Introduction to Stress

Imagine a vertical column of material. Along any imaginary horizontal plane within this, the material above the plane, because of its weight, pushes downward on the material below the plane.

Similarly, the part of the column below the plane pushes upward with a equal force on the material above the plane. The mutual action and reaction along a surface constitute a stress.

Moreover, along any imaginary plane within the column there are similar actions and reactions.

The imaginary plane may be:

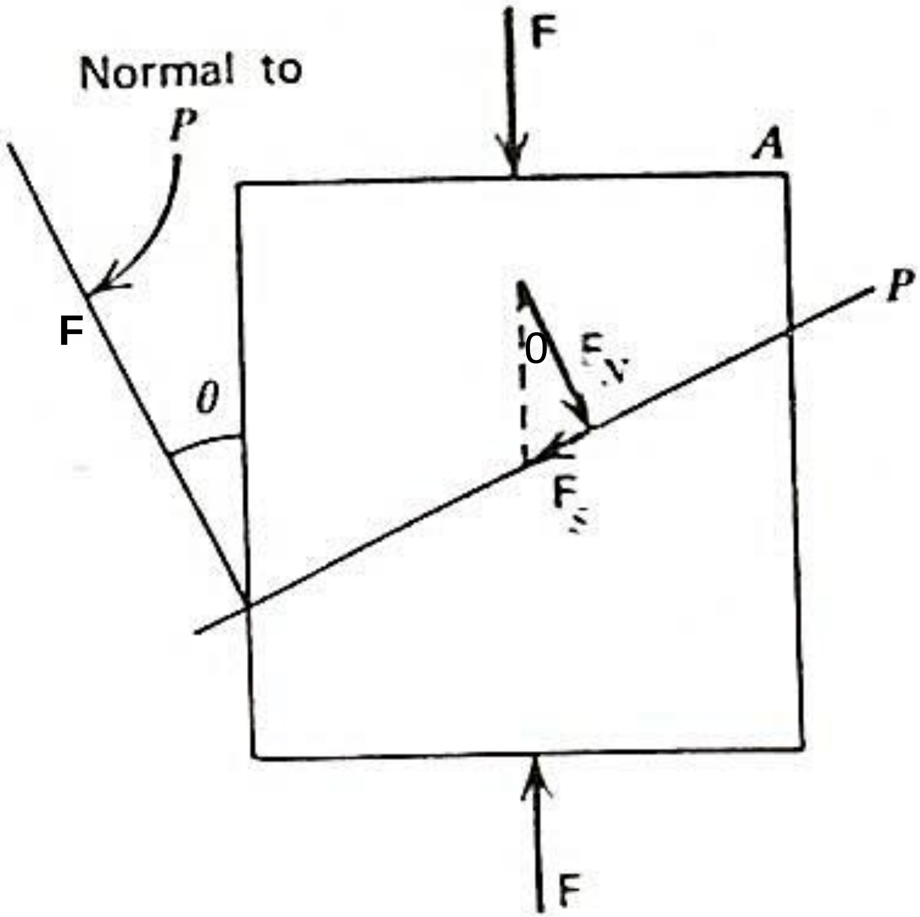
- **horizontal,**
- **vertical, or**
- **inclined at any angle.**

The force, due to the weight of that part of the column that lies above the plane, acts in a vertical direction.

Along an inclined plane, however, the vertically directed force would be resolved into a

- **A normal component and**
- **A tangential component.**

Univ. Zambia 2012



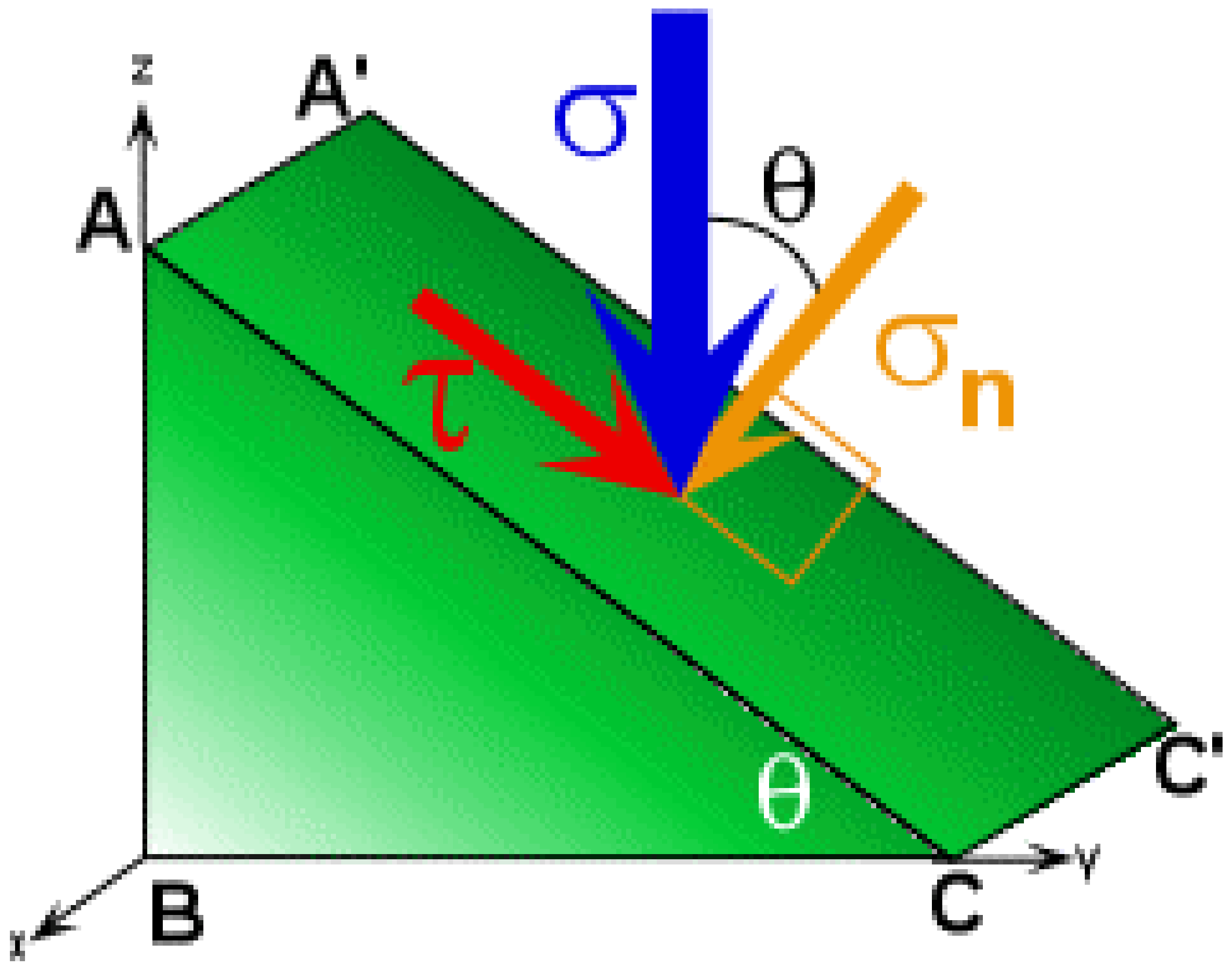
$$|F_N| = F\cos\theta$$

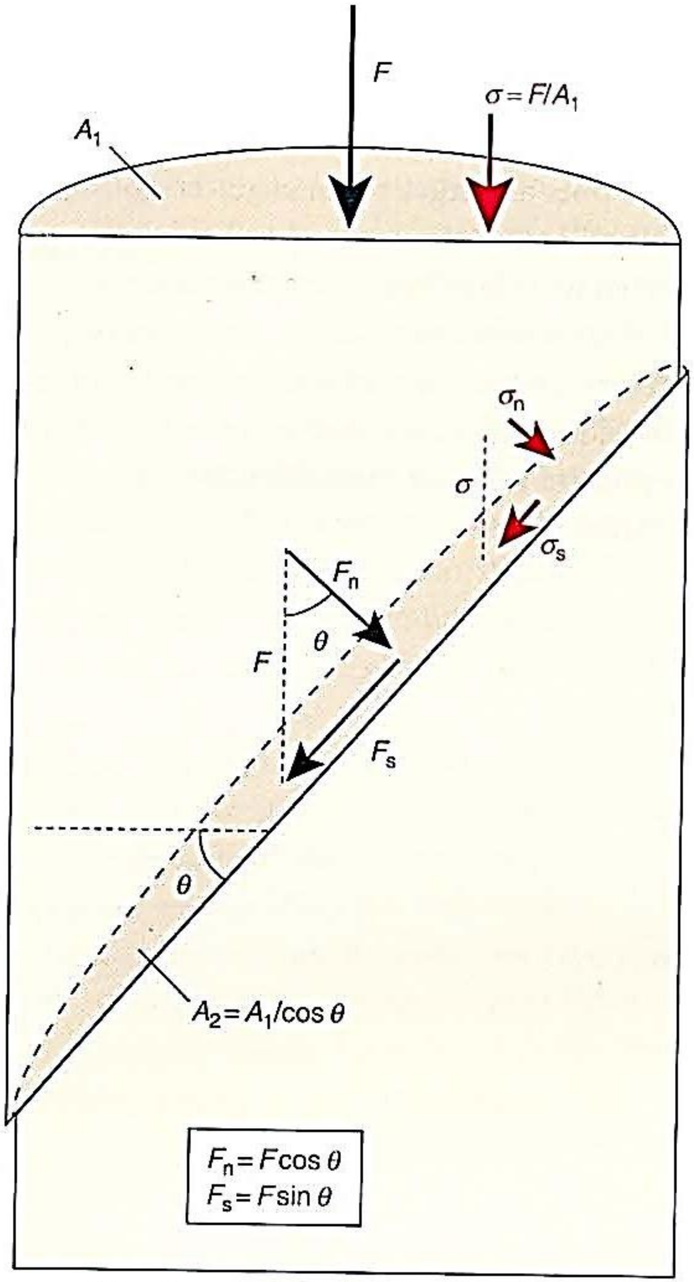
$$|F_S| = F\sin\theta$$

A Force vector F acting on a surface can be decomposed into a normal (FN) and a shear (Fs) components by simple vector addition.

The stress vector sigma (σ) can not be decomposed in this way because it depends on the area across which the force acts. Trigonometric expressions for the components σ_N and σ_S are derived.

Normal and Shear Stress





$$\sigma_n = F_n/A_2 = F\cos\theta/A_2 = F\cos^2\theta/A_1 = \sigma\cos^2\theta$$
$$\sigma_s = F_s/A_2 = F\sin\theta/(A_1/\cos\theta)$$
$$= F\sin\theta\cos\theta/A_1 = \sigma\sin\theta\cos\theta = \sigma 1/2\sin 2\theta$$

Figure 4.1 A force vector **F** acting on a surface can be decomposed into a normal (F_n) and a shear (F_s) component by simple vector addition. The stress vector σ cannot be decomposed in this way, because it depends on the area across which the force acts. Trigonometric expressions for the components σ_n and σ_s are derived.

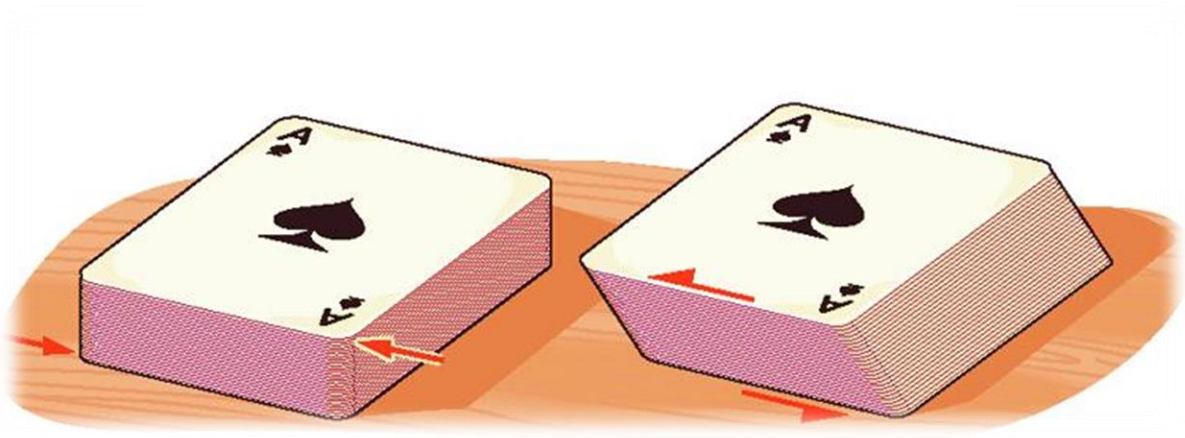
The normal component is a compressive stress if it tends to push together the material on opposite sides of the plane.

The normal component is a tensile stress if it tends to pull apart the material on opposite sides of the plane.

The tangential component is generally called a shearing stress or shear.

Shear Stress

Univ. Zambia 2012



***Shear stress*: stress that acts parallel to a plane**

It is essential to distinguish between the external force that is applied to a body and the resulting internal actions and reactions that constitute the stress.

Unit of Stress

There are several ways to express the magnitude of stress.

When describing stress levels in the Earth's crust or in specimens subjected to deformation in the laboratory, the preferred unit of major is the Pascal (Pa):

A stress of one pascal is created by the force of one Newton acting on an area of one square meter.

A newton is the force required to impart an acceleration of one meter per second per second to a body of one kilogram mass.

Because of the small magnitude of a single pascal in comparison to the greater magnitude of stresses in the earth, we commonly precede the term pascal with the prefix kilo-, mega-, or giga-, where;

One kilopascal (kPa) = 1000 pascals (10^3 Pa)
One megapascal (MPa)= 1000,000 pascals (10^6Pa)
One gigapascal (GPa) =1000,000,000 pascals (10^9Pa)

Calculation of stress

There is no direct way to measure the stress in a body, but they may be calculated if the external forces are known.

If a body is compressed or stretched, the stress is referred to a plane perpendicular to the direction in which the external forces are acting.

Thus, if a vertical square column 10 inches on a side supports a load of 5000 pounds, every horizontal plane in the column is subjected to a compressive force of 5000 pounds if we neglect the weight of the column itself.

Each square inch of these horizontal planes supports a load of 50 pounds per square inch.

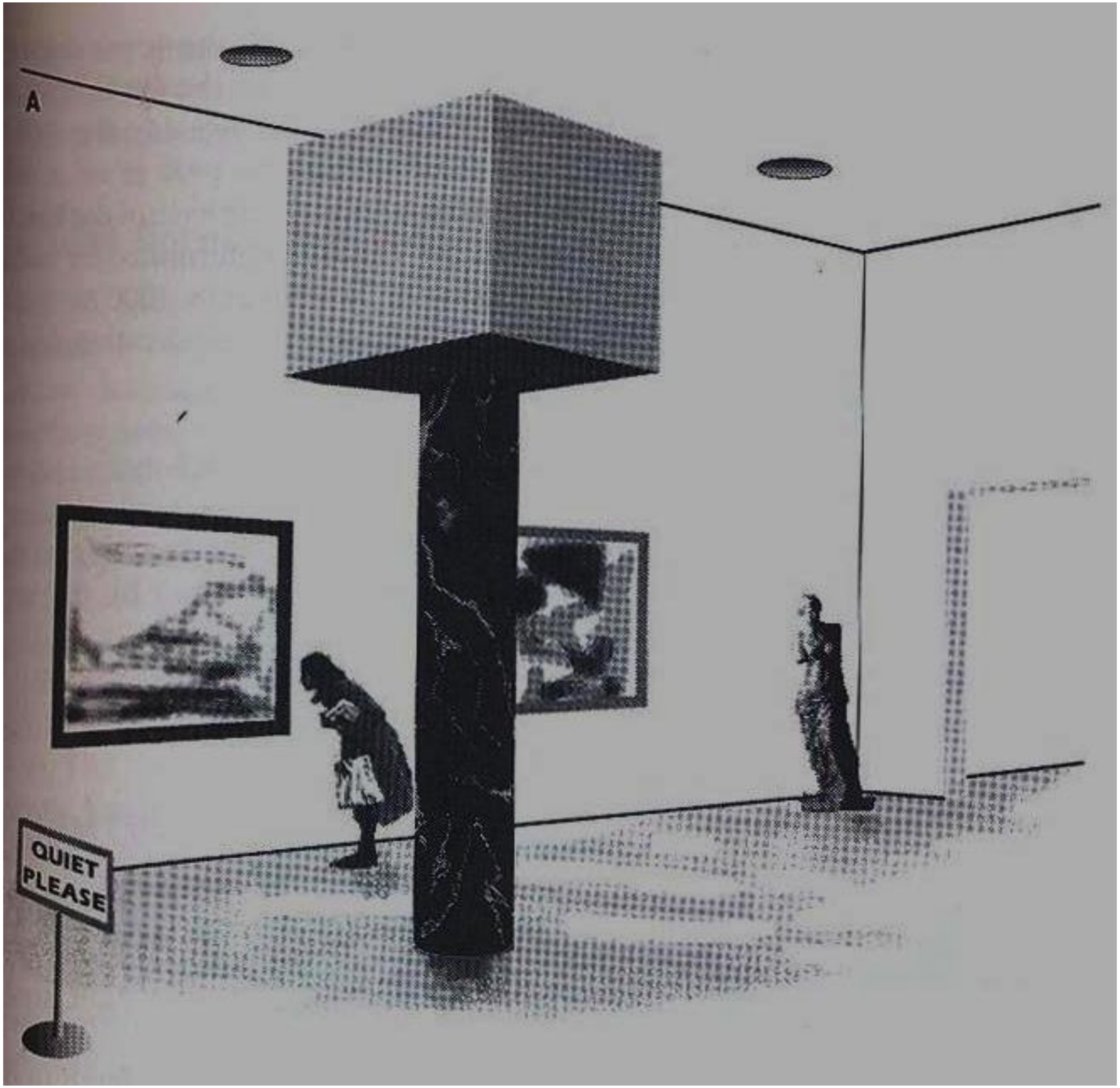
Thus, The compressive stress is said to be 50 pounds per square inch (Billings, p.17).

If a vertical rod with a cross-sectional area of 10 square inches carries a weight of 5,000 pounds at its lower end, every horizontal plane in the rod is subjected a pull of 500 pounds per square inch.

The tensile stress is said to be 500 pounds per square inch.

Imagine a large block of granite resting on a marble column. The weight of the granite block constitutes a gravitational load that imposes a stress on the marble column.

The magnitude of the stress is found by dividing the force created by the load of the granite block by the crossectional area of the top of the marble column.



Univ. Zambia 2012

The force created by the granite block is the product of mass (*m*) times acceleration (*g*) due to gravity.

