MAT1100 LECTURE NOTES

Term 1

1. Sets

1.1 Set Theory

Sets make a foundation of mathematics. The concept of a set appears in all branches of mathematics. It formalizes the idea of grouping objects together and viewing them as a single entity.

Basic Definitions

Definition 1.1 A set is any well defined list or collection of objects, called *elements* or *members*.

Sets are denoted by capital letters, say A, B, $C, \cdots$. The elements or members of a set are denoted by lowercase letters a, b, c, $\cdots$.

Example 1.1If P is a set of all vowels then it is written as $P = \{a, e, i, o, u\}$.

If A is the set of all even integers between 1 and 100, it may be written in set builder notation as $A = \{x: 1 < x < 100 \ and \ \ x \ is \ an \ even \ number\}$. In this notation x represent even numbers between 1 and 100.

If p is an element of set A we write $p \in A$. If p does not belong to A we write $p \notin A$.

For example, if B is a set of positive numbers then $5 \in B$. $-2 \notin B$ since -2 is a negative integer.

Definition 1.2 Two sets A and B are said to be **identical** or **equal**, written $A = B$, if each member of A is also a member of B and vice versa. Thus $A = B$ if $x \in A \Rightarrow x \in B$ and if $x \in B \Rightarrow x \in A$.

For example, the sets $A = \{a, u, e, i, o\}$ and $B = \{i, u, e, o, a\}$ are identical or equal. Note that the order in which the elements are written does not matter.

Definition 1.3 If all the elements of A are in B and $A \neq B$ (A is not equal to B), then A is said to be **proper subset** of B, and we write $A \subset B$.

If all the elements of A belong to B and we are not sure whether A and B are identical, we simply say that A is a **subset** of B, and we write $A \subseteq B$.

For example, if A is the set of all rain days in March and B is the set of all days in March, then clearly $A \subseteq B$.

Note: 1. A set A is a subset of B if and only if every element of A is an element of B.

2. If $A \subseteq B$ and $B \subseteq A$ then $A = B$.

Definition 1.4 All sets under investigation are subsets of a fixed set called the **universal set**. In this course, we shall denote the universal set by *U*.
On the other hand it is also possible to have a set which has no elements. This set is called an **empty set** or a **null set**, and it is denoted by $\emptyset$.

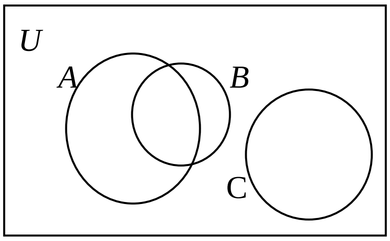
For example, the set $P = \{x: x^2 = -1 \; and \; x \; is \; a \; real \; number\} = \emptyset$.

By convention, $\emptyset$ is a subset of every set.

The number of elements in a set, say A, is denoted by $n(A)$. For example, if $A = \{a, b, c\}$, the $n(A) = 3$.

Sets can also be represented pictorially in a diagram called a **Venn diagram**, in which the sets are depicted by a enclosed areas in a plane.

For example, the universal set is a rectangle and it contains the circular subsets *A*, *B* and *C*.



Definition 1.5 A set of sets is called a **collection** or **class** or **family**.

For example, $\mathcal{B} = \{\{1,2\}, \{3\}, \{1,2,3\}\}$ is a class and the members of $\mathcal{B}$ are the sets $\{1,2\}, \{3\},$ and $\{1,2,3\}$.

Definition 1.6 The power set of set *A*, denoted by $\mathbb{P}(A)$ or 2^A, is the class of all subsets of *A*. In particular, if $A = \{a, b, c\}$, then

$$\mathbb{P}(A) = \{A, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \emptyset\}.$$

The number of elements in the power set of set, say *A* is given by $n(\mathbb{P}(A)) = 2^{n(A)}$.

For example, $n(\mathbb{P}(A)) = 2^{n(A)} = 2^3 = 8$.

Definition 1.7 A set is said to be **finite** if it has exactly *m* elements, where *m* is a positive integer. Otherwise it is said to be **infinite**.

For example, the set $A = \{a, e, i, o, u\}$ is finite while the set $\mathbb{N} = \{1,2,3, \cdots\}$ the set of all positive is infinite.

By conversion a null set $\emptyset$ is finite.

Set Operations

There are basically four set operations, namely the union, intersection ,relative complement and absolute complement or simply complement.

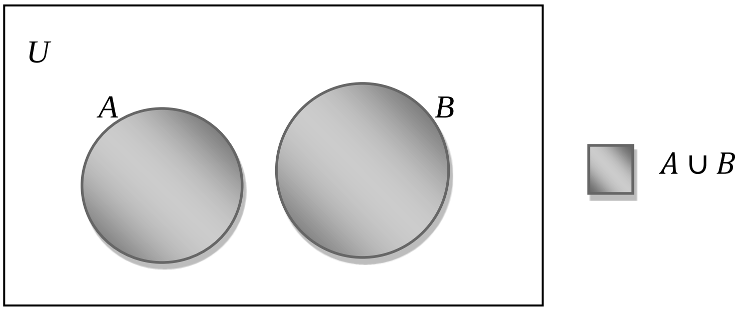
Definition 1.8 The **union** of two sets A and B, denoted by $A \cup B$, is the set of all elements which belong to A or to B, i.e.,

$$A \cup B = \{x: x \in A \; or \; x \in B\}.$$

Example 1.2 Let $A = \{a, b, c, d, e\}$ and $B = \{a, e, i, o, u\}$. Then

$$A \cup B = \{a, b, c, d, e, i, o, u\}$$

In general, $A \cup B$ can be represented in a diagram called a vein diagram, as follows:



Clearly, if A and B are finite disjoint sets, then $A \cup B$ is finite and

$$n(A \cup B) = n(A) + n(B).$$

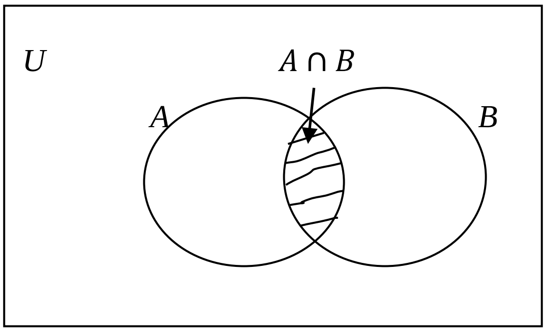
Definition 1.9 The **intersection** of two sets A and B, denoted by $A \cap B$, is the set of all elements which belong to A and to B, i.e.,

$$A \cap B = \{x: x \in A \; and \; x \in B\}.$$

Example 1.3 Let $A = \{a, b, c, d, e\}$ and $B = \{a, e, i, o, u\}$. Then

$$A \cap B = \{a, e\}$$

In a vein diagram, $A \cap B$ can be represented as follows:



Note that $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

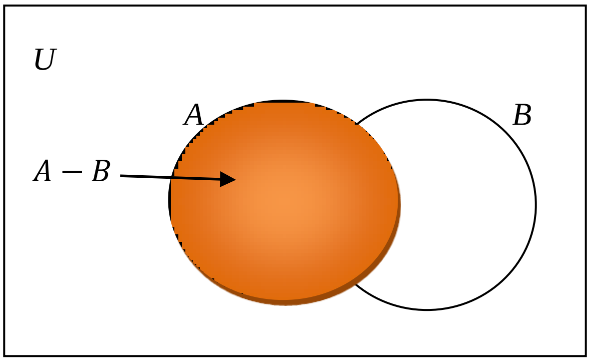
Definition 1.10 The **relative complement** of a set *B* with respect to set *A* or simply the difference of *A* and *B*, denoted by $A - B$ *or* $A \backslash B$, is the set of elements which belong to *A* and which do not belong to *B*, i.e.,

$$A - B = \{x : x \in A \text{ and } x \notin B\}.$$

Example 1.1.3 Let $A = \{a, b, c, d, e\}$ and $B = \{a, e, i, o, u\}$. Then

$$A - B = \{b, c, d\}$$

In a vein diagram, this represented as follows:



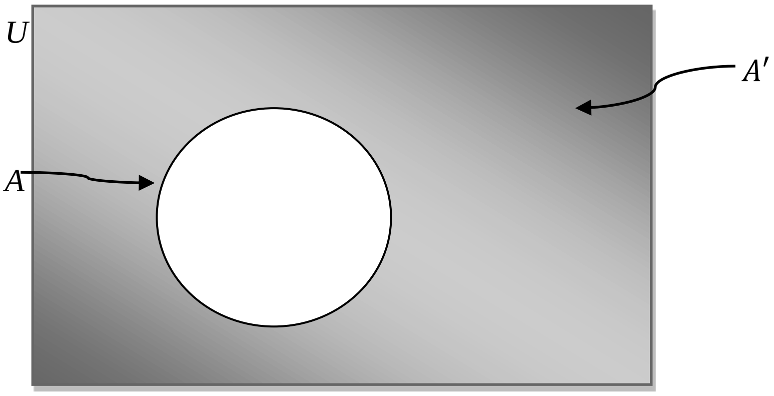
Note: 1. $A - B$ and B are disjoint sets i.e. $(A - B) \cap B = \emptyset$.

2. $A - B \subseteq A$.

Definition 1.11 The **absolute complement** or simply the complement of set *A*, denoted by A', is the set of elements which do not belong to *A* but belong to the universal set *U*, i.e.,

$$A' = \{x : x \in U, x \notin A\}.$$

Note: $A' = U - A$.



Laws of Algebra of Sets

Sets under the above operations satisfy various laws which are listed below.

1. The idempotent laws:

   (a) $A \cup A = A$

   (b) $A \cap A = A$

   The proof of this is trivial.

2. The Associative laws:

   (a) $A \cup (B \cup C) = (A \cup B) \cup C$

   Proof: Let $x \in A \cup (B \cup C)$. Then $x \in A$ or $x \in B \cup C$

   $\Rightarrow x \in A$ or $x \in B\ or\ x \in C \Rightarrow x \in A \cup B\ or\ x \in C\ x \in (A \cup B) \cup C$

   This means that $A \cup (B \cup C) \subset (A \cup B) \cup C$.

   Conversely, Let $x \in (A \cup B) \cup C$. Then $x \in (A \cup B)$ or $x \in C$

   $\Rightarrow x \in A$ or $x \in B\ or\ x \in C \Rightarrow x \in A\ or\ x \in (B \cup C) \Rightarrow x \in (A \cup B) \cup C$

   This means that $(A \cup B) \cup C \subset A \cup (B \cup C)$.

   Now since $A \cup (B \cup C) \subset (A \cup B) \cup C$ and $(A \cup B) \cup C \subset A \cup (B \cup C)$, it follows that $A \cup (B \cup C) = (A \cup B) \cup C$.

   (b) $A \cap (B \cap C) = (A \cap B) \cap C$

   Prove (b) the same way as an exercise.

3. The Commutative laws:

   (a) $A \cup B = B \cup A$

   (b) $A \cap B = B \cap A$

   The proof of this is trivial.

4. The Distributive laws:

   (a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

   Proof: (a) To prove the theorem, you must show that

   $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$ and $(A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$.

   i.e. you take an arbitrary element $x \in A \cup (B \cap C)$ and show that it is also an element of the set $(A \cup B) \cap (A \cup C)$.

   Now, let $x \in A \cup (B \cap C)$. Then

   $x \in A$ or $x \in B \cap C$

   $\Rightarrow$ $x \in A$ or $x \in B$ and $x \in C$

   $\Rightarrow$ $x \in A$ or $x \in B$ and $x \in A$ or $x \in C$

$\Rightarrow$ $x \in A \cup B$ and $x \in A \cup C$

$\Rightarrow$ $x \in (A \cup B) \cap (A \cup C)$

Thus,

$A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$

Conversely, let $x \in (A \cup B) \cap (A \cup C)$. Then $x \in A \cup B$ and $x \in A \cup C$

$\Rightarrow$ $x \in A$ or $x \in B$ and $x \in A$ or $x \in C$

$\Rightarrow$ $x \in A$ or $x \in B$ and $x \in C$

$\Rightarrow$ $x \in A$ or $x \in B \cap C$

$\Rightarrow$ $x \in A \cup (B \cap C)$.

Thus, $(A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$.

Since $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$ and

$(A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$ it follows that

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

(b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof: Complete the proof by filling in the blanks.

Let $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$

$\Rightarrow$ $x \in A$ and _______(1)___________

$\Rightarrow$ $x \in A$ and $x \in B$_____(2)_____ $x \in A$ and $x \in C$

$\Rightarrow$ ______(3)_______ or ______(4)_______

$\Rightarrow$ $x \in (A \cap B) \cup (A \cap C)$.

Thus, $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$.

Conversely, let $x \in$_______(5)_________. Then

$x \in A \cap B$ or $x \in A \cap C$

$\Rightarrow$ _____(6)______ or _____(7)_____

$\Rightarrow$ $x \in A$ and $x \in B$ or $x \in C$

$\Rightarrow$ $x \in A$ and _____(8)_____

$\Rightarrow$ $x \in A \cap (B \cup C)$.

