ZAMBIA OPEN UNIVERSITY

SCHOOL OF AGRICULTURAL SCIENCES

ABM 104

MATHEMATICS

Mr Urban N. Haankuku. B.Sc. (UNZA), M.Sc (UNZA).

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Course Outline

The course is structured as follows:

Unit 1: Set theory

Unit 2: Relations and Functions

Unit 3: Algebra

Unit 4: Logarithms and Exponentials

Unit 5: Matrices

Unit 6: Differentiations

Introduction to the Module

Aim: The aim of this module is to equip students taking Agriculture Business Management with necessary skills, and knowledge in Mathematics relevant to their field of study.

Objectives: By the end of this Module students should be able to:

- Understand and operate with sets
- Use Relations and Functions in their daily life activities
- Use algebra in their daily calculations
- Understand logarithms and Exponential functions and their uses
- Understand Matrices in their applications in real life situations
- Understand differentiations and their applications in real life situations

Introduction

This course is intended for people who completed grade 12, and are following a Degree program in ABM. This Course aims to give students Mathematical concepts, knowledge and skills which will help them in later courses in agriculture economics. The module begins by reviewing grade 12 works such as sets, relations, functions and theory of quadratic equations then the module proceeds to exponentials, logarithms, matrices, and lastly introduction to differential calculus. Therefore, it is important that we begin by introducing some notations and symbolism here which will help you understand some basic concepts when we meet them later in this module.

Objectives:

Upon completion of the module **Error! No text of specified style in document.** you will be able to:

- Understand the theory of sets and apply them in economics settings
- Write the domain, co domain and range of any given function
  Solve any quadratic equation
- Graph any quadratic function and write the max or min point of the function

Unit 1 Set Theory

1.1 Introduction

The module begins by reviewing grade 12 works such as sets, relations, functions and theory of quadratic equations. Set theory forms a basis for higher mathematics, statistics and probability theories. Therefore, it is important that we begin by introducing some notations and symbolism here which will help you understand some basic concepts when we meet them later in this module.

1.2 Objectives

Upon completion of this unit you will be able to:

- Operate sets very well
- Understand the set of real numbers as sets
- Understand binary operations
- Understand the schematic diagram of numbers and their relations

1.3 Sets

***Definition**: A set is a collection of objects which are alike.*

In this context, examples of sets are:

- Names of wild animals
- Employees of a certain firm
- Set of books in the library
- Set o f real numbers which satisfies the equation: $x^2 + 2x - 3 = 0$

Definition: A Universal set is the collection of the entire collection of objects. Notation; $U = Universal$ set. For example:

- A set of animals
- A set of books
- A set of employees

Venn Diagrams_ Venn diagrams are used to visualize sets and their relations to one another.

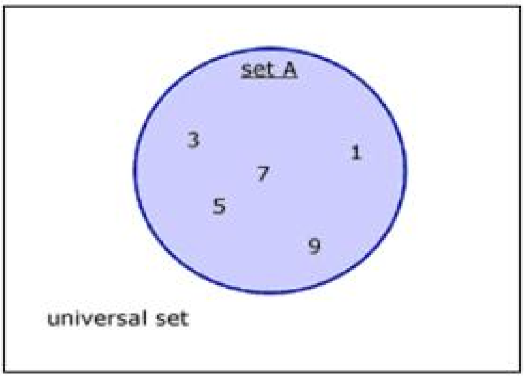


Figure 1.3.1

Above is a diagrammatic representation of set A. The set can be represented mathematically as: A={1,3,5,7,9} .

Note that set A = (the circle) is a subset of the **Universal set (U)** (the rectangle), because a circle is contained in the rectangle. In symbols we write: $A \subset U$. That is all elements in A are all elements of U.

A set can either be described or members of the set can be listed.

Example1.3.1

Let A be a set of vowels in the English alphabet. This set describes the set of vowels, but we can list the members of this set as:

$A = \{a, e, i, o, u\}$ Note that there are 5 vowels in the English alphabet. a is a member of set A in symbols we write $a \in A$ or we can say a belong to set A, or a is an element of A. But the letter c is not an element of A, in symbols $c \notin A$, reads c do not belong to A or c is not a member of A.

Description of set	Listing the elements of the set
A= set of positive multiples of 3 less than 18	$A = \{3,6,9,12,15\}$
B= set of vowels	
C= set of domestic animals	
D= set of prime numbers	$D = \{2,3,5,7,11,13,17,.....\}$
E= set of rational numbers	
F= set of irrational numbers	
G= set of real numbers	
	Dog, cow, john, Mary, car, wood
	$I = \{3,6,9,12,15\}$

Table 1.3.1

You can fill in this table as an exercise.

1.3.1 Equal Sets

Definition *Two sets A and B are said to be equal if and only if every element of A is an element of B and every element of B is an element of A. That is, they have same elements.*

Example 1.3.2

Let $A = \{1,2,3,4\}$ and $B = \{4,1,2,3\}$ then $A = B$, reads A is equal to B. If $A \subset B and B \subset A \Rightarrow A = B$

Let $A = \{1,2,3\}$ and $B = \{a,b,c\}$ then $A \approx B$, reads A is equivalent to B. Two sets are equivalent if and only if they have the same number of elements.

1.3.2 Subsets

Definition. A set A is a subset of a set B, written, $A \subset B$*, if every element of A is also an element of B.*

In fact this is a proper subset; it means B has some elements which are not elements of A. If $A \subset B$ and $B \subset A$ then $A = B$. Or we can have $A \subseteq B$ that means every elements of A are elements of B and every element of B are elements of A.

Subsets. If B is a **subset** of A. Then all of the elements of B are also in A.

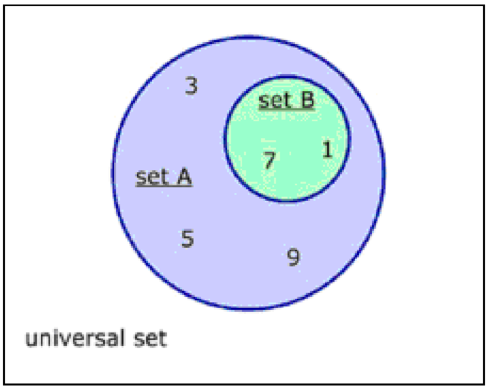


Figure 1.3.2

$$A = \{1, 3, 5, 7, 9\}$$

$$B = \{1, 7\}$$

$$\Rightarrow \quad B \subseteq A \qquad B \text{ is a subset of } A$$

These are recaps from secondary mathematics.

1.3.3 Operations on sets

Symbols

$\cup$= union, $\cap$= intersection, A' = A compliment means all elements in the universal set except those in A, ϕ = empty set or $\{\ \}$ and not both.

Just as we add, subtract numbers; we can also take the union, intersection and complements of sets.

Intersection of sets: Notation: $A \cap B$, reads A intersection B

Definition: *Intersection: is a set of elements which are found in both sets.*

$A \cap B$ *is a set of elements found in both A and B*

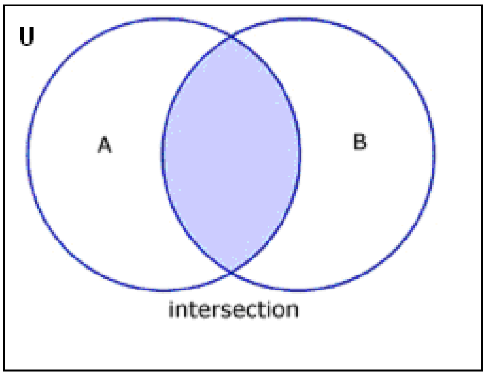


Figure 1.3.3

The **intersection** of sets A and B contains a particular group of elements that exist in set A and in set B.

Union of sets: Notation; $A \cup B$ read, A union B

Definition: Union; is a set of elements which are found in A and B.

$A \cup B$ is a set of elements which are found in A as well in B(note that elements found in both A and B are written once).

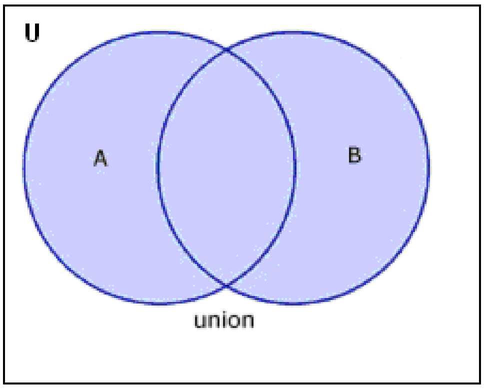


Figure 1.3.4

The **union** of sets A and B contains all the elements from set A as well as set B including elements found in both sets but written once.

Complement: Notation; A' reads, A complement

Definition: A' is the set of elements in the universal set except elements in A.

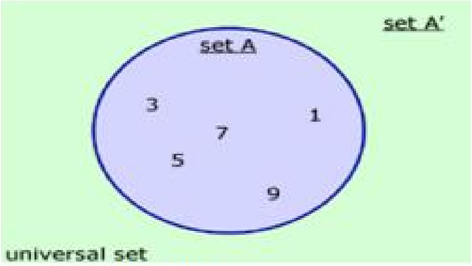


Figure 1.3.5

A' (A-dash)is called the **complement** of A. It contains all elements which are not members of A.

A and A' together make up the Universal set. That is $A \cup A' = U$

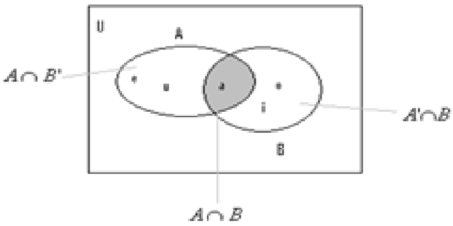


Figure 1.3.6

Example 1.3.3

Let $u = \{a, e, i, o, u\}$ and subsets $A = \{a, e, u\}$ and $B = \{a, i, o\}$

Find :

(i) $A \cap B$ (ii) $A \cup B$ (iii) $A' \cap B'$

(iv) $(A \cup B)'$

Solutions

(i) List the elements found in both A and B

$A \cap B = \{a\}$, note that a is the only element found in both A and B

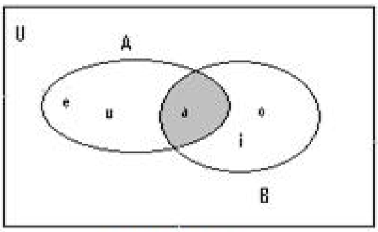


Figure 1.3.7

(ii) Here list all the elements found in A and those elements found in B

$A\cup B=\{a,e,i,o,u\}$, note that 'a' which appears in both A and B it is written once.

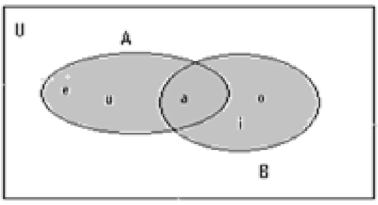


Figure 1.3.8

(iii) List the elements of A' and B' and write the elements found in both A' and B'

$A'=\{i,o\}$ and $B'=\{e,u\}$ then $A'\cap B'=\{\ \}$means there is no element common in A' and B'.

Figure 1.3.9

(iv) List the elements found in A and B then find the complement of this set.

$A=\{a,e,u\}$and $B=\{a,i,o\}$then, $A\cup B=\{a,e,i,o,u\}$, $(A\cup B)'=\phi$

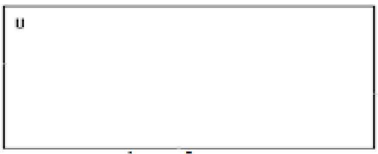


Figure 1.3.10

De Morgan's Law

$(A\cap B)' = A' \cup B'$ and $(A\cup B)' = A' \cap B'$

You can verify de Morgan's laws by showing that the LHS = RHS of the De Morgan's law.

Example 1.3.4 Let $U = \{0,1,2,3,4,5,6,7,8,9,10\}$, and $A = set of even numbers less than 10$ and B= set of prime numbers less than 9. Verify De Morgan's laws:

(i) $(A\cap B)' = A' \cup B'$

(ii) $(A\cup B)' = A' \cap B'$

Solutions

(i) You find list the elements of A and B first as; $A = \{0,2,4,6,8\}$ and $B = \{2,3,5,7\}$ and we then find $A\cap B = \{2\}$ and we find the complement of this intersection $(A\cap B)' = \{0,1,3,4,5,6,7,8,9,10\}$.

Then you find A complement and B complement as: $A' = \{1,3,5,7,9,10\}$ and $B' = \{0,1,,4,6,8,9,10\}$ and then we find $A' \cup B') = \{0,1,3,4,5,6,7,8,9,10\}$

Now you can see that $(A\cap B)' = A' \cup B' = \{0,1,3,4,5,6,7,8,9,10\}$

Hence, De Morgan's law has been verified.

(ii) You should try $(A\cup B)' = A' \cap B'$

Disjoint Sets Two sets A and B are said to be disjoint sets if and only if their intersection is empty.

That is, $A\cap B = \phi$

Example 1.3.5 Let U = the set of positive counting numbers, be the Universal set , and A = the set of even numbers and B = be the set of odd numbers. Find $A \cap B$?

Solution We know that there is no number which is both even and odd, so $A\cap B = \phi$

Equivalent sets: Two sets A and B are said to be equivalent sets if and only if they have the same number of elements, and these elements are not the same.

Example 1.3.6 Let A = { a, b , c , d} and B= { 1,2,3,4}

The two sets are equivalent and we write $A \approx B$

1.3.4 Laws of the algebraic of sets

1a. $A \cup A = A$ 1 b. $A \cap A = A$ **Idempotent laws**

2a. $(A \cup B) \cup C = A \cup (B \cup C)$ 2b. $(A \cap B) \cap C = A \cap (B \cap C)$ **Associative laws**

3a. $A \cup B = B \cup A$ 3b. $A \cap B = B \cap A$ **Commutative laws**

4a. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ 4b. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ **Distributive laws**

5a. $A \cup \phi = A$ 5b. $A \cap U = A$ **Identity laws**

6.a. $A \cup U = U$ 6b. $A \cap \phi = A$ **Identity laws**

7a. $A \cup A' = U$ 7b. $A \cap A' = \phi$ **Compliment laws**

8a. $.(A')' = A$ 8b. $U' = \phi$, $\phi' = U$

Compliment laws

9a. $(A \cup B) = A' \cap B'$ 9b. $(A \cap B) = A' \cup B'$ **De Morgan's laws**

1.4 Sets of Numbers

Notations and symbols

$N = \{1,2,3,4,.....\}$, the set of natural numbers or counting numbers

$W = \{0,1,2,3,4,5,.......\}$, the set of whole numbers

$Z = \{....,-3,-2,-1,0,1,2,3,4,......\}$, the set of integers. Note that the set of integers has three sets of numbers, namely the negative numbers, zero and positive numbers.

What this means is the this set contains the set of whole numbers W, and the set of Natural numbers N. in symbols we write, $N \subset W \subset Z$. The set of positive integers can be written as: $Z^{+} = \{1,2,3,4,5,.....\}$and the set of negative integers can be written as: $Z^{-} = \{........-3,-2,-1\}$, the two set plus 0 gives the set of integers.

The next set to this set of integers is called the set of rational numbers.

Definition 1: A rational number is defined as ; $Q = \left\{ \frac{a}{b}, b \neq 0, where\, a, b \in R \right\}$, that is the set of rational numbers can be written in the form $\frac{a}{b}$, $b \neq 0$ where a and b are integers. Note that in mathematics division by 0 is not allowed, therefore, where an unknown appears in the denominator, it is only valid in a case where it assume that the value is not zero in the denominator. A number in the form $\frac{a}{b}$ can be expressed in decimal form, and two possibilities arises: (1) either the decimal terminates such as : $\frac{1}{2}$, $\frac{3}{4}$, $\frac{2}{5}$, etc. These have terminal decimals. (2) repeated or non terminal decimals such as: $\frac{1}{3} = 0.3333........$,

$\frac{22}{7} = 3.142857142857.....$

This set of rational numbers contains all integers, in symbols we write: $Z \subset Q$, since whole numbers can be written in the form $\frac{a}{b}$, for example, $3 = \frac{3}{1}$, hence it is a rational number by definition.

Definition 2: A rational numbers can be written as a terminating decimals or decimals repeat

For example $\frac{3}{4} = 0.75$, this is an example of terminating decimals , $\frac{1}{3} = 0.33333.......$, this is an example of repeating decimals. This type of decimals can be expressed in another shorter form, that is, $\frac{1}{3} = 0.3333...... = 0.\bar{3}$, the bar on top shows that 3 repeats.

What is known is that o.75 can easily be converted to a fraction which is $\frac{3}{4}$. What would be interesting to us here is how can 0.3333…… be converted to a fraction $\frac{1}{3}$. Take for example $\frac{1}{3}$.

Let $a = 0.\bar{3}$, multiply 10 both side of the equation. We multiply by 10 because the repeating decimal has on digit after the decimal point. If the repeating digits were two we would have multiplied by 100.

$10a = 3.\bar{3}$ ……(1)

$a = 0.\bar{3}$ …….(2)

Subtract (2) from (1) $\Rightarrow 9a = 3$, Divide by 9 both sides

$\Rightarrow a = \frac{3}{9}$, $\Rightarrow a = \frac{1}{3}$

Other examples are: $2.4\overline{12}$ means 2.412121212..... that is 12 repeats, $1.543\bar{7}$=1.54377777....

If a number cannot be written in the form $\frac{a}{b}$, $b \neq 0$ where a and b are integers or cannot be written as terminating decimals or repeated decimals, then this number is an Irrational number(I). That is, $Q \cap I = \phi$, but $Q \cap I = R$,that is the set of rational numbers plus the set of irrational numbers equal the set of Real numbers. That is the set of real numbers contains rational numbers and irrational numbers. 2

Example 1.4.1 Prove that $\sqrt{2}$ is irrational.

Proof We assume that $\sqrt{2}$ is rational. Since it is rational, then we can write it in the form $\frac{a}{b}$, $b \neq 0$. That is $\sqrt{2} = \frac{a}{b}$, $b \neq 0$, Take the equation $\sqrt{2} = \frac{a}{b}$, eqn (1), where a and b are prime (that mean a and b have no common factor). And square both sides, and we get

$2 = \frac{a^2}{b^2}$, $2b^2 = a^2$, (2)

This implies a^2 is divisible by 2, then a is divisible by 2. That is a has a factor 2 say $a = 2r$, for some integer r. We substitute a = 2r in (2), and we get;

$2b^2 = (2r)^2$, this implies, $2b^2 = 4r^2$, $b^2 = 2r^2$, this implies b^2 is divisible by 2, then b is divisible by 2. Now we have found a common factor 2 this is a contradiction by (1)

Therefore, $\sqrt{2}$ is not rational, hence $\sqrt{2}$ is irrational.

Note that the set of real numbers is the union of the set of rational number and the set of irrational number. This implies that $R = Q \cup I$, a real number is either rational or irrational, that is $Q \cap I = \phi$. The set of real numbers are closed on the number line.

Real numbers

You can represent a subsets of real numbers in interval form such as: (a,b) is an open interval, [a,b] as a closed interval, (a,b] as an open-closed interval, and [a,b) as a closed- open interval. In lower mathematics these sets where represented using set builder notation for instance $(a,b) = \{x : a < x < b, x \in R\}$, while $[a,b] = \{x : a \le x \le b, x \in R\}$

Example 1.4.2 Given that the universal set is R, $A = (-1,4]$ and $B = [0,7]$. Find the following sets and illustrate them on the number line. A' (ii) $A \cap B$ (iii) $A' \cap B$

Solutions

(i) You draw a number line to help you visualize the given set. Note that the interval (-1,4] is open-closed interval and this can be illustrated on a number line as follows:

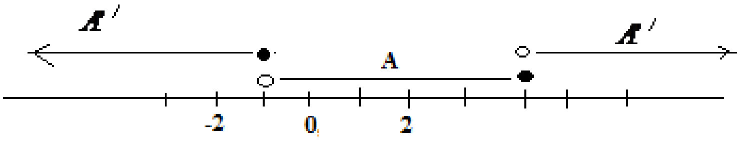


Diagram 1.4.1

Therefore $A' = (-\infty,-1] \cup (4,\infty)$

(ii) The shaded part in the number line is the intersection of A and B as shown below:

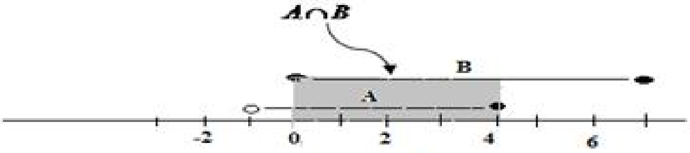


Diagram 1.4.2

That is $A \cap B = [0,4]$

1.4.1 Some properties of real numbers

1. **Identity element.**

   0 is an identity element under the operation +; 0+4=4

   1 is an identity element under the operation x, 1x4=4

2. **commutative properties**

The operation + is commutative; a+b=b+a, for real numbers a and b

The operation x is commutative; axb = bxa , for real numbers a and b

3. **Inverse element**

The operation + has an inverse element 0, for real number a

The operation x has an inverse element a^{-1}, for real number a

4. **Transitive property**

if $a > b$ and $b > c$,then $a > c$

5. **Associative property**

The operation + is associative

The operation x is associative

6. **The distributive property**

The operation x is distributive over +a(b+c)=ab+ac

SCHEMATIC DIAGRAM OF NUMBERS

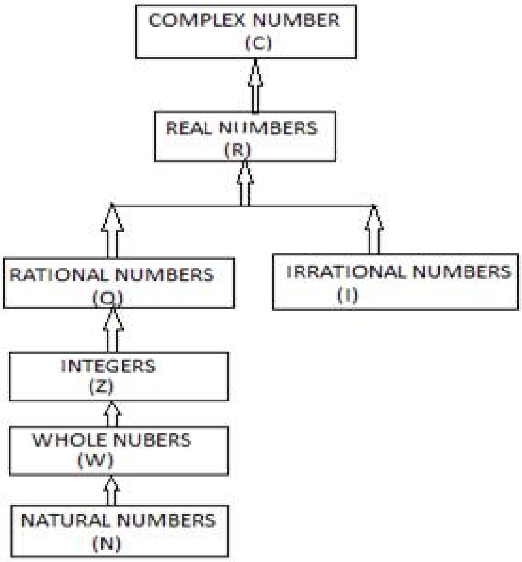


Diagram 1.4.3

1.4.2 Binary Operations

In mathematics you are interested in the study of basic operations of algebra, such as : $+,-,\times,\div$. You have studied these operations at school, but in this section, you will study the use of these operations in details.

Definition: The operation * is a binary operation in S, if and only if , for every $a,b \in S$, $the\, a*b \in S$. Otherwise the operation * is not a binary operation in S.

You know the product of 3 and 4 as 12. In this case the operation is multiplication, that is $* = \times$.

This * can be defined in any way different from the four basic operations given above as will be illustrated below.

Example 1.4.3

(a) State whether the * defined as : $a * b = \sqrt{a - b}$ is a binary operation in the set of real numbers.

Solution: All you need is to find choices of a and b were a*b fails. If you take a=1 and b=5, the $a * b = \sqrt{1 - 5} = \sqrt{-4}$, the $\sqrt{-4} \notin R$, therefore * is not a binary operation in the set of real numbers.

(b) State whether * defined as $* = \div$, in the set of real numbers. Is * commutative.

Solution: Take any two numbers in the set of real numbers, say $a, b \in R$,*then* $a * b = a \div b = \frac{a}{b} \in R$, hence $* = \div$ is a binary operation on R. But $* = \div$ is not commutative since $a * b = \frac{a}{b} \neq \frac{b}{a} = b * a$.