Or, F= mg

Mass is volume (*V*) times density (σ)

Or, (m = *V* x σ)

(because m/*V* = σ)

Univ. Zambia 2012

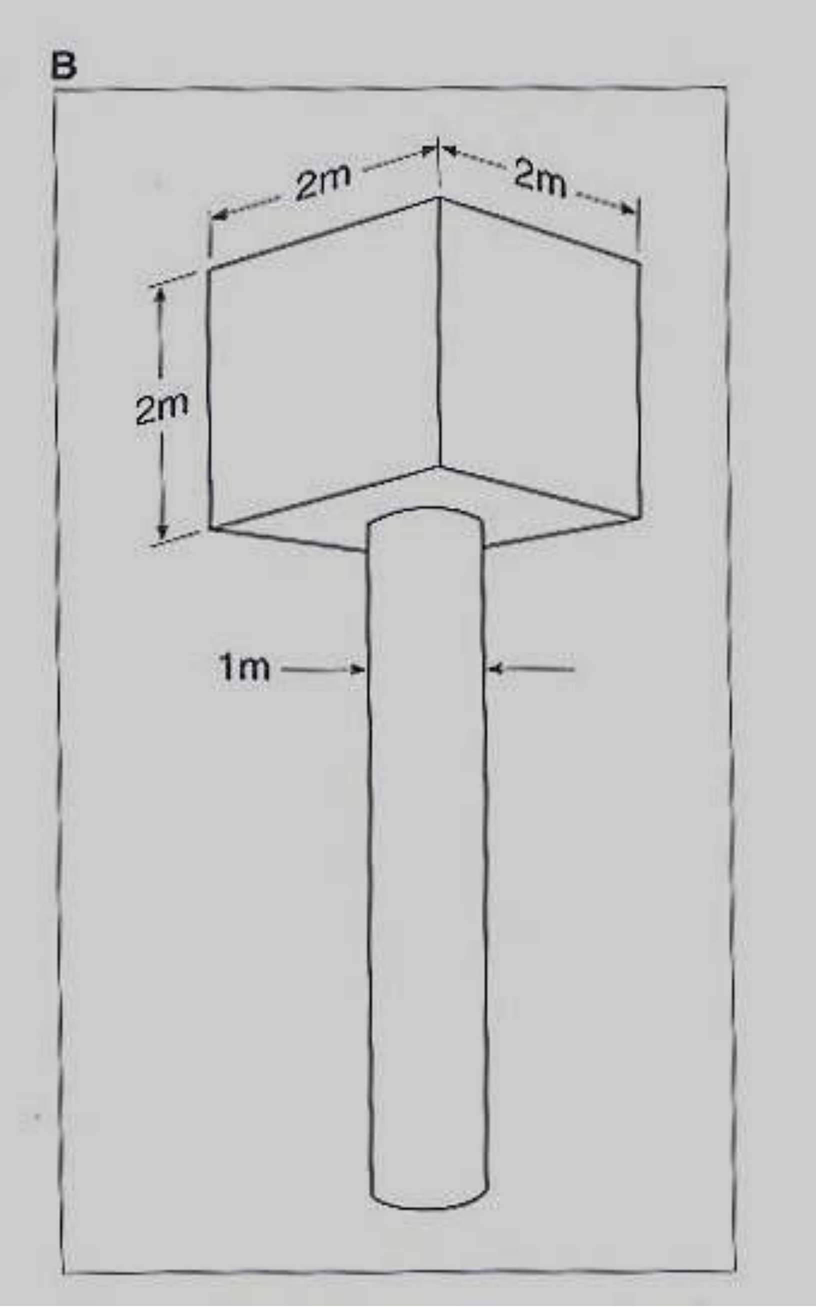


Figure 3.7 (*A*) This piece of sculpture of granite on marble creates an opportunity for calculating the force exerted by the granite block, and the stress generated by it on the top of the marble column. (*B*) Force is determined by measuring the volume of the granite block and multiplying it by its density and by the acceleration due to gravity. The stress is determined by dividing the force by the cross-sectional area of the marble column.

In this case:

Volume (*V*) = width (*W*) x (breadth (*B*) x (height (*H*) = 2 m x 2 m x 2 m

Therefore, $V = 8\ m^3$

Density (σ) 2.7 g/cm^3 = 2700 kg/m^3

Mass (*m*) = $V\sigma$ = $8\ m^3 \times 2700\ kg/m^3$ = 21,600 kg

And force, as just stated, is compounded by multiplying mass (*m*) times gravity (*g*)
Or, $(m \times g)$

Force (*F*) = mass (*m*) x acceleration (*g*) = *mg*

F = 21,600 kg x 9.8 m/s^2 = 211,680 N

The stress , represented by the Greek letter sigma (σ), created by the load of the granite block on the marble column is force dived by area:

σ = F/A

Area (A) = Лr^2 = 3.14 x (0.5) m^2 = 0.79 m^2

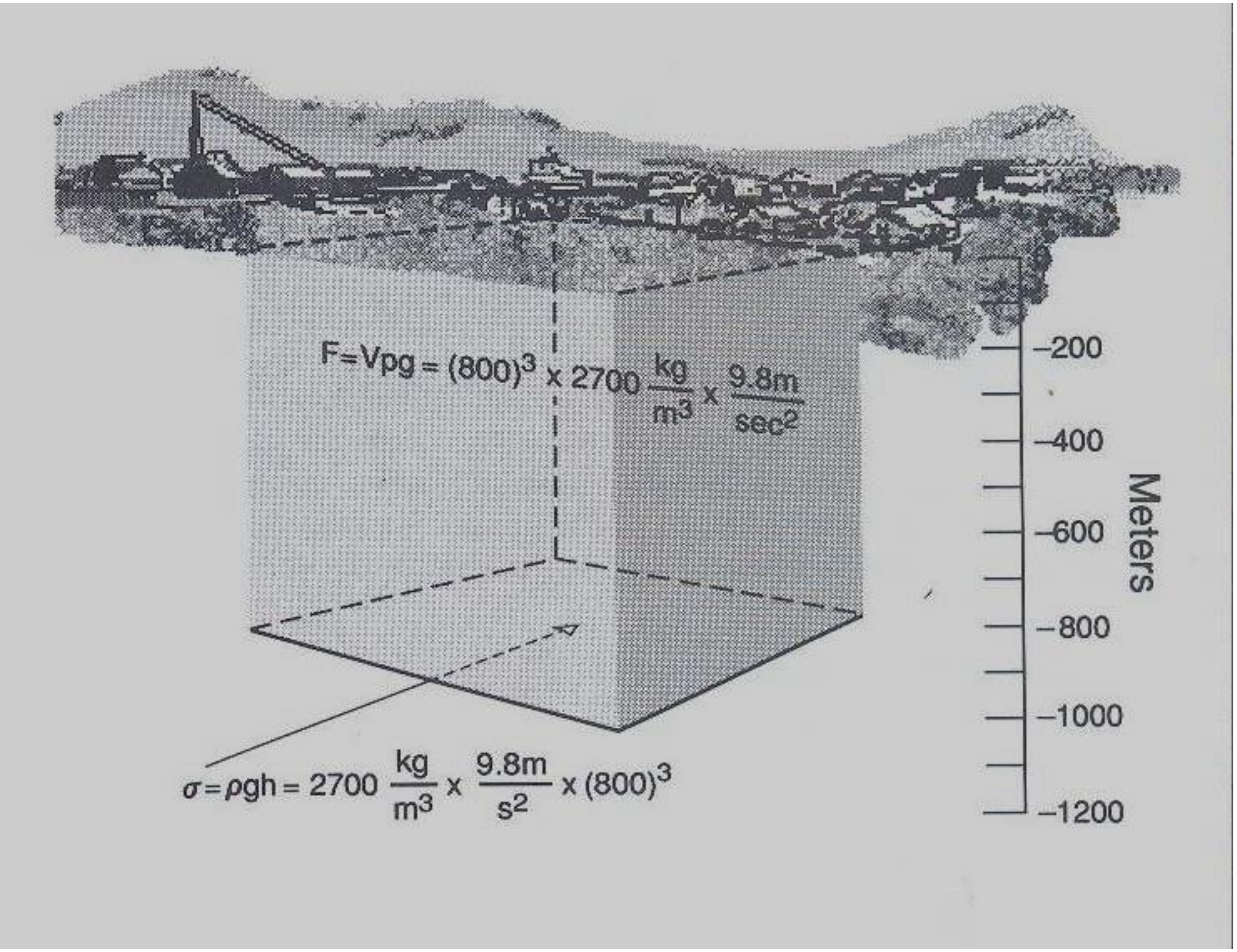
σ = F/A = 211,680 N / 0.79 m^2 = 267,949 N / m^2

= 267,949 Pa = 268 kPa

Calculating Stress underground

In a manner quite similar to the museum-piece example, we can calculate the stress created by the weight of a very large cube of granite in the upper crust.

Let us picture a region of the Earth where the upper several kilometers of the crust are entirely composed of granite. Then let us calculate, for a given depth level, the stress created by the load of the granite.



We can apply our museum-piece calculation to a huge block of granite nestled in the crust of the Earth.

We will choose -1000 m (-1 km) as the depth level of interest to us.

To set up the calculation, it is helpful to visualize the – 1000 m depth level as overlain by a giant cube of granite 1000 m on a side. Our goal is to compute the stress level at the base of the block.

The force generated by the weight of the bloc

$$F = mg$$

$$p = \frac{m}{V},$$ *and therefore,* $m = pV$

So, $F = Vpg$

It is determined by multiplying the volume of the block (V = 1000 m x 1000 m x 1000 m) times the density of the granite ($p = 2700\ kg/m^3$) times the acceleration due to gravity ($g = 9.8\ m/s^2$).

F = V (1000m x 1000m x 1000 m) x p (2700 kg/m^3) x g (9.8 m/s^2)

The stress (σ) created by the weight of the block of granite acting on the base of the block is determined by dividing the force (*F*) by the area (*A* = 1000m x 1000 m):

$$\sigma = \frac{F}{A} = \frac{1000\ m \times 1000\ m \times 1000\ m \times 2700\ kg/m^3 \times 9.8\ m/s^2}{1000m \times 1000\ m}$$

$$\sigma = 1000\ m \times 2700\ kg/m^3 \times 9.8\ m/s^2$$

$$= 26{,}460{,}000\ Pa = 26.5\ MPa$$

This tells us the *lithostatic stress gradient* at depth. Lithostatic stress increases 26.5 MPa /km, which is equivalent to 265 bars or 0.264 kbar/km.

In other words, for each 3.8 km depth, lithostatic stress increases by 1 kbar or 100 MPa.

There is s shortcut to determine the stress at any given depth in the granite. Stress (σ) is simply the product of density (*p*) times gravity (*g*) times depth (*h*) (see fig)

$\sigma = pgh = 2700\ kg/m^3 \times 9.8\ m/s^2 \times 1000\ m = 26.5\ Mpa$

The calculated stress level of 26.5 Mpa is very similar to direct '*in situ*' measurements of stress in deep mines at depth levels in the range of 1000 m.

Setting up an example of stress analysis

We now go underground again, at least in our imagination, to tiny point (P) deep in granite, located 1500 m beneath the surface (fig. 3.10).

The region of crust within which point P resides is tectonically active and is experiencing a modest east-west horizontal compression.

Thus, in addition to the vertical lithostatic compressive stress felt at point P due to the load imposed by the overlying granite, there is small horizontal externally imposed compression of tectonic origin.

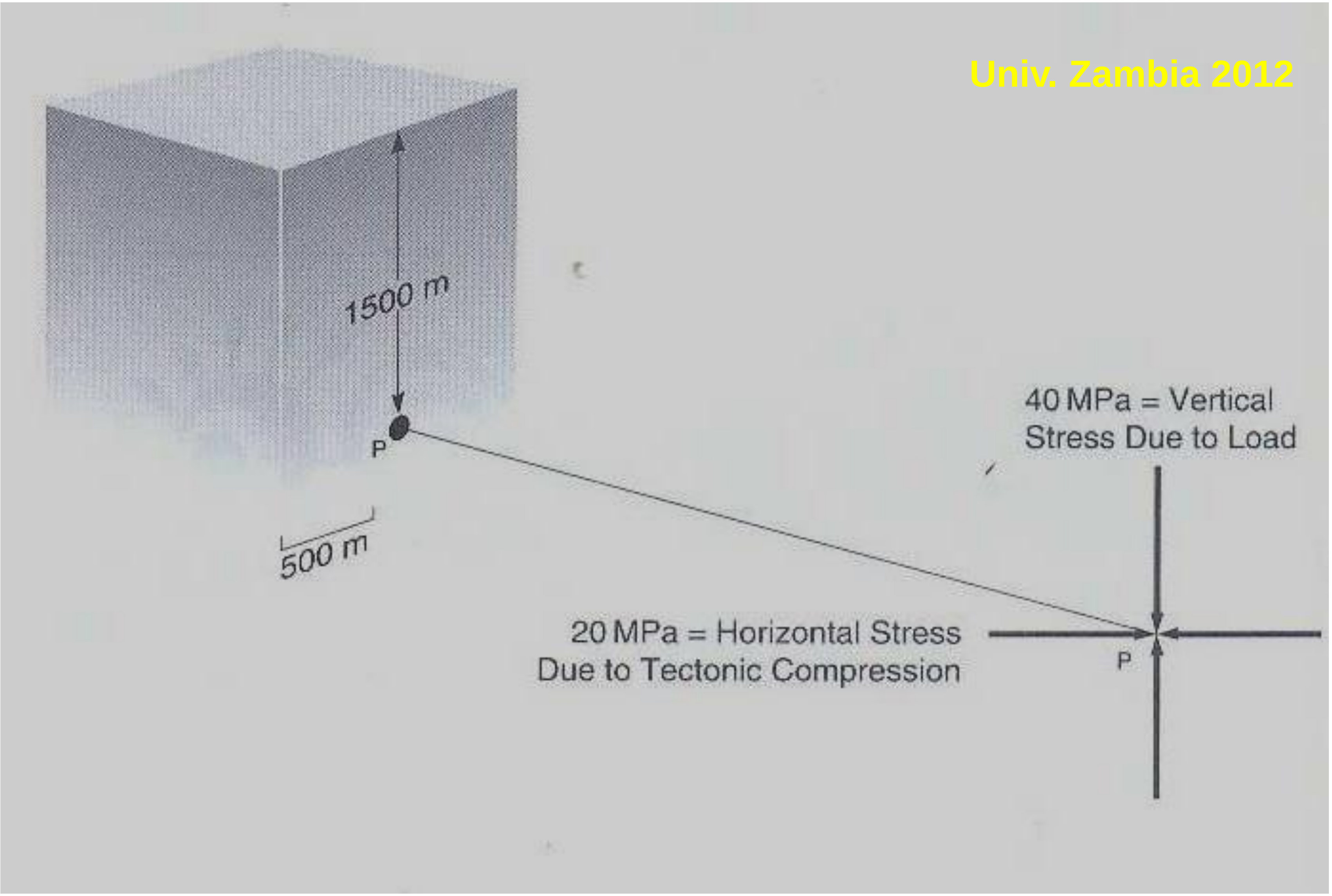


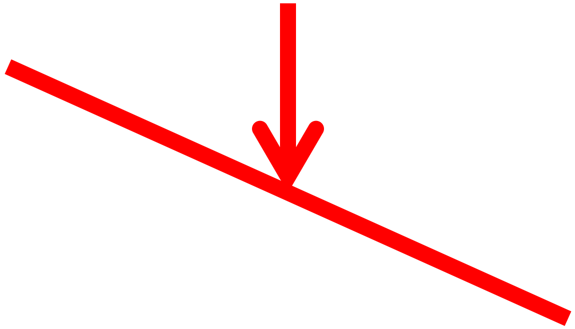
Fig. 3.10. We picture a point (*P*) deep down in granite, which for this example we assume to be an ideal elastic body of rock. The stresses acting at *P* *are generated by a combination of gravitational loading, and tectonic loading.*

Now let us picture within the interior of the granite at point P an infinite number of imaginary planes, or real physical planes, of all orientations.

The real physical planes include preexisting hairline fractures, cleavage surfaces in minerals, crystal faces, grain boundaries, and veins.

Across each one of these planes, "there will exist a field of forces equivalent to the loads exerted by material on one side of the plane on the other".

Focusing from point to point on any one of these plains, we would discover that the *forces* (*not the stress*) are not uniform in either direction or magnitude.



Yet, if we were to examine the magnitude and direction of force acting on a smaller and smaller surface area of one of the planes, cutting through point P, we would discover that the force per unit area (i.e. the stress) would approach a fixed value

… fixed in both magnitude and direction.

End of part I

Resolving Normal Stress
and
Shear Stress

Resolving Normal Stress and Shear Stress

In almost all cases a given stress can be resolved into two components: one perpendicular to the plane for which the stress has been calculated, the other parallel to the plane.

The component perpendicular to the plane is the normal stress (σ_N), and the component parallel to the plane is the shear stress (σ_S).

Normal compressive stress tends to inhibit sliding along a plane;

Whereas shear stress tends to promote sliding.

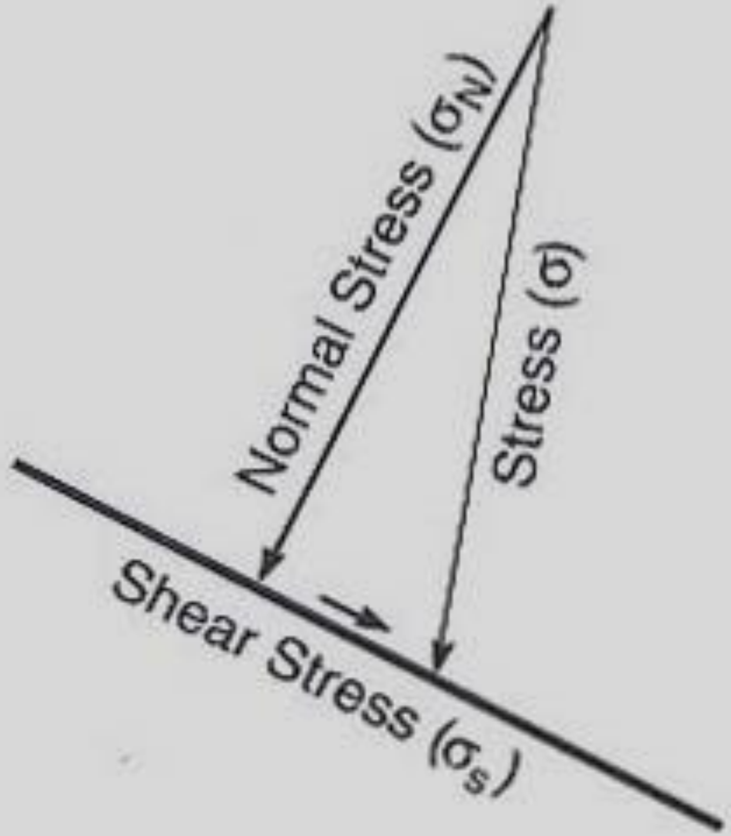


Figure 3.12 A stress (traction) can be resolved into normal (σ_N) and shear stress (σ_S) components.

Normal stresses are considered to be positive if they are compressive (i.e., directed inward), negative if they are tensile (i.e., directed outward).

Shear stresses are labeled positive or negative on the basis of their sense of shear.

Right-handed shear stresses are generally considered to be positive, whereas left-handed shear stresses are negative.