Thus, $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$.

Therefore since $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$ and

$(A\cap B)\cup(A\cap C)\subset A\cap(B\cup C)$ it follows that

$$A\cap(B\cup C)=(A\cap B)\cup(A\cap C).$$

5. The Identity laws:

   (a) $A\cup\emptyset=A$

   (b) $A\cap\emptyset=\emptyset$

   (c) $A\cup U=U$

   (d) $A\cap U=A$

   Proof: Exercise

6. The Complement laws:

   (a) $A\cup A'=U$

   (b) $A\cap A'=\emptyset$

   (c) $(A')'=A$

   (d) $U'=\emptyset$
   (e) $\emptyset'=U$

   Proof: Exercise

7. De Morgan's laws:

   (a) $(A\cup B)'=A'\cap B'$

   Proof: Let $x\in(A\cup B)'$. Then $x\notin A\cup B\Rightarrow x\notin A$ or $x\notin B$

   $\Rightarrow x\in A'$ and $x\in B'\Rightarrow x\in A'\cap B'$.

   Thus, $(A\cup B)'\subset A'\cap B'$.

   Conversely, let $x\in A'\cap B'$. Then $x\in A'$ an d $x\in B'$.

   $\Rightarrow x\notin A$ or $x\notin B\Rightarrow x\notin A\cup B\Rightarrow x\in(A\cup B)'$.

   Thus, $A'\cap B'\subset(A\cup B)'$.

   Therefore, since $(A\cup B)'\subset A'\cap B'$ and $A'\cap B'\subset(A\cup B)'$ we conclude that $(A\cup B)'=A'\cap B'$.

   (b) $(A\cap B)'=A'\cup B'$

   Proof: Complete the proof by filling in the blanks.
   Let $x\in(A\cap B)'$. Then __________(1)___________

   $\Rightarrow x\notin A$ or $x\notin B\Rightarrow$ ________(2)_________ $\Rightarrow x\in A'\cup B'$.

   Thus, $(A\cap B)'=A'\cup B'$.

   Conversely, let $x\in A'\cup B'$. Then ________(3)_________

   $\Rightarrow x\notin A$ or $x\notin B\Rightarrow$ ________(4)_________ $\Rightarrow x\in(A\cap B)'$.
   Thus, $A'\cup B'\subset(A\cap B)'$.

Therefore, since $A' \cap B' \subset (A \cup B)'$ and $(A \cup B)' \subset A' \cap B'$ we conclude that $(A \cap B)' = A' \cup B'$.

The De Morgan's laws can be generalized as follows:

(a) $(A_1 \cup A_2 \cup \cdots \cup A_n)' = A'_1 \cap A'_2 \cap \cdots \cap A'_n$

(b) $(A_1 \cap A_2 \cap \cdots \cap A_n)' = A'_1 \cup A'_2 \cup \cdots \cup A'_n$

8. The Difference law:

   $A - B = A \cap B'$

<u>Example</u> 1.4 Let $U = \{1,2,3,\cdots,8,9\}$, $A = \{1,2,3,4\}$, $B = \{2,4,6,8\}$ and $C = \{3,4,5,6\}$. Find

(i) A' (ii) $(A \cap C)'$ (iii) $B - C$ (iv) $(A \cup B)'$

<u>Solution</u>: (i) $A' = \{5,6,7,8,9\}$

(ii) $A \cap C = \{3,4\} \Rightarrow (A \cap C)' = \{1,2,5,6,7,8,9\}$

NOTE: $A' \cup C' = \{5,6,7,8,9\} \cup \{1,2,7,8,9\} = \{1,2,5,6,7,8,9\} = (A \cap C)'$

(iii) $B - C = \{2,8\}$

NOTE: $B \cap C' = \{2,4,6,8\} \cap \{1,2,7,8,9\} = \{2,8\} = B - C$

(iv) $A \cup B = \{1,2,3,4,6,8\} \Rightarrow (A \cup B)' = \{5,7,9\}$

NOTE: $A' \cap B' = \{5,6,7,8,9\} \cap \{1,3,5,7,9\} = \{5,7,9\} = (A \cup B)'$

MAT 1100 FOUNDATION MATHEMATICS TUTORIAL SHEET 1 -2020

---

1. Let the universal set $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, the sets $A = \{1, 3, 6, 8, 10\}$, $B = \{2, 4, 5, 6, 8\}$ and $C = \{1, 4, 6, 10\}$. Find the following sets:

   (a) $(A \cap B')'$ (d) $(A' \cup B)'$ (e) $(A \cap B) \cup C$

   (f) $(A \cup B) \cap C$.

2. In the problems below, one of the following relations is true: $A \subset B$, $A = B$, $A \neq B$, $B \subset A$. Write the correct relation in each case.

	A	B
(a)	$\{x \mid 2x + 3 = 11 - 2x\}$	$\{x \mid 5x + 4 = x + 12\}$
(b)	$\{x \mid x^2 + 4 = 40 - 5x\}$	$\{x \mid 3 + 2x = 11\}$

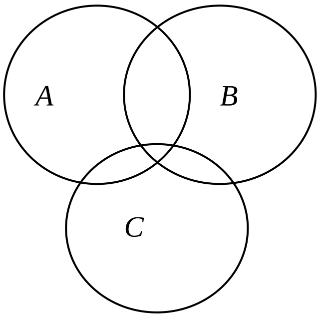
(c) $\{x|x+4 = x(x+4)\}$ $\{x|x^2+3x-4=0\}$

(d) $\{x|x-1=0\} \cup \{x|x-2=0\}$ $\{x|x^2-3x+2=0\}$

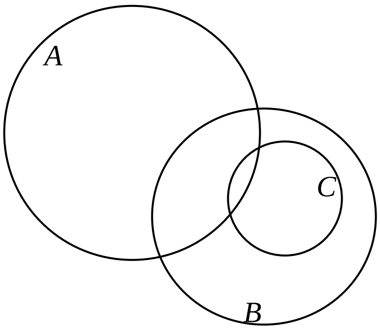
(e) $\{x|x+3=4\}$ $\{x|(x+3)^2=16\}$

3. In each of the Venn diagrams below shade: (i) $A \cap (B \cup C)$ (ii) $C - (A \cap B)$

(a)



(b)



4. Illustrate the given identities by drawing Venn diagrams, given that *A*, *B*, *C* are subsets of a universal set *U*:

(a) De Morgan's laws:

(i)$(A \cup B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$

(b) Distributive laws:

(i) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (ii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(c) $A \cup A' = U$ (d) $A \cap A' = \emptyset$

(d) Commutative laws:

(i) $A \cup B = B \cup A$ (ii) $A \cap B = B \cap A$

(e) Associative laws:

(i) $A \cup (B \cup C) = (A \cup B) \cup C$ (ii) $A \cap (B \cap C) = (A \cap B) \cap C$

(f) $A - B = A \cap B'$

5. Express each of the following in its simplest form:

(a) $[P' \cup (Q - R)]'$ (b) $X \cup (Y \cap X)$ (c) $(M \cap N) \cup (M \cap N')$

(d) $A - (A - B)$ (e) $A \cup (B - C)$(f) $[(A \cap B)' \cup (A - B)]'$

Sets of Numbers

There special symbols used for sets of numbers. These are

$\mathbb{Z}$ = the set of integers: $\{\cdots, -2, -1, 0, 1, 2, \cdots\}$.