Exercise 1.1

1. If A = { 1,2,3,4 } , B ={ 2,4,6,8 } , C ={ 3, 4 , 5, 6 } and the universal set X ={ 1,2,3,4,5,6,7,8,9 }.

   (a) Find (i) A' (ii) $(A \cap B)'$ (iii) B – C (iv) $(A \cup B)'$ (v) $A \cap (B - C)$ (vi) $(X - C)' \cap (A - B)$

   (b) Confirm

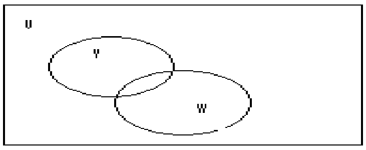
   (i) the associative laws: $(A \cap B) \cap C = A \cap (B \cap C)$ and $(A \cup B) \cup C = A \cup (B \cup C)$

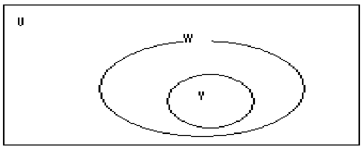
   (ii) the distributive laws: $A \cap (B \cup C) = (A \cap) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

   (iii) the De Morgan's laws: $(A \cup B)' = A' \cap B'$ and $(A \cap B)' = A' \cup B'$

2. (a) If $A \subset B$, then simplify if possible (i) $(A \cap B)$ (ii) $A' \cup B'$ (iii) $A \cup B'$ (iv) $A' \cap (A \cup B)$

3. In the Venn Diagram below , shade : (i) W – V (ii) $V^{'} \cup W$ (iii) $V \cap W^{'}$
(iv) V’ – W’





4. In problems (a) to (h) find expression of the form $\frac{a}{b}$ for the given decimal expansions, where a and b are integers.

(a) $0.\overline{5}$ (b) $3.1\overline{2}$ (c) $11.\overline{34}$ (d) $-4.3\overline{57}$ (e) $9.\overline{413}$
(f) 0.9999… (g)1.3333…. (h) 0.85

5. (a) Find an irrational number which lie between (i) 2 and 3 (ii) 19 and 19.01
(iii) -4 and -2

(b) Each of the following operation in I,II and III is a binary operation on R.

I : a*b = (a – b)(a + b) , a, b ∈ R,
II : a*b = ab , a, b ∈ R,
III: a*b = 2^{a+b} , a, b ∈ R

(i) Determine which of I, II III is commutative
(ii) For each of I , II and III , evaluate (3* 2) * 5

6. Given the sets X = { 0, 1 } and Y = { 0, 1, 2 }

(a) Check whether each of the following operations +, -, x , ÷ is a binary operation on X and Y.

(b) Also check whether the operation is commutative and associative

7. If A = [1, 4] , B = (2, 8) , C = [3, 6) and the universal set X = [1, 9] . Find each of the following sets and display it on the number line. (i) B’ (ii) $(A \cup B)^{'}$ (iii) B – C (iv) $(A \cap B)^{'}$ (v) $A \cap (B - C)$
(vi) $(X - C)^{'} \cap (A - B)$.

8. An operation * is defined on the set { 3,5,7 } in the table below as follows: a*b is the result where the row along “a” and column a long “b” meet. e.g . 5*7 = 7.

*	3	5	7
3	3	5	7
5	7	3	5
7	5	7	3

(i) Is this operation a binary operation on set { 3,5,7 }

(ii) Is this operation commutative

(iii)Evaluate (5*7)*3

Unit 2 Relations and Functions

2.1 Introduction

This unit begins with defining what a relation is and develops the concepts of domain and range which are later extended to functions. One-to-one functions are discussed and even/odd and composite functions are also discussed.

2.2 Objectives

Upon completion of this unit you will be able to:

- Define a function
- State the difference between a relation and a function
- Write the domain and range of a function
- State whether the function is 1-1 or onto
- Find the inverse of a function and graph the function and its inverse
- Calculate composite functions

2.3 Relation A relation R from a set A to a set B assigns to each pair (a,b) in AxB exactly one of the following statements:

(i) "a is related to b", written aRb
(ii) " a is not related to b" written $a \not{R} b$

A relation from a set A to the same set A is called a relation in A

Example 2.3.1

1. Marriage is a relation from the set M of men to the set W of women. For, given any man $m \in M$ and any $w \in W$, either m is married to w or m is not married to w.
2. Order, symbolized by "<" or equivalently, "x is less than y" is a relation in any set of real numbers. For, given any order pair (a,b) of real numbers, either a<b or $a \not< b$
3. Perpendicularity is a relation in the set of lines in the plane. For, given any pair of lines a and b , either a is perpendicular to b or a is not perpendicular to b.

4. X = n=set of Natural numbers and Y= n+2

This a '**notation**' for expressing a relation between two variables(say X and Y).

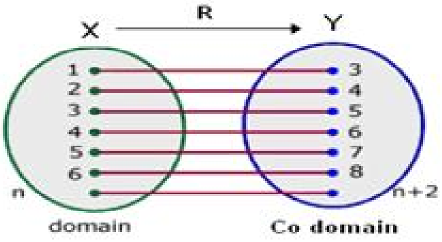


Figure 2.3.1

Individual values of these variables are called **elements** eg Domain; $X = \{1,2,3,4,5,6,....\}$ and Co domain: $Y = \{3,4,5,6,7,8,....\}$. **Note that Y values are all images of element in X, this is also called the Range**

The first set of elements (X) is called **the domain** . The second set of elements (Y) is called **the Co domain** . If all elements in Y are some images of some elements in X the co domain is called the **range.**

Example 2.3.2 A simple relation like y = x^2 can be more accurately expressed using the following format: $\{(x, y): y = x^2, x, y \in R\}$. In this example the Domain is $X = \{x : x \in R\}$, that is all real numbers is the domain. But the co domain is $Y = \{y : y \in R\}$, and the Range $Y = \{y : y \geq 0, y \in R\}$

The sketch of the graph of the function $y = x^2$

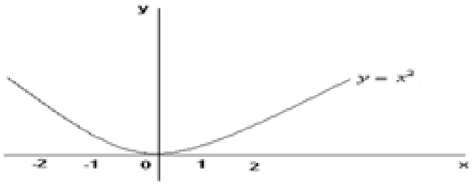


Figure 2.3.2

Here, you will observe that there are no values of x whose images are negatives. The images are always 0 or positives. This gives a good distinction example of Co domain and the Range.

Y is a set of real numbers but some real numbers are not images of the x values in the Domain.

2.3.1 Inverse Relation

Let R be a relation from A to B. The inverse of R , denoted by R^{-1}, is the relation from B to A which consists of those ordered pairs which when reversed belong to R:

$R^{-1} = \{(b,a) : (a,b) \notin R\}$

Example: 2.3.3 Consider a mapping

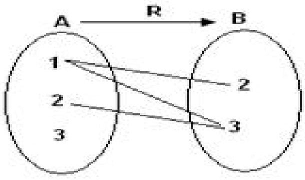


Figure 2.3.3

$$R = \{(1,2),(1,3),(2,3)\}$$

$$R^{-1} = \{(2,1),(3,1),(3,2)\}$$

Example 2.3.4 The inverse of the relations defined by; “ x is the husband of y” and “ x is taller than y” are respectively. “ y is the wife of x” and “ y is shorter than x”.

2.3.2 One-One mapping

Here one element of the domain is associated with one and only one element of the range.

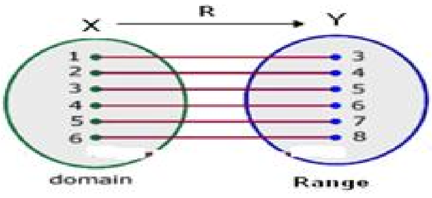


Figure 2.3.4

2.3.3 Onto

A relation is **onto** if and only if every element in Y is some image of some element in the domain X. (or if Y is filled)

Example: Let a relation be defined as given below:

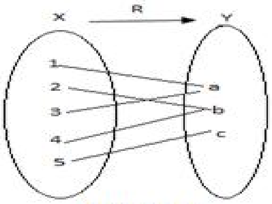


Figure 2.3.5

$$R = \{(1,a),(2,b),(3,a),(4,b),(5,c)\}$$

The co domain is $\{a,b,c\}$ and the range is $\{a,b,c\}$, that means the co domain is equal to the range , hence , the relation is **onto.**

2.3.4 Many-One mapping

Here more than one element of the domain can be associated with one particular element of the range.

Example 2.3.5 Consider the relation $\{(x,y): y = x^2, x, y \in Z,\ -4 \le x \le 4, x \ne 0\}$.

Z is the set of integers(positive & negative whole numbers not including zero)

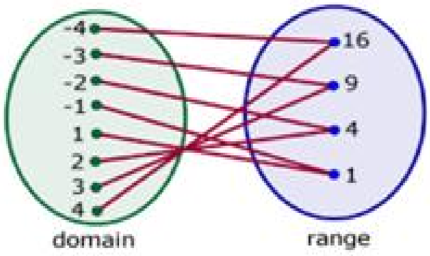


Figure 2.3.6

2.3.5 Functions

Notation: $y = f(x)$, this notation will be used throughout this module.

Definition: *If there is associated with each element of a set X exactly one element of another set Y, then this association constitutes a function from X to Y and is written* $f : X \to Y$.

The unique element in Y assigned to $x \in X$ by f denoted by f(x) , and called the image of x under f or the value of f at x. The domain of f is X, and the co domain Y. The range of f is denoted by $f(x)$ is the set of images, that is, $f(x) = \{f(x) : x \in X\}$. By definition, all functions are relations but not all relations are functions.

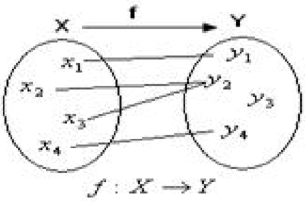


Figure 2.3.7

Domain $= \{x_1, x_2, x_3, x_4\}$, the Co domain $= \{y_1, y_2, y_3, y_4\}$, and the range = $\{y_1, y_2, y_4\}$ note that y_3 is not an image of any element in the domain. Hence by definition f is a function.

A function f is **into** if it is a function by definition, a function f is **1-1** if every element of the domain X has one and only one image in the co domain Y, a function f is **onto** if the co domain is equal to the range.

Example 2.3.6 Let a mapping be

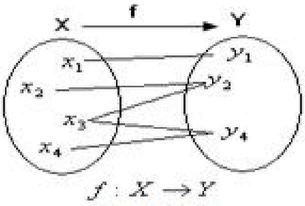


Figure 2.3.8

This is not a function because x_3 has two images y_2 *and* y_4 (refer to the definition of a function).

2.3.5 One to one functions

A function f is 1-1 if every element in the domain X is assigned to exactly one and only one element in the co domain Y.

Example2.3.7 Let the mapping be given as:

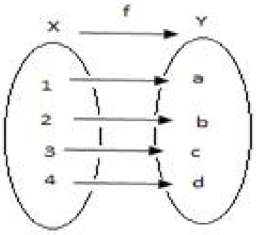


Figure 2.3.9

The function is one-to-one, since every element of X has one and only one image in the co domain Y.

A function f is one to one if for every $a, b \in X, then, f(a) = f(b) \Rightarrow a = b$. Using this condition, we can show that any given function is either one to one or not.

Example 2.3.8 Show that the function $f(x) = 3x - 1$ is one to one.

Solution: We take any $a, b \in X$, we show that for any a, b in X $f(a) = f(b) \Rightarrow a = b$

If x = a this implies $f(a) = 3a - 1$ and $f(b) = 3b - 1$, then we show $f(a) = f(b) \Rightarrow a = b$

That is f(a) = f(b) implies 3a-1 = 3b-1 $\Rightarrow 3a = 3b \Rightarrow a = b$. Hence the function $f(x) = 3x - 1$ is one to one.

Example2.3.9 Show that the function $f(x) = 3x^2 - 1$ is not one to one.

Solution: We take any $a, b \in X$, we show that for any a, b in X $f(a) = f(b) \Rightarrow a \neq b$

If x = a this implies $f(a) = 3a^2 - 1$ and $f(b) = 3b^2 - 1$, then we show $f(a) = f(b) \Rightarrow a \neq b$

That is f(a) = f(b) implies $3a^2 - 1 = 3b^2 - 1$ this implies

$3a^2 - 1 = 3b^2 - 1 \quad 3a^2 = 3b^2 \Rightarrow 3a^2 - 3b^2 = 0 \Rightarrow a^2 - b^2 = 0 \Rightarrow (a-b)(a+b) = 0 \Rightarrow a = b \ and$ $a = -b$ that we have a is equal to b and –b. Hence, by definition the function $f(x) = 3x^2 - 1$ is not a one to one function. *A property of one-one functions is that on a graph a horizontal line will only cut the graph once.*

2.3.6 Onto functions: A function f is onto if the co domain is equal to the range.

Example 2..3.10 Let a mapping be as given below:

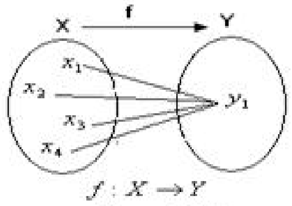


Figure 2.3.10

This relation is a function, by definition, this function is not 1 – 1, but f is onto since the co domain is equal to the range: Co domain = $\{y_1\}$ and the range = $\{y_1\}$.Hence, f is onto.

Example 2.3.11 Let a mapping be

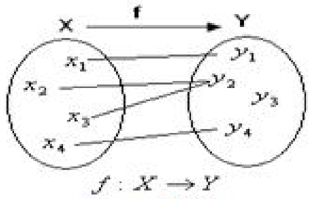


Figure 2.3.11

The function f is not onto since the co domain is not equal to the range. In this example the element y_3 is in the co domain but it is not an image of any of the element of the domain.

Example 2.3.12 Let a function be defined as **:** $\{(x, y) : y = x + 2, x, y \in R^+\}$, Where R^+ is the set of positive real numbers. Show that the function f is 1-1.

Solution: The domain is all positive real numbers; $X = \{x : x > 0, x \in R\}$ and the range is $Y = \{y : y > 2, y \in R\}$. We can use the condition , A function f is one to one if for every $a, b \in X, then, f(a) = f(b) \Rightarrow a = b$ That is, $f(a) = a + 2$ and $f(b) = b + 2$ this implies

f(a) = f(b) implies a+2= b + 2 That is, $a = b$, Hence, $\{(x, y) : y = x + 2, x, y \in R^+\}$**,** is one to one.

Example 2.3.13 Find the domain and range of the functions given below:

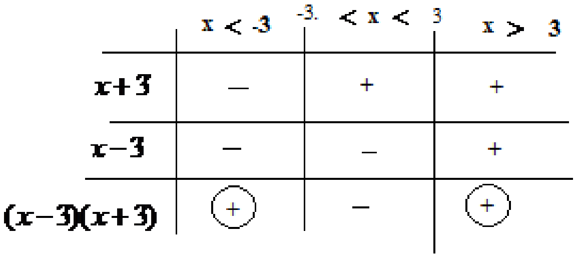
(i) $f(x) = x + 1$ (ii) $f(x) = x^2 - 1$ (iii) $f(x) = \sqrt{x^2 - 9}$

Solutions:

(i) The expressions not allowed in mathematics are : $\frac{1}{0}, \frac{0}{0}, \sqrt{-ve}$. So if none of these can be arrived at for any real number, then the function is defined (or exist) everywhere. So you should avoid these expressions.(i) The function $f(x) = x + 1$ has Domain: $D = \{x : x \in R\}$ since there is no value of x which can give you any of the expressions given above. Range: $Range = \{y : y \in R\}$. Note that the domain are the x- values while the range are the images of x which are the y-values.

(ii) The function $f(x) = x^2 - 1$ has the domain: $D = \{x : x \in R\}$, while the range is $Range = \{y : y \geq -1, y \in R\}$. There is no value of x which will give you the y value less than -1.

(iii) The function $f(x) = \sqrt{x^2 - 9}$, has a radical so, you need to avoid all values of x whose images will give you negatives. So you take what is in the radical and you want it to be greater than or equal to zero. So we set, $x^2 - 9 \geq 0$ these are the only values of x where the function exists. Therefore, $\begin{array}{l} x^2 - 9 = x^2 - 3^2 \geq 0 \\ (x-3)(x+3) \geq 0 \end{array}$ To satisfies this inequality, you solve this as an equation to find the critical values as: x = 3 or -3.



	x < -3	-3. < x < 3	x > 3
x+3	—	+	+
x−3	—	–	+
(x−3)(x+3)	⊕	–	⊕

Diagram 2.3.1

The circled signs are the intervals where the function exists. Therefore, the domain: $D = \{x : x \leq -3, x \in R\} \cup \{x : x \geq 3, x \in R\}$. The range is: $Range = \{y : y \geq 0, y \in R\}$. Note that there is no value of x which will give an image as negative, and the least image is when x is either 3 or -3 and the image is 0.

2.3.7 Inverse Function f^{-1}

The **inverse function** is obtained by interchanging x and y in the function equation and then rearranging to make x the subject. This technique was used in high schools. Given any function $f(x) = 2x + 3$ we make x the subject as follows:

$$\begin{aligned} y &= 2x + 3 \\ y - 3 &= 2x \\ x &= \frac{y-3}{2} \end{aligned}$$

Then the inverse of the given function is $f^{-1}(x) = \dfrac{x-3}{2}$. A function f has an inverse if and only if it is one to one. If f^{-1} exists then, $ff^{-1}(x) = f^{-1}f(x) = x$. It is also a condition that the two functions be 'one to one'. That is the domain of **f** is identical to the range of its inverse function f^{-1}. When graphed, the function and its inverse are reflections either side of the line y = **x**.

Example 2.3.14 Find the inverse of the function (below) and graph the function and its inverse on the same axes. $\{f : x \mapsto 2x + 3,\ x \in \mathbb{R}\}$

$$\begin{aligned} &\Rightarrow \quad y = 2x + 3 \\ &\text{interchanging } x \text{ and } y \\ &\Rightarrow \quad x = 2y + 3 \\ &\Rightarrow \quad x - 3 = 2y \\ &\Rightarrow \quad 2y = x - 3 \\ &\Rightarrow \quad y = \tfrac{1}{2}(x - 3) \end{aligned}$$

$$\Rightarrow \{f^{-1} : x \mapsto \tfrac{1}{2}(x - 3),\ x \in \mathbb{R}\}$$

You can graph f(x) and its inverse $f^{-1}(x)$ on the same coordinate system as below:

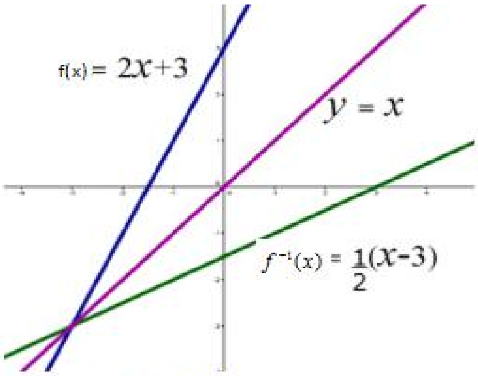


Figure 2.3.12

You can observe that the graph of an inverse of a function is simply the mirror image of the function in the line $y = x$.

2.3.8 Composite functions

Consider a mapping given below:

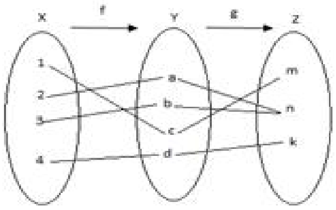


Figure 2.3.13

This is a composite function written as gof, read the composite function of f and g. $gof(x) = g(f(x))$. In this mapping above $gof(3) = g(f(3)) = g(b) = n$. Note that X is the domain of f and Y is the co domain of f, while Y is the domain of g and Z is the co domain of g. But X is the domain of gof and Z is the co domain of gof.

$gof = \{(1, m), (2, n), (3, n), (4, k)\}$.

Therefore, you can find the inverse of the composite function as;

$(gof)^{-1} = \{(m,1), (n,2), (n,3), (k,4)\}$ One useful property of inverse composite function is:

$(gof)^{-1} = f^{-1}og^{-1}$

Example 2.3.15 Let $f(x) = 3x + 1$ and $g(x) = \sqrt{x-1}$, express in terms of x the following:

(i) gof, (ii) $(gof)^{-1}$ (iii) $gof(2)$

Solutions

(i) $gof(x) = g(f(x)) = g(3x+1) = \sqrt{(3x+1)-1} = \sqrt{3x}$

(ii) Here you need to find the inverse of each function as follows:

$$\begin{aligned} f(x) &= 3x + 1 \\ y &= 3x + 1 \\ 3x &= y - 1 \\ y &= \frac{y-1}{3} \end{aligned}$$

Therefore, $f^{-1}(x) = \dfrac{x-1}{3}$

$g(x) = \sqrt{x-1}$

$y = \sqrt{x-1}$

$y^2 = x - 1$

$x = y^2 + 1$

Therefore, $g^{-1}(x) = x^2 + 1$

Therefore , using the condition, $(gof)^{-1} = f^{-1}og^{-1}$

$(gof)^{-1} = f^{-1}og^{-1} = f^{-1}(g^{-1}) = f^{-1}(x^2+1) = \frac{x^2+1-1}{3} = \frac{x^2}{3}$ Therefore, $(gof)^{-1} = \frac{x^2}{3}$

You can verify this by finding the inverse of gof using the conventional method.

(iii) $gof(x) = \sqrt{3x}$ by (i) above , $gof(2) = \sqrt{3(2)} = \sqrt{6}$

2.3.9 Even and odd functions

Definition: A function f is an even function if and only if $f(-x) = f(x)$, and a function is odd if and only if $f(-x) = -f(x)$. In establishing on whether a function is odd or even, all you need is to verify these two conditions. If both conditions fail, then a function is neither odd nor even. In fact there are some functions which are neither even nor odd.

Example 2.3.16 Verify whether each of the given function is even, odd or neither.

(i) $f(x) = x^4 + 2x^2$ (ii) $f(x) = x^3 - 2x$ (iii) $f(x) = 2x^2 + 1$

Solutions

(i) Using the conditions given above, you proceed as follows: Let $f(x) = x^4 + 2x^2$, and $f(-x) = (-x)^4 + 2(-x)^2 \Rightarrow f(-x) = x^4 + 2x^2 = f(x)$, hence $f(-x) = f(x)$ is satisfied. That means the function $f(x) = x^4 + 2x^2$, is an even function.

(ii) Let $f(x) = x^3 - 2x$, and $f(-x) = (-x)^3 - 2(-x) = -x^3 + 2x = -[x^3 - 2x] = -f(x)$, That means the function $f(x) = x^3 - 2x$, is an odd function.

(iii) Let $f(x) = 2x^2 + 1$, and $f(-x) = 2(-x)^2 + 1 = -2x^2 + 1 \neq -f(x) or - f(x)$, hence $f(x) = 2x^2 + 1$ is neither even nor odd,

If a function is neither even nor odd then it is said to be neither.

Your interest in this section is as follows:

- To be able to write the domain and the range of any given function
- To be able to state whether the function is 1-1 or onto
- To be able to show that the function is 1-1 or not
- To be able to find the inverse of a function
- To be able to write the composite functions in terms of x.
- To be able to state whether the given function is odd, even, or neither.

Exercise 2

1. The function f and g are defined by $f(x) = x^2$ and $g(x) = \frac{1}{2}x + 1$ for all real values of x in the domain $0 \le x \le 3$.

   (a) Find $f^{-1}(x)$ and $g^{-1}(x)$ in terms of x

   (b) Sketch , in the same diagram, the graphs of f, g, $f^{-1}(x)$ and $g^{-1}(x)$

2. The functions f and g are defined by $f(x) = 2x + 1, x \in R$ and $g(x) = \frac{1}{x}, x \in R, x \neq 0$

   (a) Calculate the value of gf(2)

   (b) Find $g^{-1}(x)$ in terms of x

   (c) Calculate the value of x for which fg(x) = x

3. The functions f and g are defined by $f(x) = \frac{25}{3x-2}$, $x \in R, 1 < x \le 9$ and $f(x) = x^2, x \in R, 1 < x \le 3$. Find

   (a) The range of f

   (b) The inverse function, $f^{-1}(x)$, state its domain

   (c) The composite function fg and state its domain

   (d) The solution of the equation $fg(x) = \frac{2}{x-1}$

4. Given the function $f(x) = \sqrt{1-x}$

(i) Find the domain of f(x)

(ii) Find the range of f(x)

(iii) Sketch the graph of f(x)

5. Given the function g(x) = $-\,|\,x-3\,|$

   (i) Find the domain of g(x)

   (ii) Find the range of g(x)

   (iii) Sketch the graph of g(x)

   (iv) Find the intersection of g(x) with the line $2y + x = -2$

6. The fish population y in a certain river is related to the tons of pollutants x according to the following $y = 100{,}000 - 1\,500x$

   (a) What are the domain and the range of this function?