Univ. Zambia 2012

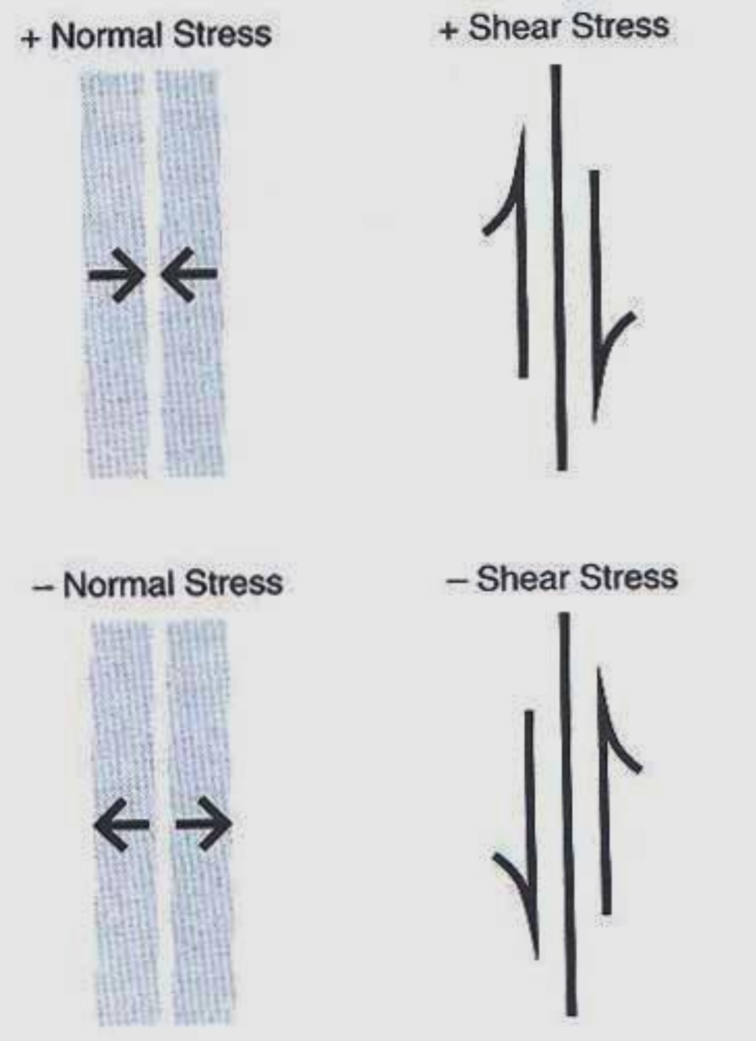


Figure 3.13 Sign conventions are necessary in describing the different kinds of stresses. Here is the convention that we use. Normal stress is positive (+) if compressive, negative (−) if tensile. Shear stress is positive (+) if right-handed, negative (−) if left-handed.

Resolving a stress into normal and shear stress components is actually quite straight forward. It can be done graphically or numerically.

We start with the orientation of the stress (σ_{XZ}) relative to the plane (XZ) on which it acts.

The acute angle (*θ*) between σ_{XZ} and the plane XZ is approximately $+78^0$.

The angle θ is positive when measured clockwise from the plane to the stress direction and negative when measured counterclockwise from the plane to the stress direction.

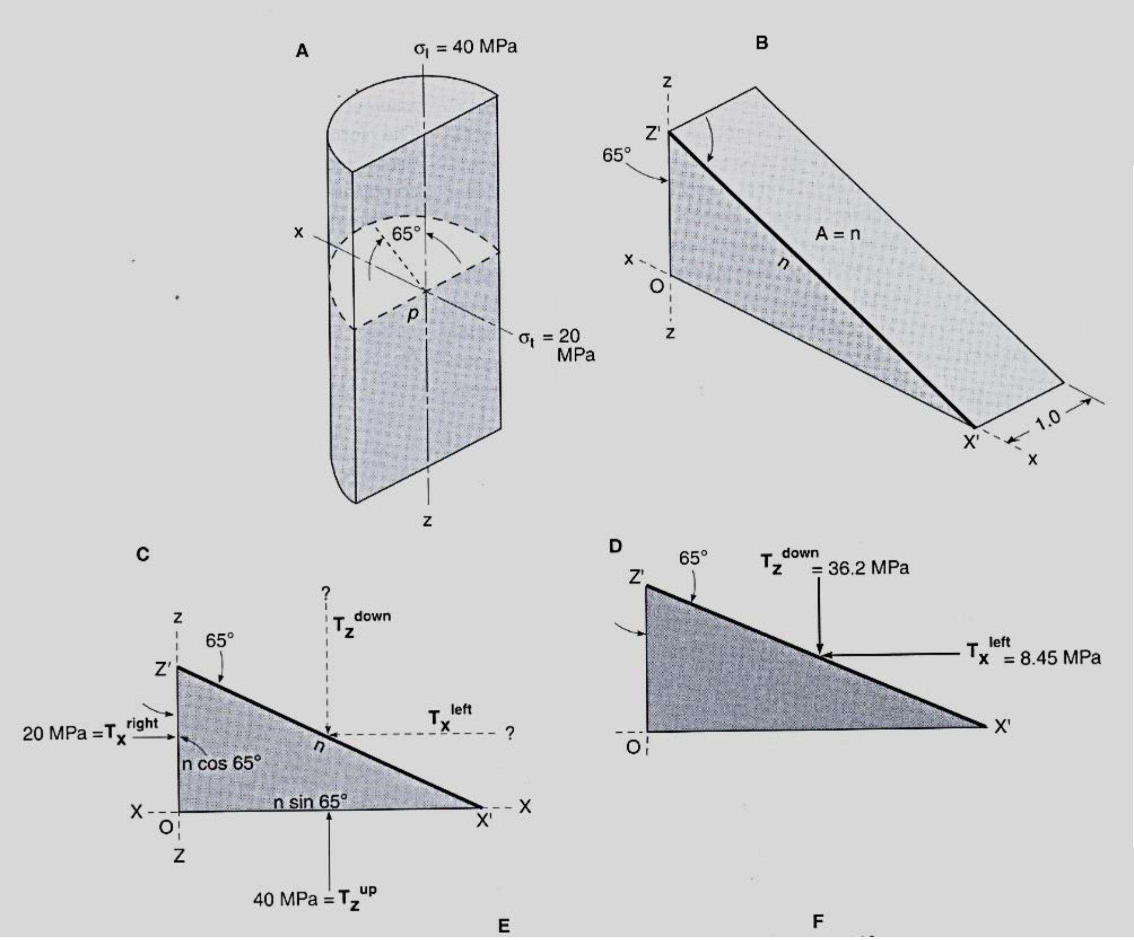


Figure 3.25 Computing traction (**T**) on a plane, which slices through point *p* deep in granite. (*A*) A look inside the granite at a plane in point *p* that is inclined 65° to the z-axis. (*B*) Block diagram view of the plane and the parcel of rock of which it is a part. (*C*) The balance of forces on the parcel of rock. (*D*) The calculated values of tractions $\mathbf{T_z}^{\mathbf{down}}$ and $\mathbf{T_x}^{\mathbf{left}}$ acting parallel to the z-axis (vertical) and x-axis (horizontal), respectively. (*E*) Graphical determination of the value of the traction ($\mathbf{T}^{\mathbf{down}}$) acting on the plane. (*F*) Calculation of the traction ($\mathbf{T}^{\mathbf{down}}$) based on application of the Pythagorean theorem. (*G*) Geometric relation of the traction ($\mathbf{T}^{\mathbf{down}}$) to the plane.

3rd Edition

Univ. Zambia 2012

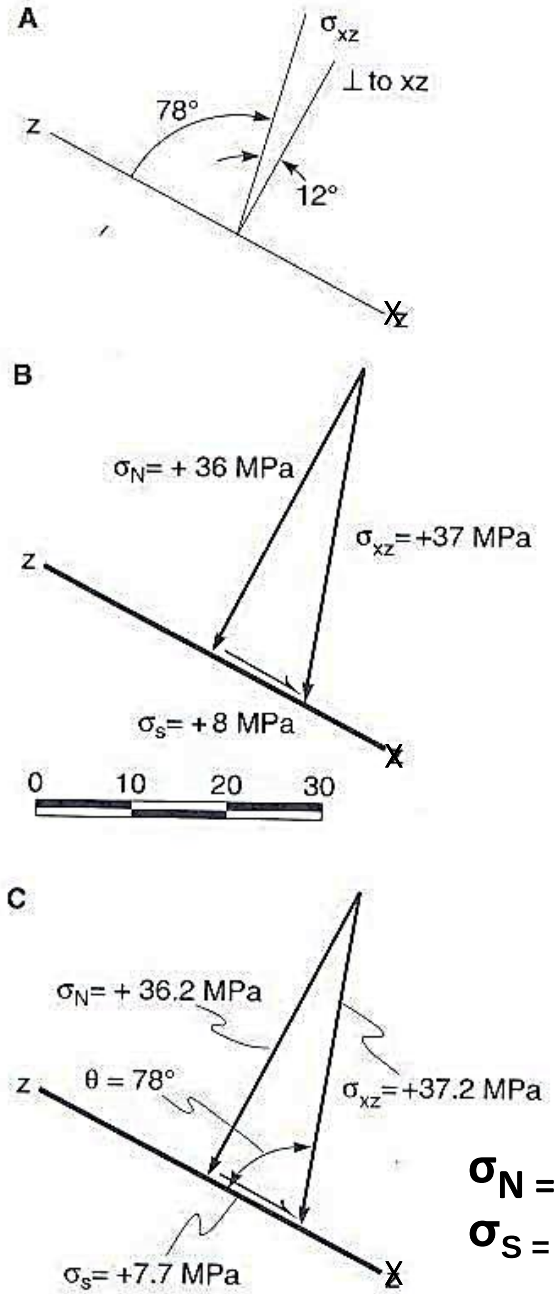


Figure 3.14 Resolving normal and shear stress. (*A*) Stress is inclined +78° to plane whose trace is *XZ*. (*B*) Graphical determination of normal stress (σ_N) and shear stress (σ_S). (*C*) Trigonometric determinaton of normal stress (σ_N) and shear stress (σ_S).

$\sigma_N = \sigma_{XZ} \sin\theta$ = 37 MPa (sin 78 $^{0=0.98)}$) = 36 Mpa

$\sigma_S = \sigma_{XZ} \cos\theta$ = 37 MPa (cos 78 $^{0\ 0.207}$) = 7.7 MPa

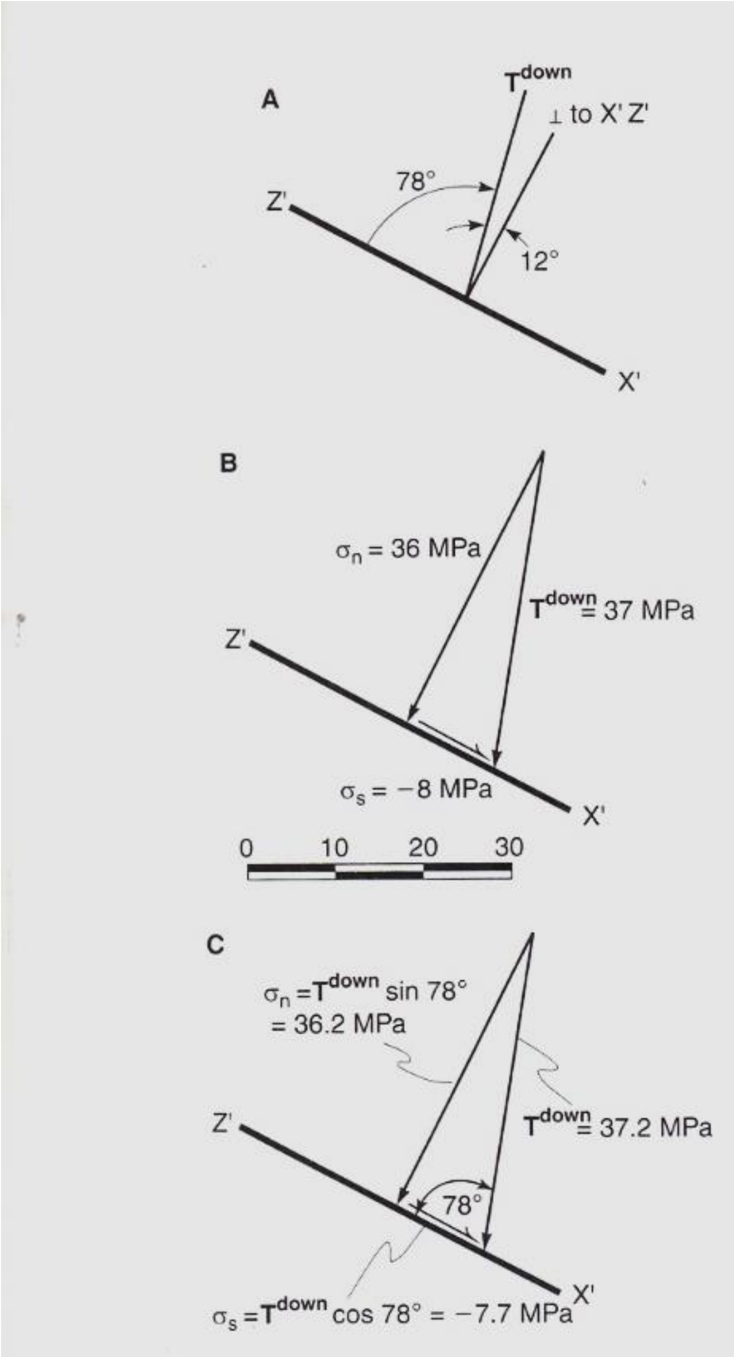
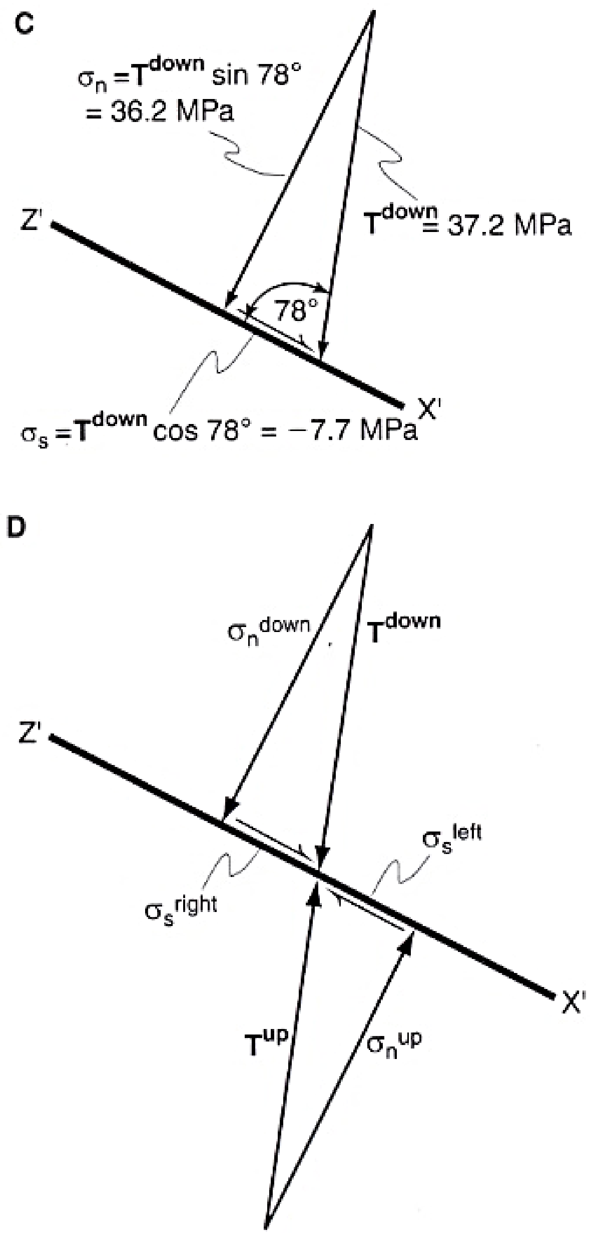


Figure 3.26 Resolving a traction ($\mathbf{T}^{down}$) into normal (σ_n) and shear (σ_s) traction components. (*A*) Traction ($\mathbf{T}^{down}$) is inclined 78° to plane whose trace is *X′Z′*. (*B*) Graphical determination of normal traction (σ_n) and shear traction (σ_s) components. (*C*) Trigonometric determination of normal traction (σ_n) and shear traction (σ_s). (*D*) The move from traction to stress is made in situations of static equilibrium by recognizing that normal stress (σ_N) is the pair of normal tractions and the shear stress (σ_S) is the pair of shear tractions.

3rd Edition



gure 3.26 Resolving a traction (T^{down}) into normal (σ_n) and shear (σ_s) traction mponents. (A) Traction (T^{down}) is inclined 78° to plane whose trace is $X'Z'$. (B) aphical determination of normal traction (σ_n) and shear traction (σ_s) components.) Trigonometric determination of normal traction (σ_n) and shear traction (σ_s). (D) ie move from traction to stress is made in situations of static equilibrium by rec- nizing that normal stress (σ_N) is the pair of normal tractions and the shear stress s) is the pair of shear tractions.

3rd Edition

Next we show the magnitude of the stress (37.2 MPa) by adding a 'map' scale to the diagram. Then, from the tail of the arrow representing the stress (σ_{XZ}), the normal stress component (σ_N) is constructed perpendicular to the plane. Its magnitude (+36 MPa) can be read directly using the scale of the drawing.

The shear stress component (σ_{S}) is considered as a vector drawn from the tip of the normal stress component (σ_N) to the tip of σ_{XZ}, and thus parallel to the plane. Its value (+8 MPa; i.e., right handed) also can be directly read as well.

The numerical solution for computing the magnitude of the normal and shear stress components is trigonometric. If $\theta = +78^0$ is the angle between the plane XZ and the stress (σ_{XZ}) that acts on it, normal stress (σ_N) can be computed as follows,

$$\sigma_N = \sigma_{XZ} \sin\theta = 37\ \text{MPa}\ (\sin 78^0) = 36\ \text{MPa}$$

The magnitude of the shear stress (σ_S) can be determined in a comparable fashion.

$$\sigma_S = \sigma_{XZ} \cos \theta = 37 \text{ MPa} (\cos 78^0) = 7.7 \text{ MPa}$$

1. Forces and stresses
2. Normal Stress and Shear Stress
3. Stress Component at a point
4. Plane stress (stress equation) and Mohr diagram of stress

End of Part II

Analysis of Stress

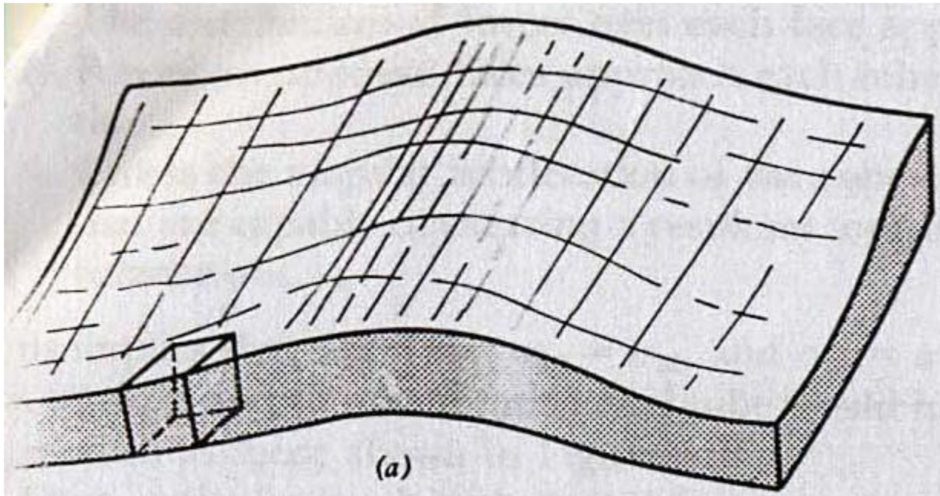
(Hobbs et al., 1976)

Analysis of Stress in three dimension

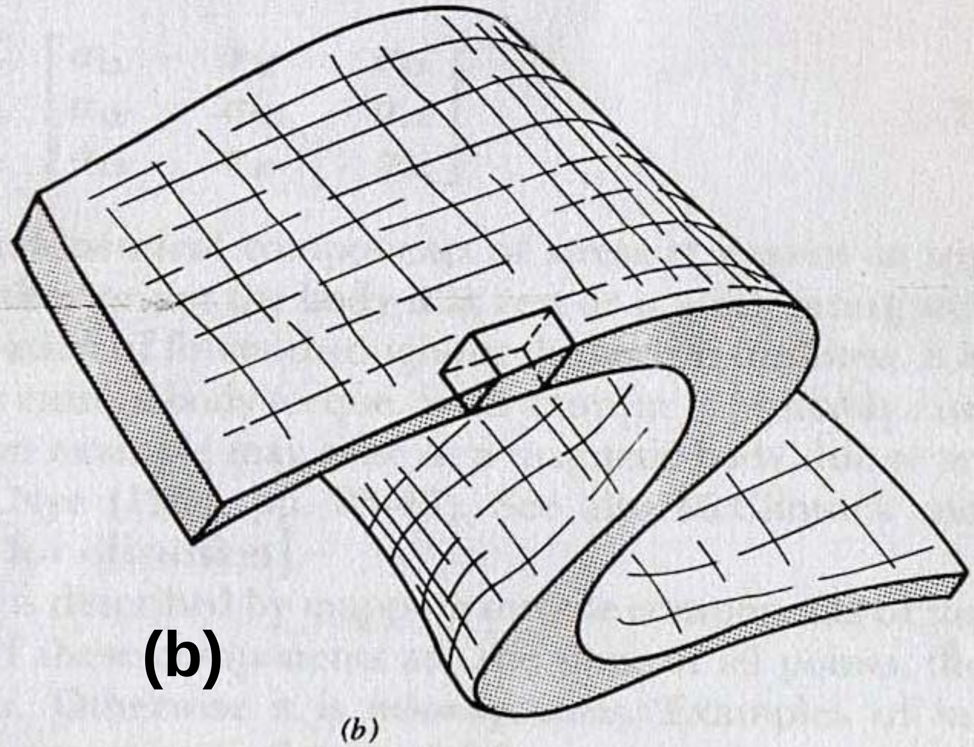
Consider a small cube of rock within a large volume of rock undergoing deformation. Fig. 1.1a shows a small cube within a large fold as an example.

