2018

MATHEMATICS 2100 ANALYTICAL GEOMETRY AND CALCULAS TUTORIAL SOLUTIONS FROM 1 TO 8

AUTHOR; FIZIE

EDITOR; MONDE LIKE MAKETO

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ABRACT

These solutions are compiled to help those doing engineering courses and actuarial science. To help people now and the next generations. The Main lecturer for 2018 mathematics 2100 second years was **Dr SABAO**. Solved and analyzed by **FIZIE.** Edited and compiled by **MONDE LIKE MAKETO**

EDITOR/SCANNED

MONDE LIKE MAKETO

CELL+260978887724

AUTHOR/SOLVED

FIZIE FULUMAKA

CELL+260976608676

33968964

THE UNIVERSITY OF ZAMBIA
SCHOOL OF NATURAL SCIENCES
DEPARTMENT OF MATHEMATICS & STATISTICS
MAT 2100–Analytic Geometry and Calculus
Tutorial Sheet 1

March, 2018

1. Find the vertex, focus, and directrix of the parabola. Then sketch the parabola.
   (a) $y^2 = -6x$
   (b) $4y = x^2 - 2x + 5$
   (c) $y^2 - 4y - 4x = 0$
   (d) $x^2 + 4x + 4y - 4 = 0$
   (e) $y^2 + 6y + 8x + 25 = 0$
   (f) $x^2 - 2x + 8y + 9 = 0$

2. Find an equation of the parabola
   (a) vertex : $(3, 2)$; focus : $(1, 2)$
   (b) vertex : $(-1, 2)$; focus : $(-1, 0)$
   (c) focus : $(2, 2)$; directrix : $x = -2$
   (d) focus : $(0, 0)$; directrix : $y = 4$

3. Find an equation of the parabola
   (a) Axis is parallel to y-axis; graph passes through $(0, 3)$, $(3, 4)$, and $(4, 11)$.
   (b) Axis is parallel to x-axis; graph passes through $(4, -2)$, $(0, 0)$, and $(3, -3)$.
   (c) Directrix: $y = 1$; length of latus rectum is 8; opens downward.
   (d) Directrix: $y = -2$; endpoints of latus rectum are $(0, 2)$ and $(8, 2)$.

4. Find the center, foci, vertices, asymptotes, and radius, as appropriate, of the following conic sections:
   (a) $x^2 + 4x + y^2 = 12$
   (b) $x^2 + 5y^2 + 4x = 1$
   (c) $x^2 + 2y^2 - 2x - 4y = -1$
   (d) $x^2 - y^2 - 2x + 4y = 4$
   (e) $y^2 - 4x^2 + 16x = 24$
   (f) $2x^2 - y^2 + 6y = 3$
   (g) $x^2 - y^2 + 4x - 6y = 6$
   (h) $2x^2 + 2y^2 - 28x + 12y = -114$
   (i) $y^2 - 4y - 8x - 12 = 0$

5. Find the standard equation of the following conic sections:
   (a) Foci : $(0, \pm 3)$; Eccentricity : 0.5
   (b) Vertices : $(0, \pm 70)$; Eccentricity : 0.1
   (c) Foci : $(\pm 8, 0)$; Eccentricity : 0.2
   (d) Vertices : $(\pm 10, 0)$; Eccentricity : 0.24
   (e) Eccentricity : 3; Vertices : $(0, \pm 1)$
   (f) Eccentricity : 2; Vertices : $(\pm 2, 0)$
   (g) Eccentricity : 3; Foci : $(\pm 3, 0)$
   (h) Eccentricity : 1.25; Foci : $(0, \pm 5)$