   (b) Sketch the graph of this function.

   (c) Suppose that 17.5 tons of pollutants is deemed critical. How many fish would be in the stream at the critical level of pollution?

7. (a) Show that the following functions are not 1 – 1 .

   (i) $y = x^2 + 3x + 2$ (ii) $y = x^2$

   (b) Show that the following functions are 1 – 1 .

   (i) $y = 3x + 4$ (ii) $y = 5$

8. For each of the following functions state whether the function is odd, even or neither of these. (i) $f(x) = 4 - 3x^2$ (ii) $f(x) = 3x^2 + x$

   (iii) $f(x) = x - \dfrac{1}{x}$ (vi) $f(x) = 2 + x^2$

3.0 Algebra

3.1 Introduction

This unit begins with linear and quadratic functions, the difference between a function and an equation are explained with the help of examples. Theory of quadratic functions is explained in relations to their turning points called maximum or minimum. Solutions of quadratic equations are explored and discuss their relationships to the coefficients of the quadratic equations. Lastly the unit discusses polynomials in relations to remainder and factor theorems, inequalities are also discussed and the unit end with binomial expansions

3.2 Objectives

Upon completion of this unit you will be able to:

- Distinguish between a linear and quadratic functions
- Solve quadratic equations using completing of square method
- Find the maximum or minimum of a quadratic function
- solve a polynomial equation
- Expand any expression raised to rational power of n.
- Expand expressions using Binomial Theorem
- Solve inequalities

3.3 Linear and quadratic functions

The simplest function defined in mathematics by means of a non trivial algebraic expression is the function defined by the equation $y = f(x) = mx + c.$

Definition: The function f , defined by the first degree equation $f = \{(x, y) : y = mx + c\}$, where m and c are constants, is called a linear function.

The function derives its name from the fact that its graph is a straight line. Moreover, any straight line other than $x = k$ (a straight line parallel to the y-axis) can be represented by such an equation with the appropriate m and c. You will consider the linear function's algebraic properties and its zeros, rather than the geometric properties. The zero of a function is the x-coordinate or x- value for which y, the value of the function, is zero. Hence, you let $y = 0$, and you can find the zero of a linear function by solving the equation $mx + c = 0$ for x. Note that $y = f(x) = mx + c.$ is a linear function while $mx + c = 0$ is called a linear equation. In general, the zeros of a function are the roots or solution of the equation $f(x) = 0$. There are many methods that you can use to solve an equation. However, any method that produces an equivalent equation, one which has the same roots, and only those roots, is allowed. Some procedures, such as squaring, may introduce new factors, and some, such as dividing, may lose some factors, so that extreme care should be exercised in using such procedures. Furthermore, it is always wise to check any purported solution, for the ultimate test of a number as a root of any equation is not how it is obtained, but whether it satisfies the equation. The following operations are called permissible, since they always result in equations which are equivalent to the original, that is, the new equation has exactly the same roots.

1. The same number or algebraic expression may be added to or subtracted from both members of an equation.
2. Both members of an equation may be multiplied or divided by any nonzero number.

Example 3.3.1 Solve the following equations $4x-5=x+7$

Solution:

$$\begin{aligned}&4x-5=x+7\\&4x-x=7-5\\&3x=2\\&x=\frac{2}{3}\end{aligned}$$

Example 3.3.2 Solve the equation $x-1=3$

Solution: This equation has solution $x=4$. But if you square both sides you will get: $(x-1)^2=9$. If this equation is solved, one of its roots i.e. -2, although it does not satisfy the original equation.

Example 3.3.3 Solve the equation $\frac{2x+5}{2}-\frac{5x}{x-1}=x$

Solution:

$$\begin{aligned}&\frac{2x+5}{2}-\frac{5x}{x-1}=x\\&\frac{(2x+5)(x-1)-5x(2)}{2(x-1)}=x\\&2x^2+5x-2x+5=2x^2-2x\\&x=-1\end{aligned}$$

3.4 Maximum and Minimum

The second type of algebraic function usually considered is one in which the defining equation is of the second degree.

Definition: The function f, defined by the second degree equation $f=\{(x,y):y=ax^2+bx+c\}$, is called a quadratic function. Note that

$y = f(x) = ax^2 + bx + c$ is a quadratic function while $ax^2 + bx + c = 0$ is called a quadratic equation. In general, the zeros of a function are the roots or solution of the equation $f(x) = 0$.

A quadratic function has on turning point called either the maximum point or a minimum point. You can determine whether the function is maximum or minimum by the coefficient of the x^2. If $a > 0$, then the function has a minimum, but if $a < 0$, then the function has a maximum.

The graph of the general quadratic function may be sketched by the more direct process of expressing the quadratic function in terms of the square of a linear function. Consider the function $f(x) = ax^2 + bx + c$, you complete square of this function as follows:

$$f(x) = ax^2 + bx + c \qquad factor\ out\ a$$

$$f(x) = a(x^2 + \frac{b}{a}x + \frac{c}{a}) \ \ take\, the\, coefficient\, of\ x, half\ it, and\ square\, it\ and\ add\ a\ zero\left(\frac{b}{a}\right)^2$$

$$f(x) = a(x^2 + \frac{b}{a}x + (\frac{b}{a})^2 - (\frac{b}{a})^2 + \frac{c}{a})$$

$$f(x) = a[(x + \frac{b}{a})^2 - (\frac{b}{a})^2 + \frac{c}{a}$$

$$f(x) = a[(x + \frac{b}{a})^2 - \frac{b^2}{a^2} + \frac{c}{a}]$$

$$f(x) = a[(x + \frac{b}{a})^2 + \frac{ac - b^2}{a^2}]$$

Since the square quantity $(x + \frac{b}{2a}) \geq 0$, the expression within the bracket has its least value when $x = -\frac{b}{2a}$. If $a > 0$, the function has its least value at $x = -\frac{b}{2a}$. This least value of the function ,

$(\frac{ac - b^2}{a})$, is called its minimum. The least point, $(-\frac{b}{a}, \frac{ac - b^2}{a})$, is called the minimum point. If, however, $a < 0$, when $x = -\frac{b}{2a}$, the function has its greatest value called its maximum, and is also equal to $(\frac{ac - b^2}{a})$. The greatest point $(-\frac{b}{a}, \frac{ac - b^2}{a})$, is called the maximum point. In either case the point $(-\frac{b}{a}, \frac{ac - b^2}{a})$, is called the vertex of the parabola, with this point found, and an additional point or two points , the graph can be sketched like the one below:

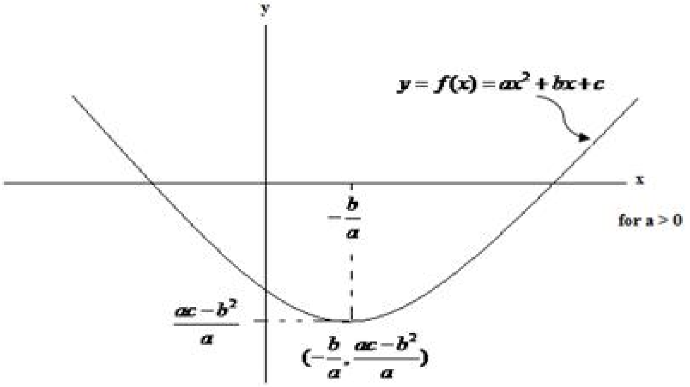


Figure 3.4.1

3.5 Theory of Quadratic equations

The general form of a quadratic equation is: $ax^2+bx+c=0$**,** where **a**, **b** & **c** are constants

The expression b^2-4ac is called the **discriminate** and it is given the letter **Δ** (delta).

All quadratic equations have two roots/solutions. These roots are real (Distinct or equal) or complex**.** *complex - involving the square root of -1

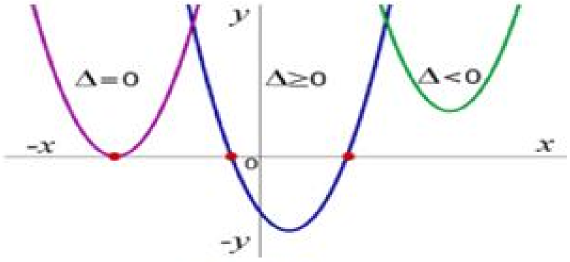


Figure 3.5.1

You can determine the nature of the roots of a given quadratic equation by calculating the value of the discriminant $b^2-4ac=\Delta$. If

(i) $b^2-4ac>0$, then the quadratic equation has two distinct real roots. That is $\Delta>0$

(ii) $b^2-4ac=0$, then the quadratic equation has one real root.

(iii) $b^2-4ac<0$, the quadratic equation has imaginary root

Solution by factorizing - This is best understood with an example. solve: $x^2-7x+12=0$

You must first ask yourself which two factors when multiplied will give **12 ?** The factor pairs of **12** are : 1 x 12, 2 x 6 and 3 x 4 You must decide which of these factor pairs added or subtracted will give **7** ?

1 : 12 ...gives 13, 11 , 2 : 6 .....gives 8, 4, 3 : 4 .....gives 7, 1 so $x^2 - 7x + 12 = (x \pm 3)(x \pm 4)$

Which combination when multiplied makes +12 and which when added gives -7?

These are the choices: (+3)(+4), (-3)(+4), (+3)(-4), (-3)(-4). Clearly, (-3)(-4) are the two factors we want. Therefore $x^2 - 7x + 12 = (x - 3)(x - 4)$

Now to solve the equation $x^2 - 7x + 12 = 0$ factorizing, as above , $(x - 3)(x - 4) = 0$ either $x - 3 = 0$ or $x - 4 = 0$ for the equation to be true. So the roots of the equation are: x = 3,

x = 4

From "O" level you know that you can solve quadratic equation by:

- Factorization method
- Using a formula-
- Graphical method
- Completing the squares- this method is not used at "O" level, so it will be explained here. The other methods are covered at secondary mathematics.

3.6 Solution of quadratic equation

You are now in a position to find the zeros of quadratic equation. Recall that the equation $f(x) = 0$, which is the values of x where $y = 0$, are what you are interested. Generally, two values of x can be found, but these values can be determined by the discriminate $b^2 - 4ac$ without solving the equation. The zeros or the solutions of a quadratic equation are values of x where the graph $f(x) = ax^2 + bx + c$ crosses or meets the x-axis. The two roots can be found graphically, by factorization, by using a formula, or by completing the squares. In this section a method of completing squares will be demonstrated below.

Example 3.6.1 Solve the following equation $ax^2 + bx + c = 0$, using the method of completing squares.

Solution: Take the equation

$$ax^2 + bx + c = 0 \quad \textit{transpose} \quad c$$

$$ax^2 + bx = -c \quad \textit{Divide by a through out}$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a} \quad \textit{take the coefficient of } x, \textit{half it and square it and add it to both sides}$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \quad \textit{take square root both sides}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a} \quad \textit{solve for } x$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \quad \textit{to give you the formula}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

There are two roots or solutions such as $x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ or $x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$. These are the roots, zeros, or solutions of the quadratic equation $ax^2 + bx + c = 0$ In solving quadratic equation, you can use any method provided the method of solving is not given in the question.

Example 3.6.2 Solve the equation $3x^2 - x = 10$, using completing square method.

Solution: you set

$$3x^2 - x = 10$$

$$x^2 - \frac{1}{3}x = \frac{10}{3}$$

$$x^2 - \frac{1}{3}x + \left(-\frac{1}{6}\right)^2 = \frac{10}{3} + \left(-\frac{1}{6}\right)^2$$

$$\left(x - \frac{1}{6}\right)^2 = \frac{10}{3} + \frac{1}{36}$$

$$\left(x - \frac{1}{6}\right)^2 = \frac{120 + 1}{36} \quad \textit{take square root both sides}$$

$$x - \frac{1}{6} = \pm\sqrt{\frac{121}{36}}$$

$$x - \frac{1}{6} = \pm\frac{11}{6} \quad \textit{solve for } x$$

$$x = \frac{1}{6} \pm \frac{11}{6}$$

$$x = 2 \ \textit{or} \ \frac{5}{3}$$

The two roots are 2 or $\frac{5}{3}$. If the method of solving is not specified in the question, you can use any method and it will give you the same solutions.

Example 3.6.3 Find the two values of x that satisfy the following quadratic equation:

$2x^2 + 5x - 4 = 0$ In this equation, you have;

$$a = 2, \quad b = 5, \quad c = -4$$

$$\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-5 \pm \sqrt{5^2 - 4(2)(-4)}}{2 \times 2} \\
&= \frac{-5 \pm \sqrt{25 + 32}}{4} \\
&= \frac{-5 \pm \sqrt{57}}{4} \\
&= -\frac{5}{4} \pm \frac{\sqrt{57}}{4} \\
&= -1.25 \pm 1.89
\end{aligned}$$

$$x = -1.25 + 1.89, \quad \underline{x = 0.64}$$

$$x = -1.25 - 1.89, \quad \underline{x = -3.11}$$

Relations between zeros and coefficients of the quadratic equations

If α and β are roots of a given quadratic equation, then the standard form of a quadratic equation is;

$x^2-(\alpha+\beta)x+\alpha\beta=0$, and the general equation of a quadratic equation is $ax^2+bx+c=0$, you divide this later equation by a throughout, you get: $x^2+\frac{b}{a}x+\frac{c}{a}=0$. On comparing the two equations, you will get:

$$\alpha+\beta=-\frac{b}{a} \text{ and } \alpha\beta=\frac{c}{a}$$

Example 3.6.4 Without solving, form a quadratic equation whose roots are the squares of the roots of the equation: $2x^2+x-6=0$

Solution: Let α and β be the roots of the given equation $2x^2+x-6=0$. You compare the equation $x^2-(\alpha+\beta)x+\alpha\beta=0$, with the given equation with 1 as the coefficient of x^2, $x^2+\frac{1}{2}x-3=0$ and then you will have; $\alpha+\beta=-\frac{1}{2}$, and $\alpha\beta=-3$.

Since $(\alpha+\beta)^2=\alpha^2+2\alpha\beta+\beta^2$, the sum of the roots for the new equation is $\alpha^2+\beta^2=\frac{25}{4}$, and the product is, $(\alpha\beta)^2=9$. Therefore, the required equation is $x^2-\frac{25}{4}x+9=0$, which is $4x^2-25x+36=0$.

3.7 Polynomials

3.7.1 Introduction A functions of the form $f(x)=a_ox^n+a_1x^{n-1}+a_2x^{n-2}+....+a_{n-1}x+a_n$, where $a_i, for\, i=1,2,3,...,n$ are constants and n is a positive integer. This is called a polynomial function of order n. If you set $f(x)=0$, then we have $a_ox^n+a_1x^{n-1}+a_2x^{n-2}+....+a_{n-1}x+a_n=0$, this called a polynomial equation of order n. If you have a function of the form: $f(x)=3x^5-2x^3+x+1$ this is a polynomial of order 5. And if you have an equation of the form $3x^4-2x^3+x+1=0$ this is a polynomial equation of order 4. Note that a polynomial of order n has a finite sum of n terms. Our task on polynomials is either to sketch if you have a polynomial function or solving for the polynomials roots if you have a polynomial equation. To do this we shall begin by looking at polynomial expression called algebraic expressions. We should look at how to divide expression by another expression.

3.7.2 Division of algebraic expressions

Just as you divide any real number by another real, such as 345 divided by 34, you carried out the operations as follows: $34\overline{)342}$ this can also be done when dividing algebraic expressions.

Long division: If $f(x)$ is divided by $d(x)$ and $d(x)$ is non zero, and the degree of $d(x) \leq$ the degree of $f(x)$, then two unique polynomials $\boldsymbol{q(x)}$ the quotient and $\boldsymbol{r(x)}$ the remainder exist, so that:

$$\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$$

Note - the degree of $r(x)$ is less than the degree of $d(x)$.

Example 3.7.1

$$\frac{5x^3 + x^2 - 3x + 2}{x^2 - 2x + 5}$$

$$\begin{array}{r l}
 & \quad 5x + 11 \\
x^2 - 2x + 5 & \overline{)5x^3 + \quad x^2 - \quad 3x \quad + 2} \\
 & \underline{-(5x^3 - 10x^2 + 25x)} \\
 & \qquad 11x^2 - 28x + 2 \\
 & \qquad \underline{-(11x^2 - 22x + 55)} \\
 & \qquad\qquad -6x - 53
\end{array}$$

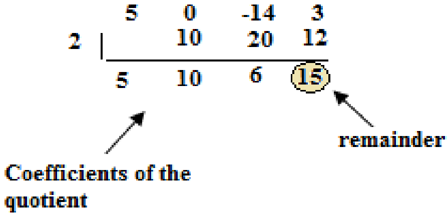
$$\begin{aligned}
\frac{5x^3 + x^2 - 3x + 2}{x^2 - 2x + 5} &= 5x + 11 + \left(\frac{-6x - 53}{x^2 - 2x + 5}\right) \\
&= 5x + 11 - \left(\frac{6x + 53}{x^2 - 2x + 5}\right)
\end{aligned}$$

Synthetic division: You can determine the quotient and remainder when an expression f(x) is divided by a factor x-a, where a is a constant using a method called synthetic division.

The process of division for polynomial in x (or any one letter) may be greatly simplified when the divisor is in the form x – a. This process, known as synthetic division, will be illustrated.

Example 3.7.2 Divide $5x^3-14x+3$ by x – 2 using synthetic division.

Solution: You write the coefficients of the expression as follows:



The procedure is as follows: after writing the coefficients of the expression, you find a, in this case a = 2. You write the division upside down, drop 5 the coefficient of the highest power, multiply this by a, in this case by 2, write the answer under the next coefficient, add write the answer below, in this case 10, multiply your answer by 2, write the answer, 20 under the next coefficient, add, your answer is 6, multiply 6 by 2 and write your answer under the constant 3, add, you will get 15. The number 15 is the remainder, while your quotient is: $5x^2 10x+6$ and remainder 6. This is easier and faster in find the quotient and remainder, than the long division.

3.7.3 The Remainder Theorem: If a polynomial f(x) is divided by (x-a), where a is any constant, until a constant remainder independent of x is obtained, this remainder is equal to f(a).

Example 3.7.2 Find the remainder when ($2x^3$+3x+x) is divided by (x+4).

$f(x)=2x^3+3x^2+x$ is divided by $(x+4)$

If a polynomial $f(x)$ is divided by $(x-a)$,
the remainder is $f(a)$.
$\Rightarrow \quad (x+4)=(x-a), \quad \therefore a=-4$
$$f(-4)=2(-4)^3+3(-4)^2+(-4)$$
$$=-128+48-4$$
$$=-84$$
<u>the remainder is -84</u>

The reader may wish to verify this answer by using algebraic division.

3.7.4 The Factor Theorem

(a special case of the Remainder Theorem) . If f(a) = R is zero , that is, a is a zero of f(x), then *(x−a)* is a factor of the polynomial *f*(*x*).

The converse is true , that is, If (x-a) is a factor of f(x), then f(a) = R = 0, and a is a zero of f(x).

Example 3.7.3 use the Factor Theorem to find factors of the function

$f(x) = x^3 + 3x^2 - x - 3$

choosing factors of the highest constant 3

1,-1,3,-3

$f(1) = (1)^3 + 3(1)^2 - (1) - 3$

$= 1 + 3 - 1 - 3 = 0 \quad \therefore \underline{(x - 1) \text{ a factor}}$

$f(-1) = (-1)^3 + 3(-1)^2 - (-1) - 3$

$= -1 + 3 + 1 - 3 = 0 \quad \therefore \underline{(x + 1) \text{ a factor}}$

$f(3) = (3)^3 + 3(3)^2 - (3) - 3$

$= 27 + 27 - 3 - 3 = 48 \quad \therefore \underline{(x - 3) \text{ not a factor}}$

$f(-3) = (-3)^3 + 3(-3)^2 - (-3) - 3$

$= -27 + 27 + 3 - 3 = 0 \quad \therefore \underline{(x + 3) \text{ a factor}}$

$\therefore \quad \underline{x^3 + 3x^2 - x - 3 = (x - 1)(x + 1)(x + 3)}$

n.b. the sign change of the constant $f(5) \Rightarrow (x - 5)$

Fundamental Theorem of algebra: Every polynomial function whose defining equation is

$$f(x) = a_o x^n + a_1 x^{n-1} + a_2 x^{n-2} + ... + a_{n-1} x + a_n$$

For $n \geq 1$ *and* $a_0 \neq 0$, has at least one (real or complex) root or at most n roots.

3.8 Inequalities:

Expressions involving inequality signs such as <, >, or ≤, ≥ are called inequalities. These type of inequalities do not demand one value as an answer but a set of values which will satisfy the inequality.

3.8.1 **Linear Inequalities**

Inequalities of the form $(x-a)\geq 0$ or $(x+a)<0$ are called linear inequalities. Our problem is to find the set of values that satisfied the inequality.

Example 3.8.1 Find the solution set for the following inequality: $(x+1)<0$.

Solution: We solve the inequality as follows:

$$(x+1)<0$$
$$x<-1$$

The solution set is $\{x : x<-1, x\in R\}$, this is written in set builder notation

3.8.2 Quadratic Inequalities.

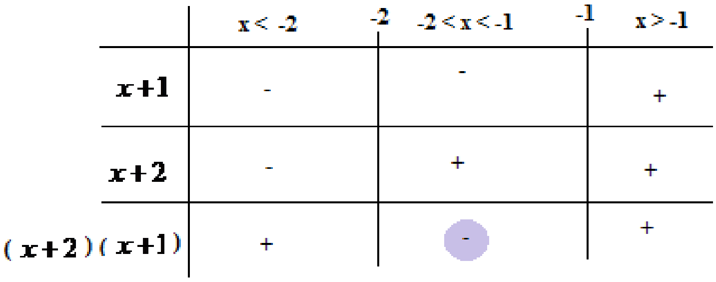
Inequalities of the form $ax^2+bx+c\geq 0$ are called quadratic inequalities because they involve quadratics. Inequalities of this form require you to factorize the expression on the left hand side. After factorizing then you follows the following procedures in order for you get the set(s) of values that satisfies the inequality.

Example 3.8.2 Find the solution set for the inequality $x^2+3x+2\leq 0$

Solution: You begin by factorizing the quadratic expression as follows;

$$x^2+3x+2\leq 0$$
$$(x+2)(x+1)\leq 0$$

To find the solution set of this inequality, you find the critical values of the inequality by solving for x. In this case x = -2 or -1. You make a table showing the critical points as shown below: Take a number in the given interval and substitute in the factor and write the sign in the box as indicated, and the last row which is the product of the two factors will determine the solution depending on the inequality given in the question. In this case, you want less that or equal so your solution will be the interval with a negative in the last row.



	x< -2	-2	-2<x<-1	-1	x>-1
x+1	-		-		+
x+2	-		+		+
(x+2)(x+1)	+		-		+

Diagram 3.8.1

The table above shows that the sets of point which satisfies the given quadratic inequality is read from the last row, ie the shaded part, which is $\{x; -2 \le x \le -1, x \in R\}$.

Example 3.8.3 Find the solution set for the following inequality $\frac{x-1}{x+1} > 2$

Solution. To find the solution set of this problem, you first transpose 2 to the left because you cannot cross multiply since the denominator is a variable. Note that when you multiply by a negative quantity the sign changes, so x is a variable and it can assume a negative number, but you do not know for which values. You proceed as follows:

$$\frac{x-1}{x+1} > 2$$

$$\frac{x-1}{x+1} - 2 > 0$$

$$\frac{x-1-2(x+1)}{x+1} > 0$$

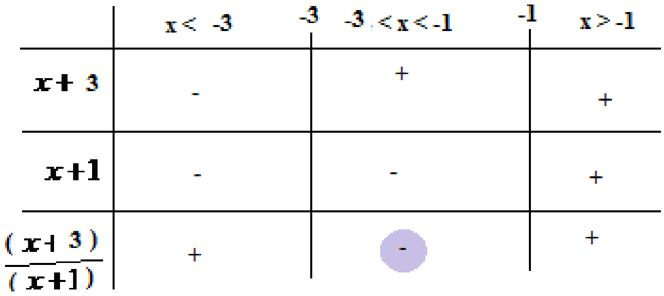
$$\frac{x-1-2x-2}{x+1} > 0$$

$$\frac{-x-3}{x+1} > 0$$

$$\frac{x+3}{x+1} < 0$$

Note the change of the sign from greater than zero to less than zero.