All six faces of this cube will be pressed on by adjacent parts of the rock, and there will be corresponding reactions from the material within the cube.

Fig. 1.1 Components of Stress: (a) Small cube in a layer of rock undergoing bending. (b) The same cube distorted after the layer has become folded



(a)



(b)

In addition, each particle within the cube is acted on by gravity.

A system of forces, therefore, exists throughout the cube, and the resultant forces acting upon the faces of the cube are illustrated in Fig. 1.1c

Univ. Zambia 2012

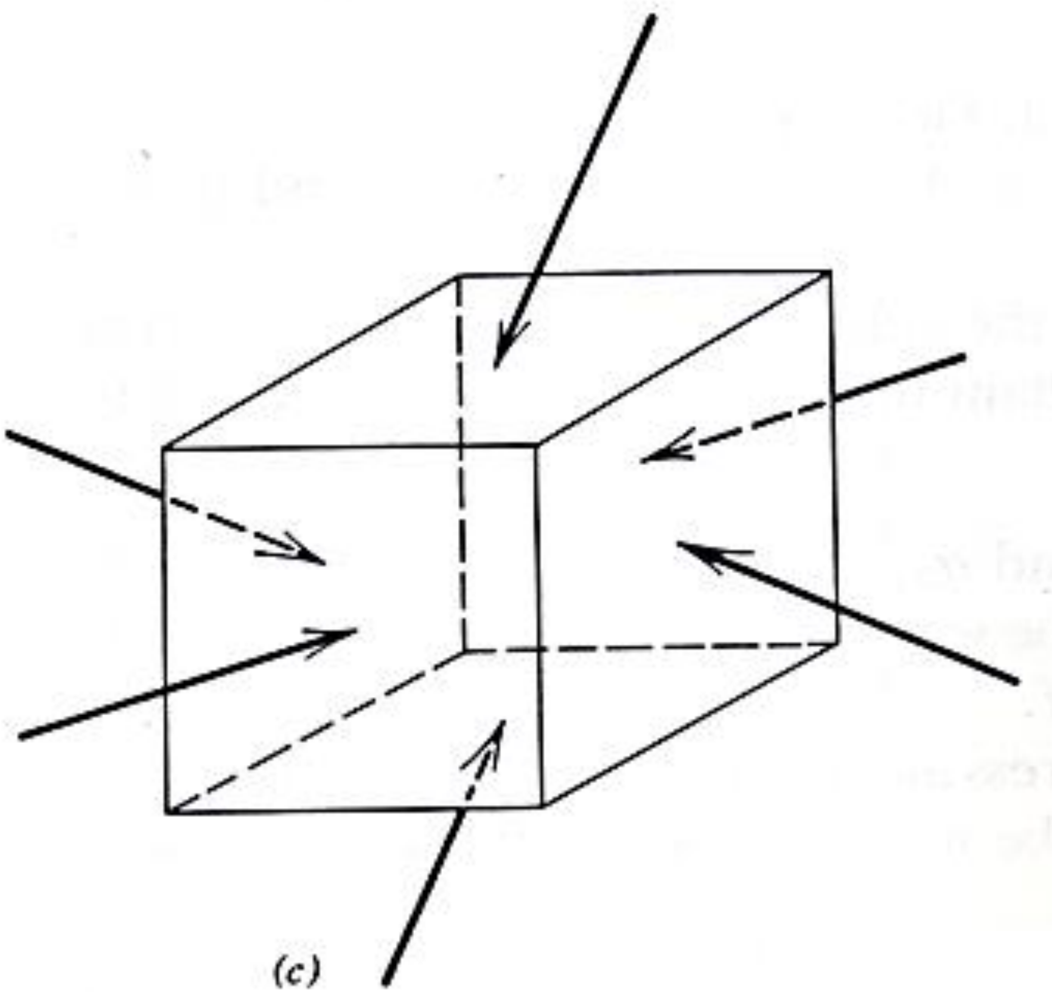


Fig. 1.1 C

Fig.1.1. Components for stress (c) Forces acting on the faces of a deforming cube.

Components of Stress
(Stress at a point)

The force on each of the cube faces in Fig.1.1c may be resolved into three orthogonal components.

One normal to the face and two parallel to the face (fig 1.1d).

If the magnitudes of each of these three components is divided by the area of the cube face then the magnitudes of three *components of stress* are obtained. (fig.1.1e).

Univ. Zambia 2012

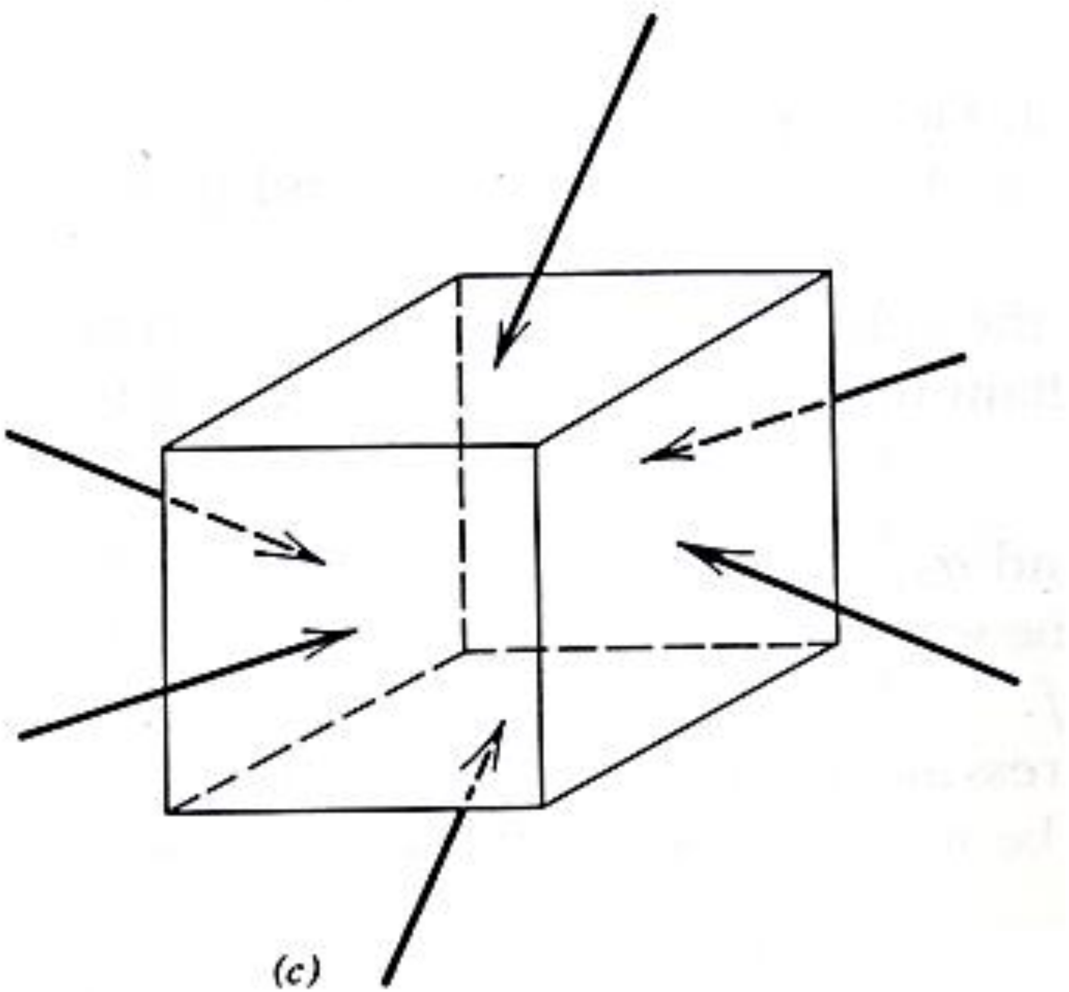
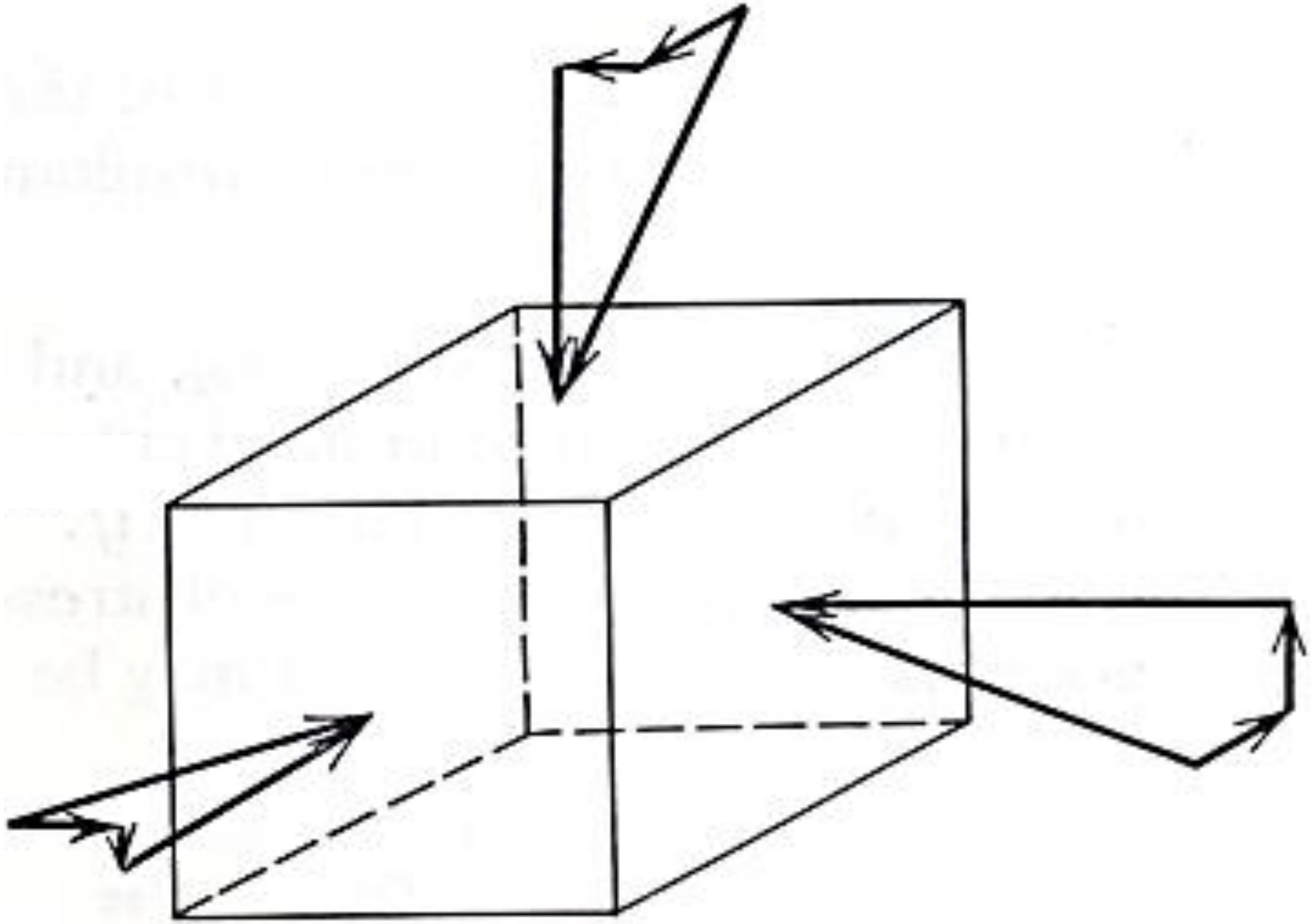


Fig. 1.1 C

Fig.1.1. Components for stress (c) Forces acting on the faces of a deforming cube.



(d)

Fig.1.1. Components for stress (d) Resolution of these forces, both normal and parallel to the faces of the cube

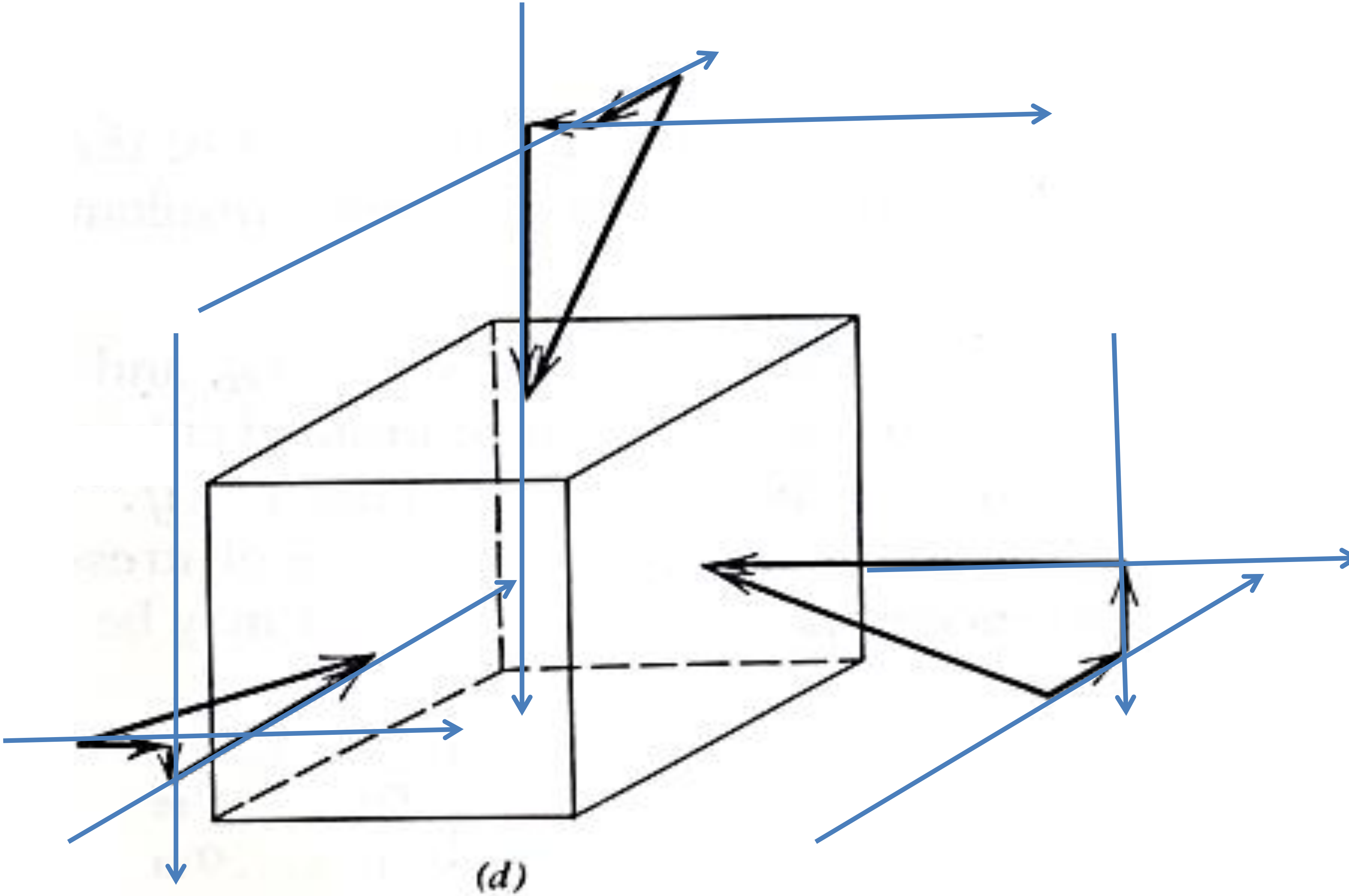


Fig.1.1. Components for stress (d) Resolution of these forces, both normal and parallel to the faces of the cube

Using the edges of the cube as a system of Cartesian coordinates (x_1,x_2,x_3) and employing the symbol σ_{ij} to denote that component of stress that acts on the face normal to x_i and in the direction of x_j (here, since there are three coordinate axes, $_i$ and $_j$ range over the values 1,2, and 3).,

- the various components of stress may be labeled as in Fig..1.1e and written down in a systematic manner in the following array:

Components of Stress

σ_{13} Stress component that acts on the face normal to **X1** and in the direction of **X3**

