3.4 Tests about variances

The test statistics and critical regions concerning variances can be summarised as follows:

H_0	Test statistic	H_1	Critical region
$\sigma^2 = \sigma_0^2$	$\chi^2 = \frac{(n-1)S^2}{\sigma_0^2}$	$\sigma^2 > \sigma_0^2$	$\chi_{obs}^2 > \chi_{\alpha,n-1}^2$
		$\sigma^2 < \sigma_0^2$	$\chi_{obs}^2 < \chi_{1-\alpha,n-1}^2$
		$\sigma^2 \neq \sigma_0^2$	$\chi_{obs}^2 < \chi_{1-\frac{\alpha}{2},n-1}^2$ or $\chi_{obs}^2 > \chi_{\frac{\alpha}{2},n-1}^2$
$\sigma_1^2 = \sigma_2^2$	$F = \frac{S_1^2}{S_2^2}$	$\sigma_1^2 > \sigma_2^2$	$f^* > f_\alpha(n_1-1, n_2-1)$
		$\sigma_1^2 < \sigma_2^2$	$f^* < f_{1-\alpha}(n_1-1, n_2-1)$
		$\sigma_1^2 \neq \sigma_2^2$	$f^* < f_{1-\frac{\alpha}{2}}(n_1-1, n_2-1)$ or $f^* > f_{\frac{\alpha}{2}}(n_1-1, n_2-1)$

Examples

1. The caffeine content of a certain brand of tea is known to be normally distributed with a variance of 1.3 mg. Test the hypothesis that $\sigma^2 = 1.3$ against the alternative that $\sigma^2 \neq 1.3$ if a random sample of 8 packets of tea has a standard deviation of 1.8. Use a 0.05 level of significance.

   Soln

$$n = 8 \quad s = 1.8 \quad \alpha = 0.05$$

   1. Testing problem
      $H_0: \sigma^2 = 1.3$
      $H_1: \sigma^2 \neq 1.3$

   2. Test statistic
      $$\chi^2 = \frac{(n-1)S^2}{\sigma_0^2}$$

   3. Critical region
      reject H_0 if
      $$\chi_{obs}^2 < \chi_{1-\frac{\alpha}{2},n-1}^2 = \chi_{0.975,7}^2 = 1.69 \ or \ \chi_{obs}^2 > \chi_{\frac{\alpha}{2},n-1}^2 = \chi_{0.025,7}^2 = 16.013$$

   4. Calculation (Observed value)
      $$\chi_{obs}^2 = \frac{(n-1)s^2}{\sigma_0^2} = \frac{7(1.8^2)}{1.3} = 17.446$$
      i.e. $\chi_{obs}^2 > 16.013$

   5. Conclusion
      Reject H_0 at the 0.05 level of significance and conclude that $\sigma^2 \neq 1.3$.

2. In the abrasive wear example, we assumed that the two unknown population variances were equal. Were we justified in making the assumption? Use a 0.10 level of significance.

$Material\ 1$: $n_1 = 12$ $\quad \bar{x}_1 = 85$ $\quad s_1 = 4$
$Material\ 2$: $n_2 = 10$ $\quad \bar{x}_2 = 81$ $\quad s_2 = 5$

Soln

$\alpha = 0.1$

1. Testing problem
   $H_0: \sigma_1^2 = \sigma_2^2$
   $H_1: \sigma_1^2 \neq \sigma_2^2$

2. Test statistic
   $F = \frac{S_1^2}{S_2^2}$

3. Critical region
   reject H_0 if

$$f^* < f_{1-\frac{\alpha}{2}}(n_1 - 1, n_2 - 1) \quad or\ f^* > f_{\frac{\alpha}{2}}(n_1 - 1, n_2 - 1)$$

$$f^* < f_{0.95}(11,9) \quad or\ f^* > f_{0.05}(11,9)$$

$$f^* < \frac{1}{f_{0.05}(9,11)} \quad or\ f^* > \frac{3.14 + 3.07}{2}$$

$$f^* < \frac{1}{2.90} \quad or\ f^* > 3.105$$

$$f^* < 0.345$$

Note that we do not have 11 degrees of freedom on our F table, so we use the average of 10 and 12 numerator degrees of freedom.

4. Calculation (Observed value)
   $f^* = \frac{s_1^2}{s_2^2} = \frac{4^2}{5^2} = 0.64$
   i.e. 0 $0.345 < f^* < 3.105$

5. Conclusion
   Do not reject H_0 at the 0.1 level of significance. The variances are not significantly different. Therefore we were justified in assuming that the variances were equal.

3.5 Tests for goodness-of-fit and independence

These tests are based on the chi-square distribution.

3.5.1 Goodness-of-fit test

This test is based on how good a fit we have between the frequency of occurrence of observations in an observed sample and the expected frequencies obtained from a hypothesized distribution.

Theorem

A goodness of fit test between observed and expected frequencies is based on

$$\chi^2 = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i}$$

where χ^2 has an approximate chi-square distribution with $k-1$ degrees of freedom with O_i and E_i being observed and expected frequencies for the i^{th} cell.

Note:

If the difference between the observed and expected frequencies is small then χ^2_{obs} will be small. If the difference is large then χ^2_{obs} is large and the hypothesis of the hypothesized distribution will be rejected.