You now find the critical values of the numerator and denominator and proceed as example 1 given above. The critical values are: $x = -3$ and $x = -1$



	x< -3	-3	-3 <x<-1	-1	x>-1
x+ 3	-		+		+
x+1	-		-		+
(x+ 3)/(x+1)	+		-		+

Diagram 3.8.2

3.9 Binomial Theorem

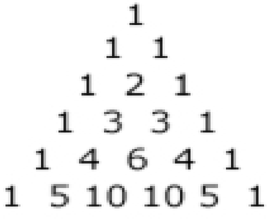
Introduction: This section of work is to do with the expansion of expressions of the forms: $(a+b)^n$, where $a, b \in R$, $n \in Z^+$. Pascal's Triangle and the Binomial Theorem give you a way of expressing the expansion as a sum of ordered terms. You will also discuss expressions of the form: $(1+x)^n$, where x is a variable and n is a rational number or negative integers.

3.9.1 Pascal's Triangle

You know the expansion of the following:

$$\begin{aligned}
(a+b)^0 &= 1 \\
(a+b)^1 &= a+b \\
(a+b)^2 &= a^2+2ab+b^2 \\
(a+b)^2 &= a^3+3a^2b+3ab^2+b^3 \\
&etc
\end{aligned}$$

See the coefficients of the terms given in each of the expressions given above. The coefficients follow the following patterns:



1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

Diagram 2.9.1

The first line represents the coefficients for n=0. $(a+b)^0 = 1$. The second line represents the coefficients for n=1. $(a+b)^1 = a + b$. The third line represents the coefficients for n=2. $(a+b)^2 = a^2+2ab+b^2$. The sixth line represents the coefficients for n=5. $(a+b)^5 = a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$.

This is a method of predicting the coefficients of the binomial series is called Pascal triangle, since the coefficients form a triangle. Coefficients are the **constants**(1,2,3,4,5,6 etc.) that multiply each variable, or group of variables. Consider $(a+b)^n$ variables a, b, and n a positive integer. The Binomial Theorem builds on Pascal's Triangle in practical terms, since writing out triangles of numbers has its limitations. It is generally difficult to use Pascal's triangle for n greater than 6, hence the use of Binomial Theorem becomes necessary. For example $(a+b)^{20}$, this would require you to tabulate the coefficients from 2 to 19 in order to obtain the coefficients of $(a+b)^{20}$. It is now necessary to state the Binomial Theorem which will help you to expand expressions like $(a+b)^{20}$. In using this theorem, you need to know combinations and to appreciate this; a theorem may make it easy to follow:

Theorem: The total number of combinations of n objects taken r at a time C(n,r) is given by the expression: $C(n,r)=\dfrac{n!}{(n-r)!r!}$. This formula is valid for positive values $n\in Z$, also $n\geq r$ always. This formula gives the coefficients of the Binomial coefficients. The following notations are use for the formula given above. $n!=n(n-1)(n-2)(n-3).......1$, that is, $3!=3.2.1=6$, $2!=2.1=2$, $1!=1$, and $0!=1$ by definition. Therefore,

$$C(5,3)=\frac{5!}{(5-3)!3!}=\frac{5!}{2!3!}=\frac{5.4.3!}{2.1.3!}=10.$$

Theorem: (The Binomial Theorem): For any positive integer n,

$$(a+b)^n=C(n,0)a^n+C(n,1)a^{n-1}b+C(n,2)a^{n-2}b^2+.....+C(n,r)a^{n-r}b^r+....+C(n,n)b^n$$

where 'n' is the power/index of the original expression and 'r' is the number order. You can immediately note the following properties of the expansion given above for any positive integer n.

(1) The number of terms in any identity is n+1.
(2) The first term in the identity is a^n and the last term is b^n
(3) The exponent of a decreases by one and that of b increases by one from any term to the next term, so that the sum of the exponents of a and b in any term is n.
(4) For the first term and last, the second and next to the last , the third and from the last , and so forth, terms of the identity, the coefficients are the same.

This is a way of finding all the terms of the expansion, the coefficients and the powers of the variables.

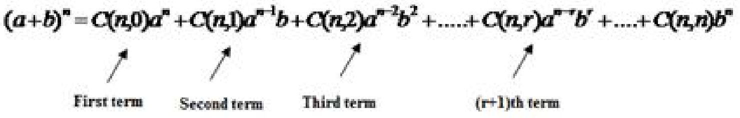


Diagram 2.9.2

Our interest is generally to find the nth term; all you need is to understand the equation given above. Note that term 1 has the power of b zero and r = 0 , second term has power of b as 1 and r = 1, third term has power of b as 2 and r = 2 etc. You can determine the nth term of the expansion using this theory.

Example 3.9.1 Expand the expression $(a+b)^5$. You proceed as follows:

$$(a+b)^5 = a^5 + {}^5C_1a^4b + {}^5C_2a^3b^2 + {}^5C_3a^2b^3 + {}^5C_4ab^4 + b^5$$

$${}^5C_1 = \frac{5!}{(5-1)!1!} = \frac{5.4.3.2.1}{(4.3.2.1)1} = 5$$
$${}^5C_2 = \frac{5!}{(5-2)!2!} = \frac{5.4.3.2.1}{(3.2.1)2.1} = \frac{5.4.}{2.1} = \frac{20}{2} = 10$$
$${}^5C_3 = \frac{5!}{(5-3)!3!} = \frac{5.4.3.2.1}{(2.1)3.2.1} = \frac{5.4.}{2.1} = \frac{20}{2} = 10$$
$${}^5C_4 = \frac{5!}{(5-4)!4!} = \frac{5.4.3.2.1}{(1)4.3.2.1} = 5$$

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

Example 3.9.2 Write down $(3x-2)^5$ as a Binomial expansion, using

$$(a+b)^5 = a^5 + C(5,1)a^4b + C(5,2)a^3b^2 + C(5,3)a^2b^3 + C(5,4)ab^4 + b^5$$

And the value of the coefficients from the previous , let a = 3x, b = -2

$$(3x-2)^5 = (3x)^5 + C(5,1)(3x)^4(-2) + C(5,2)(3x)^3(-2)^2 + C(5,3)(3x)^2(-2)^3 + C(5,4)(3x)(-2)^4 + (-2)^5$$
$$(3x-2)^5 = 243x^5 + 5(81x^4)(-2) + 10(27x^3)(4) + 10(9x^2)(-8) + 5(3x)(16) + (-32)$$
$$(3x-2)^5 = 243x^5 - 810x^4 + 1080x^3 - 720x^2 + 240x - 32$$

Example: 3.9.3 Find the 15th term in the expansion of $(3+2x)^{20}$

Solution: Using the general form $\binom{n}{r}a^{n-r}b^r$, you know $n = 20$, if you want the 15th term, this means that $r = 14$, $a = 3$ and $b = 2x$, substitute these values in the general form as:

$$\binom{20}{14}3^{20-14}(2x)^{14}$$
$$\frac{20!}{6!14!}3^6.2^{14}x^{14}$$
$$\frac{20.19.18.17.16.15.14!}{6.5.4.3.2.1.14!}.3^6.2^{14}.x^{14}$$
$$\underline{19.17.8.15.3^6.2^{14}.x^{14}}$$

This is the 15th term of the given expansion.

Example 3.9.4. Find the term independent of x in the expansion of $\left(x-\frac{1}{x}\right)^{16}$

Solution: You write the general term $\binom{n}{r}a^{n-r}b^{r}$, in this case you know the value of n as n = 16, you know the value of a as a = x and the value of b as $b=\frac{1}{x}$, but you don't know r, so you need to find r. You substitute these values in the general form and solve for r, then substitute the value of r and you will find the term which will have no x.

$$\binom{16}{r}x^{16-r}(-\frac{1}{x})^{r}$$
$$\binom{16}{r}x^{16-r}(-1)^{r}x^{-r}$$
$$\binom{16}{r}(-1)^{r}x^{16-2r}$$

But you want a term which has no x, that mean you set $16-2r=0$, because x^0 is 1. That means $r=8$. Now you substitute r in the general form and simplify as follows:

$$\binom{16}{8}(-1)^{8}x^{16-16}$$
$$\frac{16!}{8!8!}(-1)^{8}.1$$
$$\frac{16.15.14.13.12.11.10.9.8!}{8.7.6.5.4.3.2.1.8!}.1$$
$$13.11.10.9$$
$$12870$$

This is the 9th term which is independent of x in the expansion.

3.9.2 The Expansion of $(1+x)^n$

The above method of expanding expressions of the form $(a+b)^n$ is only used when n is a positive integer. For negative powers of rational numbers and negative integer, this method will not work. Instead we use the form:

$$(1+x)^n = 1+\frac{nx}{1!}+\frac{n(n-1)x^2}{2!}+\frac{n(n-1)(n-2)x^3}{3!}+.......$$ For $n \in Z^- \, and Q$ This series is valid in the region $|x|<1$. You will be expected to write this series generally up to including the x^3, and write the region where this series is valid.

Example 3.9.5 Write the expansion of the expression $\sqrt{2-3x}$, and write the region where this expansion is valid.

Solution: This expression can be written as: $(2-3x)^{\frac{1}{2}}$, so we can compare this expression with the form given above. That means we write it in the form; $(1+x)^n$ and compare, and what you see is that $(2-3x)^{\frac{1}{2}}$ can be written as; $[2(1+(-\frac{3}{2}x)]^{\frac{1}{2}}$ Now we can compare as:

$x=-\frac{3}{2}x$, and $n=\frac{1}{2}$ and substitute these in the expression

$(1+x)^n = 1+\frac{nx}{1!}+\frac{n(n-1)x^2}{2!}+\frac{n(n-1)(n-2)x^3}{3!}+.......$ and this is what we get; so using

$x=-\frac{3}{2}x$ and $n=\frac{1}{2}$ we have

$$[2(1+(-\frac{3}{2}x)]^{\frac{1}{2}} = \sqrt{2}[1+\frac{1}{2}x+\frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}x^2+\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}x^3+.........]$$

$[2(1+(-\frac{3}{2}x)]^{\frac{1}{2}} = \sqrt{2}[1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3+.........]$ This series is valid in the region $\left|\frac{3}{2}x\right|<1$ which implies $|x|<\frac{2}{3}$

Exercise 3

1. Solve the following quadratic equations by factorization method.

   (i) $x^2 - 4x - 12 = 0$ (ii) $3x^2 + 5x - 12 = 0$

   (iii) $3(4x - 1)^2 + 1 = 16$ (iv) $(x - 2)^2 = -4$

2. Use the method of completing the squares to solve each of the following equations.

   (i) $x^2 - 10x + 24 = 0$ (ii) $2x^2 + 12x + 5 = 0$

   (iii) $3x^2 + 12 - 2 = 0$ (iv) $x^2 - 3x = -1$

3. Without solving the equation, determine the nature of roots of the following equations.

   (i) $4x^2 + 20x + 25 = 0$ (ii) $x^2 + 4x + 7 = 0$

   (iii) $16x^2 = 40x - 25$ (iv) $2x^2 + 5 + 7 = 0$

4. If α and β are the roots of the equation $x^2 - 4x + 2 = 0$, find the value of :

   (i) $\frac{1}{\alpha} + \frac{1}{\beta}$ (ii) $(\alpha + 1)(\beta + 1)$ (iii) $\alpha^2 + \beta^2$

   (iv) $\alpha^2\beta + \alpha\beta^2$ (v) $(\alpha - \beta)^2$ (vi) $\frac{1}{\alpha^2 + 1} + \frac{1}{\beta^2 + 1}$

5. If α and β are the roots of the equation $x^2 - 4x + 2 = 0$, find equations whose roots are; (i) $\alpha + 3$, $\beta + 3$ (ii) $\alpha + 3\beta$, $\beta + 3\alpha$

6. (a) Find two numbers whose sum is 16 and whose product is a maximum.

   (b) A man with 180m of fencing wishes to fence off an area in the shape of a rectangle. What should be the dimension of the area if the enclosed space is to be as large as possible?

7. For each of the following functions, find the maximum or minimum point and value.

   Sketch the graph

   (i) $f(x) = x^2 + 6x + 5$ (ii) $f(x) = x^2 + x - 6$

(iii) f(x) = $2x^2 + 5x - 12$ (iv) f(x) = $-2x^2 + 11x - 15$

(iv) f(x) = $-3x^2 + 5x - 4$

8. Suppose that the equation $p(x) = -2x^2 + 280x - 1000$, where x represents the number of items sold, describes the profit function for a certain business. How many items should be sold to maximize the profit?

9. The height of a projectile fired vertically into the air (neglecting air resistance) at an initial velocity of 96 feet per second is a function of time x and is given by the equation $f(x) = 96x - 16x^2$. Find the highest point reached by the projectile.

10. Find the value of k so that the equation:

(i) $2x^2 + kx - 15 = 0$ has one root = 3

(ii) $3x^2 + kx - 2 = 0$ has roots whose sum is equal to 6

(iii) $5x^2 - 8x + k = 0$ has roots whose product is equal to $\frac{1}{5}$

(iv) $3x^2 - 5x + 8 = kx$ has roots numerically equal but opposite in sign.

11. (a) Find the range of values of k so that the equation:

(i) $3x^2 - 4kx + k = 0$ will have real roots

(ii) $kx^2 + 4\sqrt{3}x + k = 1$ will have imaginary roots

(b) Find the range of values of k or value of k so that the graph of the function where y equals:

(i) $3x^2 - 9x + k$ will touch (have its vertex on) the x axis.

(ii) $x^2 + 2kx + \frac{3}{4} - k$ will not intersect the x axis`

(iii) $4x^2 + 4\sqrt{2}kx + k + 3$ will intersect the x axis inn two real points.

12. Find the remainder when

(a) $5 + 6x + 7x^2 - x^3$ is divided by $x + 2$

(b) $6x^3 + 7x^2 - 15x + 4$ is divided by $x - 1$

(c) $x^4 - 3x^3 + 2x^2 + 5$ is divided by $x - 1$

(d) $8x^3 - 10x^2 + 7x + 3$ is divided by $2x - 1$

(e) $9x^3 + 4$ is divided by $3x + 2$

13. Factorize the following polynomials

(a) $6x^3 - 5x^2 - 17x + 6$

(b) $2x^4 + 7x^3 - 17x^2 - 7x + 15$

(c) $6x^4 + 31x^3 + 57x^2 + 44x + 12$

14. By using Synthetic Division, find the quotient and remainder when

(a) $x^4 - 2x^3 - 3x^2 - 4x - 8$ is divided by (i) $x - 2$ (ii) $x + 1$

(b) $x^3 + 3x^2 - 2x - 5$ is divided by (i) $x + 2$ (ii) $x - 3$

15. Show that $x + y$ is a factor of $x^5 + y^5$ and $x^7 + y^7$. By using synthetic division, find the quotient in each case.

16. Factorize the polynomial $x^3 - 3x^2 + 4x + 12$. Hence calculate the ranges of values of x for which $x^3 - 3x^2 > -4x - 12$.

17. Factorize the expression $6x^3 - 7x^2 - x + 12$. Hence calculate the ranges of values of x for which $6x^3 - 7x^2 > 12 - x$.

18. Given that $(x + 2)$ is a factor of $2x^3 + 6x^2 + 6x - 5$, find the remainder when the expression is divided by $(2x - 1)$.

19. The expression $3x^3 + 2x^2 - bx + a$ is divisible by $(x - 1)$ but leaves a remainder of 10 when divided by $(x + 1)$. Find the values of a and b.

20. The expression $6x^3 - 23x^2 + ax + b$ gives a remainder of 11 when divided by $(x - 3)$ and a remainder of -21 when divided by $(x + 1)$. Find the values of a and b and hence factorize the expression.

21. Find the first four terms in the series expansion of $(1-\frac{2}{x})^{\frac{1}{2}}$ in ascending power of x.

22. The series expansion of $(1+px)^q$ in ascending powers of x has coefficients -10 and 75 in x and x^2 terms respectively.

(a) Find the values of p and q

(b) Find the coefficient of the x^3 and x^4 terms in the expansion

(c) State the set of values of x for which the series is valid.

23. Expand the following up and including the term x^3

(a) $(1+2x)^{\frac{3}{2}}$ (b) $\sqrt{4+3x}$ (c) $\frac{1}{1-x}$

24. Find the 15th term in the expansion $(x-\frac{1}{2x})^{20}$. Hence find the tem independent of x.

25. Find the values of x for which the following inequalities are satisfied:

(i) $3x-7 > 0$ (ii) $5x-3 < 8x-12$ (iii) $x^2 + 2x \geq 99$

(iv) $x^2 + 2x + 4 > 0$ (v) $\frac{x-2}{x-5} > 0$ (vi) $\frac{1}{x-2} < \frac{1}{3}$

Unit 4 Logarithm and Exponential functions

4.1 Introduction

This unit introduces special functions which plays major roles in calculus. It shows the connectedness of these two functions which would help you when navigating from one base to the other. Graphs of these functions are explored and show how these functions can help certain types of equations.

4.2 Objectives

Upon completion of this unit you will be able to:

- Graph logarithms or exponential functions
- Solve equations involving logarithms and exponentials
- Apply laws of logarithms in solving logarithmic equations.

4.3 Exponential function

A function of the form $f(x)=e^x$ where x is real and a is a positive constant is called an exponential function. The word “exponential” comes from the word exponent, which is another word for “power’ or “index’. So for an exponential function $f(x)=e^x, x \in R$, the variable x is call the power , or the index, or the exponent. For a = 2, 3, etc, you can see how the graph of each of these functions look like.

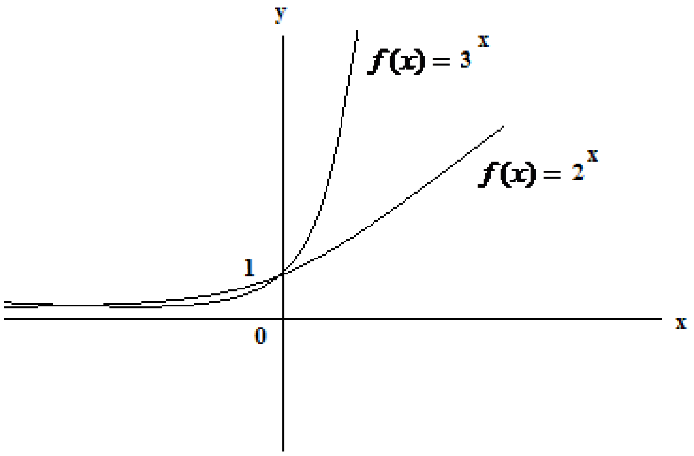


Figure 4.3.1

The Natural Logarithm function

The graph of the function $f(x)=e^x, x \in R$ is as follows:

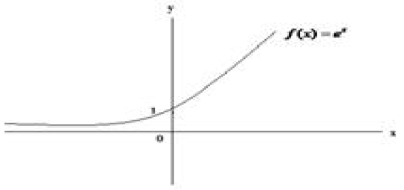


Figure 4.9.2

You can see that this function is one-to-one. So it must have an inverse. To find the inverse function, write $x = e^y$, and try to make y the subject of the equation. In the expression e^y, e is called the base and y is called the power, or index or exponent. A fourth name for y is the logarithm. So the expression $x = e^y$ can be interpreted as meaning that y is the logarithm of the number x in the base e. We write this as $y = \log_e x$ or more usually as $y = \ln x$.

Logarithm in base e is usually called ***natural logarithm***.

The function $g(x) = \ln x, x > 0$, is the inverse of the function $f(x) = e^x, x \in R$, that is $g = f^{-1}$, and the graphs of f and its inverse are shown below:

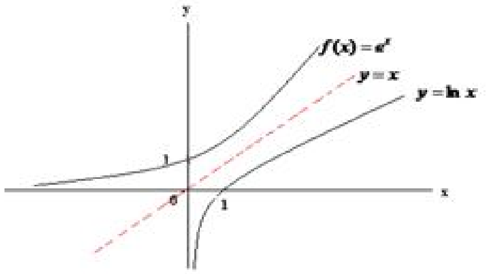


Figure 4.9.3

These are the graphs of e^x and of $\ln x$. You can see that the graph of $y = \ln x$ is the reflection of the graph of $y = e^x$ in the line $y = x$.

4.4 Logarithms

You have seen in the section above that the word power, index, exponent, and logarithm are synonymous: they are four different words to describe exactly the same thing.

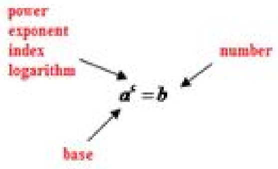


Diagram 4.9.1

In this module use the word “logarithm” and take the base to the positive $(a > 0)$, and the graph will look like:

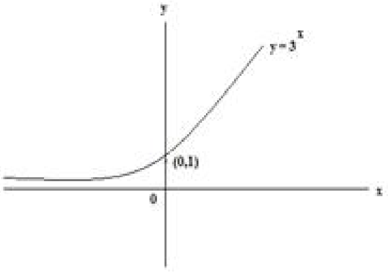


Figure 4.9.4

Now the statement $b = a^c$ reads ‘the number b is equal to the base a raised to the logarithm c’. another way of reading the statement is ‘c is the logarithm of the number b to the base a’. What this means that you can express an exponential form into logarithm and vise-visa. That is, $a^c = b \Leftrightarrow c = \log_a b$. This is a very useful relationship between exponential and logarithm when changing bases of logarithms.

Laws of logarithm

1. $\log_a MN = \log_a M + \log_a N$
2. $\log_a \frac{M}{N} = \log_a M - \log_a N$
3. $\log_a^{M^n} = n\log_a^M$
4. $\log_a^a = 1$
5. $\log_a 1 = 0$

Example 4.4.1 Find x if $\log_2^{32} = x$

Solution: You write this in exponential form: $\log_2 32 = x \Leftrightarrow 2^x = 32$ this the same as

$\Leftrightarrow 2^x = 2^5$, by inspection you have x = 5

Example 4.4.2 Find the value of $\log_3^{243}$

Solution: This can be written as: $\log_3 243 = \log_3 3^5 = 5\log_3 3 = 5$

Example 4.4.3 Simplify $\log_a 4 + 2\log_a 3 + \log_a 3^2 - \log_a 6$

Solution:: This can be written as:

$$\log_a 4 + \log_a 3^2 + \log_a 9 - \log_a 6 = \log_a \frac{4x9x9}{6} = \log_a \frac{324}{6} = \log_a 54$$

Equations of the form $a^x = b$

Although you can solve equation such as $3^x = 9$, $4^x = 64$, and so on, because the value of x is an integer, in general you cannot solve such equations by inspection. It is time consuming to find a solution to $5^x = 67$ by trial and improvement. The standard method of solving such equations is by taking logarithms with suitable base.

Example 4.4.4 Solve the equation $2(5^{2x}) - 5^x = 6$

Solution: In solving such equations, you let $z = 5^x$ and your equation becomes $2z^2 - z = 6$

Which is a quadratic equation and when you solve this equation your solutions are $z = 2, -\frac{3}{2}$. You ignore the negative since there is no logarithm of a negative number. So you use z = 2, from this value you can find the value of x as follows: $z = 5^x$ implies $2 = 5^x$ taking logarithm to base 5 will give you $x = \log_5 2$

Exercise 4

1. Solve the equations

   (a) $5^{3X+2} = 43$ (b) $4^{3x+2} = 7^{x-3}$ (c) $25^x = 5^{x+1} - 6$ (d) $2e^x + 2e^{-x} = 5$

   (e) $2(3^{2x}) - 9(3^x) + 4 = 0$ (f) $4^{2x} + 48 = 4^{x+2}$ (g) $2^{2x+1} = 3(2^x) - 1$

2. Find the values of x for which $\log_3^x - 2\log_x^3 = 1$

3. Solve the equations

   (a) $\log_3(2 - 3x) = \log_9(6x^2 - 19x + 2)$ (b) $\log_2^x + 4\log_x^2 = 5$

4. (i) Find the values of x for which $2\ln 2x - 6\ln 2 = \ln(x-3)$

(ii) Given that x + y = 2 and $3^x = 4^y$ Show that $x = \dfrac{\ln 16}{\ln 12}$, and find y in a similar form.