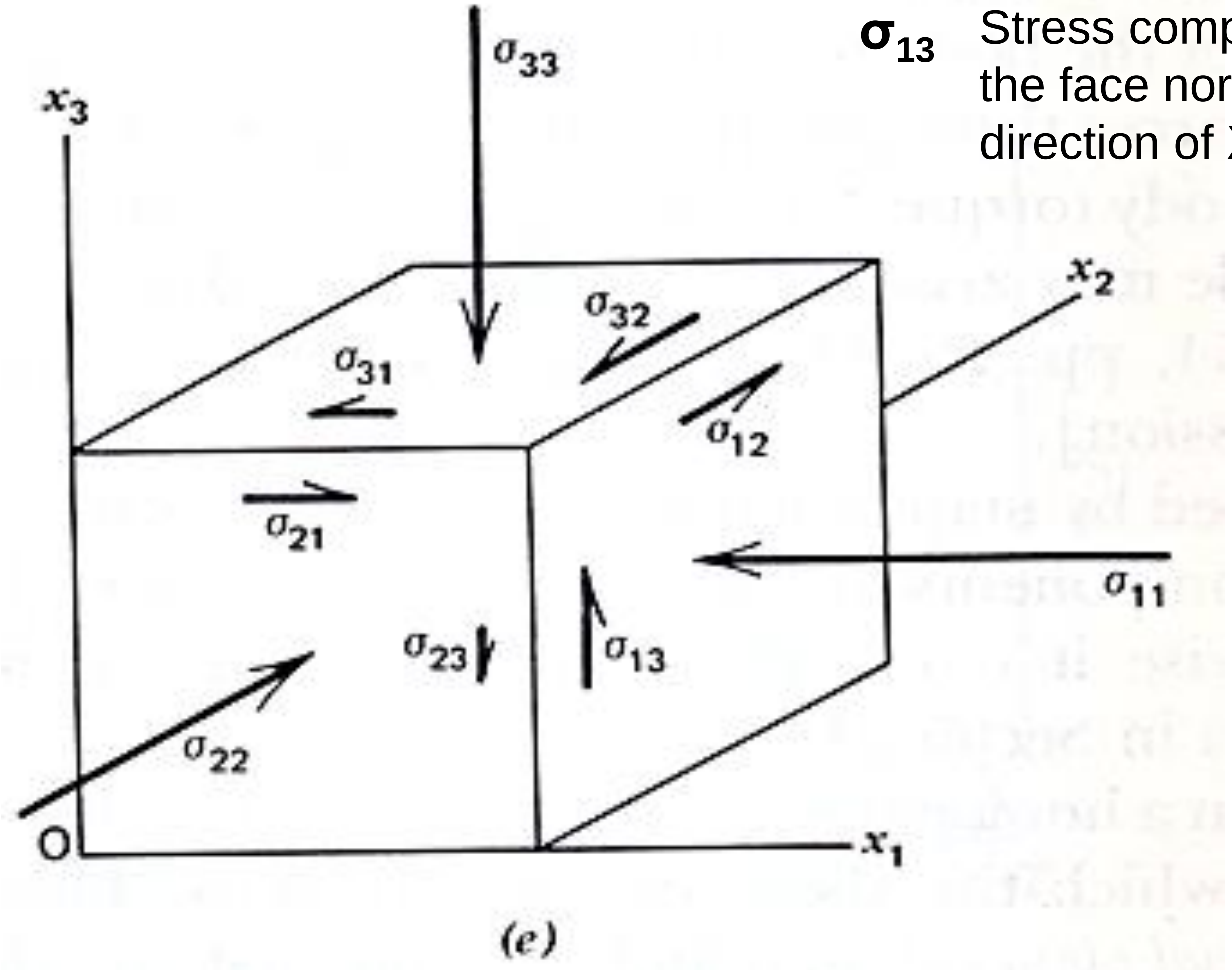


Fig.1.1. (e) Stress system acting upon the faces of the cube referred to the coordinate system Ox_1, Ox_2, Ox_3

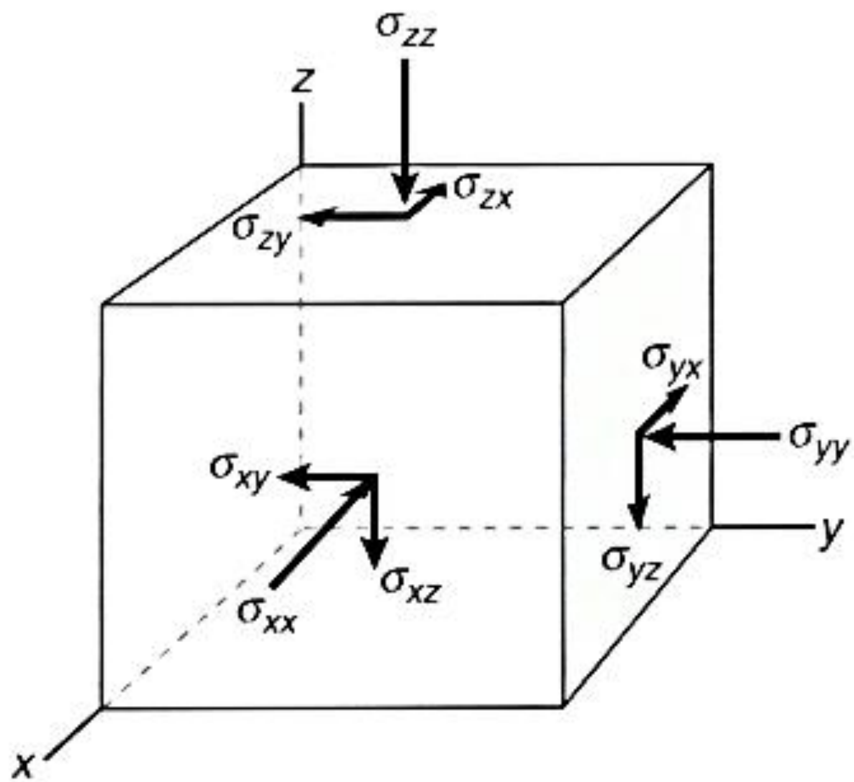


Figure 4.5 The stress components acting on the faces of a small cube. Positive stress components are shown – corresponding stress components exist on the negative and hidden faces of the cube. σ_{xx}, σ_{yy} and σ_{zz} are normal stresses, the others are shear stresses and are parallel to the edges of the cube.

Components of Stress

σ_{13} Stress component that acts on the face normal to **X1** and in the direction of **X3**

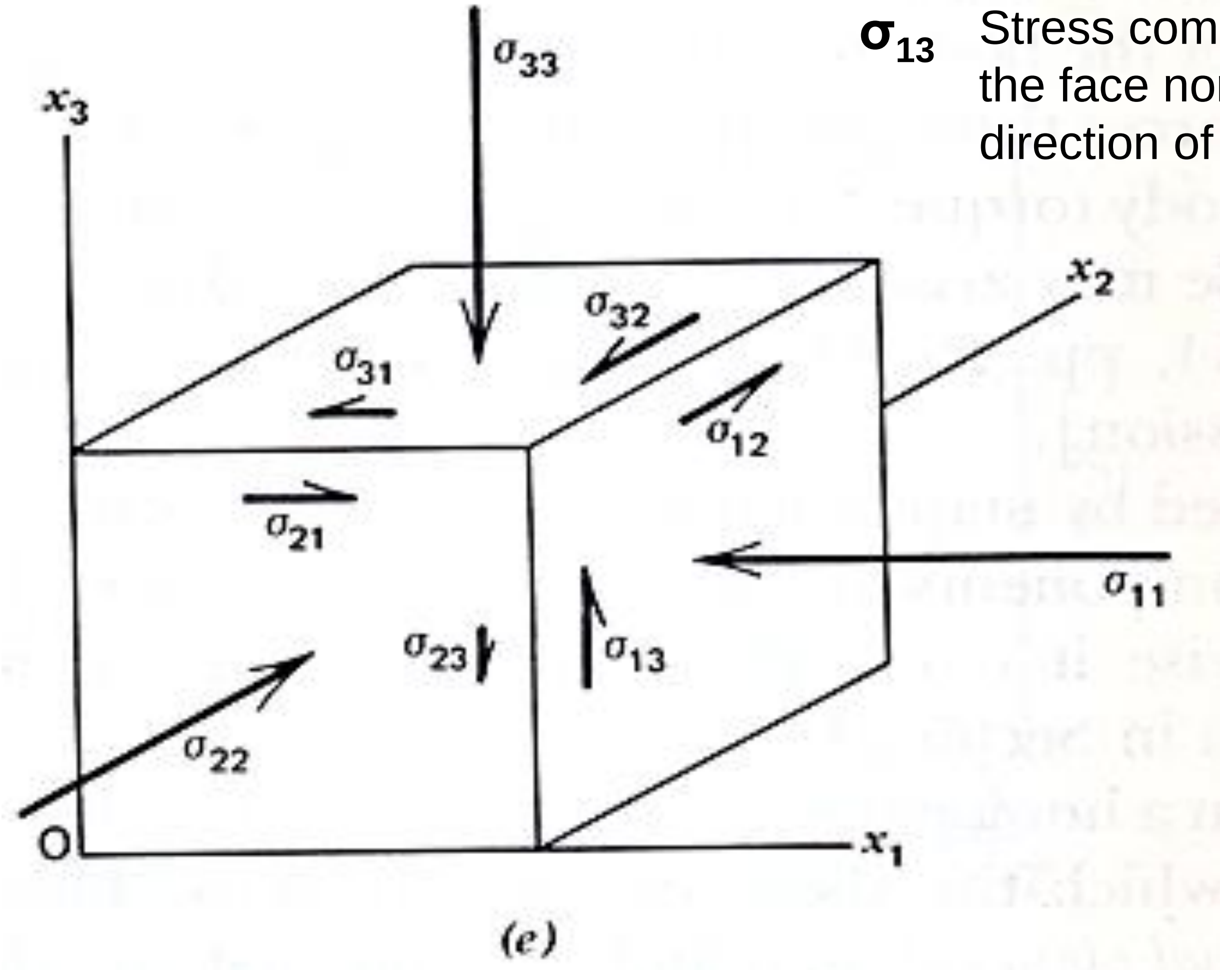


Fig.1.1. (e) Stress system acting upon the faces of the cube referred to the coordinate system Ox_1, Ox_2, Ox_3

$$\begin{vmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{vmatrix} \qquad (1.1)$$

Thus σ_{13} is a stress component that acts on the face normal to x_1 and in the direction of x_3, where, σ_{31} acts on the face normal to x_3 and in the direction of x_1.

Stress components represented by σ_{ij} where the two subscripts are the same, (*$i=j$*), and which act normal to a cube face, are known as *normal stresses*.

Those stress components represented by σ_{ij} where the two subscripts are different ($i \neq j$), and which act parallel to a cube face, are known as *shear stresses*.

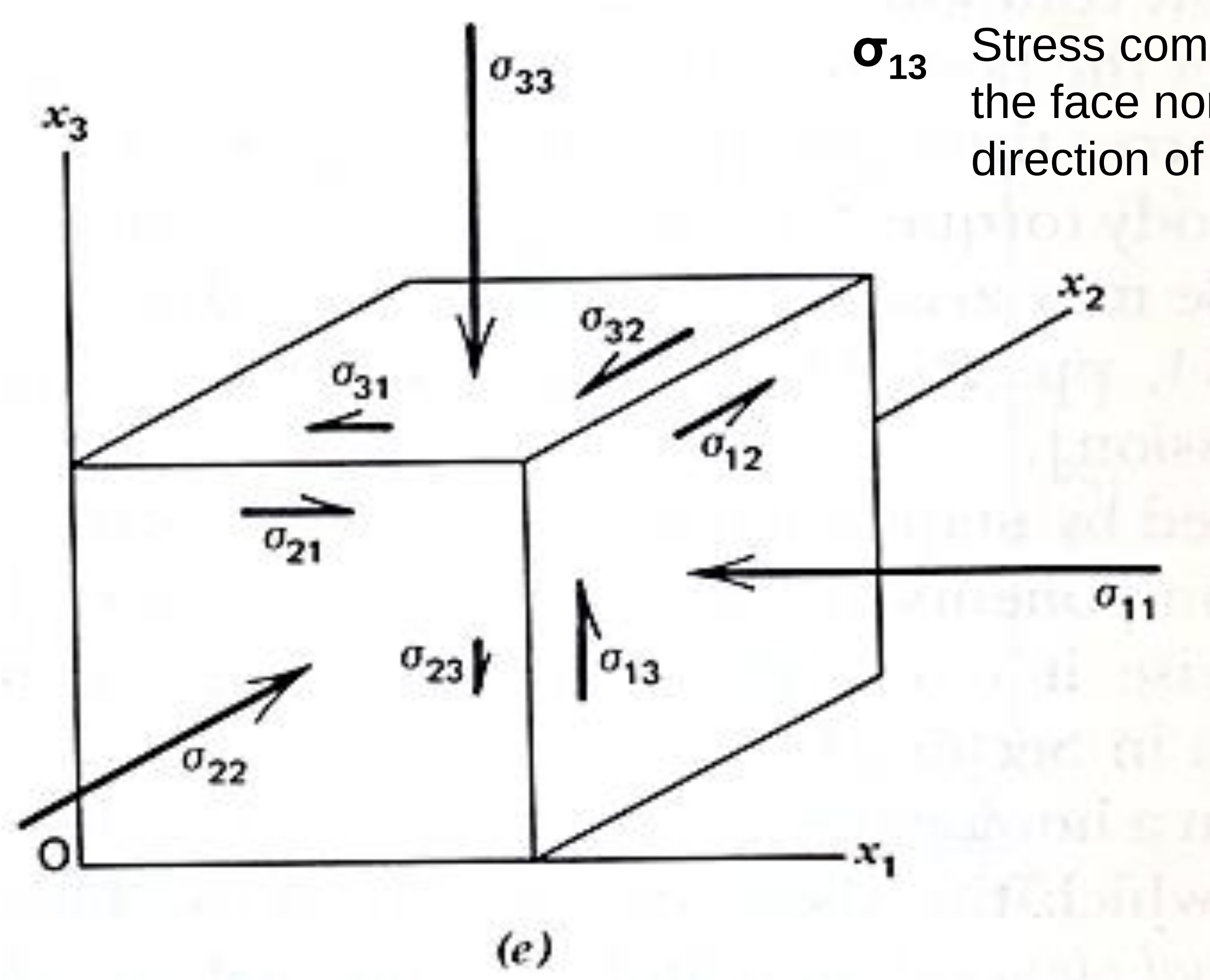
In some text books normal stresses are represented by symbols such as σ_{ij} ($i=j$), where as a different symbol *T* ij with ($i \neq j$) is used to represent the *shear stresses*

The situation illustrated in Figure 1.1a is likely to be complicated by variation in the magnitude and direction of force over each cube face, and it becomes convenient in such inhomogeneous situations to consider the state of stress at a point.

This is achieved by imagining the cube of Fig 1.1e to shrink to a point, the stress at a point being defined as the limiting ration of force to area as the area of the face approaches zero.

Components of Stress

Univ. Zambia 2012



σ_{13} Stress component that acts on the face normal to **X1** and in the direction of **X3**

Fig.1.1. (e) Stress system acting upon the faces of the cube referred to the coordinate system *Ox_1*, *Ox_2*, *Ox_3*

Three important features arise as this limiting situation is approached:

1. **The distribution of forces over each face approaches uniformity.**

2. **Forces on opposite faces approach each other in magnitude and direction.**

3. Unless the angular acceleration of the cube is to become infinite, forces that are capable exerting a resultant toque on the cube must tend to balance out.

This implies that $\sigma_{12} = \sigma_{21}$, $\sigma_{23} = \sigma_{32}$, and $\sigma_{31} = \sigma_{13}$ in the limit and that an infinitesimal cube would have stresses acting upon its faces such as those shown in fig. 1.1f.

Univ. Zambia 2012

Components of Stress

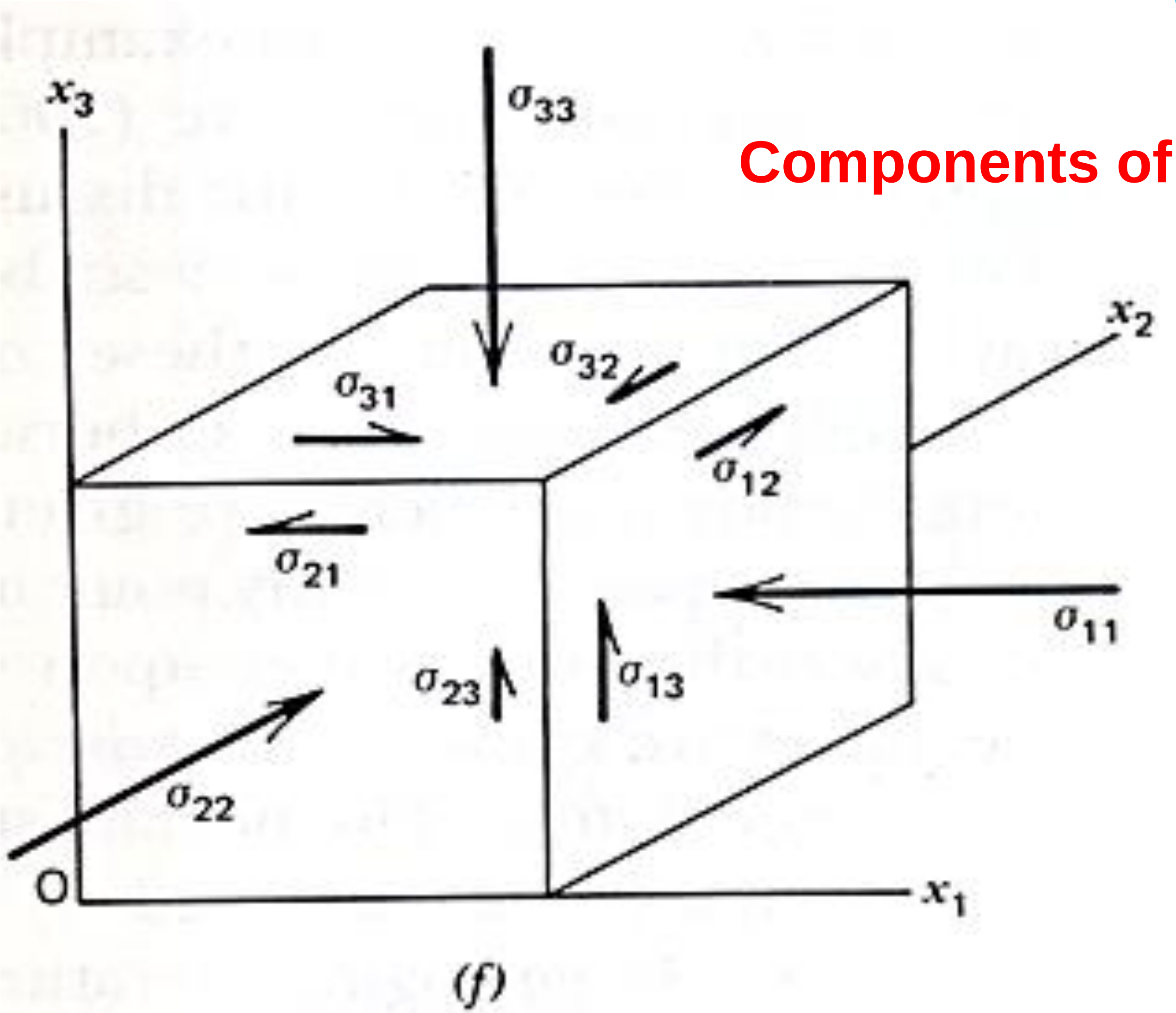


Fig.1.1. (f) Stress system acting on the faces of an infinitasimal cube.