$\mathbb{N}$ = the set of natural numbers (positive integers or counting numbers): $\{1,2,3,\cdots\}$.

$\mathbb{Q}$ = the set of rational numbers: numbers which can be expressed as ratios of Integers.

$\mathbb{Q}'$= the set of irrational numbers: numbers which cannot be expressed as ratios of integers.

$\mathbb{R}$ = the set of real numbers.

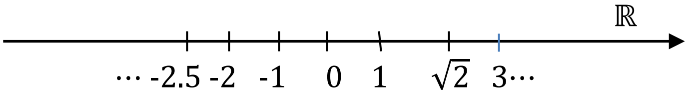
$\mathbb{C}$ = the set of complex numbers.

NOTE: 1. $N \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.

2. $\mathbb{Q}'$= the set of irrational numbers, like the numbers $\sqrt{2}, \sqrt{3}, \pi$, etc.

3. $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}'$.

4. The set of real numbers can be represented graphically by points on a straight line as follows:



Intervals

Lat *a* and *b* be distinct real numbers with, say, $a < b$. Then intervals with endpoints *a* and *b* are denoted and defined as follows:

(i) $(a, b) = \{x \in \mathbb{R}: a < x < b\}$, open interval from *a* to *b*.

Note: 1. Here the numbers $a, b \notin (a, b)$

2. (a, b) is a set which contains all the numbers between *a* and *b*, and it is an infinite set. It contains an infinite number of elements. For example, for the set $(0,2)$, $0,2 \notin (0,2)$. However it contains all the numbers between 1 and 2, like $\frac{1}{10}, \frac{1}{2}, 1, \sqrt{2}, \sqrt{3}, \frac{10}{9}$, etc.

(ii) $[a, b] = \{x \in \mathbb{R}: a \leq x \leq b\}$, closed interval from *a* to *b*.

Note: 1. Here the numbers $a, b \in [a, b]$

2. $[a, b]$ is a set which contains the numbers a, b and the other numbers between a and b. It is also an infinite set. It contains an infinite number of elements.

(iii) $(a, b] = \{x \in \mathbb{R}: a < x \leq b\}$, open- closed interval from a to b.

Note: 1. Here the number $a, \notin (a, b]$ but $b \in (a, b]$

(iv) $[a, b) = \{x \in \mathbb{R}: a \leq x < b\}$, closed-open interval from a to b.

Similarly, here the number $a \in [a, b)$ but $b \notin [a, b)$.

It should also be noted that

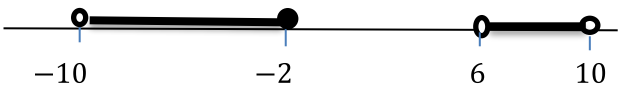
1. An interval is a subset of $\mathbb{R}$ and it is infinite.
2. An interval is open if it does not include its end points and is closed if it includes its endpoints.
3. A parenthesis "("or ")" is used to indicate that an endpoint does not belong to the interval, and the bracket "["or "]" is used to indicate that an endpoint does belong to the interval.

Example 1.5 Given the sets $A = (-2, 6]$, $B = [-5, 3]$, $C = [-1, 8)$ and $X = (-10, 10)$ is the universal set. Find each of the following sets and display it on the number line:

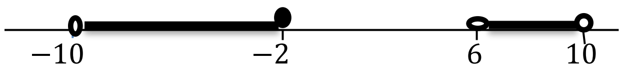
(i) A' (ii) $X - A$ (iii) $(A \cap C)'$ (iv) $(B - A) \cap C$

Solutions:

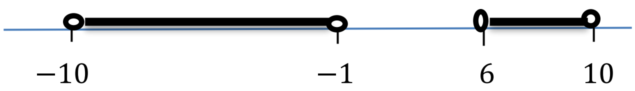
(i) $A' = (-10, -2] \cup (6, 10)$



(ii) $X - A = (-10, -2] \cup (6, 10)$



(iii) $A \cap C = [-1, 6] \Rightarrow (A \cap C)' = (-10, -1) \cup (6, 10)$



(iv) $B - A = [-5, -2]$, $(B - A) \cap C = \emptyset$

Now we shall introduce the symbol ∞, called infinity. It is not a number itself. Thus $\infty \notin \mathbb{R}$.

However, it is perceived to be greater than any real number, whereas the symbol $-\infty$ (minus infinity) is perceived to be less than any real number.

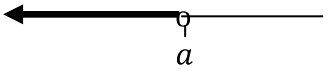
Definition 1.12 Let a be any real number. Then the set of real numbers x satisfying $x < a,\ x \leq a, x > a\ or\ x \geq a$, is called an **infinite interval** with endpoint a.
The infinite interval is said to be open or closed according as whether the endpoint a does or does not belong to the interval.

The four intervals may also be denoted and defined as follows:

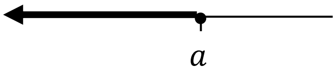
$(-\infty, a) = \{x: x < a, x \in \mathbb{R}\}$, $(-\infty, a] = \{x: x \leq a, x \in \mathbb{R}\}$,

$(a, \infty) = \{x;: a > x, x \in \mathbb{R}\}$ and $[a, \infty) = \{x: a \geq x, x \in \mathbb{R}\}$.

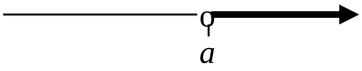
Graphically, the infinity intervals $x < a, x \leq a, x > a\ and\ x \geq a$ are represented as follows:



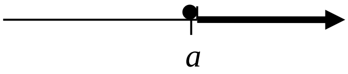
$$\{x \in \mathbb{R}: x < a\} = (-\infty, a)$$



$$\{x \in \mathbb{R}: x \leq a\} = (-\infty, a]$$



$$\{x \in \mathbb{R}: x > a\} = (a, \infty)$$



$$\{x \in \mathbb{R}: x \geq a\} = [a, \infty)$$

Note that the set of real numbers,$\mathbb{R}$, as an infinite interval can be written as $(-\infty, \infty)$.