Unit 5 Matrices

5.1 Introduction

This unit begins by reviewing basic operations of matrices; these are addition, subtraction, and multiplications of matrices. The 'O 'level knowledge of matrices is extended from a 2x2 matrices to a 3x3 matrices. The unit discusses determinants, transpose, and inverses of a 3x3 matrices. Lastly, the unit explores the methods of solving systems of equations using matrices.

5.2 Objectives

Upon completion of this unit you will be able to:

- Add, subtract, and multiply matrices
- Find the inverse of a matrix
- Solve systems of equations using matrix methods

5.3 Operations on matrices

A matrix is an array of numbers arranged in rows and columns, such as

$$\begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 3 & 4 & -2 \\ -1 & 0 & 3 \\ 2 & 1 & 2 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$$

Such an array of numbers is called a Matrix. The numbers such as; 2, 3 for first one are called the elements or entries of the matrix. The horizontal line 2 , 3 are called rows while the vertical line are called columns. In general, if a rectangular array has m rows and n columns, you call it an m x n matrix. Therefore the matrix given above is a 2x2 matrix, it means it has two rows and two columns. The number of rows is stated first, followed by the number of columns. Then you have the first matrix a 2x2 matrix, the second as a 3x3 matrix and the last one is a 3x1 matrix.

5.3.1 Basic properties of Matrices

You use capital letters to denote matrices, and you enclose actual matrix in square brackets.

Definition: Two matrices A and B are equal if and only if :

(i) The two matrices have the same number of rows and columns

(ii) Their corresponding elements are equal.

Definition: The sum of two matrices A and B denoted by A + B, is the matrix such that each of its elements is the sum of the corresponding elements of A and B.

Example 5.3.1: If $A = \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$, find A + B.

Then $A + B = \begin{bmatrix} 2+1 & 3+2 \\ 1+0 & 5+4 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 1 & 9 \end{bmatrix}$ with the definition of any operation, you can now think of the possibility of an "identity matrix" of the operation addition.

Definition: If all of the entries of a matrix are zeros, the matrix is called the zero matrix, and is denoted by O.

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ That is A+O = A for any matrix A. Furthermore, there is the inverse matrix with respect to addition called the negative of A and denoted by –A, such that A +(-A) = O. Note that you can only add matrices of the same size or matrices of the same order.

Definition: The product of a scalar k and a matrix A , denoted by kA, is the matrix in which each entry is k times the corresponding element of A. For example, if $A = \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$, then $2A = \begin{bmatrix} 4 & 6 \\ 2 & 10 \end{bmatrix}$. Where k = 2.

Definition: The product of two matrices A and B is the matrix AB whose entry in the ith row and jth column is the sum of the products formed by multiplying each entry in the ith row of A by the corresponding entry in the jth column of B.

Example 5.3.2 If $A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 2 & 4 \end{bmatrix}$, and $B = \begin{bmatrix} 0 & 2 \\ 1 & 3 \\ 5 & -2 \end{bmatrix}$, find AB and BA

Solution: You can multiply matrices if they are of the same order, or the number of elements of the rows of the first matrix is equal to the number of elements of the columns of the second matrix. For instance, a 2x3 matrix can be multiplied by a 3x3 matrix and the product will be a 2x3 matrix. In this example you are multiplying a 2x3 matrix by a 3x2 matrix and the product will be a 2x2 matrix.

$$\begin{pmatrix} 3 & 2 & -1 \\ 1 & 2 & 4 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 1 & 3 \\ 5 & -2 \end{pmatrix} = \begin{bmatrix} 3x0+2x1+(-1)x5 & 3x2+2x3+(-1)x(-2) \\ 1x0+2x1+4x5 & 1x2+2x3+4x(-2) \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 14 \\ 22 & 0 \end{bmatrix}$$

Note that matrices are not commutative. $AB \neq BA$

5.4 Determinant of a matrix

You may have learnt how to find a determinant of a 2x2 matrix a ‘O ‘level mathematics. In this section you will learn how to find the determinant of a 3x3 matrix. Note that determinant can only be found in a square matrix. You cannot find a determinant for a rectangular matrix.

If a matrix A= $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2x2 matrix, the determinant of A,

written $\det A = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

To find the determinant of a 3x3 matrix is not as easy as this one. Take for example a general 3x3 matrix like $B = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$, then $|B| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$

The 3x3 matrix is decomposed in determinants of 2x2 matrix which you can easily evaluate each of the 2x2 matrices and add the to find the determinant of a 3x3 matrix.

Example 5.4.1 Given that $A = \begin{pmatrix} 1 & 2 & 4 \\ -1 & 3 & 0 \\ 0 & 1 & 5 \end{pmatrix}$, find $|A|$.

Solution:

$$|A| = \begin{vmatrix} 1 & 2 & 4 \\ -1 & 3 & 0 \\ 0 & 1 & 5 \end{vmatrix} = 1\begin{vmatrix} 3 & 0 \\ 1 & 5 \end{vmatrix} - 2\begin{vmatrix} -1 & 0 \\ 0 & 5 \end{vmatrix} + 4\begin{vmatrix} -1 & 3 \\ 0 & 1 \end{vmatrix} = 1(15-0) - 2(-5-0) + 4(-1-0)$$

$$= 15 + 10 - 4$$

$$= 21$$

That means the determinant of A is 21.

5.5 Transpose of a Matrix

If you have a matrix and take its first row and write this as the first column, then take its second row and write this as the second column, and so no, the resulting matrix is called the transpose of the first matrix.

If $A = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 4 & 6 \\ 1 & -1 & 2 \end{pmatrix}$, then the transpose of A, written $A^T = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 4 & -1 \\ 2 & 6 & 2 \end{pmatrix}$

5.6 The inverse of a matrix

Just as you were able to find inverse of functions, you learnt that a function has an inverse if and only if the function is 1-1. Also here, the inverse of matrix exists if and only if the determinant of the matrix is non zero. Note that you can only find an inverse of a square matrix. You will write the inverse of a matrix A as" A^{-1} ".

Definition: An inverse of matrix A is defined as:

$$A^{-1} = \frac{1}{|A|} adjo\,\text{int}.A^T .$$

Read ad joint A matrix transpose divided by the determinant of A. What you need to learn is how to find this ad joint matrix which has to be transposed. This ad joint matrix is a matrix of co-factors called minors which will be demonstrated here on how to get it. A general approach will be used here on how to get the ad joint matrix.

Take for instance the matrix

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

The minor of this matrix are: a_{11} reads the element of the first row and first column, a_{12}, reads the element of the first row and second column, a_{21}, reads element of the second row and first column, and so on.

Therefore, the minors are computed as follows:

$$a_{11} = \begin{vmatrix} e & f \\ h & i \end{vmatrix} = ei - fh \quad a_{12} = \begin{vmatrix} d & f \\ g & i \end{vmatrix} = di - fg \qquad a_{13} = \begin{vmatrix} d & e \\ g & f \end{vmatrix} = df - eg$$

$$a_{21} = \begin{vmatrix} b & c \\ h & i \end{vmatrix} = bi - ch \qquad a_{22} = \begin{vmatrix} a & c \\ g & i \end{vmatrix} = ai - cg \qquad a_{23} = \begin{vmatrix} a & b \\ g & h \end{vmatrix} = ah - bg$$

$$a_{31} = \begin{vmatrix} b & c \\ e & f \end{vmatrix} = bf - ce \quad a_{32} = \begin{vmatrix} a & c \\ d & f \end{vmatrix} = af - cd \qquad a_{33} = \begin{vmatrix} a & b \\ d & e \end{vmatrix} = ae - bd$$

Now you impose negative positive to each of these minors stating from a_{11} as positive and write them in as matrix as shown below:

$$AdJ.A = \begin{pmatrix} (ei - fh) & -(ai - cg) & (bi - ch) \\ -(bi - ch) & (ai - cg) & -(ah - bg) \\ (bf - ce) & -(af - cd) & (ae - bd) \end{pmatrix}$$ This is the ad joit matrix A

$$Ad\,jo\operatorname{int} A^T = \begin{pmatrix} (ei - fh) & -(bi - ch) & (bf - ce) \\ -(ai - cg) & (ai - cg) & -(af - cd) \\ (bi - ch) & -(ah - bg) & (ae - bd) \end{pmatrix}$$

Therefore,

$$A^{-1} = \frac{1}{|A|} Adjo\operatorname{int} A^T = \frac{1}{|A|} \begin{pmatrix} (ei - fh) & -(bi - ch) & (bf - ce) \\ -(ai - cg) & (ai - cg) & -(af - cd) \\ (bi - ch) & -(ah - bg) & (ae - bd) \end{pmatrix}$$

Example 5.6.1 Let $A = \begin{pmatrix} 1 & 2 & 4 \\ -1 & 3 & 0 \\ 0 & 1 & 5 \end{pmatrix}$, Find the inverse of A.

Solution: You find the determinant of the matrix A.

$$|A| = \begin{vmatrix} 1 & 2 & 4 \\ -1 & 3 & 0 \\ 0 & 1 & 5 \end{vmatrix} = 1\begin{vmatrix} 3 & 0 \\ 1 & 5 \end{vmatrix} - 2\begin{vmatrix} -1 & 0 \\ 0 & 5 \end{vmatrix} + 4\begin{vmatrix} -1 & 3 \\ 0 & 1 \end{vmatrix} = 15 + 10 - 4 = 21$$

$$|A| = 21$$

$$a_{11} = \begin{vmatrix} 3 & 0 \\ 1 & 5 \end{vmatrix} = 15 \qquad a_{12} = \begin{vmatrix} -1 & 0 \\ 0 & 5 \end{vmatrix} = 5 \qquad a_{13} = \begin{vmatrix} -1 & 3 \\ 0 & 1 \end{vmatrix} = -1$$

$$a_{21} = \begin{vmatrix} 2 & 4 \\ 1 & 5 \end{vmatrix} = 10 - 4 = -6 \qquad a_{22} = \begin{vmatrix} 1 & 4 \\ 0 & 5 \end{vmatrix} = 5 \qquad a_{23} = \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = -1$$

$a_{31} = \begin{vmatrix} 2 & 4 \\ 3 & 0 \end{vmatrix} = -12$ $\quad a_{32} = \begin{vmatrix} 1 & 4 \\ -1 & 0 \end{vmatrix} = -4$ $\quad a_{33} = \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} = 3 + 2 = 5$

$$AdjA^T = \begin{pmatrix} 15 & -6 & -12 \\ 5 & 5 & -4 \\ -1 & -1 & 5 \end{pmatrix}$$

Therefore, $A^{-1} = \frac{1}{21}\begin{pmatrix} 15 & -6 & -12 \\ 5 & 5 & -4 \\ -1 & -1 & 5 \end{pmatrix}$

Note that $A^{-1}A = I$

5.7 Simultaneous equations

You came across simultaneous equations in two equations in two unknown in junior mathematics and solved these equations using; elimination method, substitution method, and graphical methods. In this section you will learn how to solve systems of equations in three equations in three unknown using matrix methods. Two methods will be demonstrated here;

(a) Inverse Method

In this method you will use the fact that $A^{-1}A = I$ in solving system of equations. To do that you need to express the system of equation in the matrix form like $AY = B$. Where A is the matrix of the coefficients of the variables, Y is the column vector of the variables and B the column vector of constants.

Given a system of equation as:

$$\begin{aligned} ax + by + cz &= m \\ dx + ey + fz &= n \\ gx + hy + jz &= p \end{aligned}$$

You can express this system of equation in matrix form as;

$AX = D$ where the matrix A is

$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, and $D = \begin{pmatrix} m \\ n \\ p \end{pmatrix}$ and your system of equation can be written as:

$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} m \\ n \\ p \end{pmatrix}$. You can solve this system of equation using either Inverse of a matrix or using crammers' rule.

(a) Inverse of a matrix method

You need to find the determinant of the matrix A, and get its inverse A^{-1}, and using the relationship $A^{-1}A = I$, you will be able to find the values of x, y, and z.

Example 5.7.1 Solve the following system of equations using inverse of a matrix method.

$$2x + y - z = 1$$
$$x + 3z = 2$$
$$-3x - y + z = -1$$

Solution: You write this system in matrix form as: $AY = B$

$$\begin{pmatrix} 2 & 1 & -1 \\ 1 & 0 & 3 \\ -3 & -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \quad (1)$$

Where $A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 0 & 3 \\ -3 & -1 & 1 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, and $D = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

There $|A| = -3$, you can verify this value.

You now find the adjoint matrix of co factor minors:

$$a_{11} = \begin{vmatrix} 0 & 3 \\ -1 & 1 \end{vmatrix} = 3 \quad a_{12} = \begin{vmatrix} 1 & 3 \\ -3 & 1 \end{vmatrix} = 10 \qquad a_{13} = \begin{vmatrix} 1 & 0 \\ -3 & -1 \end{vmatrix} = -1$$

$$a_{21} = \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} = 0 \qquad a_{22} = \begin{vmatrix} 2 & -1 \\ -3 & 1 \end{vmatrix} = -1 \qquad a_{23} = \begin{vmatrix} 2 & 1 \\ -3 & -1 \end{vmatrix} = 1$$

$$a_{31} = \begin{vmatrix} 1 & -1 \\ 0 & 3 \end{vmatrix} = 3 \quad a_{32} = \begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} = 7 \qquad a_{33} = \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} = -1$$

Therefore, the ad joint matrix transpose is;

$$Adj.A^T = \begin{pmatrix} 3 & 0 & 3 \\ -10 & -1 & -7 \\ -1 & -1 & -1 \end{pmatrix}, \text{ therefore, } A^{-1} = \frac{1}{-3}\begin{pmatrix} 3 & 0 & 3 \\ -10 & -1 & -7 \\ -1 & -1 & -1 \end{pmatrix} \quad (2)$$

Multiply (2) to (1)

$$\frac{1}{-3}\begin{pmatrix}3 & 0 & 3\\-10 & -1 & -7\\-1 & -1 & -1\end{pmatrix}\begin{pmatrix}2 & 1 & -1\\1 & 0 & 3\\-3 & -1 & 1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\frac{1}{-3}\begin{pmatrix}3 & 0 & 3\\-10 & -1 & -7\\-1 & -1 & -1\end{pmatrix}\begin{pmatrix}1\\2\\-1\end{pmatrix}$$

The left hand side of the equation is $A^{-1}A=I$

$$\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\frac{1}{-3}\begin{pmatrix}3 & 0 & 3\\-10 & -1 & -7\\-1 & -1 & -1\end{pmatrix}\begin{pmatrix}1\\2\\-1\end{pmatrix}$$

$$\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\frac{1}{-3}\begin{pmatrix}0\\-5\\-2\end{pmatrix}$$

$\begin{pmatrix}x\\y\\z\end{pmatrix}=\frac{1}{-3}\begin{pmatrix}0\\-5\\-2\end{pmatrix}$ This implies, $x=0, y=\frac{5}{3}, z=\frac{2}{3}$

(b) Using Cramer's rule

You can solve the same equation using crammer's rule, where you use determinants only.

Take the general form $\begin{pmatrix}a & b & c\\d & e & f\\g & h & j\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}m\\n\\p\end{pmatrix}$, in here you take your $A=\begin{pmatrix}a & b & c\\d & e & f\\g & h & j\end{pmatrix}$ and

$|A|$, define the following determinants as follows:

$|A_1|=\begin{vmatrix}m & b & c\\n & e & f\\p & h & j\end{vmatrix}$, $|A_2|=\begin{vmatrix}a & m & c\\d & n & f\\g & p & j\end{vmatrix}$, and $|A_1|=\begin{vmatrix}a & b & m\\d & e & n\\g & h & p\end{vmatrix}$ That is you replace the i^th^ column by D. Then your x, y, and z values are found by:

$x=\frac{|A_1|}{|A|}$, $y=\frac{|A_2|}{|A|}$, and $z=\frac{|A_3|}{|A|}$

Example 5.7.2 Solve the following system of equations using inverse of a matrix method.

$$2x+y-z=1$$
$$x+3z=2$$
$$-3x-y+z=-1$$

Solution: This system is of the form $\begin{pmatrix} 2 & 1 & -1 \\ 1 & 0 & 3 \\ -3 & -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 0 & 3 \\ -3 & -1 & 1 \end{pmatrix}$

We know $|A| = -3$ and you find the determinants of the A_i.

$$|A_1| = \begin{vmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ -1 & -1 & 1 \end{vmatrix} = 0 \qquad |A_2| = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 2 & 3 \\ -3 & -1 & 1 \end{vmatrix} = -5 \qquad |A_3| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 0 & 2 \\ -3 & -1 & -1 \end{vmatrix} = -2$$

Therefore, using Cremer's rule the solutions are:

$x = \frac{|A_1|}{|A|} = \frac{0}{-3} = 0$, $y = \frac{|A_2|}{|A|} = \frac{-5}{-3} = \frac{5}{3}$, and $z = \frac{|A_3|}{|A|} = \frac{-2}{-3} = \frac{2}{3}$

Exercise 5

1. Evaluate

(a) $\begin{pmatrix} -7 & 6 \\ -5 & 4 \end{pmatrix} + \begin{pmatrix} -9 & 2 \\ 1 & -3 \end{pmatrix}$ (b) $\begin{pmatrix} -8 & -7 \\ -5 & 4 \end{pmatrix} + \begin{pmatrix} -9 & 3 \\ 4 & -9 \end{pmatrix}$

(c) $\begin{pmatrix} 3 & -5 & 4 \\ 0 & -6 & 3 \end{pmatrix}\begin{pmatrix} 1 & -3 & 2 \\ -1 & 0 & 4 \\ 0 & 0 & -2 \end{pmatrix}$ (d) $\begin{vmatrix} 2 & -3 & 6 \\ -2 & 4 & 5 \\ -1 & 0 & -5 \end{vmatrix}$

2. Find the inverse of

(a) $\begin{pmatrix} -3 & 2 \\ -1 & 7 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 1 & -5 \\ 1 & 0 & -2 \\ 0 & 0 & 3 \end{pmatrix}$ (c) $\begin{pmatrix} 2 & 2 & 1 \\ 4 & 1 & 5 \\ -1 & 1 & 7 \end{pmatrix}$

3. Given that $A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 4 & -7 \\ 0 & 6 & -3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & 4 \\ 3 & 1 & -2 \\ -1 & 1 & 3 \end{pmatrix}$

find (a) A^T (b) B^T

Hence find (c) $(AB)^T$ (d) $(BA)^T$

4. Solve the system of equations.

(a) $4x + 3y + 5z = 11$

$9x + 4y + 15z = 13$

$12x + 10y - 3z = 4$

(b) $5x + 4y + 2z = 16$

$7x - 8y + 3z = -45$

$x + 6y - 4z = 16$

5. Solve the system of equations

(a) Using matrix method.

(b) Using Cramer's rule.

(i) $x + y + 2z - 2 = 0$

$-3x + y + 2z - 1 = 0$

$6x + 2y + z + 4 = 0$

(ii) $x + y + z = 2$

$x + 2y + 3z = 6$

$3x + 2y - z = -4$

Unit 6 Differentiations

6.1 Introduction

This unit begins with the background of the development of calculus by Isaac Newton (1642-1727) and Gottfried Wilhelm Leibniz (1646-1716). Their investigation on the slope of a tangent on curves at a given point and finding the areas of irregular shapes in a plane gave birth to differential and integral calculus respectively.

The unit also gives an overview of the linkage between pre calculus mathematics with differential calculus just to show you that differential calculus is not independent of pre calculus mathematics.

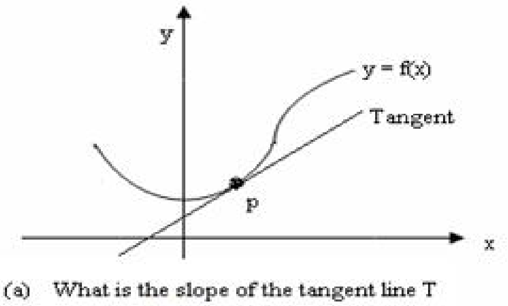
6.2 Objectives

Upon completion of this unit you will be able to:

- Explain how differential calculus was developed
- Find the limit of a function as x approaches a given number
- Apply rules of derivatives
- Find derivatives of a function from first principles
- Apply differentiation to curve sketching
- Find maximum and minimum of the curve
- Apply derivatives to tangents and normal to curves
- Apply derivatives to related rates of change

6.3 Back ground

Historically, the development of calculus by Isaac Newton (1642-1727) and Gottfried Wilhelm Leibniz (1646-1716) resulted from the investigation of the following problems:

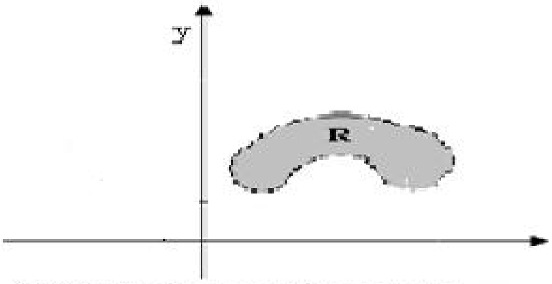


(a) What is the slope of the tangent line T

Figure 6.3.1

(i) Finding the tangent line to a curve at a given point on the curve

(ii) Finding the area of a planar region bounded by an arbitrary cure



(b) What is the area of the region R
Figure 6.3.2

The tangent line problem might appear to be unrelated to any practical applications of mathematics, but as you will see later, the problem of finding the rate of change of one quantity with respect to another is mathematically equivalent to the geometric problem of finding the slope of the tangent line to a curve at a given point on the curve. It is precisely the discovery of the relationship between these two problems that spurred the development of calculus in the seventeenth century and made it such an indispensable tool for solving practical problems. The following are the few examples of such problems:

- Finding the rate of change of a bacteria population with respect to time
- Finding the velocity of an object
- Finding the rate of change of a company's profit with respect to time
- Finding the rate of change of a travel agency's revenue with respect to the agency's expenditure for advertising.

The study of tangents-line problem led to the creation of *differential calculus*, which relies on the concept of the derivative of a function. The study of the area problem led to the creation of *integral calculus*, which relies on the concept of the ant-derivative or integral, of function. Both the derivative of a function and the integral of a function are defined in terms of a more fundamental concept called – *Limit.*

What is Calculus?

Calculus can be referred to as a " Limit Machine" that involves three stages. The first stage is pre-calculus mathematics such as the slope of a straight line or area of a rectangle. The second stage is the limit process and the third stage is new calculus formulation, such as derivative of integral.

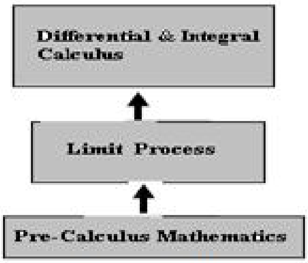
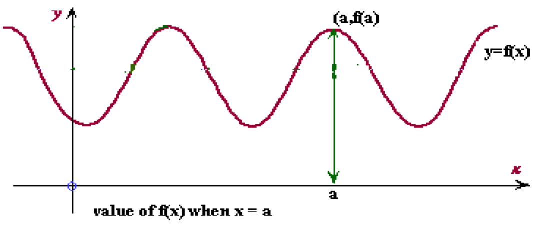


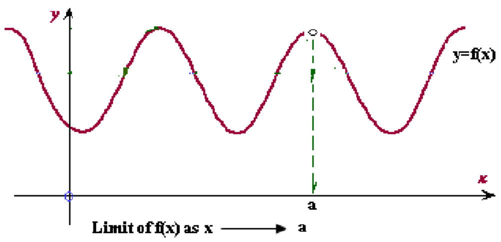
Diagram 6.3.1

Students who learn calculus as a collection of new formulas, and reduce calculus to the memorization of differentiation and integral formulas miss a great deal of understanding and lacks self confidence and satisfaction of the subject matter. As a result they may not appreciate the applications of calculus in modeling real life situations in economics, business and engineering. In this text, our goal is to learn how pre calculus formulas and techniques are used as building blocks to produce the more general calculus formulas and techniques.

