qquad (11.3.1)$$

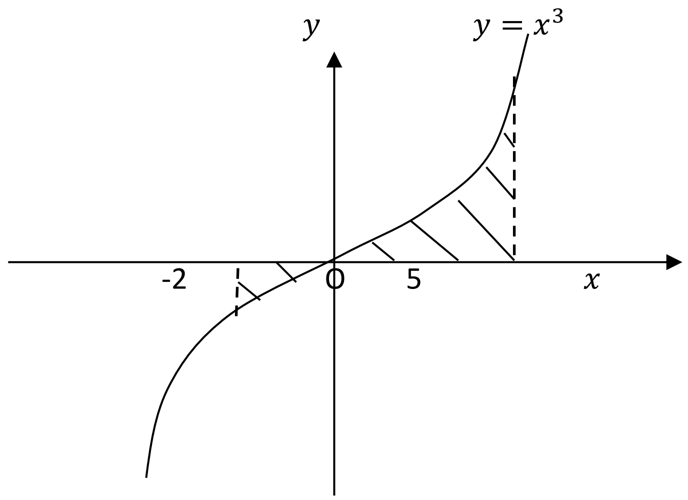
(b)) If $f(x)$ is continuous and $f(x) < 0$ on the interval $[a,b]$, then the area *A* of the region *R* between the curve and the *x* – axis is given by

$$A = -\int_a^b f(x)dx = \int_b^a f(x)dx \qquad (11.3.2)$$



Example 11.3.1 Find the area bounded by the curve $y = x^3$ and the *x* – axis from $x = -2$ to $x = 5$

Solution



$$Area = -\int_{-2}^{0} x^3\,dx + \int_0^5 x^3\,dx$$

$$= -\tfrac{1}{4}(x^4)_{-2}^{0} + \tfrac{1}{4}(x^4)_0^5$$

$$= \tfrac{1}{4}((-2)^4 - 0^4) + \tfrac{1}{4}(5^4 - 0^4)$$

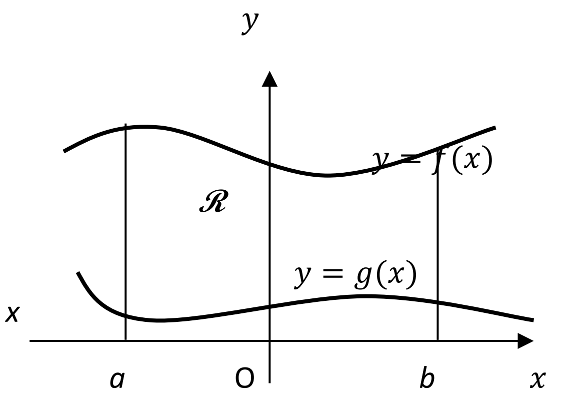
$= \frac{1}{4}(16 + 625) = \frac{641}{4}$ sq. Unit

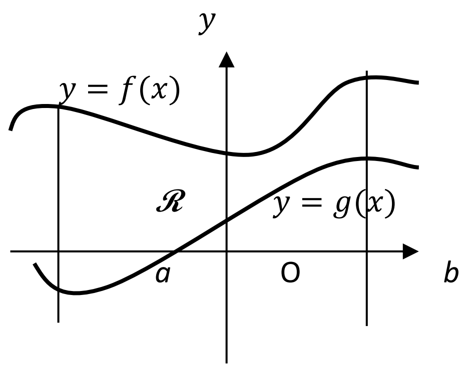
2 Area between two Curves

If we assume that $f(x)$ and $g(x)$ are continuous functions such that $0 \le g(x) \le f(x)$ for $a \le x \le b$, then the area *A* of the region $\mathscr{R}$ between the curves $y = f(x)$ and $y = g(x)$ from $x = a$ and $x = b$, is given by

$$A = \int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b [f(x) - g(x)]dx \qquad (11.3.3)$$

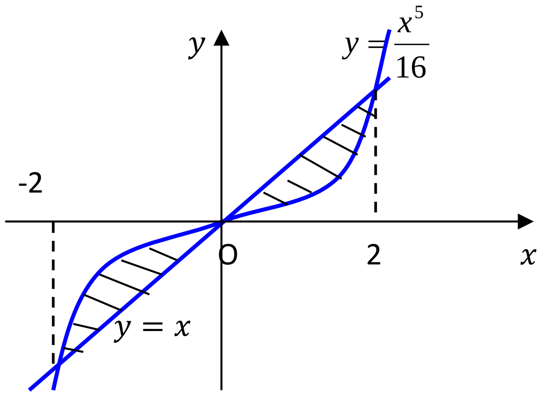
The formula (11.3.3) holds when one or both curve $y = f(x)$ and $y = g(x)$ lie partially or completely below the *x* – axis.





Example 11.3.2 Find the area of the region bounded by the curves $y = x$ and $y = \frac{x^5}{16}$.