By convention, this interval is both open and closed.

Graphically, $\mathbb{R}$ can be represented as follows:

$\mathbb{R} = (-\infty, \infty)$

Rational Numbers

A rational number is a real number which can be expressed in the form $\frac{p}{q}$ where p and q are integers and $q \neq 0$. In view of this, any rational number can be represented as a decimal. Some representations terminate at after a finite number of steps, i.e. all later terms in the expansion are zero. For example,

$$\frac{1}{2} = 0.5000 \cdots$$

$$\frac{1}{4} = 0.2500 \cdots$$

But other expansions never terminate, such as

$$\frac{1}{3} = 0.3333 \cdots$$

$$\frac{8}{7} = 1.14285714257 \cdots$$

In the expansion of $\frac{1}{3}$, 3 is repeated after the decimal point and in $\frac{8}{7}$, 142857 is repeating after the decimal point. This is always true for rational numbers.
Now it is awkward to express non terminating decimals such as $\frac{1}{3}$ and $\frac{8}{7}$ in the form given above. To remove this ambiguity, we place a bar over the set of numbers which is to be repeated indefinitely. In this notation we write, for example,

$$0.5\overline{0}$$

$$0.25\overline{0}$$

$$\frac{1}{3} = 0.\overline{3}$$

$$\frac{8}{7} = 1.\overline{142857}$$

NOTE: Every repeating decimal expansion is a rational number.

Example 1.6 Show that each of the following numbers is a rational number:

(a) $3.\overline{3}$ (b) $25.\overline{12}$ (c) $0.29\overline{432}$

Solution: (a) Let $a = 3.\overline{3}$. Multiplying by 10 both sides we have

$$10a = 33.\overline{3}$$

$$\Rightarrow 10a - a = 33.\overline{3} - 3.\overline{3}$$

$$\Rightarrow 9a = 30 \quad \Rightarrow a = \frac{30}{9}$$

$$\therefore \quad 3.\overline{3} = \frac{30}{9}$$, which is a rational number.

(b) Let $p = 25.\overline{12}$. Multiplying both sides by 100, yields

$$100p = 2512.\overline{12}$$

$$\Rightarrow 100p - p = 2512.\overline{12} - 25..\overline{12}$$

$$\Rightarrow 99p = 2487 \quad \Rightarrow p = \frac{2487}{99}$$

$$\therefore \quad 25.\overline{12} = \frac{2487}{99} = \frac{829}{33}$$, which is a rational number.

(c) Let $t = 0.29\overline{432}$. Multiplying both sides by 1000, yields

$$10000t = 29432.\overline{432}$$

$$\Rightarrow 100000t - 100t = 29432.\overline{432} - 29.\overline{432}$$

$$\Rightarrow 99900t = 29403 \quad \Rightarrow t = \frac{29403}{99900} = \frac{3267}{11100} = \frac{1089}{3700}$$, which is a rational number.

1.1.7 Irrational Numbers

An irrational number cannot be expressed as a ratio of two integers.
For example, $\sqrt{2}$ is an irrational number and it cannot be express in the form $\frac{p}{q}$, $p, q \in \mathbb{Z}$ and $q \neq 0$.

To prove this we shall require the following preliminary result:

Theorem 1.1.1 If a^2 is divisible by 2, then a is also divisible by 2.

Proof: Every integer a can be written in one of the forms

$$a = \begin{cases} 2n \\ 2n+1 \end{cases}, \text{ where } n \text{ is an integer.}$$

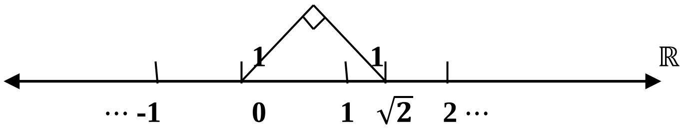
Hence, $a^2 = \begin{cases} 4n^2 \\ 4n^2 + 4n + 1 \end{cases}$

Since a^2 is divisible by 2, $a^2 = 4n^2$, since $4n^2 + 4n + 1$ is not a multiple of 2 for all $n \in \mathbb{Z}$. This means that $a = 2n$, which is divisible by 2. Hence the theorem.

Theorem 1.1.2 $\sqrt{2}$ is not a rational number.

Proof: We shall prove this by contradiction. Suppose that $\sqrt{2}$ is a rational number. Then it can be expressed in the form $\frac{p}{q}$, where $p, q \in \mathbb{Z}$, $q \neq 0$ such that p and q have no common factor. Now, if $\sqrt{2} = \frac{p}{q}$, then $2 = \frac{p^2}{q^2}$ or $p^2 = 2q^2$. Thus, p^2 is divisible by 2. From theorem 1.1.1 it follows that p is also divisible by 2, i.e. $p = 2r$, where r is an integer. When we replace $p = 2r$ in $p^2 = 2q^2$ we have $4r^2 = 2q^2$ or $q^2 = 2r^2$. Hence q^2 is divisible by 2 $\Rightarrow q$ is also divisible by 2. This means that p and q have common factor 2, contrary to our assumption. Therefore, $\sqrt{2}$ cannot be expressed as a ratio of two integers, implying that it is not a rational number. Therefore it is an irrational number.

Other examples of irrational numbers are $\sqrt{3}$, π, e, $\sqrt{5}$ etc.
Irrational numbers are also real numbers. For example, $\sqrt{2}$ is a real number and has a specific position on the number line.



By construction above, $\sqrt{2}$ lies on the number line. Hence $\sqrt{2} \in \mathbb{R}$.

In general, every irrational number is also a real number.

Complex Numbers

Some problems cannot be solved using real numbers alone. For example, we cannot find a real number x such that $x^2 = -1$. To handle such problems, the new symbol i had to be introduced with the property $i = \sqrt{-1}$ or $i^2 = -1$. i is called an **imaginary number**.

For an example to solve the quadratic equation $x^2 - 2x + 2 = 0$, we need to use the imaginary number $i = \sqrt{-1}$. Note that by using the quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm \sqrt{4 \times (-1)}}{2} = \frac{2 \pm 2\sqrt{-1}}{2} = \frac{2(1 \pm \sqrt{-1})}{2}$$

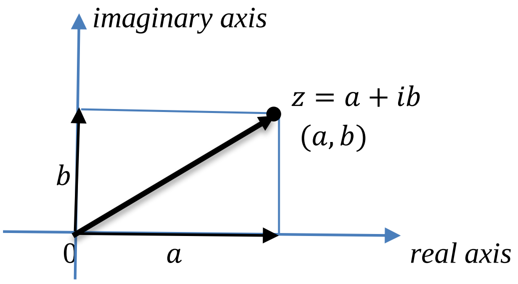
$= 1 \pm i$, since $i = \sqrt{-1}$.