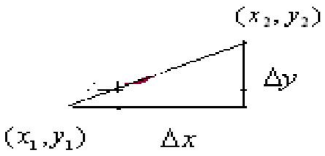
Pre calculus Mathematics **With Differential calculus**



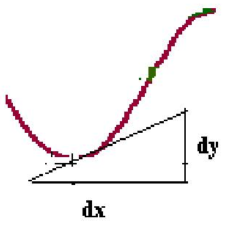
value of f(x) when x = a



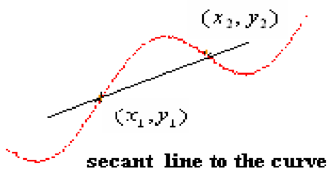
Limit of f(x) as x ⟶ a



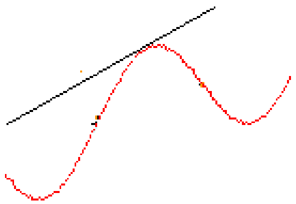
$$m = \frac{y_1 - y_2}{x_1 - x_2}$$



slope of a curve



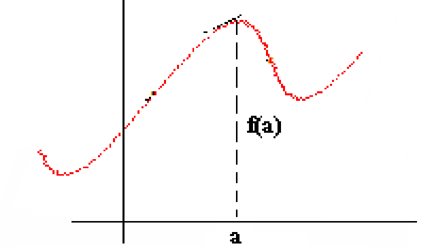
secant line to the curve



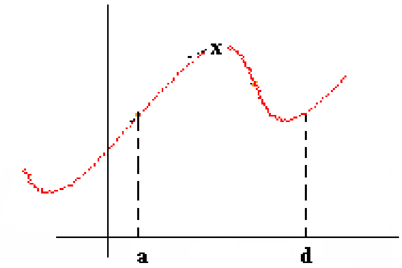
Tangent line to a curve

Average rate of change between t=a and t=b **Instantaneous rate of change at t=a**

Curvature of a circle **Curvature of a curve**



Height of a curve when x = a



Maximum height of a curve on an interval

Tangent plane to a sphere **Tangent plane to a surface**

Direction of a motion **Direction of motion along**

Along a straight line **A curved line.**

Figure 6.3.3(a) *Figure 6.3.3(b)*

These are some of the examples of pre calculus formulas required to understand differential calculus formulas. In the study of calculus, what happens to the value of a function as the independent variable *gets very close* to a particular value is very important. We came across this concept in the introduction where you zoomed in on a curve to get an approximation for the slope of that curve. This lead to the theory of limits and continuity as x approaches a particular number or approaches a number from either the left or right.

6.4 Limits

In this unit you will learn how to take limits. As consecutive points, given by the terms of the sequence $2-\frac{1}{n}$ as shown on a number scale, it is noted that they cluster about the point 2.

Diagram 6.9.1

In such a way that there are points of the sequences whose distance from 2 is less than any pre assigned positive number, however small. In this unit we shall be interested in the limit of a function. Let $x \to 2$, then $f(x) = x^2 \to 4$ as limit. Under this assumption, we say "the limit , as x approaches 2, of x^2 is 4" and write $\lim_{x \to 2} x^2 = 4$

Right and Left limits: as $x \to 2$ over the sequence given above, its value is always less than 2. We say that x approaches 2 from the left and write $x \to 2^-$. Similarly, as $x \to 2$ over the given sequence $2 + \frac{1}{n}$, its value is always greater than 2. We say that x approaches 2 from the right and we write $x \to 2^+$. Clearly, the statement $\lim_{x \to a} f(x)$ exists implies that both the left $\lim_{x \to a^-} f(x)$ and the right limit $\lim_{x \leftarrow a^+} f(x)$ exists and are equal. However, the existence of the right (left) limits does not imply the existence of the left (right) limit.

Example 6.4.1 The function $f(x) = \sqrt{9 - x^2}$ has the interval $-3 \le x \le 3$ as domain of definition. If a is any number on the open interval $-3 < x < 3$, then $\lim_{x \to a} \sqrt{9 - x^2}$ exists and is equal to $\sqrt{9 - a^2}$. Now consider a = 3. First , let x approaches 3 from the left, then $\lim_{x \to 3^-} \sqrt{9 - x^2} = 0$. Next, let x approaches 3 from the right, then $\lim_{x \to 3^+} \sqrt{9 - x^2}$ does not exist since for x>3, $\sqrt{9 - x^2}$ is imaginary. Thus, $\lim_{x \to 3} \sqrt{9 - x^2}$ does not exist.

Theorems: The following theorems on limits are listed for future reference.

1. If f(x)=c, a constant , then $\lim_{x \to a} f(x) = c$

   If $\lim_{x \to a} f(x) = A$ and $\lim_{x \to a} g(x) = B$, then

2. $\lim_{x \to a} k.f(x) = kA$, k being a constant

3. $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \mp \lim_{x \to a} g(x) = A \pm B$

4. $\lim_{x \to a} [f(x).g(x)] = \lim_{x \to a} f(x).\lim_{x \to a} g(x) = A.B$

5. $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{A}{B}, B \neq 0$

6. $\lim_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{A}$

These results may be very useful in the next sections.

6.4.1 Limits as *x* Approaches a Particular Number

Sometimes, finding the limiting value of an expression means simply substituting a number directly in the function..

Example 6.4.2 Find the limit as *t* approaches 10 of the expression $P = 3t + 7$.

Answer , We write this using limit notation **as:** $\lim_{t\to 10}(3t+7)$

In this example there is no complication - we simply substitute and write $\lim_{t\to 10}(3t+7=37$

There is no complication because $f(t) = 3t + 7$ is a <u>continuous function</u>. But there are cases where we cannot simply substitute like this.

Example 6.4.3 We know that *x* cannot equal 3 in the following expression (because we cannot have a denominator equal to zero): $f(x) = \frac{x^2 - 2x - 3}{x - 3}$

However, we can see that the function approaches a particular value as *x* approaches 3 from the left:

x	2.5	2.6	2.7	2.8	2.9
$f(x)$	3.5	3.6	3.7	3.8	3.9

Continuing, we get closer and closer to $x = 3$:

x	2.9	2.92	2.94	2.96	2.97	2.98	2.99

$f(x)$	3.9	3.92	3.94	3.96	3.97	3.98	3.99

x	3.5	3.1	3.01	3.00001
$f(x)$	4.5	4.1	4.01	4.00001

Table 6.9.1

Likewise, approaching $x = 3$ from the right gives the same limit value: We note that the function value is getting closer and closer to 4. We write: $\lim\limits_{x\to3}\frac{x^2-2x-3}{x-3}=4$

NOTE: We could have evaluated this limit by *factorizing* first:

CAUTION: The factorizing process is only possible in this example because we have:

$x \neq 3$. This is a typical problem in the study of introductory limits. It appears to be a bit trivial , in that we could have factored it, cancelled and substituted $x = 3$ like we just saw. But the example is important for the concept that there is no actual value of the function when $x = 3$, but if we get really, really close to 3, the function value is really close to some value (4, in this case).

Note that if you get the form ${}^{0}\!/_{0}$ when you substitute directly in the function, this form is called indeterminate, means you cannot determine the limit , so you use factorization method to cancel one of the zeros, generally the zero in the denominator but not always.

Example 6.4.3 Find the limit $\lim\limits_{x\to5}\frac{x^2-25}{x^2+x-30}$

Solution Although the limit in question is the ratio of two polynomials, x = 5 makes both the numerator and denominator equal to zero. We need to factor both numerator and denominator as shown below.

$$\lim_{x\to5}\frac{\cancel{(x-5)}(x+5)}{\cancel{(x-5)}(x+6)}$$

Simplify to obtain

$$= \lim_{x \to 5} \frac{(x+5)}{(x+6)}$$

$$= \frac{10}{11}$$

Example 6.4.4 Find the limit $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$

Solution If you substitute x = -1 directly in the function, you will get $0/0$ and this form is called indeterminate, meaning you cannot determine the limit of the function as x approaches -1.

So we factorize the numerator and simplify the expression and then take limit.

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x-1)(x+1)}{(x-1)}$$

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}}$$

$$\lim_{x \to 1} (x+1) = 2$$

6.4.2 Limits as *x* Approaches Infinity

When taking limits as x approaches infinity, written as; $\lim_{x \to \infty} f(x)$, and we find the form ∞/∞ this form is also indeterminate. In this case we cannot factorize into factors as shown in the two examples above, what we need to do is factor out the highest power of x, and use the fact that $\lim_{x \to \pm\infty} \left(\frac{1}{x}\right) = 0$.

Example 6.4.5 Find the limit $\lim_{x \to \infty} (\frac{5 - 3x}{6x + 1})$**.**

Solution: This time it is not so obvious what the limit value is. We could substitute larger and larger values of x until we could see what was happening (try 100, then 1000, then 1000000 and so on). Or, we could rearrange the expression and use the fact that $\lim_{x \to \infty} \left(\frac{1}{x}\right) = 0$,to find the limiting value. We divide throughout by x to get the expression in a form where we can evaluate it.

$$\lim_{x\to\pm\infty}\left(\frac{5-3x}{6x+1}\right)=\lim_{x\to\infty}\left(\frac{5/x-3}{6+1/x}\right)$$

$$=\frac{0-3}{6+0}$$

$$=-\frac{1}{2}$$

Notice that we cannot substitute ∞ into the fraction $\left(\frac{5-3x}{6x+1}\right)$, because this does not make mathematical sense. Please do not write $\left(\frac{5-3(\infty)}{6(\infty)+1}\right)$. It has no meaning

Example 6.4.6 Find the limit $\lim_{x\to\infty}\frac{3x^2}{4x^2+2x-1}$

Solution . Factor x ^{2} in the denominator and simplify.

$$\lim_{x\to\infty}\frac{3\cancel{x^2}}{\cancel{x^2}(4+\frac{2}{x}-\frac{1}{x^2})}$$

As x takes large values (infinity), the terms 2/x and 1/x ^{2} approaches 0 hence the limit is

$$=\frac{3}{4}$$

Example 6.4.7 Find the $\lim_{x\to\infty}\frac{x-1}{2x^2+3}$

Solution . Factor x ^{2} in the numerator and denominator and simplify.

$$=\lim_{x\to\infty}\frac{\cancel{x^2}(\frac{1}{x}-\frac{1}{x^2})}{\cancel{x^2}(2+\frac{3}{x^2})}$$

As x takes large values (infinity), the terms 1/x and $1/x^2$ and $3/x^2$ approaches 0 hence the limit is

$$=\frac{0}{2}=0$$

6.5 Differentiation

Introduction. In this unit we begin by introducing the concept of differentiation using the concept of slope of a straight line. This will be developed to derivative of a function from first principles which illustrate the ideas of limits which we discussed in the previous unit. Then we will discuss differentiation of functions using basic rules, and finally we shall discuss some examples of the applications of differentiations in economics and businesses. Differentiations can also be applied in areas such as: Temperature change at a particular time, Velocity of a falling object at a particular time, Current through a circuit at a particular time, Variation in stock market prices at a particular time , Population growth at a particular time , Temperature increase as density increases in a gas .

4.5.1 The Slope (gradient) of a Tangent to a Curve (Numerical Approach)

Since we can model many physical problems using curves, it is important to obtain an understanding of the **slopes** (gradient) of curves at various points and what a slope *means* in real applications. **Remember:** We are trying to find the **rate of change** of one variable compared to another. In this section, we show you one of the historical approaches for finding slopes(gradient) of tangents, before differentiation was developed. This is to give you an idea of how it works. Before we proceed, let look at a numerical example given below just as a motivation into this unit.

Example6.5.1 Find the slope (gradient) of the curve $f(x) = x^2$ at the point (2,4), using a *numerical* method.

Solution. We start with a point $Q(1, 1)$ which is near $P(2,4)$:

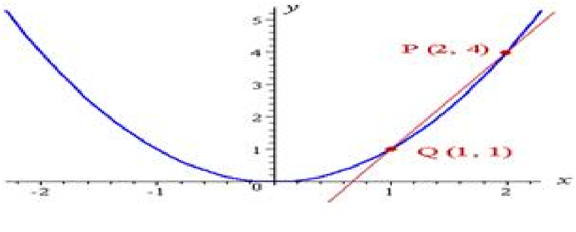


Figure 6.5.1

The slope (gradient) of *PQ* is given by: $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-1}{2-1} = 3$

Now we move *Q* further around the curve so it is closer to *P*. Let's use *Q*(1.5,2.25) which is closer to *P*(2,4):

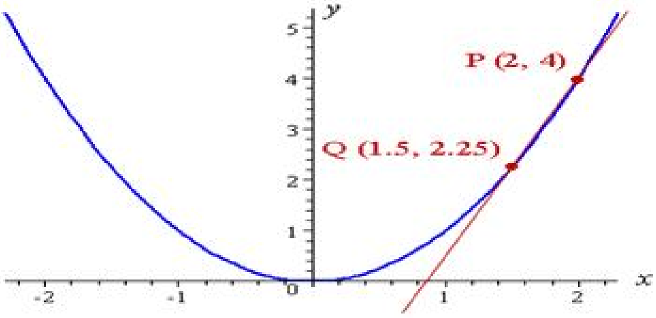


Figure 6.5.2

The slope (gradient) of *PQ* is now given by: $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-2.25}{2-1.5} = 3.5$

We see that this is already a pretty good approximation to the tangent at *P*, but not good enough.

Now we move *Q* even closer to *P*, say *Q*(1.9,3.61). Now we have:

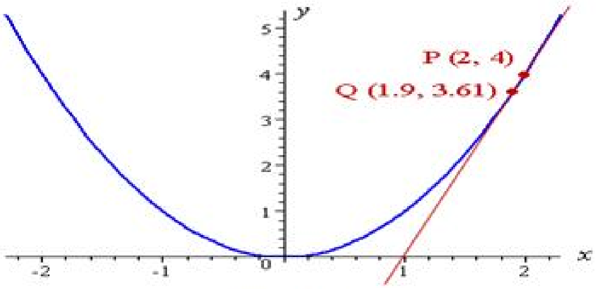


Figure 6.5.3

So $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-3.61}{2-1.9} = 3.9$

We can see that we are very close to the required slope (gradient). Now if Q is moved to (1.99,3.9601), then slope PQ is 3.99. If Q is (1.999,3.996001), then the slope is 3.999. Clearly, as $x \to 2$, the slope of $PQ \to 4$. But notice that we cannot actually let $x = 2$, since the fraction for m would have 0 on the bottom, and so it would be undefined.

We have found that the rate of change of y with respect to x is 4 units at the point $x = 2$. We will now extend this numerical approach so that we can find the slope of any continuous curve if we know the function. We will see an algebraic approach that can be used for most functions.

6.5.2 The Derivative from First Principles

In this section, we will differentiate a function from "first principles". This means we will start from scratch and use algebra to find a general expression for the slope of a curve, at any value x.

A first principle is also known as "delta method", since many texts use Δx (for "change in x) and Δy (for "change in y"). This makes the algebra appear more difficult, so here we use h for Δx instead. We still call it "delta method". We look at the general case and write our functions involving the familiar x (independent) and y (dependent) variables.

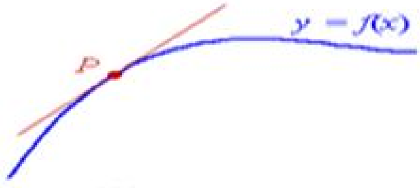


Figure 6.9.4

We wish to find an **algebraic method** to find the slope of $y = f(x)$ at P, to save doing the numerical substitutions that we saw in the last section (Slope of a Tangent to a Curve - Numerical approach). We can approximate this value by taking a point somewhere near to $P(x, f(x))$, say $Q(x + h, f(x + h))$.

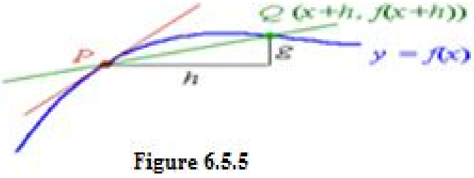


Figure 6.5.5

The value $\frac{g}{h}$ is an approximation to the slope of the tangent which we require. We can also write this slope as "change in *y* / change in *x*" or: $m = \frac{\Delta y}{\Delta x}$. If we move *Q* closer and closer to *P*, the line *PQ* will get closer and closer to the tangent at *P* and so the slope of *PQ* gets closer to the slope that we want.

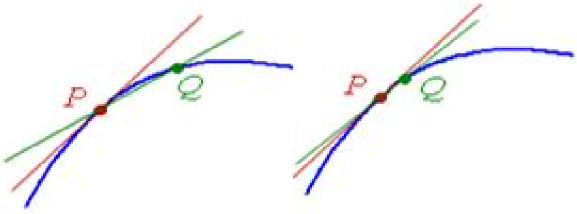


Figure **6.5.6**

If we let *Q* go all the way to touch *P* (i.e. $h = 0$), then we would have the **exact** slope of the tangent.

Consider figure 6.5.4 given below:

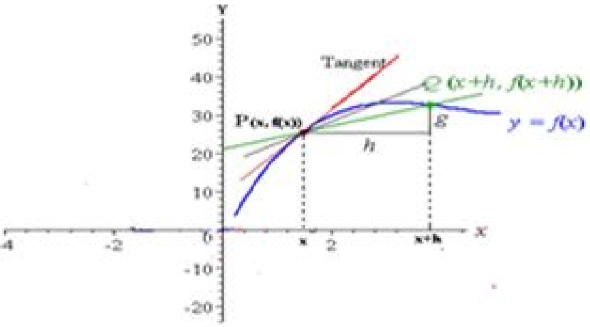


Figure 6.5.7

In this work, we write

- ***change in y*** as Δy
- ***change in x*** as Δx

Now, $\frac{g}{h}$ can be written: $\frac{g}{h}=\frac{f(x+h)-f(x)}{h}$. So also, the slope *PQ* will be given by: $m=\frac{y_2-y_1}{x_2-x_1}=\frac{\Delta y}{\Delta x}=\frac{f(x+h)-f(x)}{h}$. But we require the slope **at** *P*, so we let $h \to 0$ (that is let *h* approach 0), then in effect, *Q* will approach *P* and $\frac{g}{h}$ will approach the required slope.

Putting this together, we can write the slope of the tangent at *P* as: $\frac{dy}{dx}=\lim_{x\to 0}\frac{f(x+h)-f(x)}{h}$

This is called **differentiation from first principles,** (or the **delta method**). It gives the instantaneous rate of change of *y* with respect to *x*. This is equivalent to the following (where before we were using *h* for Δx): $\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}$

You will also come across the following for delta method: $\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{f(x+h)-f(x)}{\Delta x}$

The slope (gradient) of a **curve** at the point *P* means the slope (gradient) of the **tangent** at the point *P*. Note that the slope (gradient) of a straight line is the same along the line while the slope (gradient) of a curve is different at every point along the curve. We need to find this slop (gradient) to solve many applications since it tells us *the rate of change* at a particular instant.

And we write $\frac{\Delta y}{\Delta x}=\lim_{x\to 0}\frac{f(x+h)-f(x)}{h}$ and this is called the derivative of f(x) from first principles and this becomes $\frac{dy}{dx}=\lim_{x\to 0}\frac{f(x+h)-f(x)}{h}$

Notation for the derivatives

IMPORTANT: The *derivative* (also called **differentiation**) can be written in several ways. This can cause some confusion when we first learn about differentiation. The following are equivalent ways of writing the first derivative of $y=f(x)$: $\frac{dy}{dx}$ or $f'(x)$ or y'.

Example 6.5.2 Find $\frac{dy}{dx}$ from first principles if $y = 2x^2 + 3x$.

Solution: $f(x) = 2x^2 + 3x$ **so**

$$\begin{aligned} f(x+h) &= 2(x+h)^2 + 3(x+h) \\ &= 2(x^2 + 2xh + h^2) + 3x + 3h \\ &= 2x^2 + 4xh + 2h^2 + 3x + 3h \end{aligned}$$

We now need to find:

$$\begin{aligned} \frac{dy}{dx} &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{[2x^2 + 4xh + 2h^2 + 3x + 3h] - [2x^2 + 3x]}{h} \\ &= \lim_{h\to 0} \frac{4xh + 2h^2 + 3h}{h} \\ &= \lim_{h\to 0} 4x + 2h + 3 \\ &= 4x + 3 \end{aligned}$$

We have found an expression that can give us the slope of the tangent anywhere on the curve.

If $x = -2$, the slope is $4(-2) + 3 = -5$ (red, in the graph below)

If $x = 1$, the slope is $4(1) + 3 = 7$ (green)

If $x = 4$, the slope is $4(4) + 3 = 19$ (black)

We can see that our answers are correct when we graph the curve (which is a parabola) and observe the slopes of the tangents.

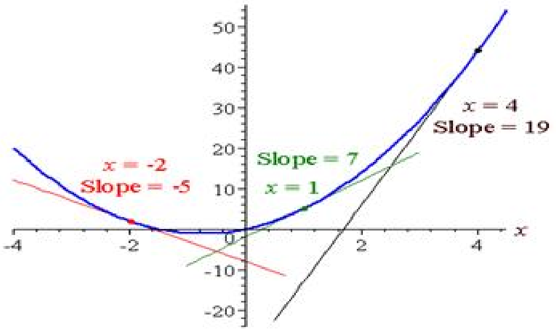


Figure 6.5.8

This is what makes calculus so powerful. We can find the slope anywhere on the curve (i.e. the rate of change of the function anywhere).

Example 6.5.3

a. Find y' from first principles if $y = x^2 + 4x$.

b. Find the slope of the tangent where $x = 1$ and also where $x = -6$.

c. Sketch the curve and both tangents.

Solution:

a. **Note:** $\boldsymbol{y'}$ means "the first derivative". This can also be written *dy/dx*.

Now $f(x) = x^2 + 4x$

$$\begin{aligned} f(x+h) &= (x+h)^2 + 4(x+h) \\ &= x^2 + 2xh + h^2 + 4x + 4h \end{aligned}$$

So

$$\begin{aligned} \frac{dy}{dx} &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{[(x+h)^2 + 4(x+h)] - [x^2 + 4x]}{h} \\ &= \lim_{h \to 0} \frac{[x^2 + 2xh + h^2 + 4x + 4h] - [x^2 + 4x]}{h} \\ &= \lim_{h \to 0} \frac{2xh + h^2 + 4h}{h} \\ &= \lim_{h \to 0} (2x + h + 4) \\ &= 2x + 4 \end{aligned}$$

b. When $x = 1$, $m = 2(1) + 4 = 6$

When $x = -6$, $m = 2(-6) + 4 = -8$

c. Sketch:

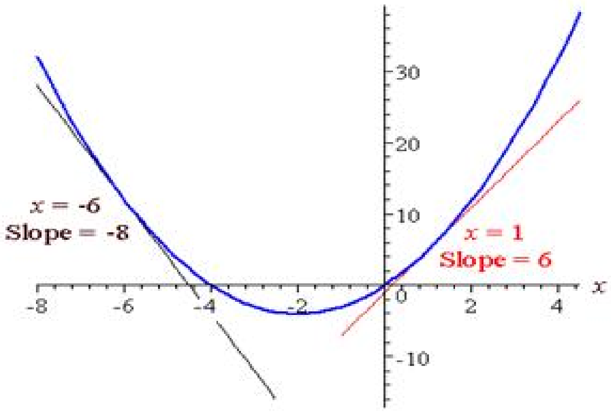


Figure 6.5.9

Example 6.5.4 Find y' from first principles if: $f(x) = x^2$,

$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{(x^2 + 2(\Delta x)x + \Delta x^2) - x^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{2(\Delta x)x + \Delta x^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} (2x + \Delta x) \\ &= 2x \end{aligned}$$

as expected.

Example 6.5.5 From first principles find $\frac{dy}{dx}$ of $f(x) = \frac{1}{x}$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{\frac{1}{x+\Delta x} - \frac{1}{x}}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\frac{x-(x+\Delta x)}{(x+\Delta x)(x)}}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\frac{-\Delta x}{(x+\Delta x)(x)}}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{-1}{(x+\Delta x)(x)} \\ &= -\frac{1}{x^2} \end{aligned}$$

6.5.3 Rules of Derivatives

The good news is that we can find the derivatives of polynomial expressions without using the delta method that we met in the derivative from first principles. They follow from the "first principles" approach to differentiating, and make life much easier for us.