6. Use the discriminant to determine whether the equations of the graph represent parabolas, ellipses, or hyperbolas.

Tot

(a) $16x^2 - 24xy + 9y^2 - 30x - 40y = 0$
(b) $x^2 - 4xy - 2y^2 - 6 = 0$
(c) $13x^2 - 8xy + 7y^2 - 45 = 0$
(d) $2x^2 + 4xy + 5y^2 + 3x - 4y - 20 = 0$
(e) $x^2 - 6xy - 5y^2 + 4x - 22 = 0$
(f) $36x^2 - 60xy + 25y^2 + 9y = 0$
(g) $x^2 + 4xy + 4y^2 - 5x - y - 3 = 0$
(h) $x^2 + xy + 4y^2 + x + y - 4 = 0$

7. Rotate the axes to eliminate the xy-term. Sketch the graph of the resulting equation, showing both sets of axes.

(a) $xy = 2$
(b) $x^2 + xy + y^2 = 1$
(c) $3x^2 + 2\sqrt{3}xy + y^2 - 8x + 8\sqrt{3}y = 0$
(d) $x^2 - \sqrt{3}xy + 2y^2 = 1$
(e) $x^2 - 2xy + y^2 = 2$
(f) $3x^2 - 2\sqrt{3}xy + y^2 = 1$
(g) $\sqrt{2}x^2 + 2\sqrt{2}xy + \sqrt{2}y^2 - 8x + 8y = 0$
(h) $xy - y - x + 1 = 0$
(i) $3x^2 + 2xy + 3y^2 = 19$
(j) $3x^2 + 4\sqrt{3}xy - y^2 = 7$
(k) $x^2 + y^2 + 2xy + 2\sqrt{2}x - 2\sqrt{2}y + 2 = 0$
(l) $9x^2 + 6y^2 + 4xy - 20 = 0$
(m) $x^2 - 10xy + y^2 + 1 = 0$
(n) $5x^2 - 2xy + 5y^2 - 12 = 0$
(o) $9x^2 + 24xy + 16y^2 + 80x - 60y = 0$
(p) $xy - 2y - 4x = 0$

8. Graph the equation $\sqrt{x} + \sqrt{y} = 1$ and show that it is a portion of a parabola.

9. The eccentricities of conic sections with one focus at the origin, along with the directrix corresponding to that focus are given. Find a polar equation for each conic section.

(a) $e = 1, x = 2$
(b) $e = 5, y = -6$
(c) $e = \frac{1}{2}, x = 1$
(d) $e = \frac{1}{5}, y = -10$
(e) $e = 1, y = 2$
(f) $e = 2, x = 4$
(g) $e = \frac{1}{4}, x = -2$
(h) $e = \frac{1}{3}, y = 6$

10. Sketch and identify the graph.

(a) $r = \dfrac{-1}{1 - \sin\theta}$
(b) $r = \dfrac{6}{1 + \cos\theta}$
(c) $r = \dfrac{6}{2 + \cos\theta}$
(d) $r = \dfrac{3}{3 + 2\cos\theta}$
(e) $r = \dfrac{5}{-1 + 2\cos\theta}$
(f) $r(2 + \sin\theta) = 4$
(g) $r(3 - 2\cos\theta) = 6$
(h) $r = \dfrac{-3}{2 + 4\sin\theta}$
(i) $r = \dfrac{3}{2 + 6\sin\theta}$
(j) $r = \dfrac{4}{1 + 2\cos\theta}$

11. Write the polar form of the equation of the conic

(a) Ellipse: focus at $(4, 0)$; vertices at : $(5, 0), (5, \pi)$
(b) Hyperbola: focus at $(5, 0)$; vertices at : $(4, 0), (4, \pi)$
(c) $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$
(d) $\dfrac{x^2}{4} + y^2 = 1$

$3(2, \frac{\sqrt{2}}{4}$

ellipse $r^2 = \dfrac{b^2}{1 - e^2\cos^2\theta}$

$\frac{2}{3}$

$\frac{\sqrt{2}}{4} - x'$

$x' = \frac{\sqrt{2}}{4} + \sqrt{2}$

Hyp $r^2 = \dfrac{-b^2}{1 - e^2\cos^2\theta}$

$y' = \frac{-3\sqrt{2}}{4}$

$\sqrt{2} =$

MAT 2100 – ANALYTIC GEOMETRY AND CALCULUS

TUTORIAL SHEET 1

(1) (a) $y^2 = -6x$

$y^2 = -4px$ (standard form)