Example

A die is thrown 120 times and the number of times each outcome appears is recorded. The following results are obtained:

Outcome	1	2	3	4	5	6
Observed	20	22	17	18	19	24

Is the die balanced? Use $\alpha = 0.05$.

Soln

$$\alpha = 0.05$$

1. Testing problem
   $H_0: the\ die\ is\ balanced$
   $H_1: the\ die\ is\ not\ balanced$

2. Test statistic
   $$\chi^2 = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i}$$

3. Critical region
   reject H_0 if
   $$\chi^2_{obs} > \chi^2_{\alpha,k-1} = \chi^2_{0.05,5} = 11.071$$

4. Calculation

Outcome	1	2	3	4	5	6
Observed	20	22	17	18	19	24
Expected	20	20	20	20	20	20

Note that if the die is balanced then we expect each outcome to appear the same number of times i.e. $\frac{120}{6} = 20$

$$\begin{aligned}\chi^2_{obs} &= \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i} = \frac{(20-20)^2}{20} + \frac{(22-20)^2}{20} + \frac{(17-20)^2}{20} \\ &+ \frac{(18-20)^2}{20} + \frac{(19-20)^2}{20} + \frac{(24-20)^2}{20} \\ &= 1.7\end{aligned}$$

i.e. $\chi^2_{obs} < 11.071$

5. Conclusion
Fail to reject H_0 at the 0.05 level of significance. Therefore the die is balanced.

3.5.2 Test for independence

Suppose a random sample is cross classified based on two variables and the observed frequencies recorded as follows:

		Variable 2				
		1	2		c	Total
Variable 1	1	O_{11}	O_{12}	$\cdots$	O_{1c}	$O_{1\bullet}$
	2	O_{21}	O_{22}	$\cdots$	O_{2c}	$O_{2\bullet}$
	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
	r	O_{r1}	O_{r2}	$\cdots$	O_{rc}	$O_{r\bullet}$
Total		$O_{\bullet 1}$	$O_{\bullet 2}$	$\cdots$	$O_{\bullet c}$	$O_{\bullet\bullet}$

Grand total

We can test whether variable 1 and variable 2 are independent by comparing the observed and expected frequencies and use the χ^2 statistic

$$\chi^2 = \sum_{i=1}^{r}\sum_{j=1}^{c} \frac{\left(O_{ij} - E_{ij}\right)^2}{E_{ij}} \sim \chi^2_{(r-1)(c-1)}$$

where

$$\begin{aligned}E_{ij} &= \frac{O_{i\bullet} \times O_{\bullet j}}{O_{\bullet\bullet}} \\ &= \frac{(row\ total) \times (column\ total)}{grand\ total}\end{aligned}$$

We reject the null hypothesis of independence if χ^2_{obs} is large $\left(\chi^2_{obs} > \chi^2_{\alpha,(r-1)(c-1)}\right)$.
χ^2 is an approximate χ^2 distribution with $\nu = (r-1)(c-1)$ degrees of freedom.

Example

In an experiment to study the dependence of hypertension on smoking habits, the following data were collected on 180 individuals

	Nonsmokers	Moderate smokers	Heavy smokers
Hypertension	21	36	30
No hypertension	48	26	19

Test the hypothesis that presence or absence of hypertension is independent of smoking habits. Use a 0.05 level of significance.

Soln

1. Testing problem

   $H_0: smoking\ and\ hypertension\ are\ independent$

   $H_1: smoking\ and\ hypertension\ are\ not\ independent$

2. Test statistic

$$\chi^2 = \sum_{i=1}^{r}\sum_{j=1}^{c}\frac{\left(O_{ij}-E_{ij}\right)^2}{E_{ij}}$$

3. Critical region

   reject H_0 if

$$\chi^2_{obs} > \chi^2_{\alpha,(r-1)(c-1)} = \chi^2_{0.05,(2-1)(3-1)} = \chi^2_{0.05,2} = 5.991$$

4. Calculation

$$\chi^2_{obs} = \sum_{i=1}^{2}\sum_{j=1}^{3}\frac{\left(O_{ij}-E_{ij}\right)^2}{E_{ij}}$$

		1	2	3	
		N	MS	HS	Total
1	H	21 $E_{11} = 33.35$	36 $E_{12} = 29.97$	30 23.68	87
2	N H	48 35.65	26 32.03	19 25.32	93
	Total				

$$E_{11} = \frac{O_{1\bullet} \times O_{\bullet 1}}{O_{\bullet\bullet}} = \frac{87 \times 69}{180} = 33.35$$

$$E_{12} = \frac{O_{1\bullet} \times O_{\bullet 2}}{O_{\bullet\bullet}} = \frac{87 \times 62}{180} = 29.97$$

$\vdots$

And so on you, can complete the table….

$$\chi^2_{obs} = \frac{(21-33.35)^2}{33.35} + \frac{(36-29.97)^2}{29.97} + \frac{(30-23.68)^2}{23.68} + \frac{(48-35.65)^2}{35.65} + \frac{(26-32.03)^2}{32.03} + \frac{(19-25.32)^2}{25.32} = 14.46$$

i.e. $\chi^2_{obs} > 5.991$

5. Conclusion
   Reject H_0 at the 0.05 level of significance. Therefore smoking and hypertension are not independent.