Constant:	$\frac{d(c)}{dx} = 0$	This is basic. In English, it means that if a quantity has a constant value, then the rate of change is zero.
n-th power of *x*:	$\frac{d}{dx}x^n = nx^{n-1}$	This follows from the delta method.
Constant product:	$\frac{d}{dx}(cy) = c\frac{d}{dx}(y)$	Here, *y* is some function of *x*. It means that if we are finding the derivative of a constant times that function, it is the same as finding the derivative of the function first, then multiplying by the constant.
Derivative of sum or difference:	$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$	Here, *u* and *v* are functions of *x*. The derivative of the sum is equal to the derivative of the first plus derivative of the second.
Derivative of product	$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$	Here, *u* and *v* are functions of *x*. The derivative of the product is equal to the first x derivative of the second plus the second x derivative of the first.
Derivative of quotient	$\frac{d}{dx}(\frac{u}{v}) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$	Here, *u* and *v* are functions of *x*. The derivative of the quotient is equal to the denominator x derivative of the numerator minus the numerator x derivative of the denominator divided by the denominator squared.

$y = e^x$	$\frac{dy}{dx} = e^x$	Special function
$y = \ln\lvert x\rvert$	$\frac{dy}{dx} = \frac{1}{x}$, $x \neq 0$	Special function

6.5.4 Function of a Function

If y is a function of u, and u is a function of x, then we say

"y is a function of the function u".

Example 6.5.6 Consider the function

$y = (5x + 7)^{12}$. If we let $u = 5x + 7$ (the inner-most expression), then we could write our original function as $y = u^{12}$

We have written y as a function of u, and in turn, u is a function of x.

This is a vital concept in differentiation, since many of the functions we meet from now on will be functions of functions, and we need to recognise them in order to differentiate them properly.

6.5.5 Chain Rule

To find the derivative of a function of a function, we need to use the Chain Rule:

$$\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$$

This means we need to :

1. Recognize u (always choose the inner-most expression, usually the part inside brackets, or under the square root sign).
2. Then we need to re-express y in terms of u.

3. Then we differentiate y (with respect to u), then we re-express everything in terms of x.

4. The next step is to find *du/dx*.

5. Then we multiply *dy/du* and *du/dx*.

Example 6.5.7 Find $\frac{dy}{dx}$ if $y = (x^2 + 3)^5$.

Solution In this case, we let $u = x^2 + 3$ and then $y = u^5$. We see that: y is a function of u. u is a function of x and this means we use the chain rule; For the **chain rule**, we firstly need to find $\frac{dy}{dx} = 5u^4 = 5(x^2+3)^4$ and $\frac{dy}{dx} = 2x$. So

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$

$$= 5(x^2+3)^4(2x)$$

$$= 10x(x^2+3)^4$$

Example 6.5.8 Find $\frac{dy}{dx}$ if $y = \sqrt{4x^2 - x}$.

Solution: We write the given function as $y = (4x^2 - x)^{1/2}$, we let $u = 4x^2 - x$, and our function now becomes, $y = u^{1/2}$. Applying the chain rule $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$, $\frac{dy}{du} = \frac{1}{2}u^{-1/2}$ and $\frac{du}{dx} = 8x - 1$. Therefore, $\frac{dy}{dx} = \frac{1}{2}u^{-1/2}.(8x-1)$

$$= \frac{1}{2}(4x^2 - x)^{-1/2}(8x-1), \text{ since } u = 4x^2 - x.$$

Example 6.5.9 Find $\frac{dy}{dx}$ if $y=(x^2+5x)^6$

differentiate $(x^2+5x)^6$

let $y=(x^2+5x)^6$ and $t=x^2+5x$

then $y=t^6$

$\frac{dt}{dx}=2x+5$, $\frac{dy}{dt}=6t^5$

using the Chain Rule

$\frac{dy}{dx}=\frac{dy}{dt}.\frac{dt}{dx}$, $\frac{dy}{dx}=6t^5\,2x+5$

$$\frac{d\{(x^2+5x)^6\}}{dx}=6(x^2+5x)^5(2x+5)$$

$$\underline{=6(2x+5)(x^2+5x)^5}$$

6.5.6 The Derivative of a Power of a Function (Power Rule)

An extension of the **chain rule** is the **Power Rule** for differentiating. We are finding the **derivative of** $x^n, n \in R$ (a power of a function): Let $y=x^n$, therefore, $\frac{dy}{dx}=nx^{n-1}\frac{dx}{dx}$, by the chain rule, give $\frac{dy}{dx}=nx^{n-1}$

Example6.5.10 Find the derivative of $y=(2x^3-1)$

In the case of $y=(2x^3-1)$ we have a power of a function**. So** $\frac{dy}{dx}=6x^2-0=6x^2$

4.5.7 The Product Rule Equation

If **u** and **v** are two differentiable functions of *x*, then the derivative of ***uv*** is given by:

$y=uv$, $\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$,this can also be written, using '**prime notation**' as :

$y^{'}=(uv)^{'}=u.v^{'}+vu^{'}$

Example 6.5.11 Find $\frac{dy}{dx}$ of the given function : $y = (x^2+1)^3(x^3+1)^2$;

Solution: Let $u = (x^2+1)^3$ and $v = (x^3+1)^2$; so we have,

$$y = u.v$$

$$\frac{du}{dx} = 3(x^2+1)^2.2x \qquad \frac{dv}{dx} = 2(x^3+1).3x^2$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\begin{aligned}\frac{dy}{dx} &= (x^2+1)^3.2(x^3+1).3x^2 + (x^3+1)^2.3(x^2+1)^2.2x \\ &= 6x^2(x^2+1)^3.(x^3+1) + 6x(x^2+1)^2(x^3+1)^2 \\ &= 6x(x^2+1)^2(x^3+1)\{x(x^2+1) + (x^3+1)\} \\ &= 6x(x^2+1)^2(x^3+1)\{x^3+x+x^3+1\} \\ &= \underline{6x(x^2+1)^2(x^3+1)(2x^3+x+1)}\end{aligned}$$

Example 6.5.12 Find $\frac{dy}{dx}$ of the given function : $y = (x^2-4)(x+3)^2$.

Solution. Let $u = x^2-4$ and $v = (x+3)^2$, so we have;

$$y = u.v$$

$$\frac{du}{dx} = 2x \qquad \frac{dv}{dx} = 2(x+3)$$

$$\text{using} \quad \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\begin{aligned}\frac{dy}{dx} &= (x^2-4).2(x+3) + (x+3)^2.2x \\ &= 2(x+3)(x^2-4) + 2x(x+3)^2 \\ &= 2(x+3)\{x^2-4+x^2+3x\} \\ &= \underline{2(x+3)(2x^2+3x-4)}\end{aligned}$$

Example 6.5.13 Find $\frac{dy}{dx}$ of the given function : $y = (x^2 + 3)\sqrt{(2 + x)}$.

Solution. Let $u = (x^2 + 3)$ and $v = (2 + x)^{1/2}$, so we have,

$$y = u.v$$

$$\frac{du}{dx} = 2x \qquad \frac{dv}{dx} = \frac{1}{2}(2 + x)^{-1/2}$$

$$\text{using} \quad \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{dy}{dx} = (x^2 + 3).\frac{1}{2}(2 + x)^{-1/2} + (2 + x)^{1/2}.2x$$

$$= \frac{(x^2 + 3)}{2(2 + x)^{1/2}} + \frac{2x(2 + x)^{1/2}.}{1}$$

$$= \frac{(x^2 + 3) + 2(2 + x)^{1/2}.2x(2 + x)^{1/2}}{2(2 + x)^{1/2}}$$

$$= \frac{(x^2 + 3) + 4x(2 + x)}{2(2 + x)^{1/2}}$$

$$= \frac{(x^2 + 3) + 8x + 4x^2}{2(2 + x)^{1/2}}$$

$$= \frac{5x^2 + 8x + 3}{2(2 + x)^{1/2}}$$

$$\underline{\frac{dy}{dx} = \frac{(5x + 3)(x + 1)}{2(2 + x)^{1/2}}}$$

6.5.8 The quotient Rule Equation

As with the Product Rule, , if **u** and **v** are two differentiable functions of *x*, then the derivative of ***u/v*** is given by: $y = \frac{u}{v}$ $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dxv}}{v^2}$ .This can also be written, using '**prime notation**' as : $y' = \frac{vu' - uv'}{v^2}$ .This rule must be remembered, and note that you cannot interchange the positions of these letters from where they are placed, otherwise the derivative will be wrong. The choice of u and v can be any of the two given functions. The examples below illustrate this rule:

Example 6.5.14 Find $\frac{dy}{dx}$ of the given function : $y = \frac{(x-3)^2}{(x+2)^2}$

Solution; Let $u = (x-3)^2$ and $v = (x+2)^2$, so we have;

$$y = \frac{u}{v}$$

$$\frac{du}{dx} = 2(x-3) \qquad \frac{dv}{dx} = 2(x+2)$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{(x+2)^2.2(x-3) - (x-3)^2.2(x+2)}{(x+2)^4}$$

$$= \frac{(x+2)^2.2(x-3) - (x-3)^2.2(x+2)}{(x+2)^4}$$

$$= \frac{2(x+2)(x-3)\{(x+2) - (x-3)\}}{(x+2)^4}$$

$$= \frac{2(x-3)\{x+2-x+3\}}{(x+2)^3}$$

$$= \frac{2(x-3)(5)}{(x+2)^3}$$

$$\underline{\frac{dy}{dx} = \frac{10(x-3)}{(x+2)^3}}$$

Example 6.5.15 Find $\frac{dy}{dx}$ of the given function $y = \frac{x}{\sqrt{(1+x^2)}}$.

Solution: Le$u = x$ t and $v = (1+x^2)^{1/2}$

$$y = \frac{u}{v}$$

$$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = \frac{1}{2}.2x(1+x^2)^{-1/2}$$

$$= x(1+x^2)^{-1/2}$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(1+x^2)^{1/2}.1 - x.x(1+x^2)^{-1/2}}{(1+x^2)}$$

$$= \frac{(1+x^2)^{1/2} - x^2(1+x^2)^{-1/2}}{(1+x^2)}$$

mult. top & bottom by $(1+x^2)^{1/2}$

$$= \frac{(1+x^2) - x^2}{(1+x^2)^{3/2}}$$

$$\underline{\frac{dy}{dx} = \frac{1}{(1+x^2)^{3/2}}}$$

Example 6.5.16 Find y' of the following; $y = \dfrac{1-x^2}{1+x^2}$

Solution: Let $u = 1 - x^2$ and $v = 1 + x^2$, so

$$y = \frac{u}{v}$$

$$\frac{du}{dx} = -2x \qquad \frac{dv}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

$$= \frac{(1+x^2).(-2x) - (1-x^2).2x}{(1+x^2)^2}$$

$$= \frac{-(2x+2x^3) - (2x-2x^3)}{(1+x^2)^2}$$

$$= \frac{-2x - 2x^3 - 2x + 2x^3}{(1+x^2)^2}$$

$$\underline{\frac{dy}{dx} = -\frac{4x}{(1+x^2)^2}}$$

6.6 Applications of Differentiation; Curve Sketching, Determination of Maximum or Minimum of a function.

In this unit we further explore the power of the derivatives, which we use to help analyse the properties of functions. The information obtained can then be used to accurately sketch graphs of functions. As we have seen on numerous occassions, the graph of a function is a useful aid for visualizing the function's properties from a practical point of view. The graph of a function also gives at one glance a complete summary of all the information captured by the function.

6.6.1 **Curve Sketching**

In curve sketching you need to be Out Comes

Upon completion of this unit you will be able to:

Sketch the curve, showing important features. Avoid drawing *x-y* boxes and just joining the dots. In this section we will be using calculus to help find important points on the curve. The kinds of things we will be searching for in this section are indicated in table 4.1.1:

x - intercept	Use y = 0. Note : In many cases, finding x-intercepts is not easy.
y- intercept	Use x = 0
Relative (local) Maximum	Use $\frac{dy}{dx} = 0$, sign: $x \to -$
Relative (local) minimum	Use $\frac{dy}{dx} = 0$, sign: $- \to +$
Points of inflection	$\frac{d^2y}{dx^2} = 0$, and sign of $\frac{d^2y}{dx^2}$ changes

Table 6.6.1

The maximum or minimum of any curve is the turning point of the curve to give the highest point for the maximum or the lowest point for the minimum. The turning point can be referred to as ; critical value, stationary point, extrema. At this point the derivative of the function is equal to zero. Therefore, when looking for the turning point we solve the equation $\frac{dy}{dx} = 0$. The solution(s) of this equation gives the value(s) of x for which the curve has its maximum or minimum. However, if we solve the equation $\frac{d^2y}{dx^2} = 0$, the solution(s) will give you the values of x where the curve changes from concave down to concave up, and this point is called point of inflection(s).

6.6.2 Increasing and Decreasing Functions and the First Derivative Test

We begin by discussing how derivatives can be used to classify relative extrema as either relative minima or relative maxima. We begin by defining increasing and decreasing functions.

Definition: Increasing and Decreasing Functions

A function f is increasing on an interval if for any two numbers x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1 < f(x_2)$. A function f is decreasing on an interval if for any two numbers

x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1 > f(x_2)$. A function is increasing if, as x moves to the right, its graph moves up, and is decreasing if its graph moves down.

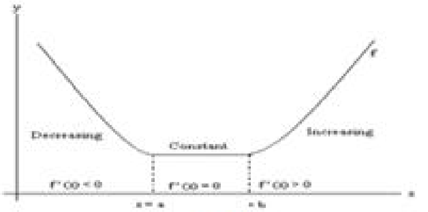


Figure 6.6.1

For example, Fig.6.6.2 above the function is decreasing on the interval $(-\infty, a)$, is constant on the interval (a,b), and is increasing on the interval (b, ∞). The theorem below states that , a positive derivative implies that the function is is increasing; a negative derivative implies that the function is decreasing; and zero derivative on an entire interval implies that the function is constant on that interval.

Theorem 6.6.1 Test for Increasing and Decreasing Functions

Let f be a function that is continuous on the closed interval [a,b] and differentiable on the open interval (a,b).

1. If f ‘ (x) > 0 for all x in (a,b), then f is increasing on [a,b]
2. If f’ (x) < 0 for all x in (a,b) , then f is decreasing on [a,b]
3. If f ‘ (x) = 0 for all x in (a,b), then f is constant on [a,b]

Example: 6.6.1 Find the open intervals on which $f(x) = x^3 - \frac{3}{2}x^2$, is increasing or decreasing.

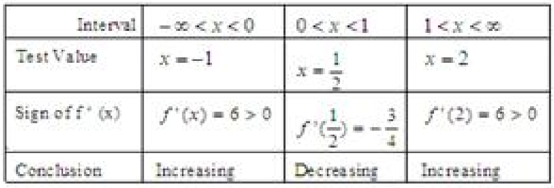
Solution Note that f is continuous on the entire real line. To determine the critical numbers of f, we find f ‘ (x) and set it to zero.

$$f'(x) = 3x^2 - 3x = 0$$

$$3x(x-1) = 0$$

X = 0, 1 Critical numbers

Because there are no points for which f ‘ is undefined, you can conclude that x = 0 and x = 1 are the only critical numbers. The following table summarizes the testing of the three intervals determined by these two critical numbers.



Interval	$-\infty < x < 0$	$0 < x < 1$	$1 < x < \infty$
Test Value	$x = -1$	$x = \frac{1}{2}$	$x = 2$
Sign of f ' (x)	$f'(x) = 6 > 0$	$f'(\frac{1}{2}) = -\frac{3}{4}$	$f'(2) = 6 > 0$
Conclusion	Increasing	Decreasing	Increasing

Table: **6.6.2**

We can show this graphically as follows:

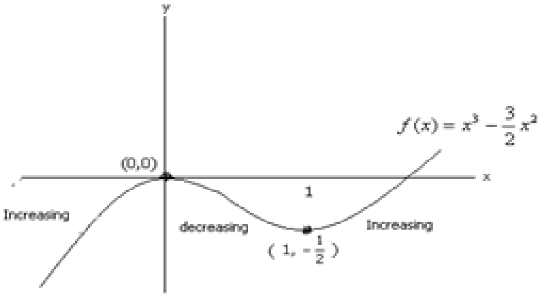


Figure **6.6.2**

You can see that f is increasing on the interval $(-\infty, 0)$ and $(1, \infty)$ and decreasing on the interval (0, 1) as shown on fig 6.6.2.

Guidelines for finding intervals on which a Function is Increasing or Decreasing.

Let f be continuous on the interval (a,b). To find the open intervals on which f is increasing or decreasing, use the following steps.

1. *Locate the critical numbers of f in (a,b), and use these numbers to determine test intervals.*
2. *Determine the sign of f ‘ (x) at one test value in each of the intervals.*
3. *Use theorem 4.1.1 to decide whether f is increasing or decreasing on each interval.*

These guidelines are also valid if the interval (a,b) is replaced by an interval of the form $(-\infty, b)$, (a, ∞), or $(-\infty, \infty)$.

6.6.2 First Derivative Test

After you have determined the intervals on which a function is increasing or decreasing, it is not difficult to locate the relative extrema of the function. In Fig. 4.1.2 above, the function $f(x) = x^3 - \frac{3}{2}x^2$ has a relative maximum at the point (0,0) because f is increasing immediately to the left of x = 0 and decreasing immediately at the point $(1, -\frac{1}{2})$ because f is decreasing immediately to the left at x = 1 . The following Theorem called the first Derivative Test makes it clear;

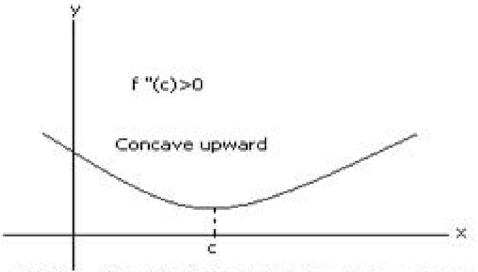
Theorem: 6.6.2 The First Derivative test

Let c be a critical number of a function f that is continuous on an open interval I containing c. If f is differentiable on the interval, except possibly at c, then f9c) can be classified as follows:

1. If f ‘ (x) changes from negative to positive at c, then f(c) is a *relative minimum* of f.
2. If f ‘ (x) changes from positive to negative at c, then f(c) is a *relative maximum* of f.

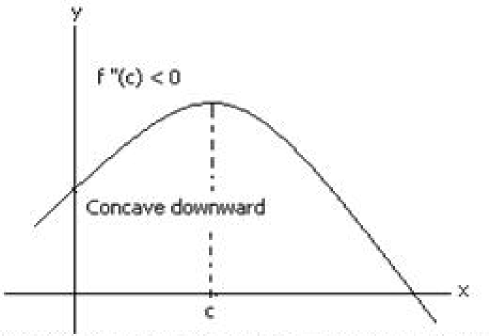
In determining the relative maximum or minimum of a function, we can use another approach, called the second Derivative Test. This test is based on the fact that if the graph of a function f is concave upward on an open interval containing c, and f ‘(c) = 0, f(c) must be a relative minimum of f. Similarly, if the graph of a function f is concave downward on an open interval containing c, and f ‘(c) = 0, f (c) must be a relative maximum of f. The second derivative test can be stated without a proof as:

(c.f. Fig. 6.6.3(a),(b) below)



If f ' (c) = 0 and f " (c) > 0, f(c) is a relative minimum

Figure 6.6.3a



If f ' (c) = 0 and f " (c) < 0, f(c) is a relative maximum
Figure 6.6.3b

If these two graphs are put together it will look like the figure 6.6.4 below:

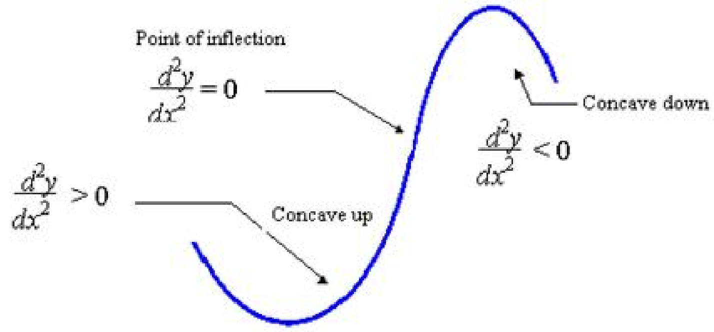


Figure 6.6.4

This method involves finding $\frac{dy}{dx}$ of the curve. The equation $\frac{dy}{dx}$=0 will give you the turning point(s) of the curve, when the value(s) of x is/ are substituted in the original equation y = f(x) , you will get the turning point, since it will have x- value and y-value , this is a point in the plane. To check whether a particular point at x = a is a local (relative) minimum or maximum, we use the second derivative test as follows: Find $\frac{d^2y}{dx^2}$. At the point $x = a$;

Theorem: 6.6.3 Second Derivative Test

Let f be a function such that $f'(a) = 0$ and the second derivative exists on an open interval containing a.

(i) If $\frac{d^2y}{dx^2} > 0$ at x = a, then the point at x = a is a local minimum.

(ii) If $\frac{d^2y}{dx^2} < 0$ at x = a, then the point at x = a is a local maximum

(iii) If $\frac{d^2y}{dx^2} = 0$ at x = a, the Test Fails.

When the test fails, it means at x = a the point is neither a local minimum nor local maximum, this point is called an ***inflexion point***., a point where the function changes from concave downward to concave upward. In this case, you can use the first derivative test.

Example:6.6.2 Find the relative extrema for

$$f(x) = -3x^5 + 5x^3$$

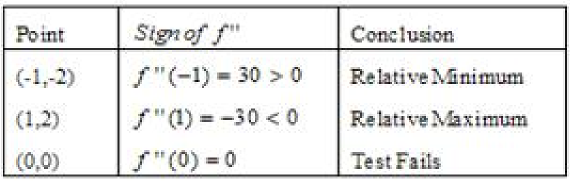
Solution We begin by finding the critical values of f by solving the equation f '(a) = 0.

$$f'(x) = -15x^4 + 15x^2 = 15x^2(1 - x^2) = 0$$
$$x = -1, 0, 1$$

We now find the second derivative

$$f''(x) = -60x^3 + 30x = 30(-2x^3 + x)$$

You can now apply the Second Derivative Test as follows:



Point	*Sign of f''*	Conclusion
(-1,-2)	$f''(-1) = 30 > 0$	Relative Minimum
(1,2)	$f''(1) = -30 < 0$	Relative Maximum
(0,0)	$f''(0) = 0$	Test Fails

Table 6.6.3

Because the second derivative test fails at (0,0) , you can use the first derivative test and observe that f increases to the left and right of x = 0. That is, (0,0) is neither a relative minimum nor a relative maximum.

The second derivative can tell us the *shape* of a curve at any point.

- If $\frac{d^2y}{dx^2} > 0$, the curve will have a **minimum**-type shape (called *concave up*)

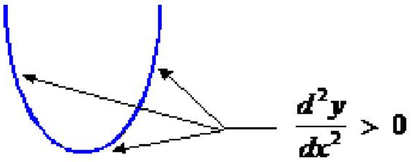


Figure 6.6.5 (a)

Example:6.6.3 $y = x^2 + 3x - 2$ has $\frac{dy}{dx} = 2x + 3$ and $\frac{d^2y}{dx^2} = 2 > 0$, for all values of x. So it has a concave up shape for all x. If $\frac{d^2y}{dx^2} < 0$, the curve will have a **maximum**-type shape (called *concave down*)

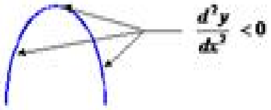


Figure 6.6.5b

Example: 6.6.4

$y = x^3 - 2x + 5$ has $\frac{dy}{dx} = 3x^2 - 2$ and $\frac{d^2y}{dx^2} = 6x < 0$ for all values of $x < 0$. *So it has a concave down shape for all* $x < 0$.

Determining Maximum/Minimum using gradient method

A **local maximum** occurs when $y' = 0$ and y' changes sign from positive to negative (as we go left to right).

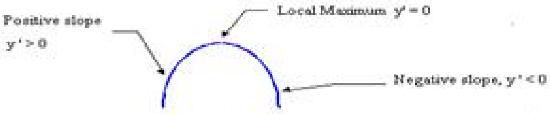


Figure 6.6.6a

A **local minimum** occurs when $y' = 0$ and y' changes sign from negative to positive.

