3.2.3 Two population means

Test statistics and critical regions for two population means can be summarised as follows:

H_0	Test statistic	H_1	Critical region
$\mu_1 - \mu_2 = \mu_0$	$Z = \dfrac{\bar{X}_1 - \bar{X}_2 - \mu_0}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}}$ σ_1^2 and σ_2^2 known	$\mu_1 - \mu_2 < \mu_0$ $\mu_1 - \mu_2 > \mu_0$ $\mu_1 - \mu_2 \neq \mu_0$	$z^* < -z_\alpha$ $z^* > z_\alpha$ $\lvert z^* \rvert > z_{\frac{\alpha}{2}}$
$\mu_1 - \mu_2 = \mu_0$	$T = \dfrac{\bar{X}_1 - \bar{X}_2 - \mu_0}{\sqrt{S_p^2\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}}$ $\nu = n_1 + n_2 - 2$ $\sigma_1^2 = \sigma_2^2$ but unknown	$\mu_1 - \mu_2 < \mu_0$ $\mu_1 - \mu_2 > \mu_0$ $\mu_1 - \mu_2 \neq \mu_0$	$t^* < -t_{\alpha,\nu}$ $t^* > t_{\alpha,\nu}$ $\lvert t^* \rvert > t_{\frac{\alpha}{2},\nu}$
$\mu_1 - \mu_2 = \mu_0$	$T = \dfrac{\bar{X}_1 - \bar{X}_2 - \mu_0}{\sqrt{\dfrac{S_1^2}{n_1} + \dfrac{S_2^2}{n_2}}}$ $\nu = \dfrac{\left(\dfrac{S_1^2}{n_1} + \dfrac{S_2^2}{n_2}\right)^2}{\dfrac{\left(\dfrac{S_1^2}{n_1}\right)^2}{n_1 - 1} + \dfrac{\left(\dfrac{S_2^2}{n_2}\right)^2}{n_2 - 1}}$ $\sigma_1^2 \neq \sigma_2^2$ and unknown	$\mu_1 - \mu_2 < \mu_0$ $\mu_1 - \mu_2 > \mu_0$ $\mu_1 - \mu_2 \neq \mu_0$	$t^* < -t_{\alpha,\nu}$ $t^* > t_{\alpha,\nu}$ $\lvert t^* \rvert > t_{\frac{\alpha}{2},\nu}$
$\mu_D = \mu_0$	$T = \dfrac{\bar{D} - \mu_0}{\dfrac{S_d}{\sqrt{n}}}$ $\nu = n - 1$ paired observations	$\mu_D < \mu_0$ $\mu_D > \mu_0$ $\mu_D \neq \mu_0$	$t^* < -t_{\alpha,\nu}$ $t^* > t_{\alpha,\nu}$ $\lvert t^* \rvert > t_{\frac{\alpha}{2},\nu}$

Examples

1. An experiment was performed to compare the abrasive wear of two different laminated materials. Twelve pieces of material 1 gave an average (coded) wear of 85 units with a standard deviation of 4, while 10 pieces of material 2 gave an average of 81 and a standard deviation of 5. Can we conclude at the 0.05 level of significance that the abrasive wear of material 1 exceeds that of material 2 by more than 2 units? Assume the populations to be normally distributed with equal variances.

Soln

$$n_1 = 12 \qquad \bar{x}_1 = 85 \qquad s_1 = 4$$
$$n_2 = 10 \qquad \bar{x}_2 = 81 \qquad s_2 = 5$$
$$\alpha = 0.05$$

1. Testing problem
   $H_0: \mu_1 - \mu_2 = 2$
   $H_1: \mu_1 - \mu_2 > 2$

2. Test statistic
   $$T = \frac{\bar{X}_1 - \bar{X}_2 - \mu_0}{\sqrt{S_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