$y^2 = -4(\frac{3}{2})x$

Focus $(-\frac{3}{2}, 0)$

Vertex $(0,0)$

Directrix : $x = \frac{3}{2}$

(b) $4y = x^2 - 2x + 5$

$(x-1)^2 - 1 + 5 = 4y$

$(x-1)^2 = 4y - 4$

$(x-1)^2 = 4(y-1)$

Vertex ; $(1,1)$

focus ; $(1,2)$

directrix ; $y = 0$

(c) $y^2 - 4y - 4x = 0$

$(y-2)^2 - 4 - 4x = 0$

$(y-2)^2 = 4x + 4$

$(y-2)^2 = 4(x+1)$

vertex ; $(-1, 2)$

focus ; $(0, 2)$

directrix ; $x = -2$

(d) $x^2 + 4x + 4y - 4 = 0$

$(x+2)^2 - 4 + 4y - 4 = 0$

$(x+2)^2 = -4y + 8$

$(x+2)^2 = -4(y-2)$

vertex ; $(-2, 2)$

focus ; $(-2, 1)$

directrix ; $y = 3$

(e) $y^2 + 6y + 8x + 25 = 0$

$(y+3)^2 - 9 + 8x + 25 = 0$

$(y+3)^2 + 8x + 16 = 0$

$(y+3)^2 = -8x - 16$

$(y+3)^2 = -4(2)(x+2)$

vertex ; $(-2, -3)$

focus ; $(-4, -3)$

directrix ; $x = 0$

(f) $x^2 - 2x + 8y + 9 = 0$

$(x-1)^2 - 1 + 8y + 9 = 0$

$(x-1)^2 = -8y - 8$

$(x-1)^2 = -8(y+1)$

$(x-1)^2 = -4(2)(y+1)$

vertex ; $(1, -1)$

focus ; $(1, -3)$

directrix ; $y = 1$

(2) (a) vertex : $(3,2)$ focus : $(1,2)$

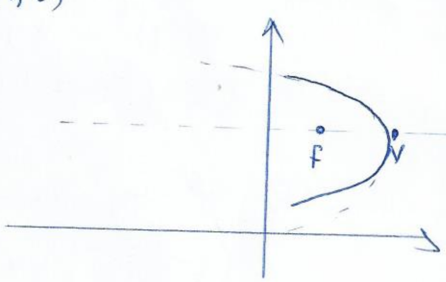
this opens to LHS :