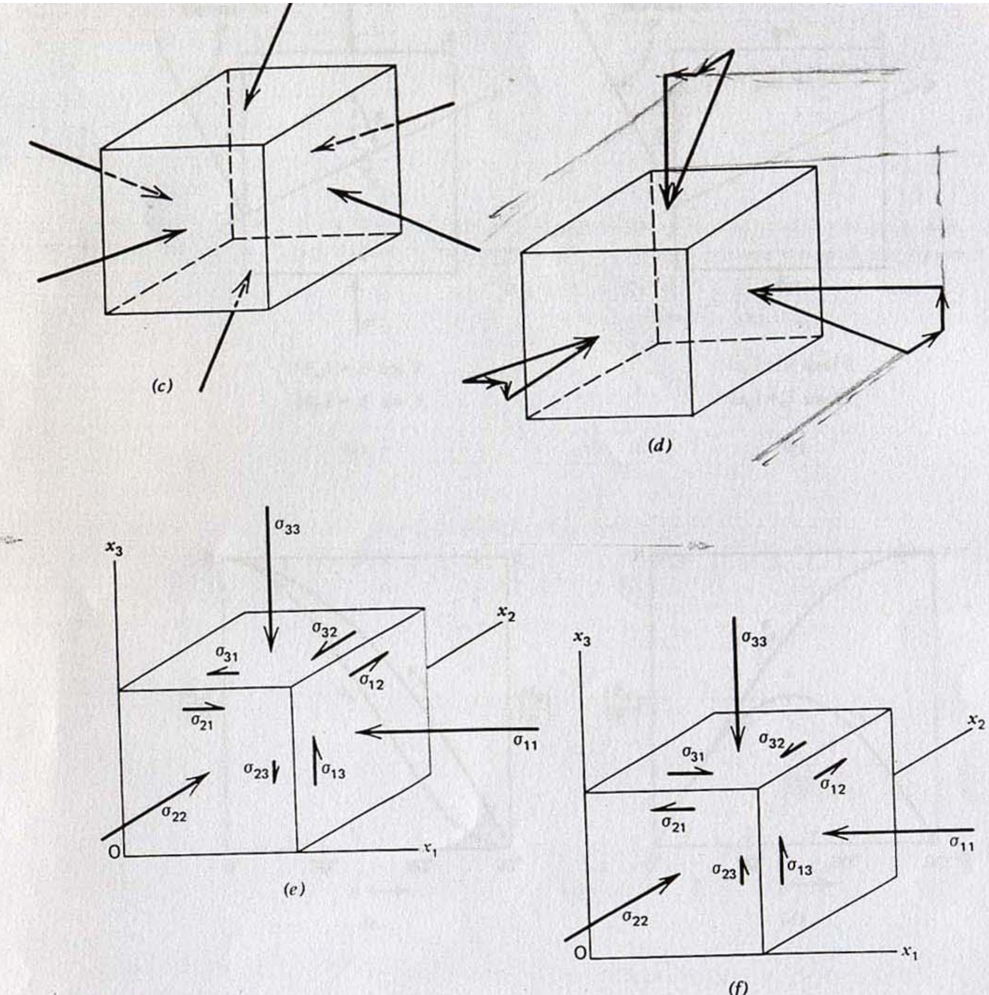
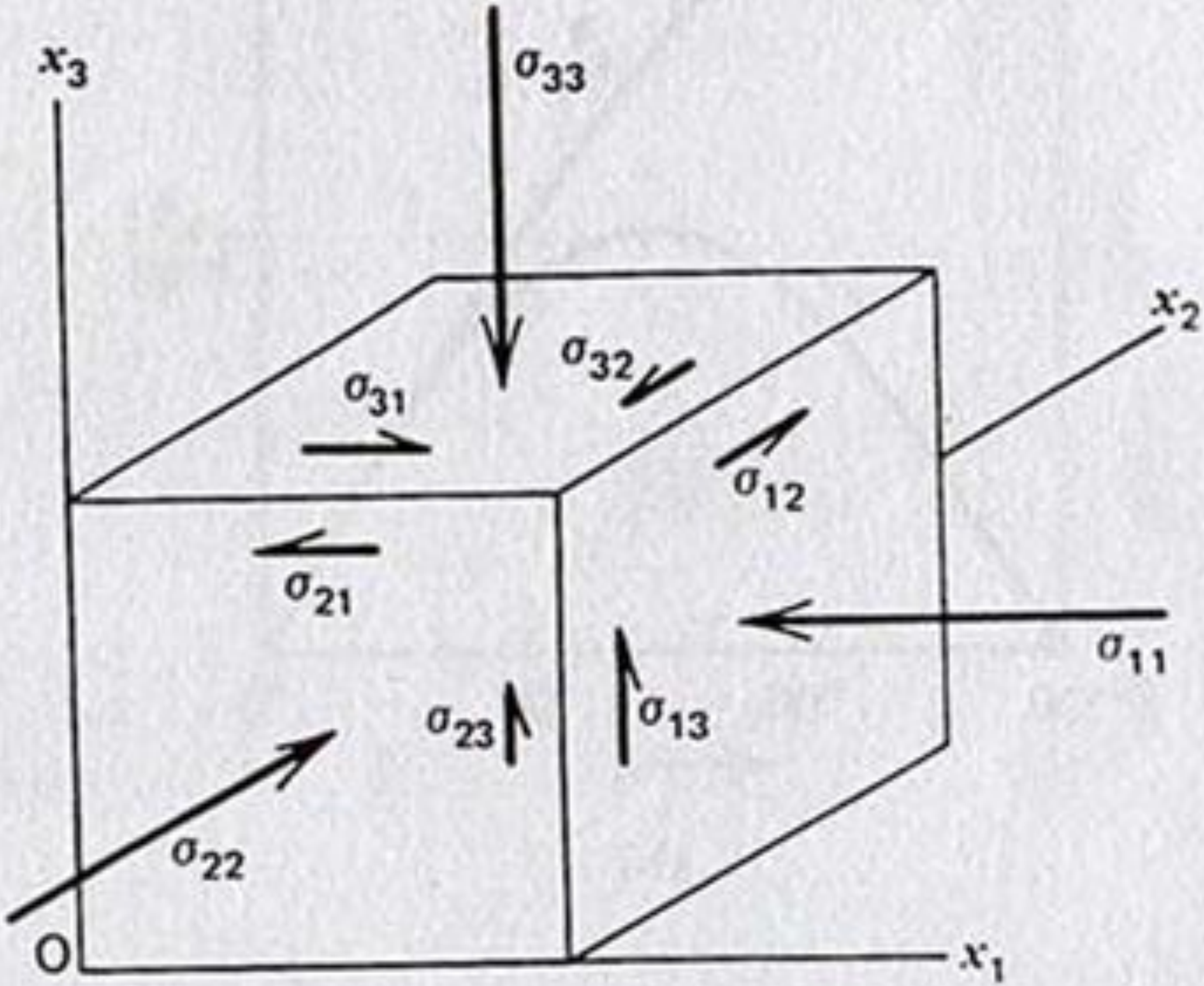


FIGURE 1.1 (Continued)

The situation illustrated in Figure 1.1*a* is likely to be complicated by variation in the magnitude and direction of force over each cube face, and it becomes convenient in such inhomogeneous situations to consider the *state of stress at a point*. This is achieved by imagining the cube of Figure 1.1*e* to shrink to a point, the stress at a point being defined as the limiting ratio of force to area as the area of the face approaches zero. Three important features arise as this limiting situation is approached:



x_3
σ_{33}
x_2
σ_{32}
σ_{31}
σ_{12}
σ_{21}
σ_{11}
σ_{23}
σ_{13}
σ_{22}
O
x_1

(f)

Thus in the limit when the state of stress at a point is considered the array of stress components at that point may be written in a symmetrical manner:

$$\begin{vmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{vmatrix} \quad \sigma_{12} = \sigma_{21},\ \sigma_{13} = \sigma_{31} \text{ and } \sigma_{33} = \sigma_{11} \qquad (1.2)$$

Thus, there are just six independent components of stress at a point in any material. This is true whether or not the body is at rest or is accelerating and whether or not the distribution of forces throughout the body is uniform.

It is not true; however, if there exists a body torque. Such a torque is probably rare in geological bodies.

Univ. Zambia 2012

$$\begin{vmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{vmatrix}$$

$$\downarrow$$

$$\begin{vmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{vmatrix}$$

The stress field in a body is described by mapping out the components of the array (1.2) at all points.

If these components are the same at all points, the stress field is homogeneous. Otherwise it is inhomogeneous.

It is always possible, at any point in a homogeneous stress field, to find three mutually orthogonal planes upon which the shear stresses are zero. These three planes are known as the:
principal planes of stress, and their normals are the principal axes of stress,

The normal stresses across the principal planes are the principal stresses, often denoted by σ_1, σ_2 and σ_3 with the convention that $\sigma_1 > \sigma_2 > \sigma_3$.

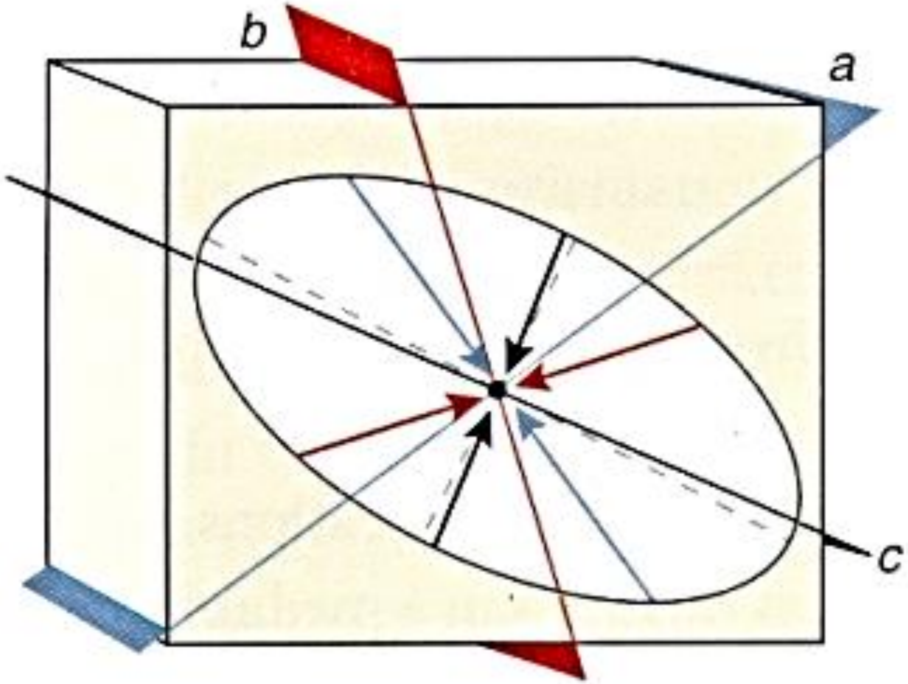


Figure 4.3 Two-dimensional illustration of stress at a point. Three planes (a, b and c) are oriented perpendicular to the section in question, and their normal stresses are represented in the form of vectors (corresponding colors). The stress vectors define an ellipse, whose ellipticity depends on the state of stress.

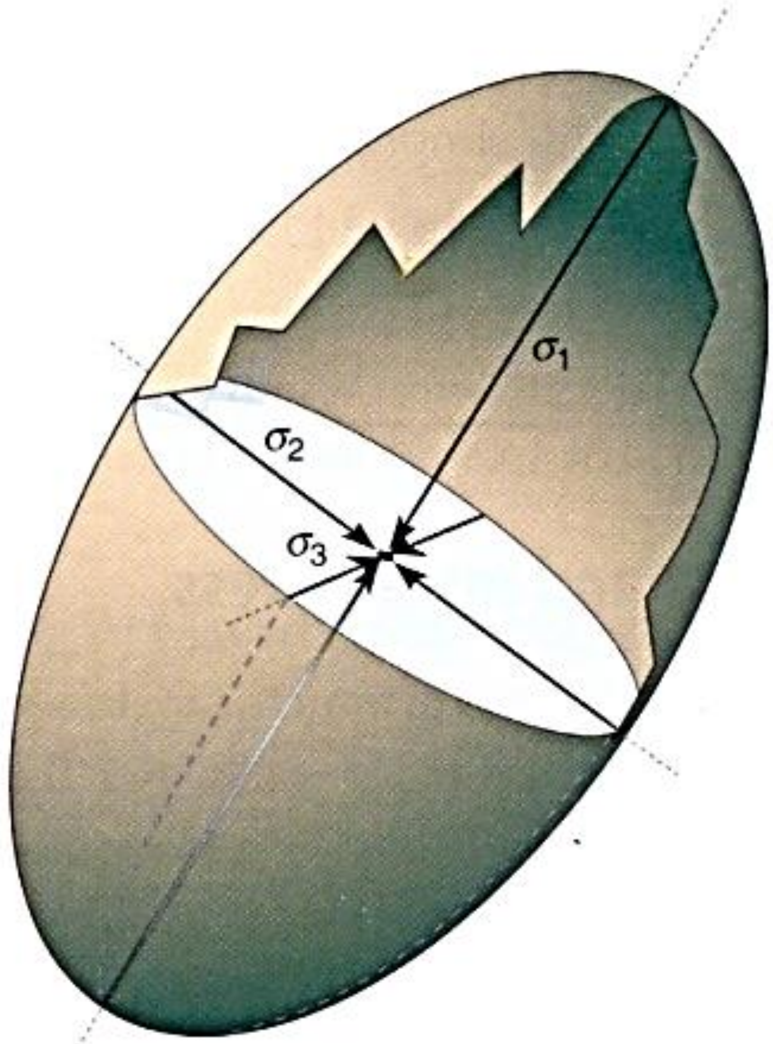


Figure 4.4 The stress ellipsoid.

In geological literature compressive stresses are commonly taken as positive.

The maximum, intermediate, and minimum principal stresses are $\sigma_{1,,}$ σ_2 and σ_3 respectively.

The state of stress at a point may, therefore, be characterized by giving these three principal stresses and their directions, or by giving the six independent stress components contained in the array (1.2) when the faces of the reference cube are not parallel to the principal planes of stresses.

End of Part III

Resolution of Stress

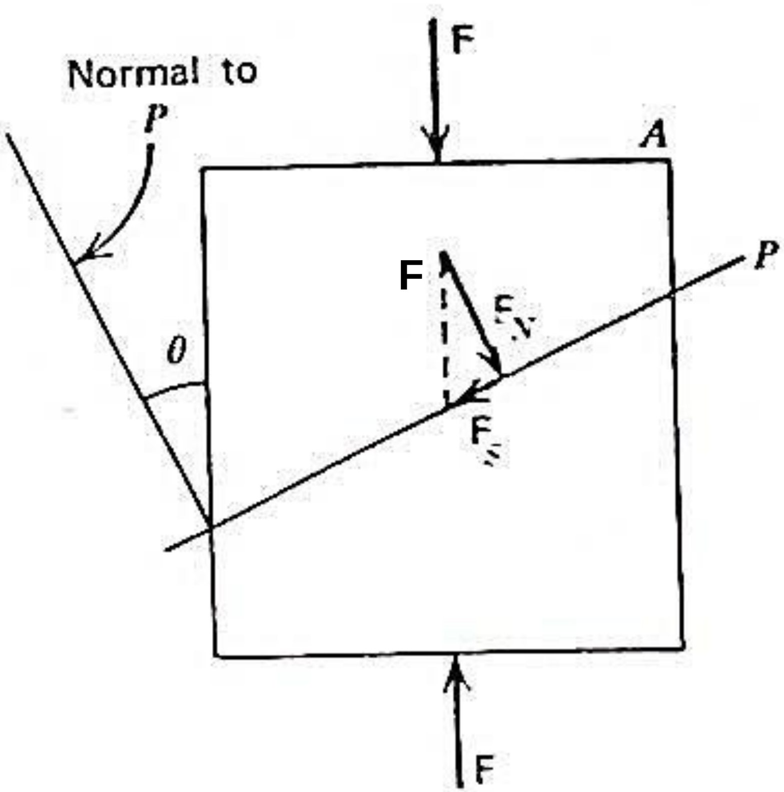
It is important to note that although stress has many of the physical characteristics of force associated with it, the concept always contains the additional physical association with area.

Hence, the value of a stress not only varies with the orientation and magnitude of the imposed force but varies also as the area and inclination changes orientation and magnitude.

This may be seen more clearly in figure 1.2, where crossections of cubes are shown with a force of magnitude *F* acting normally to one cube face of area A, (fig. 1.2 a).

Cutting across the cube is another plane, *P*, whose normal is inclined at an angle θ to *F*. We ask: what are the normal and shear components of force across the plane *P*, and how do they differ in magnitude from the normal and shear components of stress across *P*.

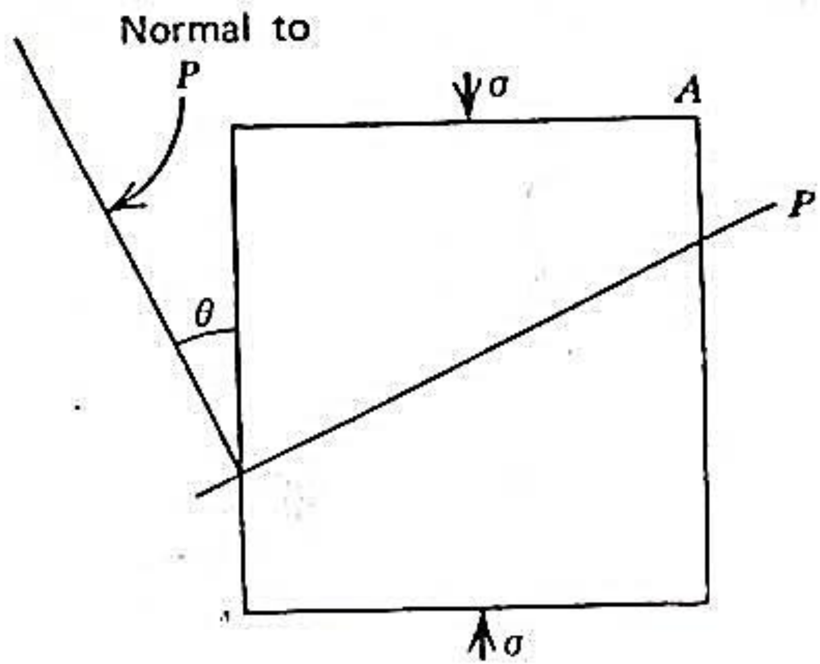
Univ. Zambia 2012



$|F_N| = F \cos \theta$

$|F_S| = F \sin \theta$

a



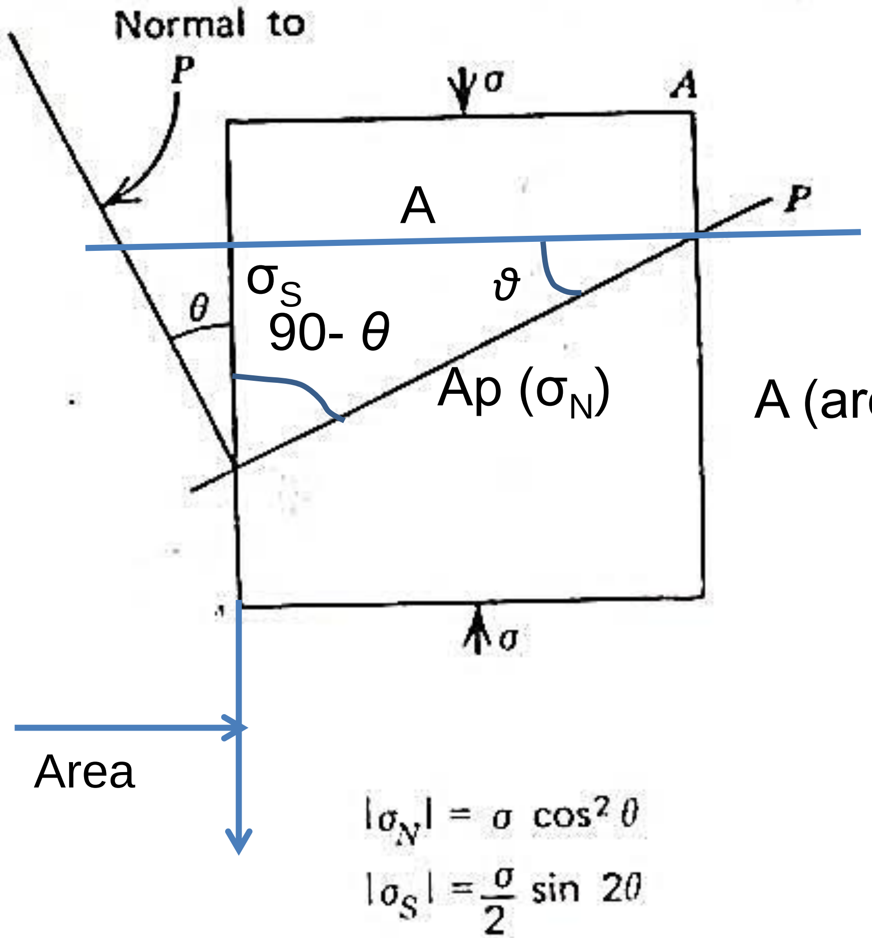
$|\sigma_N| = \sigma \cos^2 \theta$

$|\sigma_S| = \frac{\sigma}{2} \sin 2\theta$

b

Fig. 1.2

Univ. Zambia 2012



(b)