3. Critical region
   reject H_0 if
   $$t^* > t_{\alpha, n_1+n_2-2} = t_{0.05,20} = 1.725$$

4. Observed value
   $$t^* = \frac{\bar{x}_1 - \bar{x}_2 - \mu_0}{\sqrt{s_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

   $$\begin{aligned} s_p^2 &= \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2} \\ &= \frac{11(4)^2 + 9(5)^2}{12 + 10 - 2} \\ &= 20.05 \end{aligned}$$

   $$t^* = \frac{\bar{x}_1 - \bar{x}_2 - \mu_0}{\sqrt{s_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{85 - 81 - 2}{\sqrt{20.05\left(\frac{1}{12} + \frac{1}{10}\right)}} = 1.043$$

   i.e. $t^* < 1.725$

5. Conclusion
   We fail to reject H_0 at the 0.05 level of significance and conclude that there's insufficient evidence to conclude that the average wear for material 1 exceeds that of material 2 by more than 2 units.

2. Business schools A and B reported the following summary GMAT (Graduate Management Aptitude Test) verbal scores

	n	$\bar{x}$	s^2
A	201	34.75	48.59
B	115	33.74	30.68

At a 5% level of significance, is there sufficient evidence to believe there is a difference in the GMAT scores of the two schools?

Soln

$$n_A = 201 \quad \bar{x}_A = 34.75 \quad s_A^2 = 48.59$$
$$n_B = 115 \quad \bar{x}_B = 33.74 \quad s_A^2 = 30.68$$
$$\alpha = 0.05$$

1. Testing problem
   $H_0: \mu_A - \mu_B = 0$
   $H_1: \mu_A - \mu_B \neq 0$

2. Test statistic
   $$Z = \frac{\bar{X}_A - \bar{X}_B - \mu_0}{\sqrt{\frac{S_A^2}{n_A} + \frac{S_B^2}{n_B}}}$$

3. Critical region
   reject H_0 if
   $$|z^*| > z_{\frac{\alpha}{2}} = z_{0.025} = 1.96$$

4. Observed value
   $$z^* = \frac{\bar{x}_A - \bar{x}_B - \mu_0}{\sqrt{\frac{s_A^2}{n_A} + \frac{s_B^2}{n_B}}} = \frac{34.75 - 33.74 - 0}{\sqrt{\frac{48.59}{201} + \frac{30.68}{115}}} = 1.42$$
   i.e. $|z^*| < 1.96$

5. Conclusion
   Do not reject H_0 at the 0.05 level of significance. Therefore the GMAT scores in the two schools are not significantly different.

3.3 Tests about proportions

Tests about population proportions are based on the sampling distribution of $\hat{P}$ or $\hat{P}_1 - \hat{P}_2$. The test statistics and critical regions are given below:

H_0	Test statistic	H_1	Critical region
$p = p_0$	$Z = \dfrac{\hat{P} - p_0}{\sqrt{\dfrac{p_0(1-p_0)}{n}}}$	$p < p_0$ $p > p_0$ $p \neq p_0$	$z^* < -z_\alpha$ $z^* > z_\alpha$ $\lvert z^*\rvert > z_{\frac{\alpha}{2}}$
$p_1 - p_2 = 0$	$Z = \dfrac{\hat{P}_1 - \hat{P}_2}{\sqrt{\hat{P}(1-\hat{P})\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}}$ where $\hat{P} = \dfrac{X_1 + X_2}{n_1 + n_2}$	$p_1 - p_2 < 0$ $p_1 - p_2 > 0$ $p_1 - p_2 \neq 0$	$z^* < -z_\alpha$ $z^* > z_\alpha$ $\lvert z^*\rvert > z_{\frac{\alpha}{2}}$
$p_1 - p_2 = p_0$	$Z = \dfrac{\hat{P}_1 - \hat{P}_2 - p_0}{\sqrt{\dfrac{\hat{P}_1(1-\hat{P}_1)}{n_1} + \dfrac{\hat{P}_2(1-\hat{P}_2)}{n_2}}}$	$p_1 - p_2 < p_0$ $p_1 - p_2 > p_0$ $p_1 - p_2 \neq p_0$	$z^* < -z_\alpha$ $z^* > z_\alpha$ $\lvert z^*\rvert > z_{\frac{\alpha}{2}}$