Solution



Solution: First we must find the points of intersection of the two curves by solving the two equations simultaneously.

$$x = \frac{x^5}{16}.$$

$$x - \frac{x^5}{16} = 0$$

$$x\left(1 - \frac{x^4}{16}\right) = 0$$

$$x\left(1 - \frac{x^2}{4}\right)\left(1 + \frac{x^2}{4}\right) = 0$$

$$x\left(1 - \frac{x}{2}\right)\left(1 + \frac{x}{2}\right)\left(1 + \frac{x^2}{4}\right) = 0$$

$\Rightarrow \qquad x = 0,\ x = 2,\ x = -2$

Therefore, the area bounded by the two curves is

$$A = \int_{-2}^{0}\left(\frac{x^5}{16} - x\right)d + \int_{0}^{2}\left(x - \frac{x^5}{16}\right)dx$$

$$= \left(\frac{x^6}{96} - \frac{x^2}{2}\right)_{-2}^{0} + \left(\frac{x^2}{2} - \frac{x^6}{96}\right)_{0}^{2}$$

$$= 0 - \left(\frac{(-2)^6}{96} - \frac{(-2)^2}{2}\right) + \left(\frac{2^2}{2} - \frac{2^6}{96}\right) - 0$$

$$= -\frac{64}{96} + 2 + 2 - \frac{64}{96} = \frac{8}{3} \text{ square units.}$$

3 More Applications of Integration

Example 11.3.3. The rate of increase of a population *P* of microorganisms at time *t* is given by $\frac{dP}{dt} = kP$, where *k* is a positive constant. Given that at $t = 0$ the population was of size 8, and at $t = 1$ the population is 56, find the size of the population at time $t = 2$.

Solution: Here we are asked to find *P*. This can be found by using the fact that integration is the reverse operation of differentiation.

Thus, to find *P* we first separate the variables of $\frac{dP}{dt} = kP$ and integrate both sides.

i.e. $\qquad \int \frac{1}{P} dP = k \int dt$

$\Rightarrow \qquad ln|P| = kt + c. \qquad$ (11.3.4)

Next we need to use the given conditions to find the values of *k* and *c*.

For $t = 0, P = 8 \Rightarrow ln8 = k(0) + c \Rightarrow c = ln8$.

This means that (11.3.4) becomes

$$ln|P| = kt + ln8. \qquad (11.3.5)$$

Also, replacing $t = 1$ *and* $P = 56$ *in* (11.3.5) *we have*

$\Rightarrow ln56 = k(1) + ln8 \Rightarrow k = ln56 - ln8 = ln\frac{56}{8} = ln7$.

Hence, (11.3.5) becomes

$$ln|P| = tln7 + ln8.$$

or $$ln|P| = ln8(7^t).$$

or $$P = 8 \times 7^t$$

Therefore, at time $t = 2$,

$$P = 8 \times 7^2 = 392$$

Example 11.3.4 The mass *M* at time *t* of the leaves of a certain plant caries according to the differential equation

$$\frac{dM}{dt} = M - M^2.$$

(a) Given that $t = 0, M = 0.5$, find an expression for *M* in terms of *t*.
(b) Find a value for *M* when $t = ln2$.

(c) Explain what happens to the value of *M* as *t* increases.

Solution: (a) First we must separate the variables in the given differential equation and integrate both sides:

$$\int \frac{1}{M-M^2} dM = \int dt$$

$$\int \frac{1}{M(1-M)} dM = \int dt$$

$$\int \left(\frac{1}{M} + \frac{1}{1-M}\right) dM = \int dt$$

$$ln|M| - ln|1 - M| = t + c$$

$$ln\left|\frac{M}{1-M}\right| = t + c \qquad (11.3.6)$$

When $t = 0$ *and* $M = 0.5$, we have

$$ln1 = 0 + c \Rightarrow c = 0$$

Therefore, (11.3.6) becomes

$$ln\left|\frac{M}{1-M}\right| = t$$

or $$\frac{M}{1-M} = e^t$$

$$M = \frac{e^t}{1+e^t} \quad or \; M = 1 - \frac{1}{1+e^t} \qquad (11.3.7)$$

(b) When $t = ln2, M = \frac{e^{ln2}}{1+e^{ln2}} = \frac{2}{3}$.

(c) As t increases, M approaches 1.

<u>Example</u> 11.3.5 The rate of depreciation $\frac{dV}{dt}$ of a machine is inversely proportional to the square of $t + 1$, where V is the value of the machine t years after it has been purchased. If the initial value of the machine was worth \$500,000, and its value decreased by \$100,000 in the first year, estimate its value after 4 years.

<u>Solution</u>: $$\frac{dV}{dt} \alpha \frac{1}{(t+1)^2} \quad \Rightarrow \frac{dV}{dt} = \frac{k}{(t+1)^2}$$

Separating the variables in the differential equation and integrating both sides we have

$$\int dV = k \int \frac{1}{(t+1)^2} dt$$

$$\Rightarrow \qquad V = \frac{-k}{t+1} + c \qquad (11.3.8)$$

Note that when $t = 0, V = 500{,}000$.