A (area) for σ_N = Ap cosθ

A (area) for σ_s also will be = Ap cosθ

In Figure 1.2a the force has been resolved into components normal and parallel to the plane *p*. The components have magnitude:

$$F_N = F\cos\theta \qquad F_S = F\sin\theta \qquad (1.3)$$

Now in Fig. 1.2b, the stress σ on the cube face has the magnitude F/A whereas the area of the inclined plane *P* is

$$A_P = \frac{A}{Cos\theta} \qquad (1.4)$$

(Because: $Cos\theta = A/A_P$**) or** $A = A_p \, Cos\,\theta$

Hence,

$$|F_N| = F\cos\theta = A\,\sigma\cos\theta = A_P\,\sigma\cos^2$$

(Because **σ=F/A, therefore F= A.σ) and A=Ap Cos *θ)***

And **(1.5)**

$$|F_S| = F\sin\theta = A_P\,\sigma\sin\theta\cos\theta$$

Thus, the magnitude of the normal and shear components of stress across *P* are:

$$\sigma_N = \frac{|F_N|}{A_P} = \sigma\cos^2\theta = \frac{F}{A}\cos^2\theta$$

$$\sigma_S = \frac{|F_S|}{A_P} = \frac{\sigma}{2}\sin 2\theta = \frac{F}{A}\sin\theta.\cos\theta \quad (1.6)$$

(or, $\sigma_{S= Fs/Ap=Ap}\sigma\sin\theta.\cos\theta/Ap = \sigma\ \sin\theta.\cos\theta = F/A\sin\theta.\cos\theta$)

In Fig. 1.2a (explanation)

$$A_P = \frac{A}{\cos\theta},$$

Therefore, $A = A_P \cos\theta$

$$\sigma = \frac{F}{A} \quad \text{(by definition)}$$

Therefore, $F = A.\sigma$

Normal component of Force:

$$F_N = F.\cos\theta = A.\sigma.\cos\theta = A_P \cos\theta\, \sigma \cos\theta$$

$$F_N = A_P\, \sigma \cos^2\theta$$

In Summary:

σ = Principal stress

σ_N = Normal stress

σ_S = Shear stress

A =Area of horizontal plane

A_p = Area of inclined plane

Now, $\sigma_N = F_N / A_p$, ...(i)

$F_N = F \cos\theta$ ....(ii)

$A_p = A/\cos\theta$ .... (iii)

and therefore, $A = A_p \cos\theta$..........(iv)

Now, subsituting values in equation (i)

$\sigma_N = F_N / A_p$, ...(i)

$$\sigma N = \frac{F\,Cos\theta}{A/Cos\theta} = = \frac{F\,Cos\theta}{A} \,.\, Cos\theta = \frac{F}{A} Cos^2\theta$$

Similarly,

$\sigma S = F_S / A_p$, (Also $F_s = F sin\,\theta$)

$$\sigma_S = \frac{F\,sin\,\theta}{A/Cos\theta} = = \frac{F}{A} Sin\theta\,.Cos\theta$$

The manner in which the magnitudes of F_N and σ_N and F_S and σ_S vary with θ is shown in Figures 1.2c and d, respectively.

A comparison of equations 1.3 and 1.6 shows that stresses may not be resolved as though they were forces and that the change in the magnitude of the area of action must be considered also.

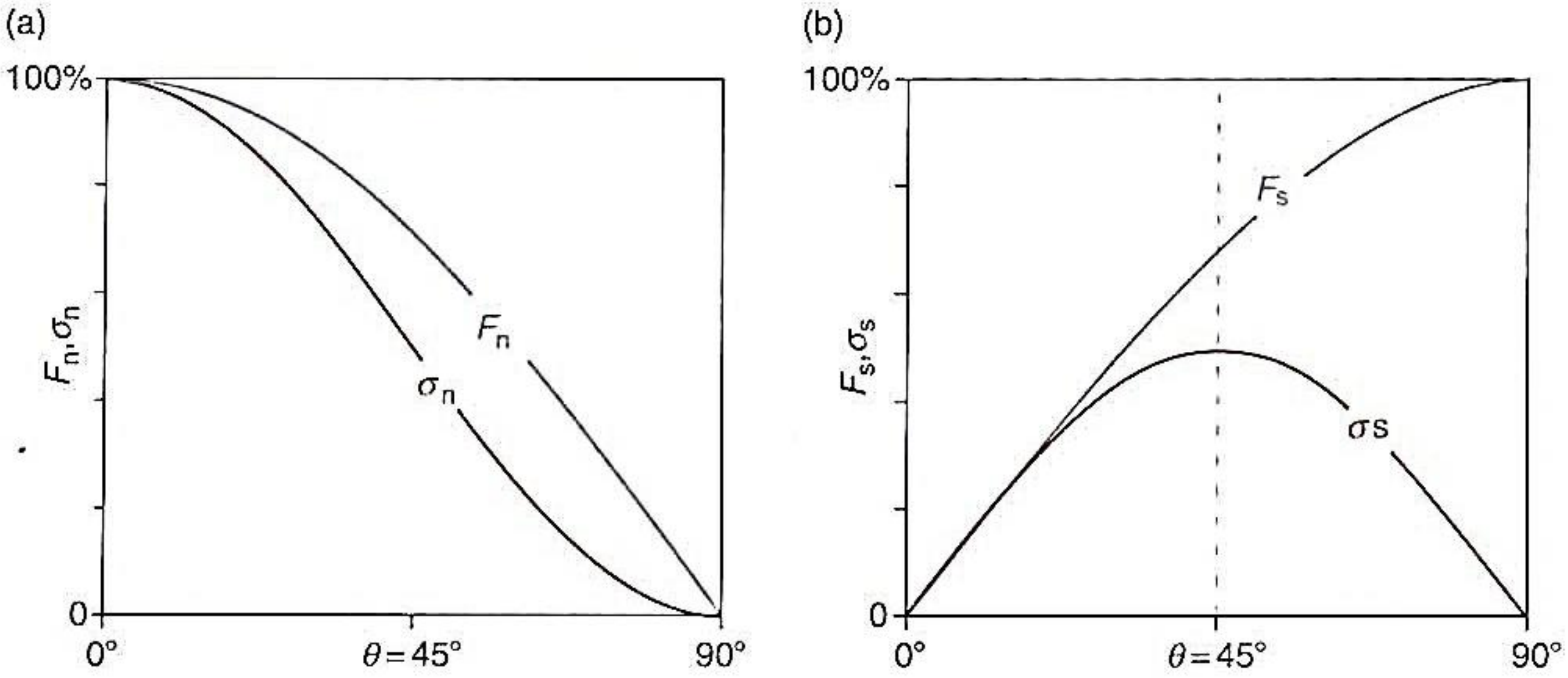


Figure 4.2 (a) The normal components of the force (F_n) and stress (σ_n) vectors acting on a surface, plotted as a function of the orientation of the vectors relative to the surface (θ, see Figure 4.1). Note the difference between the two. (b) The same for the shear components. Note that the shear stress is at its maximum at 45° to the surface while maximum shear force is obtained parallel to the surface.

Cos 0 degree = 1
Cos 90 degree =0

Where the principal stresses are σ_1 and σ_2 the equations for the normal and shear stresses across a plane whose normal is inclined at θ *to* σ_1 are:

$$\sigma_N = \tfrac{1}{2}(\sigma_1 + \sigma_3) + \tfrac{1}{2}(\sigma_1 - \sigma_3)\cos 2\theta$$

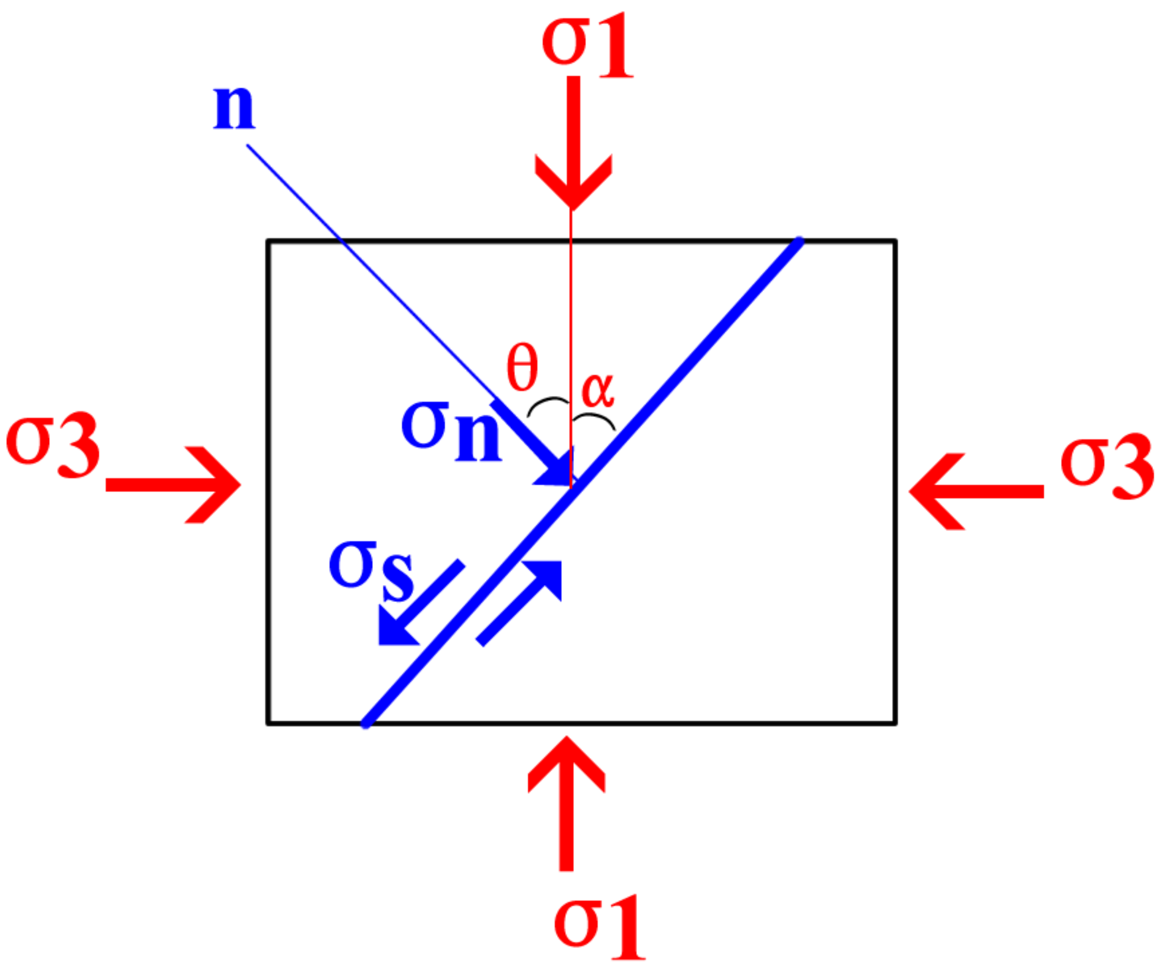
$$\sigma_S = \tfrac{1}{2}(\sigma_1 - \sigma_3)\sin 2\theta \qquad (1.7)$$

respectively. We can also consider for three dimensional case (Jaeger, 1969). Note that (1.7) reduces to (1.6) when σ_2 is zero.

Comparison of equations 1.3 and 1.6 shows that the stresses may not be calculated as though they were forces. Stress is an example of another type of quantity knows as a second –order tensor.

Univ. Zambia 2012

Normal and Shear Stress



Computing the stress Tensor

Examples

Computing the Stress Tensor

Stresses, normal stresses, and shear stresses can be calculated for planes of all orientations that pass through point *P*.

For example, the magnitude of the stress to a plane that makes an angle *(θ)* of 25° with the z-axis is +24.6 MPa (Figure *3.15A*).

The stress is inclined +55° to the plane and can be resolved into normal stress (σ_N) and shear stress (σ_s) components of +23.6 and +7.6 MPa, respectively (Figure *3.15B*).

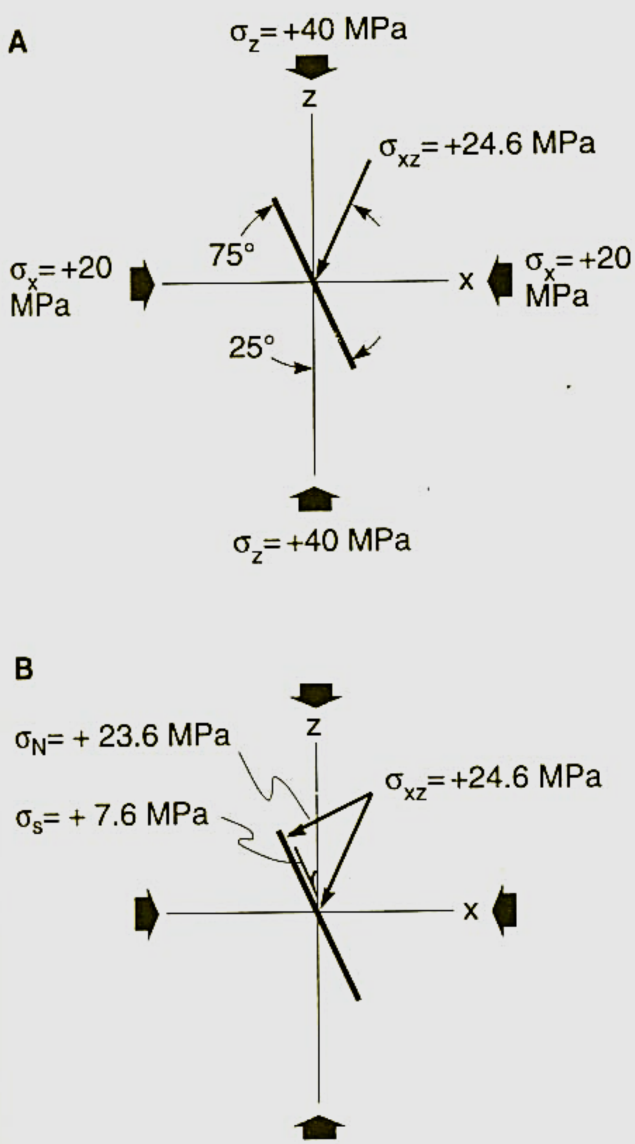


Figure 3.15 Yet another example of (*A*) stress on a plane and (*B*) resolution of the stress into normal and shear stress components.

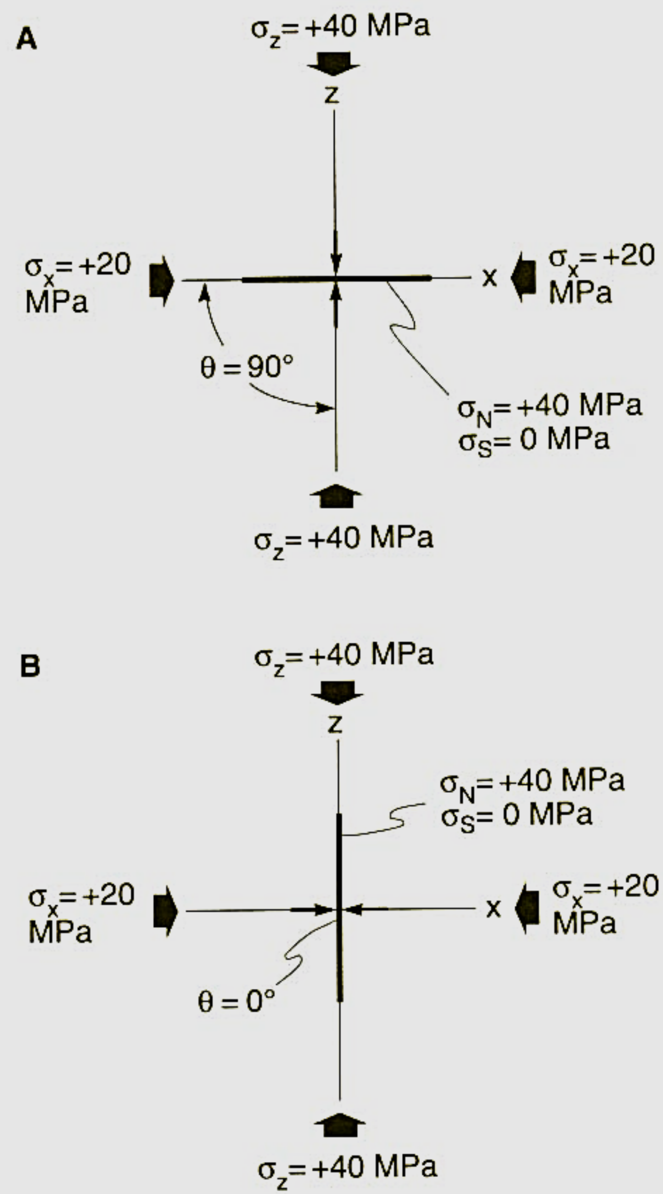


Figure 3.16 The special case of stresses parallel and perpendicular to planes. Purely normal stresses result. Shear stresses are zero.

Stresses to planes inclined at θ = 0° and θ= 90° are special cases.

The plane marked by an orientation of θ = 90° is horizontal, parallel to *x* (Figure *3.16A*)*.* The stress calculated for this plane is perpendicular to it and has the same magnitude as σ_z namely +40 MPa. This stress is purely a normal stress and has no shear stress component.

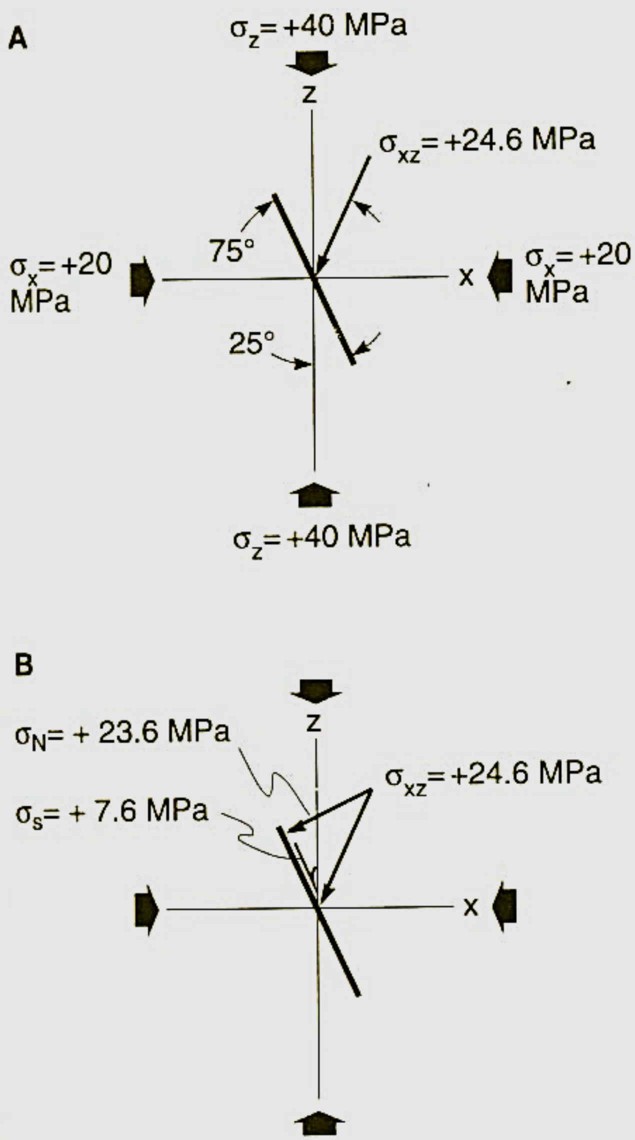


Figure 3.15 Yet another example of (*A*) stress on a plane and (*B*) resolution of the stress into normal and shear stress components.