So $(y-k)^2 = -4p(x-h)$

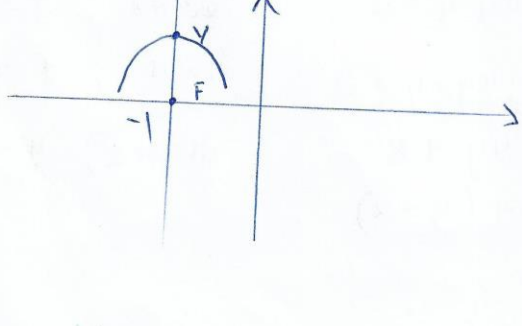
$(y-2)^2 = -4(2)(x-3)$

$y^2 - 4y + 4 = -8x + 24$

$y^2 - 4y + 8x - 20 = 0$



(b) vertex : $(-1, 2)$, focus $(-1, 0)$

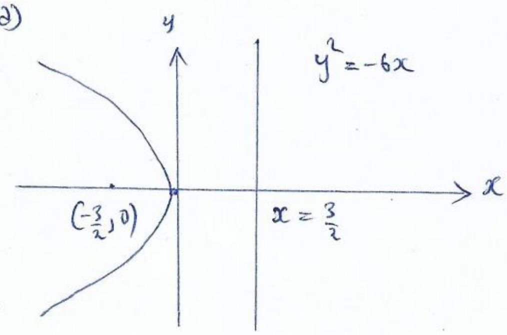


∴ The graph opens downwards

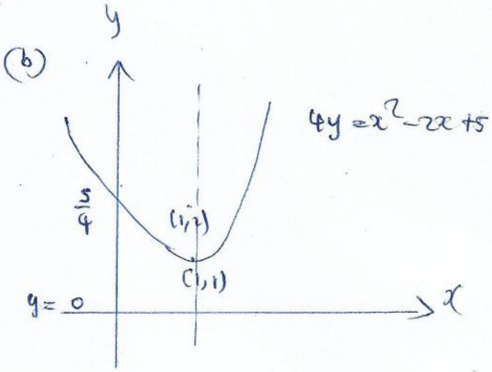
(2) T1

Sketches of parabolas for Q 1

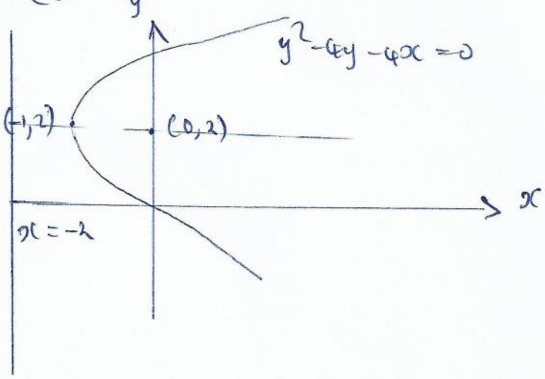
(a)



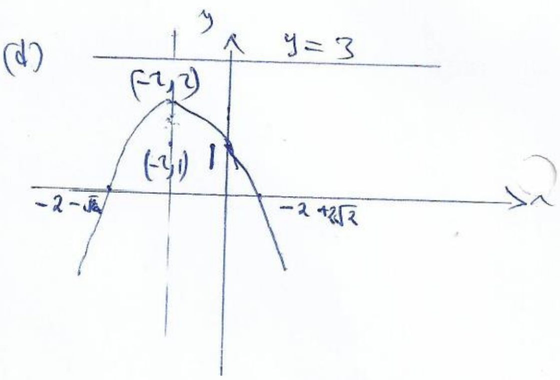
(b)



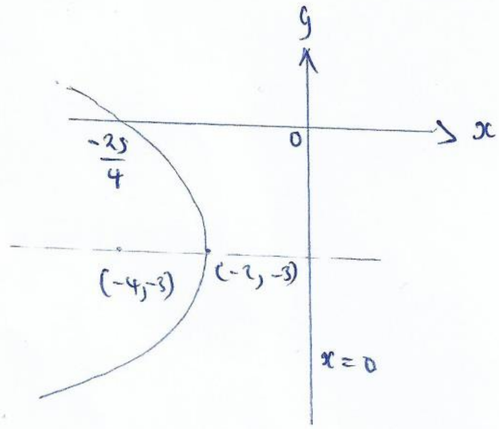
(c)



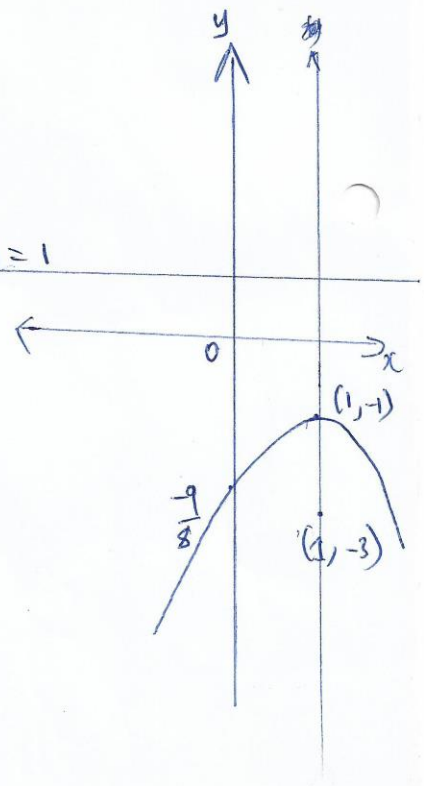
(d)