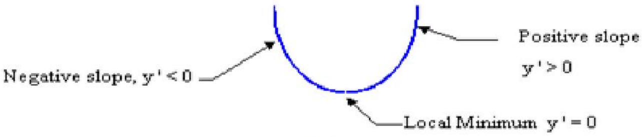


Figure 6.6.6b

6.7 Finding Points of Inflection

A point of inflection is a point where the shape of the curve changes from a **maximum**-type shape ($\frac{d^2y}{dx^2} < 0$) to a **minimum**-type shape ($\frac{d^2y}{dx^2} > 0$). Clearly, the point of inflection will occur when

$\frac{d^2y}{dx^2} = 0$ **and** when there is a change in sign (from plus → minus or minus → plus) of $\frac{d^2y}{dx^2}$.

Taking the example 1 above, we want to check which point is maximum and which one is minimum, we proceed as follows: Take the points (-I,8) ans (3,-24) which are the turning points:

Using the **second derivative test** , we find $\frac{d^2y}{dx^2}$. Here $\frac{d^2y}{dx^2} = 6x - 6$

At (-1,8), we substitute x = -1 in the second derivative and get 6(-1)-6= -12 <0, therefore, the point (-1,8) is a maximum. Check the conditions given above.

At (3,-24) , we substitute x = 3 in the second derivative and we get 6(3)-6 = 12 >0, therefore the point (3,-24) is a minimum.

Example : 6.7.1. Find the stationary points of the curve whose function is
$f(x) = x^3 - 3x^2 - 9x + 3$

Solution We first find $\frac{dy}{dx}$ of the given function and then solve the equation $\frac{dy}{dx} = 0$. The solutions to this equation will give us the values of x where the function has a maximum and a minimum. When these values of x are substituted in the original function the points obtained are called turning points respectively. $\frac{dy}{dx} = 3x^2 - 6x - 9$, to solve the equation $\frac{dy}{dx} = 0$, we set; $3x^2 - 6x - 9 = 0$, the solution to this quadratic equation is $x = -1 \; or \; 3$
Therefore, the turning points occurs at $x = -1 \; or \; 3$.We get the turning points by substituting these values in f(x). The turning points are (-1,8) and (3,-24). The next step is to check which

point is a maximum and which one is a minimum. There are two methods of checking the nature of the turning points, these are:

(a) We can use the second Derivative Test to determine the nature of these points as follows:

Using theorem 4.2.1, given above, we find the second derivative as follows;

$$\frac{d^2y}{dx^2} = 6x - 6$$

At x = -1, $\frac{d^2y}{dx^2} = 6x - 6 = 6(-1) - 6 = -12 < 0$, this implies that at (-1, 8), the function has a relative Maximum. At x = 3, $\frac{d^2y}{dx^2} = 6x - 6 = 6(3) - 6 = 18 - 6 = 12 > 0$, this implies that at (3, -24), the function has a minimum.

(b) We can determine the nature of the turning points by using the gradient method as follows;

A **local maximum** occurs when $y' = 0$ and y' changes sign from positive to negative (as we go left to right) and

A **local minimum** occurs when $y' = 0$ and y' changes sign from negative to positive.

(i) So, we take the point (-1,8) and the first derivative is- $\frac{dy}{dx} = 3x^2 - 6x - 9$, we take a point to the left of the critical value x = -1, which is x = -2, and substitute in the first derivative as; $\frac{dy}{dx} = 3x^2 - 6x - 9 = 3(-2)^2 - 6(-2) - 9 = 12 + 12 - 9 = 15$, **implies the gradient is Positive.**

Then take another value of x on the right of the critical value, say 0. Substitute it in the first derivative as : $\frac{dy}{dx} = 3x^2 - 6x - 9 = 3(0)^2 - 6(0) - 9 = -9$, **implies the gradient is negative.**
This means the function is increasing up to (-1,8) and then decrease after this point. That is the point is a relative maximum point.

(ii) Now we take the point (3,-24), we take appoint to the left of the value x = 3, say, 0. We substitute this value x = 0 in the first derivative as follows:

$\frac{dy}{dx} = 3x^2 - 6x - 9 = 3(0)^2 - 6(0) - 9 = -9 < 0$, **implies the gradient is negative.**

Then, we take a value to the right of x = 3, say x = 4, and substitute it in the first derivative as follows; $\frac{dy}{dx} = 3x^2 - 6x - 9 = 3(4)^2 - 6(4) - 9 = 15 > 0$**, implies the gradient is Positive.**

This means the function is decreasing up to (3,-24) and then increases. That is, the point (3,-24) is a relative Minimum point.

The shape of the graph of this function, would look like Figure 4.3.1 given below.

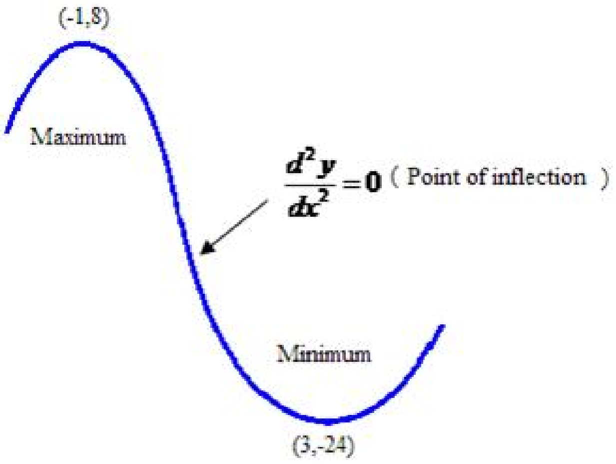


Figure 6,7.1

6.8 Tangents & Normals

6.8.1 Tangents

The gradient of the tangent to the curve **y = f(x)** at the point **(x_1, y_1)** on the curve is given by: the value of **dy/dx**, when **x = x_1** and **y = y_1**

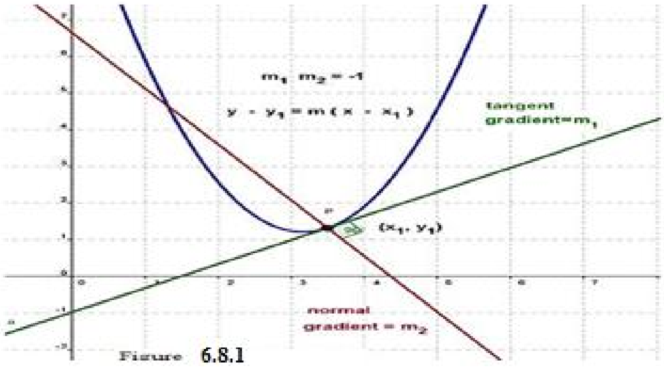


Figure 6.8.1

6.8.2 Normal Two lines of gradients m_1, m_2 respectively are perpendicular to each other if the product, $\mathbf{m_1} \times \mathbf{m_2} = -1$

Equation of a tangent The equation of a tangent is found using the equation for a straight line of gradient **m**, passing through the point **(x_1, y_1)** $y - y_1 = m(x - x_1)$. To obtain the equation we substitute in the values for **x_1** and **y_1** and **m** $(\frac{dy}{dx})$ and rearrange to make y the subject.

Example: 8.8.1 Find the equation of the tangent to the curve $y = 2x^2$ at the point (1,2).

Solution: The gradient of the tangent line at the point $(1,2)$ to the function $y = 2x^2$ is :

$$m = \frac{dy}{dx} = 4x = 4(1) = 4$$

Using the general equation of a straight line, $y - y_1 = m\ (x - x_1)$, where (x_1, y_1) is a point (1,2). We have

$$\begin{aligned}
&y - y_1 = m(x - x_1) \\
&y - 2 = 4(x - 1) \\
&y - 2 = 4x - 4 \\
&y = 4x - 6 \\
&y - 4x + 6 = 0
\end{aligned}$$

This is the equation of the tangent line to the function $y = 2x^2$ at the point (1,2).

Equation of a normal

The equation of a normal is found in the same way as the tangent. The gradient (**m_2**) of the normal is calculated from; $m_1 m_2 = -1$ (where **m_1** is the gradient of the tangent) so

$$m_2 = -\frac{1}{m_1}$$

Example: 8.8.2

Find the equation of the normal to the curve: $y = x^2 + 4x + 3$, at the point (-1,0).

Solution: Therefore gradient (m_1), for the function $y = x^2 + 4x + 3$ is; $\frac{dy}{dx} = 2x + 4$, at the point (-1, 0), implies you substitute these values in the first derivative as;

$m_1 = \frac{dy}{dx} = 2x + 4 = 2(-1) + 4 = 2$ that means the gradient of this function at the point (-1,0) is 2.

Let the gradient of then normal line be (m_2), we know that the product of the tangent and normal gradients is linked by the following relationship: $m_1.m_2 = -1$

But we know $m_1 = 2$, therefore,

$$2m_2 = -1$$
$$m_2 = -\frac{1}{2}$$

Using the general equation of a straight line, $y - y_1 = m_2(x - x_1)$, where (x_1, y_1) is a point (-1,0).

$$y - 0 = -\frac{1}{2}(x - (-1))$$
$$y = -\frac{1}{2}(x + 1)$$

So our normal line is $y = -\frac{1}{2}x - \frac{1}{2}$

$$2y = -x - 1$$
$$2y + x + 1 = 0$$

This is the equation of the line which is normal to a tangent line at the point (-1,0).

6.9 Related Rates

You have seen how the chain rule can be used to find $\frac{dy}{dx}$ implicitly. Another important use of the chain rule is to find the rates of change of two or more related variables that are changing with respect to time.

For example, when water is drained out of a conical tank, the figure below; the volume V, the radius r, and the height h of the water level are all functions of time t. Knowing that these variables are related by the equation $V = \frac{\pi}{3} r^2 h$

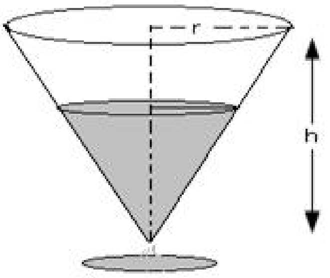


Figure 6.9.1

You can differentiate implicitly with respect to t to obtain the ***related rate*** equation

$$\begin{aligned} \frac{dV}{dt} &= \frac{\pi}{3}\left[r^2 \frac{dh}{dt} + h(2r\frac{dr}{dt}) \right] \\ &= \frac{\pi}{3}\left(r^2 \frac{dh}{dt} + 2rh\frac{dr}{dt} \right) \end{aligned}$$

From this equation you can see that the rate of change of V is related to the rates of change of both h and r. Finding the Related Rate. In the conical tank shown above, suppose that the height is changing at a rate of -0.2 cm per minute and the radius is changing at the rate of -0.1 cm per minute. What is the rate of change in the volume when the radius is r = 1cm and the height is h = 2 cm? Does the rate of change in volume depend on the values of r and h?

Guidelines for solving related rate problems

1. Identify all given quantities and quantities to be determined. Make a sketch and label the quantities.

2. Write an equation involving the variables whose rates of change either are given or are to be determined.
3. Using the chain rule, implicitly differentiate both sides of the equation with respect to time t.
4. After completing step 3, substitute into the resulting equation all known values for the variables and their rates of change. Then solve for the required rate of change.

Example: 6.9.1

Air is being pumped into a spherical balloon at a rate of 4.5 cubic cm per minute. Find the rate of change of the radius when the radius is 2 cm.

Solution Let V be the volume of the balloon and let r be its radius. Since the volume is increasing at a rate of 4.5 cubic cm per minute, you know that at time t the rate of change of the volume is $\frac{dV}{dt} = \frac{9}{2}$.

Now the problem can be stated as follows: *Given rate*: $\frac{dV}{dt} = \frac{9}{2}$ (constant rate). *Find;*

$\frac{dr}{dt}$ when r = 2

To find the rate of change of the radius, you must find an equation that relates the radius r to the volume V. $V = \frac{4}{3}\pi r^3$ (Volume of sphere). Implicit differentiation with respect to t produces

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \qquad \text{(Differentiation with respect to t)}$$

$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\left(\frac{dV}{dx}\right) \qquad \left(\text{ Solve for } \frac{dr}{dt}\right)$$

Finally, when r = 2, the rate of change of the radius is

$$\frac{dr}{dt} = \frac{1}{16\pi}\left(\frac{9}{2}\right) = 0.09\, cm\ per \min ute$$

In this example note that the volume is increasing at a constant rate but the radius is increasing at a variable rate. Just because two rates are related does not mean that they are proportional. In this example, the radius is growing more and more slowly as t increases. Why?

The Chain Rule is a means of connecting the rates of change of dependent variables.

The **derivative** tells us the rate of change of one quantity compared to another at a particular instant or point (so we call it "instantaneous rate of change"). This concept has many applications in electricity, dynamics, economics, fluid flow, population modeling, queuing theory and so on. Wherever a quantity is always changing in value, we can use **calculus** (differentiation and integration) to model its behaviour.

In this section, we will be talking about events at certain times, so we will be using Δt instead of the Δx that we saw in the last section Derivative from first principles.

Note: This section is part of the introduction to differentiation. We learn some (much easier) rules for differentiating in the next section, Derivatives of polynomial.

Example: 6.9.2 If air is blown into a spherical balloon at the rate of 10 cm^3 how quickly will the radius grow?

if the radius of the balloon is r

then the volume $V = \frac{4}{3}\pi r^3$

and $\frac{dV}{dr} = 4\pi r^2$

the rate of change of volume with time

is given by: $\frac{dV}{dt} = 10\text{cm}^3\ /\ \text{sec}.$

using the Chain Rule

$$\frac{dV}{dt} = \frac{dV}{dr}.\frac{dr}{dt} \quad \text{and} \quad \frac{dV}{dr} = 4\pi r^2$$

$$\therefore \frac{dr}{dt} = \frac{dV}{dt}.\frac{1}{\frac{dV}{dr}} = \frac{dV}{dt}.\frac{dr}{dV} = 10.\frac{1}{4\pi r^2}$$

$$= \frac{5}{2\pi r^2}$$

i.e. rate of change of radius is $\frac{5}{2\pi r^2}$ cm/sec.

Example: 2.9.7 A spherical raindrop is formed by condensation. In an interval of 10 sec. its volume increases at a constant rate from 0.010mm^3 to 0.500mm^3 Find the rate at which the surface area of the raindrop is increasing, when its radius is 1.0mm

radius r mm

volume V is given by: $V = \frac{4}{3}\pi r^3$

$\therefore \quad \frac{dV}{dr} = \frac{4\pi}{3}.3r^2 = 4\pi r^2$

also, area A is given by: $A = 4\pi r^2$

$\therefore \quad \frac{dA}{dr} = 4\pi.2r = 8\pi r$

vol. increases at a constant rate by:

$0.5 - 0.010 = 0.490$ mm^3 in 10 sec.

so $\frac{dV}{dt} = \frac{0.49}{10} = 0.049$ mm^3.s^{-1}.

we are required to find $\frac{dA}{dt}$ when $r = 1.0$mm

using the Chain Rule, $\frac{dA}{dt} = \left(\frac{dA}{dV}\right).\frac{dV}{dt}$

$$= \left(\frac{dA}{dr}.\frac{dr}{dV}\right).\frac{dV}{dt}$$

$$\frac{dA}{dt} = \left(8\pi r.\frac{1}{4\pi r^2}\right)0.049 = \left(\frac{2\times 0.049}{r}\right) = \frac{0.098}{r}$$

when $r = 1.0$mm, $\frac{dA}{dt} = \frac{0.098}{1} = 0.098$mm^2.s^{-1}.

$\therefore$ surface area, for a radius of 1mm, increases by 0.098mm^2.s^{-1}.

Summary

In this unit you learned the following concepts:

1. $\frac{dy}{dx} = \lim\limits_{x \to x} \frac{f(x+h) - f(x)}{h}$ is the derivative of y w.r.t.x from first principles.

2. For any functions uv, then

$$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$$ **derivatives of sum and difference**

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$ **product of functions- product rule**

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2},$$ **rational functions- quotient rule**

The Chain Rule

If h(x) = g[f(x)], then $h'(x) = \frac{d}{dx} g[f(x)] = g'[f(x)]f'(x)$

Equivalently, if we write y = h(x) =g(u), where u = f(x), then

General power Rule

If the function f is differentiable and $h(x) = [f(x)]^n$ (n, a real number), then

$$h'(x) = \frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1} f'(x)$$

We observe that h(x) = g[f(x)], where $g(x) = x^n$, so that, by virtue of the chain rule, we have

$$\begin{aligned} h'(x) &= g'[f(x)]f'(x) \\ &= n[f(x)]^{n-1} f'(x) \end{aligned}$$

Since $g'(x) = nx^{n-1}$, y'', $f''(x)$, $\frac{d^2y}{dx^2}$

These are the forms of writing the second derivative of y = f(x)

Guidelines for solving related rate problems

1. Identify all given quantities and quantities to be determined. Make a sketch and label the quantities.
2. Write an equation involving the variables whose rates of change either are given or are to be determined.
3. Using the chain rule, implicitly differentiate both sides of the equation with respect to time t.
4. After completing step 3, substitute into the resulting equation all known values for the variables and their rates of change. Then solve for the required rate of change.

Increasing and Decreasing Functions

A function f is increasing on an interval if for any two numbers x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1 < f(x_2)$.

A function f is decreasing on an interval if for any two numbers x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1 > f(x_2)$

Test for Increasing and Decreasing Functions

Let f be a function that is continuous on the closed interval [a,b] and differentiable on the open interval (a,b).

1. If f ‘ (x) > 0 for all x in (a,b), then f is increasing on [a,b]
2. If f’ (x) < 0 for all x in (a,b) , then f is decreasing on [a,b]
3. If f ‘ (x) = 0 for all x in (a,b), then f is constant on [a,b]

Guidelines for finding intervals on which a Function is Increasing or Decreasing.

Let f be continuous on the interval (a,b). To find the open intervals on which f is increasing or decreasing, use the following steps.

1. *Locate the critical numbers of f in (a,b), and use these numbers to determine test intervals.*
2. *Determine the sign of f ‘ (x) at one test value in each of the intervals.*
3. *Use theorem 4.1.1 to decide whether f is increasing or decreasing on each interval.*

The First Derivative test

Let c be a critical number of a function f that is continuous on an open interval I containing c. If f is differentiable on the interval, except possibly at c, then f9c) can be classified as follows:

1. If f ‘ (x) changes from negative to positive at c, then f(c) is a relative minimum of f.
If f ‘ (x) changes from positive to negative at c, then f(c) is a relative maximum of f.

Second Derivative Test

Let f be a function such that $f'(a) = 0$ and the second derivative exists on an open interval containing a.

(i) If $\frac{d^2y}{dx^2} > 0$ at x = a, then the point at x = a is a local minimum.

(ii) If $\frac{d^2y}{dx^2} < 0$ at x = a, then the point at x = a is a local maximum

(iii) If $\frac{d^2y}{dx^2} = 0$ at x = a, the Test Fails.

Exercise: 6

1. Calculate the following limits

(i) $\lim\limits_{x\to 2}\frac{x^2-1}{x-1}$ (ii). $\lim\limits_{x\to\infty}\frac{3-x}{\sqrt{x^2+3x}}$ (iii) $\lim\limits_{x\to 3}\frac{x^2-x-6}{x-3}$ (iv) $\lim\limits_{x\to\infty}\left(\frac{1-x^2}{8x^2+5}\right)$

2. (a) From first principles find $\frac{dy}{dx}$ of :

(i) y=2x+1 (ii) $y = x^3+3x^2-2$

(iv) $y=\sqrt{2x-1}$ (v) $y=\frac{1}{x+1}$ (vi) $y=\frac{1}{\sqrt{x+1}}$

(vii) $y=\frac{x+1}{x-1}$ (viii) y = 3

(b) Find $\frac{dy}{dx}$ of the following ;

(i)$y = -7x^6$ (ii) $y = 3x^5 - 1$ (iii) $y = 13x^4 - 6x^3 - x - 1$

(iv) $y = -\frac{1}{4}x^8 + \frac{1}{2}x^4 - 3x^2$ (v) $y = x^4 - 9x^2 - 5x$, at the point (3,15).

(vi) $y = x^{\frac{1}{4}} - \frac{2}{x}$

3. Sketch the following curve by finding intercepts, maxima and minima and points of inflection:

(a) $y = x^3 - 9x$

(b) $y = x^4 - 6x^2$

(c) $y = x^5 - 5x^4$

4. Find the critical numbers (if any) and the open intervals on which the function is increasing or decreasing.

(a) $y = (x-1)^2(x-3)$

(b) $y = \sqrt{x}(x-3)$

(c) $f(x) = -2x^2 + 4x + 3$

(d) $f(x) = x^2 - 6x$

(e) $f(x) = 2x^3 + 3x^2 - 12x$

(f) $f(x) = \dfrac{x^5 - 5x}{5}$

(g) $f(x) = x^{1/3} + 1$

(h) $f(x) = 5 - |x - 5|$

(i) $f(x) = \dfrac{x^2}{x^2 - 9}$

(j) $f(x) = x^3 - 6x^2 + 15$

(k) $f(x) = (x-1)^{2/3}$

(l) $f(x) = x + \dfrac{1}{x}$

(m) $f(x) = \dfrac{x^2 - 2x + 1}{x + 1}$

5. Let $f(x) = \dfrac{1}{4}x^4 - 8x$, use the first derivative test to find any relative extrema of the function. Use your graph to verify your answers.

6. Let $(x) = x + \cos x$, find any points of inflection of the function in the interval $0 \le x \le 2\pi$.

7. Find the second derivative of the function

(a) $y = 2x^2 + \sin 2x$ (b) $y = \cot x$ (c) $y = \dfrac{x}{(1-x)^2}$ (d) $y = x\tan x$

8. Find the equation of the tangent and normal to the curve of the equation at the indicated point. Graph the function, the tangent and the normal lines.

(a) $y = (x+3)^3$ at (-2,1)

(b) $y = \sqrt[3]{(x-2)^2}$ at (3,1)

(c) $x^2 + y^2 = 20$, at (2, 4)

9. Find the points on the graph of $y = \frac{1}{3}x^3 + x^2 - x - 1$ when the slope is (a) -1 (b) 2 (c) 0

10. Sketch the graph of $f(x) = 4 - |x-2|$

(a) Is f continuous at x = 2?

(b) Is f differentiable at x = 2? Explain

11. Show that the function satisfies the given differential equation.

Function	Equation
$y = 2\sin x + 3\cos x$	$y'' + y = 0$
$y = \dfrac{10 - \cos x}{x}$	$xy' + y = \sin x$

12. The cross section of a 5 meter trough is an isosceles trapezoid with a 2 meter lower base, a 3 meter upper base , and an altitude of 2 meters. Water is running into the trough at a rate of 1 cubic meter per minute. How fast is the water level rising when the water level is i meter deep?

13. Find all relative extrema. Use the second derivative test where applicable.

(a) $f(x)=x+\frac{4}{x}$ (b) $f(x)=x^4-4x^3+2$ (c) $f(x)=\cos x-x$, $[0,4\pi]$

14. The radius r of a circle is increasing at a rate of 2 centimetres per minute. Find the rate of change of the area when (a) r = 6 cm and (b) r = 24 cm. (13)

15. A spherical balloon is inflated with gas at the rate of 500cubic cm per minute. How fast is the radius of the balloon increasing at the instant the radius is (a) 30 cm and (b) 60 cm

16. A all edges of a cube are expanding at a rate of 3cm per second.

(i) How fast is the volume changing when each edge is 10 cm

(ii) How fast is the volume changing when each edge is (i) 1 cm and (ii) 10 cm

17. At a sand and gravel plant, sand is falling off a conveyor and onto a conical pile at a rate of 10 cubic meters per minute. The diameter of the base of the cone is approximately three times the altitude. At what rate is the height of the pile changing when the pile is 15 meters high? (21)

18. Find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$ for the following functions:

(i) $x^2+2x-2xy^2=2$

(ii) $y=\frac{x^2(3x+1)}{x^4+2}$

Bibiography

1. Buckhouse, J.K and Houldsworth, S.P.T.1985.Pure Mathematics 1.Longman

2. Roland E. Larson , Robert P. Hostetler, Bruce H Edward: 1998: Calculus with Analystical Geometry. Houghton Mifflin Co. Boston. New York

3. S.T.Tan: 2004: College Mathematics .Thomson,Books/cole

4. www.mathstutor.com.