Thus, the equation $x^2 - 2x + 2 = 0$ has two solutions which in this case are complex numbers $1 + i$ and $1 - i$.

Definition 1.13 A **complex number** is an ordered pair $(a, b)$of real numbers a and b, and is written $a + ib$. The number a is called the *real part* of $a + ib$, and b is its *imaginary part*.

Note that a complex number at the point (a, b) is the vector sum of the two numbers a and bj.

Usually a complex number is denoted by the letter z. Thus, $z = a + ib$.



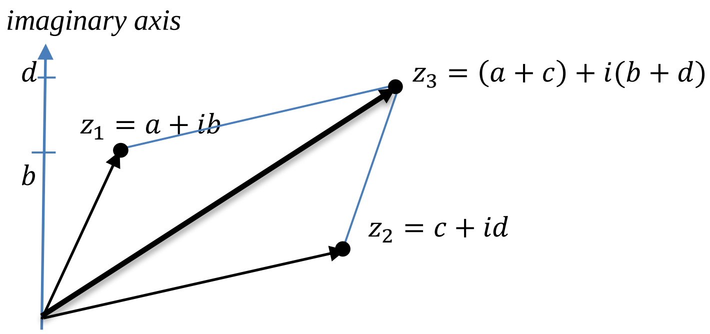
As opposed to a real number which lies on the real line, a complex number lies on the Cartesian plane.

Definition 1.14 The arithmetic operations on complex numbers are defined as follows:

(a) Two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ are equal if and only if $a = c$ and $b = d$.

(b) Addition: By vector addition in the diagram,

$$\boxed{z_1 + z_2 = (a + ib) + (c + id) = (a + c) + i(b + d)}$$



(c) Multiplication: Multiplication of two complex numbers is just ordinary algebraic expansion of brackets, and replacing i^2 by -1.

$$\begin{aligned}(a+ib)\times(c+id) &= a(c+id)+ib(c+id)\\ &= ac+iad+ibc+i^2bd\\ &= ac+iad+ibc-bd\\ &= (ac-bd)+i(bc+ad).\end{aligned}$$

Example 1.7 Evaluate each of the following:

(a) $(4+i)+(3-5i)$

Solution: $(4+i)+(3-5i)=(4+3)+(i-5i)=7-4i.$

(b) $(2+3i)-(7+4i)$

Solution: $(2+3i)-(7+4i)=2+3i-7-4i=2-7+3i-4i=-5-i.$

(c) $(2-3i)(2+5i)$

Solution: $(2-3i)(2+5i)=2(2+5i)-3i(2+5i)=4+10i-6i-15i^2$

$$\begin{aligned}&=4+10i-6i-15i^2=4+10i-6i-15(-1)\\ &=4+15+10i-6i=19+4i.\end{aligned}$$

Definition 1.15 The complex numbers $a+ib$ and $a-ib$ are said to be *conjugates* of each other. The conjugate of a complex number z is denoted by $\bar{z}$. i.e. if $z=a+ib$ then $\bar{z}=a-ib$.

Note that $z\bar{z}=(a+ib)(a-ib)=a(a-ib)+ib(a-ib)=a^2-abi+abi-i^2b^2$

$$=a^2-(-1)b^2=a^2+b^2.$$

i.e. $z\bar{z}=(a+ib)(a-ib)=a^2+b^2=\bar{z}z$.

Example 1.8 The conjugate of $-3+7i$ is $-3-7i$ and the conjugate of $-3-7i$ is $-3+7i$ and $(-3+7i)(-3-7i)=(-3)^2+(7)^2=9+49=58$

Definition 1.16 To divide a complex number $a+ib$ by $c+id$ i.e. to evaluate $\dfrac{a+ib}{c+id}$, we multiply both the numerator and the denominator by the conjugate $c-id$ and obtain

$$\frac{a+ib}{c+id}\times\frac{c-id}{c-id}=\frac{(ac+bd)+(bc-ad)i}{c^2+d^2}=\frac{ac+bd}{c^2+d^2}+i\frac{bc-ad}{c^2+d^2}.$$

Example 1.9 Express the given complex numbers in the form $a+id$, where $a,b\in\mathbb{Q}$.

(a) $\dfrac{4+7i}{5-i}$ (b) $\dfrac{-4+2i}{-6-5i}$ (c) $\dfrac{6-i}{2i}$

Solutions: (a) $\dfrac{4+7i}{5-i}=\dfrac{4+7i}{5-i}\times\dfrac{5+i}{5+i}=\dfrac{4(5+i)+7i(5+i)}{(5-i)(5+i)}=\dfrac{20+4i+35i-7}{5^2+1^2}$

$$= \frac{13+39i}{25+1} = \frac{13+39i}{26} = \frac{13}{26} + \frac{39}{26}i = \frac{1}{2} + \frac{3}{2}i.$$

(b) $\frac{-4+2i}{-6-5i} = \frac{-4+2i}{-6-5i} \times \frac{-6+5i}{-6+5i} = \frac{-4(-6+5i)+2i(-6+5i)}{(-6-5i)(-6+5i)} = \frac{24-20i-12i-10}{(-6)^2+5^2}$

$$= \frac{14-32i}{36+25} = \frac{14-32i}{61} = \frac{14}{61} - \frac{32}{61}i.$$

(c) $\frac{6-i}{2i} = \frac{6-i}{0+2i} \times \frac{0-2i}{0-2i} = \frac{6(0-2i)-i(0-2i)}{(0+2i)(0-2i)} = \frac{0-12i+0-1}{0^2+2^2} = \frac{-1-12i}{4}$

$$= -\frac{1}{4} - \frac{12}{4}i = -\frac{1}{4} - 3i.$$

Surds and Manipulation of Surds

A surd is a number that cannot be simplified to remove a square root or cube root etc. Surds are used to write numbers exactly. For example, $\sqrt{2}, \sqrt{3}+4, \sqrt{11}$ exact the way they are and cannot be simplified. Surds cannot be expressed exactly as decimal fractions because they give never-ending, non repeating decimal fractions, for example, $\sqrt{2} = 1.414213562\cdots$.

However, we can manipulate surds using the following rules:

1. $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$
2. $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

Note: $\sqrt{a} \times \sqrt{a} = a$, since $\left(\sqrt{a}\right)^2 = a$.

Example 1.10 Simplify:

$\sqrt{28}$ 2. $\frac{\sqrt{72}}{2}$ 3. $5\sqrt{6} - 2\sqrt{24} + \sqrt{294}$.