Thus, substituting these in (11.3.8) we have

$$500{,}000 = -k + c$$

$$\Rightarrow \qquad c = 500{,}000 + k \qquad (11.3.9)$$

Also, when $t = 1,\; V = 400{,}000$. Substituting in (11.3.8) we have

$$400{,}000 = \frac{-k}{2} + c$$

$$\Rightarrow \qquad 400{,}000 = \frac{-k}{2} + (500{,}000 + k)$$

$$\Rightarrow \qquad k = -200{,}000$$

$$\Rightarrow \qquad c = 300{,}000$$

Hence, (11.3.8) becomes

$$V = \frac{-200{,}000}{t+1} + 300{,}000 \qquad (11.3.10)$$

Therefore, using ((11.3.10)) we find V after t = 4 years:

$$V = \frac{-200{,}000}{4+1} + 300{,}000 = \$250{,}000.$$

TUTORIAL SHEET 15

1. Integrate the following functions with respect to *x*:

   (a) $(2+7x)^3$ (b) $3\sqrt{2-5x}$ (c) $\dfrac{1}{(4x+5)^3}$ (d) $4\sqrt{x}+\sqrt{4x+1}-4(1-3x)^3$.

2. Evaluate the following integrals:

   (a) $\int x(x^2+3)^3\,dx$ (b) $\int x(x^2+3)^3\,dx$ (c) $\int \dfrac{2x}{\sqrt{x^2+1}}dx$ (d) $\int \dfrac{\cos x}{1+\sin x}dx$

   (e) $\int \dfrac{2x+3}{(x^2+3x+4)^3}dx$ (g) $\int \dfrac{x^2}{1+x^3}dx$ (h) $\int \cos^3 x\,dx$.

3. Evaluate the following integrals:

   (a) $\int xe^{-x}dx$ (b) $\int x^3 \ln 3x\,dx$ (c) $\int (1+x)e^x dx$ (d) $\int x^2 \sin x\,dx$

   (e) $\int \tan^{-1} x\,dx$.

4. Use partial fractions to find the following integrals:

   (a) $\int \dfrac{x}{x+1}dx$ (b) $\int \dfrac{x^2-2}{x^2-1}dx$ (c) $\int \dfrac{3}{(x+1)(x+2)}dx$ (d) $\int \dfrac{x^2+x+5}{x(x+1)^2}dx$

   (e) $\int \dfrac{12x}{(2-x)(3-x)(4-x)}dx$ (f) $\int \dfrac{x^2+2x+4}{(2x-1)(x^2-1)}dx$

6. Evaluate the following definite integrals:

   (a) $\int_2^3 (x^2+2x-1)dx$ (b) $\int_1^2 xe^{x^2}dx$ (c) $\int_{\pi/6}^{\pi/3} cos2xdx$

   (d) $\int_0^{\pi/6} sin^2 xdx$ (e) $\int_1^2 \frac{x}{x^2+1}dx$ (f) $\int_0^1 x^2(x^3+1)^4 dx.$

7. Find the areas bounded by the specified lines and curves in the following questions:

   (a) The *x*- and *y*-axes, the line $x = 3$ and the curve $y = x^2 + 1$.

   (b) The *x*-axis, the lines $x = 1$ and $x = 4$ and the curve $xy = 2$.

   (c) The *x*-axis, and the curve $y = 1 - x^2$.

   (d) The *y*-axis, and the curve $x = 9 - y^2$.

   (e) The curve $y = x(x-1)(x-2)$, and the *x*-axis.

8. Compute the area of the region for which $x \in [-1,2]$ and which is bounded by the graphs of the function *f* and g where

   $f(x) = -\frac{1}{4}x^2 + 1$ and $g(x) = \frac{1}{2}x^2 - 3$.

9. The Goldegg Chicken Farm produces $E(t)$ dozen eggs in *t* days. Its production rate (in dozen/day) is given by

   $$E'(t) = 150 + \frac{2}{5}t,$$

   find

(a) the number of eggs produced in *t* days assuming $E(0) = 0$

(b) the number of eggs produced in one year (365 days)

(c) the average daily production.

10. Consider an (idealized) experiment in which a colony of live bacteria is introduced to a limited food supply. Suppose the rate of change in the number of *N* of live bacteria with respect to time is given by

$$N'(t) = 6000t^2 - 75t^4.$$

Find the size $N(t)$ of the population of the bacteria at time *t* if initially 1000 bacteria were introduced to the food supply.

11. An object on the ground is projected vertically with initial velocity of 32 m/sec. If the acceleration $a(t) = -10.6$ m/sec^2, find

(a) the velocity function,

(b) the distance at time *t*,

(c) the height the object will attain,

(d) the height of the object in 5 seconds.