Note

1. There is a difference in the standard errors used in confidence intervals and hypothesis testing e.g. for a single proportion we use the standard error $\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$ for the confidence interval and $\sqrt{\frac{p_0(1-p_0)}{n}}$ for testing hypothesis.
2. In hypothesis testing the test statistic assumes the null hypothesis H_0 is true, that is why we use p_0 instead of $\hat{p}$ in the standard error for a single proportion. Similarly, to test $H_0: p_1 - p_2 = 0$ we assume the null hypthesis is true and estimate a common estimate of p $\left(\hat{p} = \frac{x_1+x_2}{n_1+n_2}\right)$. If $H_0: p_1 - p_2 = p_0$ then under H_0 we have different proportions and hence the standard error uses different estimates of the two proportions.

Examples

1. A poll is taken to compare the proportion of town and country voters favouring a proposal of constructing a chemical plant. Is the proportion of town voters favouring the proposal higher than the proportion of country voters, if 120 of 200 town voters and 240 of 500 country residents favour the proposal? Use a 0.025 level of significance.

Soln

Let $p_1 = proportion\ of\ town\ voters\ favouring\ the\ proposal$
$p_2 = proportion\ of\ country\ voters\ favouring\ the\ proposal$

$$n_1 = 200 \qquad x_1 = 120 \qquad \hat{p}_1 = \frac{x_1}{n_1} = \frac{120}{200} = 0.6$$
$$n_2 = 500 \qquad x_2 = 240 \qquad \hat{p}_2 = \frac{x_2}{n_2} = \frac{240}{500} = 0.48$$
$$\alpha = 0.025$$

1. Testing problem
   $H_0: p_1 = p_2\ (p_1 - p_2 = 0)$
   $H_1: p_1 > p_2\ (p_1 - p_2 > 0)$

2. Test statistic
   $$Z = \frac{\hat{P}_1 - \hat{P}_2}{\sqrt{\hat{P}(1-\hat{P})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

3. Critical region
   reject H_0 if
   $$z^* > z_\alpha = z_{0.025} = 1.96$$

4. Observed value
   $$z^* = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

   $$\hat{p} = \frac{x_1 + x_2}{n_1 + n_2} = \frac{120 + 240}{200 + 500} = \frac{360}{700} = 0.51$$

   $$z^* = \frac{0.6 - 0.48}{\sqrt{0.51(1-0.51)\left(\frac{1}{200} + \frac{1}{500}\right)}} = 2.87$$

   i.e. $z^* > 1.96$

5. Conclusion
   Reject H_0 at the 0.025 level of significance. Therefore the proportion of town voters favouring the proposal is higher.

2. A distributor of washing powder claims that 20% of women in Lusaka prefer Boom washing powder. To test this claim, 20 women are selected at random and asked what brand they prefer. If 6 out of 20 name Boom as their preference, what conclusion do we draw? Use a 0.05 level of significance.

   Soln

$$n = 20 \qquad x = 6 \qquad \hat{p} = \frac{x}{n} = \frac{6}{20} = 0.3$$
$$\alpha = 0.05$$

   1. Testing problem
      $H_0: p = 0.20$
      $H_1: p \neq 0.20$

   2. Test statistic
      $$Z = \frac{\hat{P} - p_0}{\sqrt{\frac{p_0(1 - p_0)}{n}}}$$

   3. Critical region
      reject H_0 if
      $$|z^*| > z_{\frac{\alpha}{2}} = z_{0.025} = 1.96$$

   4. Observed value
      $$z^* = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1 - p_0)}{n}}} = \frac{0.3 - 0.2}{\sqrt{\frac{0.2(1 - 0.2)}{20}}} = 1.12$$

      i.e. $|z^*| < 1.96$

   5. Conclusion
      Do not reject H_0 at the 0.05 level of significance. Therefore the claim is valid.