Solutions:

1. $\sqrt{28} = \sqrt{4 \times 7} = \sqrt{4} \times \sqrt{7} = 2 \times \sqrt{7} = 2\sqrt{2}$.

Note: $\sqrt{4} = 2$ and **not** -2 i.e. $\sqrt{4} \neq \pm 2$.

The square root a positive real number is positive.

2. $\frac{\sqrt{72}}{2} = \frac{\sqrt{36\times 2}}{2} = \frac{\sqrt{36}\times\sqrt{2}}{2} = \frac{6\sqrt{2}}{2} = 3\sqrt{2}.$

3. $5\sqrt{6} - 2\sqrt{24} + \sqrt{294} = 5\sqrt{6} - 2\sqrt{4\times 6} + \sqrt{49\times 6} = 5\sqrt{6} - 2(2\sqrt{6}) + 7\sqrt{6}$
$= 5\sqrt{6} - 4\sqrt{6} + 7\sqrt{6} = 8\sqrt{6}.$

Example 1.11 Evaluate:

1. $\left(2\sqrt{7}\right)\left(3\sqrt{7}\right)$ 2. $(\sqrt{5}+\sqrt{2})^2$ 3. $\left(2\sqrt{3}-\sqrt{5}\right)\left(2\sqrt{3}+2\sqrt{5}\right)$

4. $(2\sqrt{5}-3\sqrt{2})(2\sqrt{5}+3\sqrt{2})$

Solutions:

1. $(2\sqrt{7})(3\sqrt{7}) = 2\times 3\times(\sqrt{7})^2 = 2\times 3\times 7 = 42.$

1. $(\sqrt{5}+\sqrt{2})^2 = (\sqrt{5})^2 + 2\sqrt{5}\sqrt{2} + (\sqrt{2})^2 = 5 + 2\sqrt{10} + 2 = 7 + 2\sqrt{10}.$

2. $(2\sqrt{3}-\sqrt{5})(2\sqrt{3}+2\sqrt{5}) = 2\sqrt{3}(2\sqrt{3}+2\sqrt{5}) - \sqrt{5}(2\sqrt{3}+2\sqrt{5})$

$$\begin{aligned}&= 4(\sqrt{3})^2 + 4\sqrt{3}\sqrt{5} - 2\sqrt{5}\sqrt{3} + 2(\sqrt{5})^2\\ &= 12 + 4\sqrt{15} - 2\sqrt{15} + 10\\ &= 22 + 2\sqrt{15}.\end{aligned}$$

Rationalization of the Denominator of a Fraction involving Surds

To rationalize the denominator of a fraction involving surds is to make the denominator free of the root sign.

The rules to rationalize the denominator are as follows:

1. Fractions in the form of $\frac{1}{\sqrt{a}}$, multiply the numerator and the denominator by $\sqrt{a}$ so that the fraction become $\frac{\sqrt{a}}{a}$.
2. Fractions in the form $\frac{1}{\sqrt{a}+\sqrt{b}}$, multiply the numerator and the denominator by the conjugate $\sqrt{a}-\sqrt{b}$ of the denominator. Thus

$$\frac{1}{\sqrt{a}+\sqrt{b}} = \frac{1}{\sqrt{a}+\sqrt{b}}\times\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}-\sqrt{b}} = \frac{\sqrt{a}-\sqrt{b}}{(\sqrt{a})^2-(\sqrt{b})^2} = \frac{\sqrt{a}-\sqrt{b}}{a-b}.$$

Example 1.12 Rationalize the denominator of each of the following:

1. $\frac{3}{\sqrt{5}+\sqrt{2}}$ 2. $\frac{1+\sqrt{2}}{1-\sqrt{2}}$ 3. $\frac{\sqrt{3}-2}{\sqrt{3}-2\sqrt{5}}$.

Solutions:

1. $\frac{3}{\sqrt{5}+\sqrt{2}} = \frac{1}{\sqrt{5}+\sqrt{2}}\times\frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}-\sqrt{2}} = \frac{\sqrt{5}-\sqrt{2}}{(\sqrt{5})^2-(\sqrt{2})^2} = \frac{\sqrt{5}-\sqrt{2}}{5-2} = \frac{\sqrt{5}-\sqrt{2}}{3}.$

   Note that in the form $p\sqrt{5}+q\sqrt{2}$ where p and q are integers,

   $\frac{3}{\sqrt{5}+\sqrt{2}} = \frac{1}{3}\sqrt{5} - \frac{1}{3}\sqrt{2}.$

2. $\frac{1+\sqrt{2}}{1-\sqrt{2}} = \frac{1+\sqrt{2}}{1-\sqrt{2}}\times\frac{1+\sqrt{2}}{1+\sqrt{2}} = \frac{(1+\sqrt{2})+\sqrt{2}(1+\sqrt{2})}{1^2-(\sqrt{2})^2} = \frac{1+\sqrt{2}+\sqrt{2}+2}{1-2} = \frac{3+2\sqrt{2}}{-1}$

   $= -(3+2\sqrt{2}) = -3-2\sqrt{2}.$

3. $$\frac{\sqrt{3}-2}{\sqrt{3}-2\sqrt{5}} = \frac{\sqrt{3}-2}{\sqrt{3}-2\sqrt{5}} \times \frac{\sqrt{3}+2\sqrt{5}}{\sqrt{3}+2\sqrt{5}} = \frac{\sqrt{3}\left(\sqrt{3}+2\sqrt{5}\right)-2\left(\sqrt{3}+2\sqrt{5}\right)}{\left(\sqrt{3}\right)^2-\left(2\sqrt{5}\right)^2}$$

$$= \frac{\left(\sqrt{3}\right)^2+2\sqrt{3}\sqrt{5}-2\sqrt{3}-4\sqrt{5}}{3-4(5)} = \frac{3+2\sqrt{15}-2\sqrt{3}-4\sqrt{5}}{-17}.$$

Binary Operations

The term ‘binary’ emanates from the fact that the operation acts on two elements. The usual operations of arithmetic $+$, $-$ and $\times$ are some of the examples of binary operations because when we choose any **two** numbers, each operation will generate a third number.

Other examples of binary operations are the set union and intersection, $A \cup B = C$ and $A \cap B = D$.

We shall use the symbol $*$ to represent a generic binary operation.

Definition 1.17 Let $S = \{a, b, c, \cdots\}$ be any set. The operation $*$ is a binary operation on S if and only if to every ordered pair (a, b), where $a, b \in S$, there is assigned unique element $a * b \in S$. We indicate this assignment using the notation $(a, b) \to a * b$.