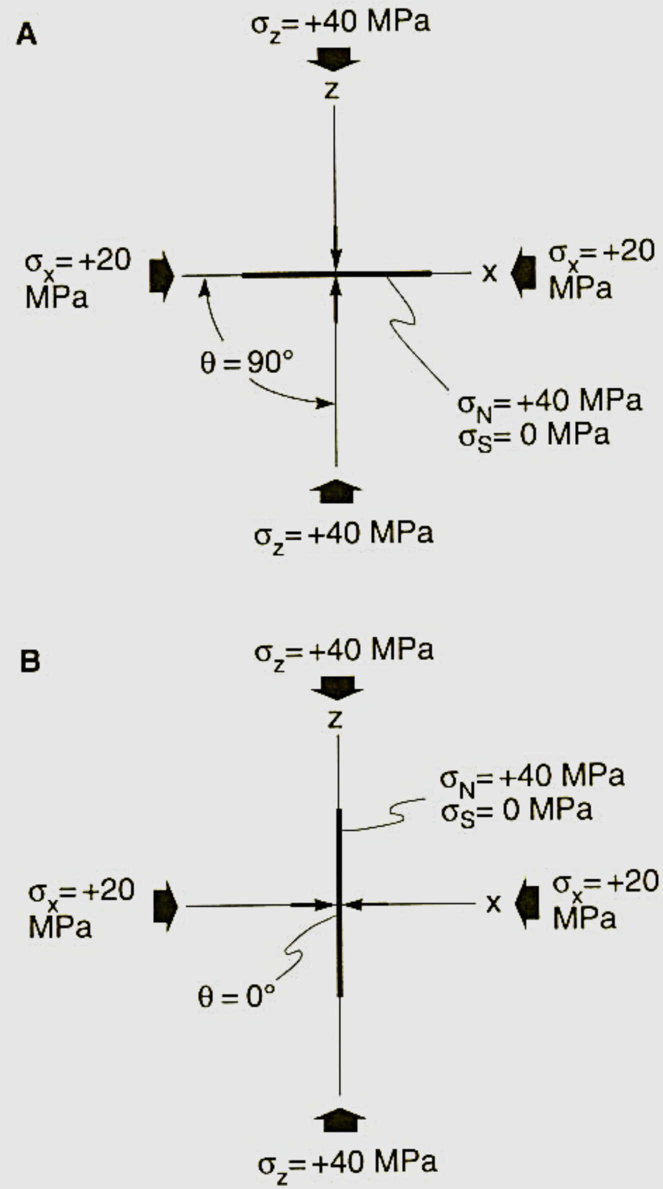


Figure 3.16 The special case of stresses parallel and perpendicular to planes. Purely normal stresses result. Shear stresses are zero.

Similarly, the stress to a plane whose orientation is defined by $\theta = 0°$ (in this example, a vertical plane) is a normal stress of 20 MPa (Figure *3.16B*).

Using a calculator, or better yet a computer spreadsheet that summarizes the mathematics of these calculations, we can determine the orientations and absolute values of stresses, normal stresses, and shear stresses for a host of planes inclined at 5° intervals through point O.

We have done this for stress conditions of σ_z = 40 MPa and σ_x = 20 MPa (Figure 3.17).

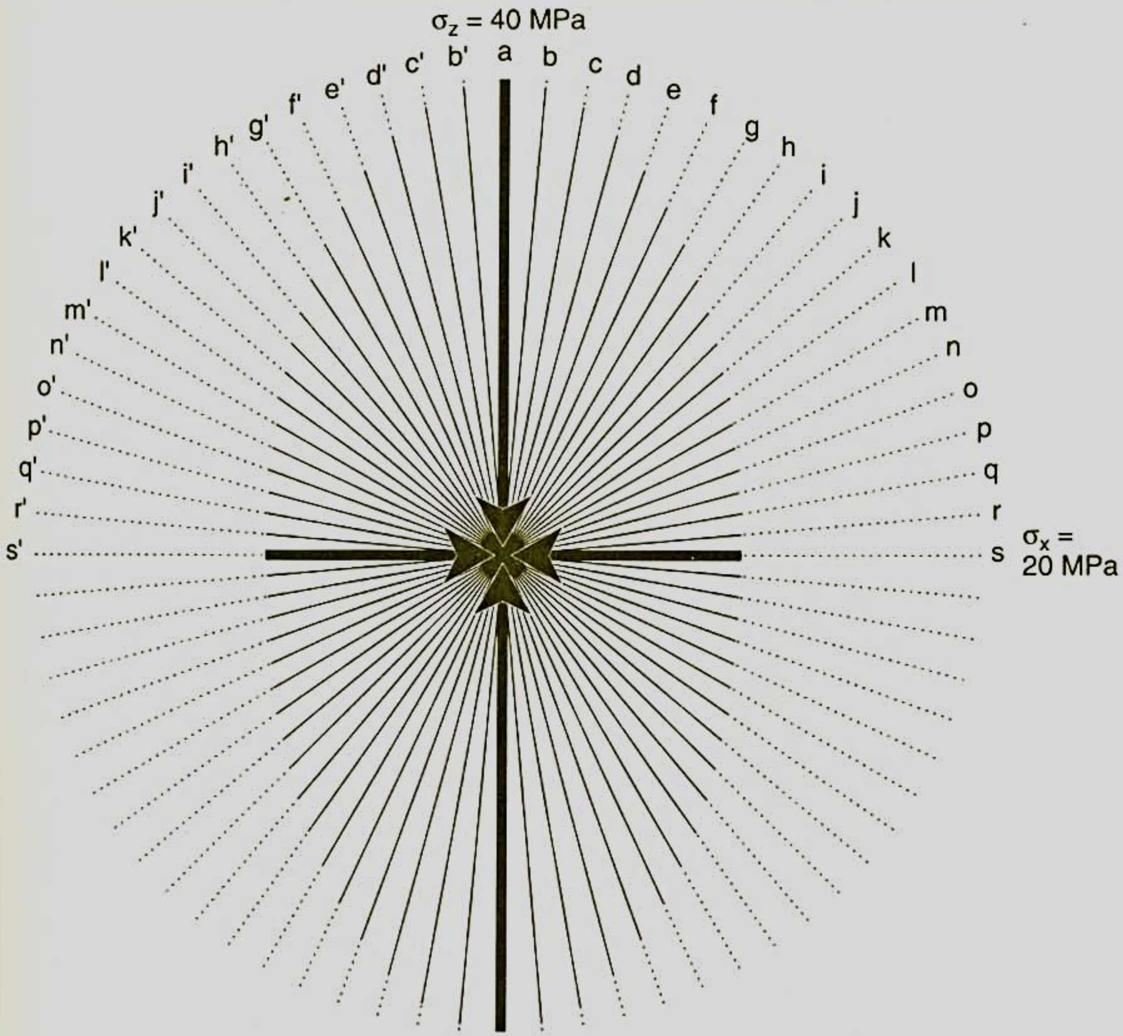


Figure 3.17 Pictured here is a vertical stress (σ_z) of magnitude 40 MPa and a horizontal stress (σ_x) of magnitude 20 MPa. The lines labeled *a–s* and *b′* to *r′* are traces of planes at 5° intervals. Stress (σ), normal stress (σ_N), and shear stress (σ_S) have been calculated for each of the planes to show the systematic variations.

Plane	∠ to Z	σ (MPa) 20	σ_N (MPa) 20	σ_S (MPa) 20
a	0°	20.00	20.00	0.00
b	−5°	20.23	20.15	−1.47
c	−10°	20.89	20.60	−3.42
d	−15°	21.92	21.34	−5.00
e	−20°	23.25	22.34	−6.43
f	−25°	24.79	23.57	−7.66
g	−30°	26.46	25.00	−8.66
h	−35°	28.19	26.58	−9.40
i	−40°	29.93	28.26	−9.85
j	−45°	31.62	30.00	−10.00
k	−50°	33.23	31.74	−9.85
l	−55°	34.72	33.42	−9.40
m	−60°	36.06	35.00	−8.66
n	−65°	37.22	36.43	−7.66
o	−70°	38.21	37.66	−6.43
p	−75°	38.98	38.66	−5.00
q	−80°	39.55	39.40	−3.42
r	−85°	39.89	39.85	−1.74
s	−90°	40.00	40.00	0.00
a'	0°	20.00	20.00	0.00
b'	5°	20.23	20.15	1.47
c'	10°	20.89	20.60	3.42
d'	15°	21.92	21.34	5.00
e'	20°	23.25	22.34	6.43
f'	25°	24.79	23.57	7.66
g'	30°	26.46	25.00	8.66
h'	35°	28.19	26.58	9.40
i'	40°	29.93	28.26	9.85
j'	45°	31.62	30.00	10.00
k'	50°	33.23	31.74	9.85
l'	55°	34.72	33.42	9.40
m'	60°	36.06	35.00	8.66
n'	65°	37.22	36.43	7.66
o'	70°	38.21	37.66	6.43
p'	75°	38.98	38.66	5.00
q'	80°	39.55	39.40	3.42
r'	85°	39.89	39.85	1.74
s'	90°	40.00	40.00	0.00

It is clear that normal stress and shear stress magnitudes vary systematically as a function of orientation: shear stress is zero for planes oriented parallel to the *x*- and z-axes (θ = 0° and 90°, respectively).

Shear stress (σ_s) steadily increases from 0.0 MPa to a maximum of 10 MPa as θ increases from 0° to 45°, decreasing again to zero from θ = 45° and θ = 90°.

Stress Ellipse
and
Stress Ellipsoids

The Stress Ellipse and the Stress Ellipsoid

The stress data posted in Figure 3.17 are incredibly systematic. This can be appreciated by plotting all the stresses such that their tails meet at a common point, the tiny point O containing the planes for which the stresses were computed.

When the stresses are plotted to scale in this fashion, a stress tensor of elliptical form is generated, called the stress ellipse (Fig 3.18).

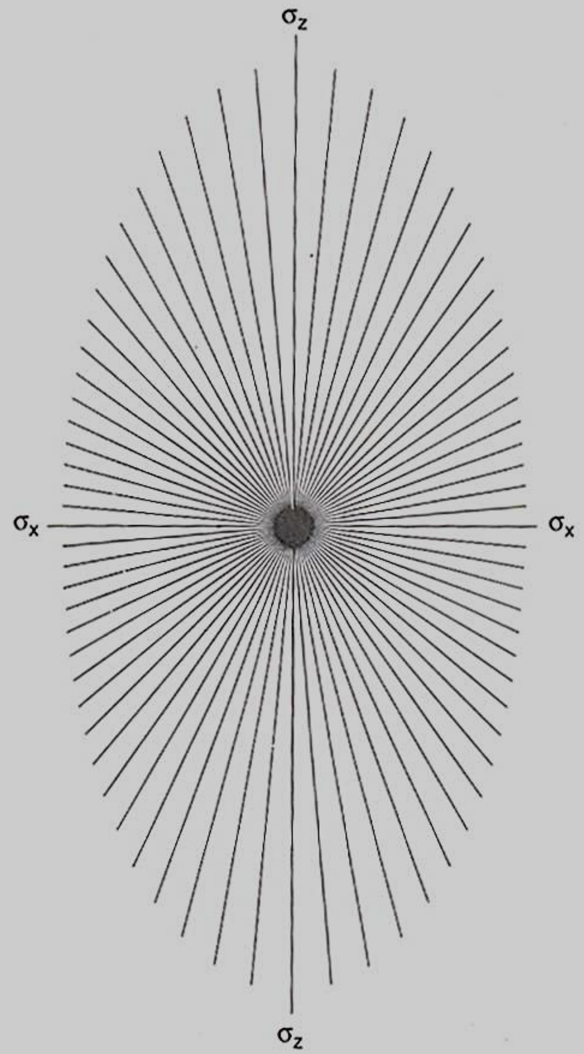


Figure 3.18 Stress ellipse created by arranging all of the stresses (σ) calculated in the preceding figure and arranging them in such a way that their tips meet at a point. Schematic.

The ellipse graphically portrays the stress tensor at a single point in a body. The stress tensor is not a single vector. To describe the stress tensor requires describing the orientation, size, and shape of the stress ellipse.

We accomplish this by determining the orientations and lengths of the principal axes of the stress ellipse.

The axes of the stress ellipse are called the principal stress directions (Figure 3.19). They are mutually perpendicular.

- **The long axis of the ellipse is the axis of greatest principal stress (σ_1);**

- **the short axis is the axis of least principal stress *(σ_3).***

These axes define the stress ellipse.

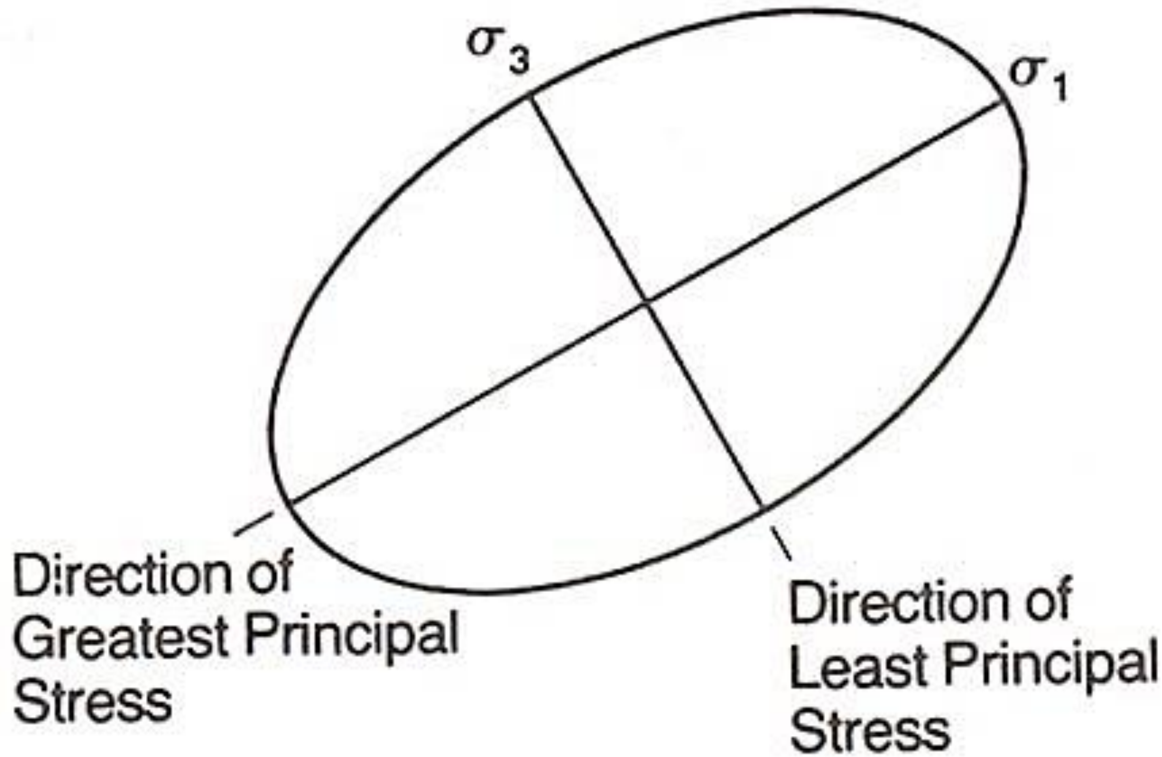


Figure 3.19 The stress ellipse, an image that shows the directions and ratios of the greatest and least principal stresses. It is a two-dimensional portrayal of a stress tensor.

Of all the values of normal stress (σ_N) computed for stresses operating about some point, the maximum normal stress component is parallel to the greatest principal stress direction (σ_1).

Indeed this stress is strictly a normal stress. It has no shear stress component. Similarly, the direction of least principal stress (σ_3) is marked by a finite normal stress, by a zero shear stress.

The stress operating parallel to σ_3 is the smallest of all the normal stress values. The form (i.e., aspect ratio) of the stress ellipse, of course, changes according to the absolute and relative values of the principal stresses.

The ratio of the greatest to least principal stress values will reflect the local stress state.

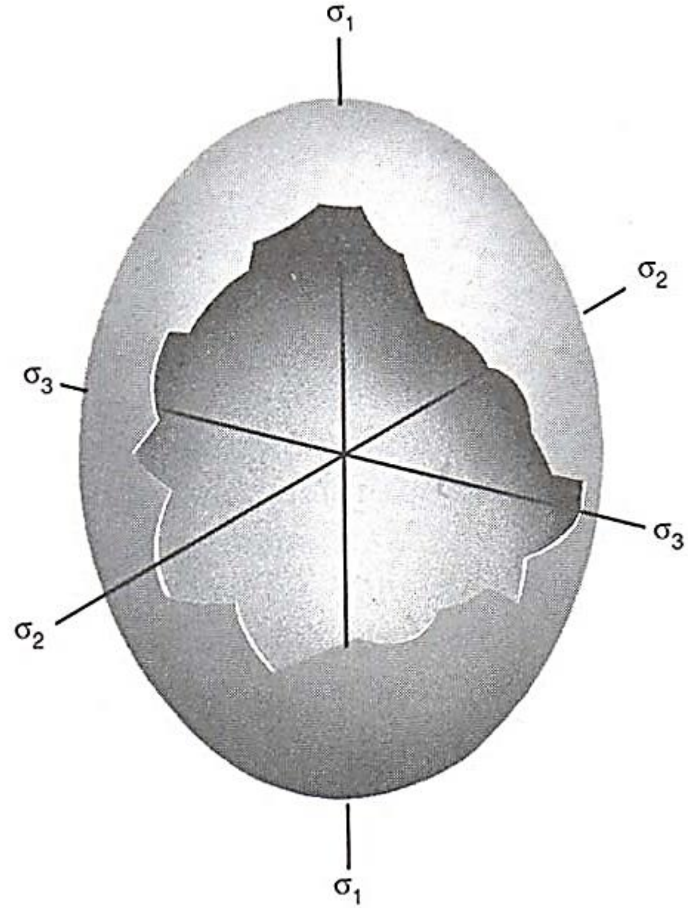


Figure 3.20 The stress ellipsoid, an image that shows the directions and ratios of the greatest, intermediate, and least principal stresses. It is the full three-dimensional portrayal of a stress tensor.

End of Lecture-2

Structural geology Assignment questions

1. Define Force and stress. Calculate the stress condition one km underground under the hydrostatic condition in a granitic terrain.
2. Write the stress equation and show how to construct a Mohr's circle. Construct Mohr's circle for stress on lithostatic condition. Determine the normal stress (σ_N) and shear stress (σ_s) values for a plane that makes an angle (θ) = 30° with (σ_N) (σ_1 = 40 MPa and σ_3 = 20 MPa).
3. Define Force and Stress. Calculate the stress condition at a depth of 1500 m with vertical force of 40 MPa and horizontal stress of 20 MPA.
4. Define stress. What are the normal and shear stress? Calculate σ_N and σ_s stress conditions on a plane inclined at an angle θ to F.