Example 1.10.1 Let the binary operation $*$ be defined on the set $\mathbb{R}$ as

$$a * b = a + b + 2ab.$$

Find (a) $2 * 5$ (b) $-5 * 3$

The following are the delicate points we need to observe in the definition of binary operation:

1. The order of a and b may be important, for (a, b) is an ordered pair and it may happen that $a * b \neq b * a$. For example, if A and B are matrices, then $A \times B \neq B \times A$.
2. For $a, b \in S$ the operation $*$ must be defined for every pair (a, b).
3. The output $a * b$ must be an element of S.

Why is the operation $\div$ not a binary operation on $\mathbb{Z}$.

Definition 1.18 The binary operation $*$ on a set is called **commutative** if and only if for every ordered pair (a, b) of elements in S $a * b = b * a$.

Example 1.13 Let S be the set of real numbers, $\mathbb{R}$. Then the binary operations $+$ and $\times$ are commutative on $\mathbb{R}$, since for every $a, b \in S$

$$a + b = b + a \text{ and } a \times b = b \times a,$$

but the operation – is not commutative, since real numbers e.g. $7, 5 \in \mathbb{R}$,

$$7 - 5 \neq 5 - 7.$$

Exercise 1.2.1 Are set union and intersection commutative?

Definition 1.19 The binary operation $*$ on a set S is **associative** if and only if for every triple $a, b, c \in S$

$$a * (b * c) = (a * b) * c.$$

Example 1.14 (a). Addition and multiplication are associative binary operations on $\mathbb{R}$.

(b) Subtraction is not associative on $\mathbb{R}$, since for example,

$$12 - (8 - 2) = 6 \neq (12 - 8) - 2 = 2.$$

(c) Division is not associative on $\mathbb{R}$, since, for example,

$$24 \div (6 \div 2) = 8 \neq (24 \div 6) \div 2 = 2$$

Example: 1.15 The binary operation * on $\mathbb{R}$ is defined by

$$a * b = (a - b)^2 - ab.$$

(a) Determine whether the operation * is commutative.

(b) Evaluate (i) $(-1*3)*2$ (ii) $-1 * (3 * 2)$

Hence, state whether the binary operation * is associative or not.

Example 1.16 (a) Let $*$ be a binary operation on the set of integers $\mathbb{Z}$ defined by

$$a * b = a + b - ab.$$

Is the operation $*$ is associative on $\mathbb{Z}$?

(b) Evaluate (i) $(-3 * 2) * 5$ (ii) $-3 * (2 * 5)$

Definition 1.20 A binary operation $*$ on a set *S* has an **identity** element, denoted by *e*, if and only if *e* is an element of *S*, and for all elements *a* of *S*,

$$a * e = a = e * a.$$

For example, the identity element for $+$ on a set of real numbers is 0, since

$$a + 0 = a = 0 + a,$$

and the identity element for × on a set of real numbers is 1, since

$$a \times 1 = a = 1 \times a.$$

Definition 1.21 If $*$ is a binary operation on *S* which has an identity element *e*, and if *a* is any given element of *S*, then the element, denoted by a^{-1}, of *S* is called the **inverse** of *a* if and only if

$$a * a^{-1} = e = a^{-1} * a.$$

For example, the additive inverse of any real number *a* is $-a$, since

$$a + (-a) = 0 = -a + a.$$

The multiplicative inverse of any real number *a* is $\frac{1}{a}$, since

$$a \times \frac{1}{a} = 1 = \frac{1}{a} \times a.$$

NOTE: For an operation $*$ on a set *S*, the inverse a^{-1} of a in *S* in not necessarily $\frac{1}{a}$.

MAT 1100 FOUNDATION MATHEMATICS TUTORIAL SHEET 2 -2020

---

1. Rewrite each interval in set builder form:

   (a) $A = (-10,10]$ (b) $B = [4,9]$ (c) $C = (-1,5)$ (d) $[0,-2)$.

2. Let $\mathbb{R}$ be the universal set and $A = (-1,6]$, $B = (0,4)$, $C = \{x: x \geq 4, x \in \mathbb{R}\}$ be its subsets. Illustrate each set on the same number line. Hence, find each of the following sets:

   (a) $A - C$ (b) $(B' \cap C)'$ (c) $(C - A) \cap B$.

3. Express the each of the following numbers as a fraction in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$, in its lowest terms:

   (a) $4.\overline{3}$ (b) $-0.2\overline{55}$ (c) $12.34\overline{11}$.

4. Prove that each of the following is an irrational number:

   (a) $\sqrt{3}$ (b) $\sqrt{2} - 1$.

5. Simplify each of the following:

   (a) $2\sqrt{27} - 3\sqrt{48} + \sqrt{75}$ (b) $\sqrt{80} - \sqrt{20} - 2\sqrt{45}$ (c) $\frac{\sqrt{28}}{\sqrt{175}}$.

6. Rationalize the denominator of each of the following:

   (a) $\frac{\sqrt{2}-1}{1+\sqrt{2}}$ (b) $\frac{\sqrt{3}-2}{\sqrt{3}-1}$ (c) $\frac{2+\sqrt{2}}{(\sqrt{2}-1)^2}$.

7. Simplify each of the following, giving your answer in the form $a + bi$, where $a, b \in \mathbb{R}$:

   (a) $(2-3i) - (1+2i)$ (b) $(3-i)^2$ (c) $\frac{(1+2i)^2}{2+i}$.

8. Given that $z_1 = 8 + 2i$, $z_2 = 2 + i$, $z_3 = 3 + i$, find the answer to each of the following in the form $a + bi$, where $a, b \in \mathbb{R}$:

   (a) $\frac{z_1 z_2}{z_3}$ (b) $\frac{z_1 - z_2}{z_1 + z_3}$ (c) $\frac{(z_1 - 2z_3)^2}{(3z_2)^2}$.

9. Express $z = \sqrt{4 + 3i}$ in the form of $p + qi$, where $p, q \in \mathbb{R}$.

10. State whether each of the following operation is a binary operation on $\mathbb{Z}$, the set of integers:

    (a) $a * b = a - 2b$ (b) $a * b = \sqrt{a+b}$ (c) $a * b = (a-b)^2$ (d) $a * b = a^2 - b^2$.

11. Let $*$ be a binary operation on the set of real numbers and a, b, c be real numbers. Given that the operation is defined by $a * b = (a-b)^2 - 3ab$.

    (a) Does the operation posses the commutative and/or associative?

    (b) For this binary operation calculate (i) $-2 * (3 * 4)$ (ii) $(-4 * 3) * 2$.

12. The binary operation $*$ is defined on the set of real numbers by

$a * b = 3(a - b)^2$.

Show that the binary operation is commutative.