(e)



(f)

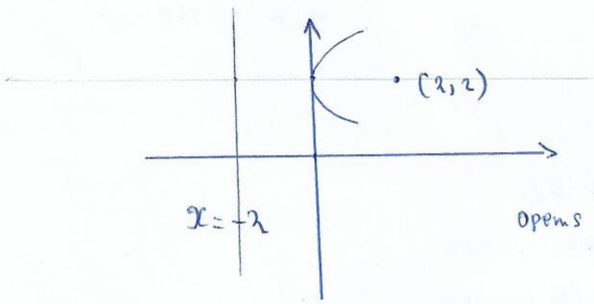


3

T1

$$(x-h)^2 = -4p(y-k) \qquad p = 2$$
$$(x+1)^2 = -4(2)(y-2) \qquad h = -1$$
$$x^2+2x+1 = -8y+16 \qquad k = 2$$
$$x^2+2x+8y-15 = 0$$

(c) focus : $(2,2)$; directrix : $x = -2$

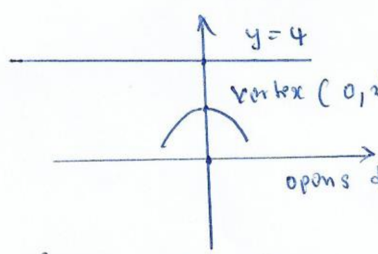


vertex ; $(0,2)$

opens to the RHS.

$$(y-k)^2 = 4p(x-h) \qquad p = 1$$
$$(y-2)^2 = 4x \qquad h = 0$$
$$y^2-4y+4-4x = 0 \qquad k = 2$$

(d) focus : $(0,0)$; directrix : $y = 4$

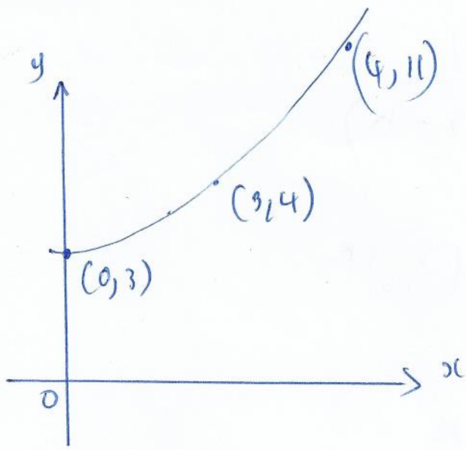


opens downwards

$$(x-h)^2 = -4p(y-k)$$
$$(x-0)^2 = -4(2)(y-2)$$
$$x^2 = -8y+8$$

(4) T

3(a)



This curve opens upwards

$(x-h)^2 = 4p(y-k)$

$h^2 = 4p(3-k) \quad \cdots\cdots (1) \quad | \ (0,3)$

$(3-h)^2 = 4p(4-k) \quad \cdots\cdots (2) \quad | \ (3,4)$

$(4-h)^2 = 4p(11-k) \quad \cdots\cdots (3) \quad | \ (4,11)$

$9 - 6h + h^2 = 16p - 4pk \quad \cdots (2i)$

$16 - 8h + h^2 = 44p - 4pk \quad \cdots (3i)$

$9 - 6h + 12p - \cancel{4pk} = 16p - \cancel{4pk}$

$16 - 8h + 12p - \cancel{4pk} = 44p - \cancel{4pk}$

$4p + 6h = 9$

$32p + 8h = 16$

$4p + 6h = 9$

$-\quad 4p + h = 2$

$5h = 7$

$h = \frac{7}{5}$

$4p = 2 - \frac{7}{5} \Rightarrow p = \frac{3}{20}$

(5) T1

$(x-h)^2 = 4p(y-k)$

$\left(x-\frac{7}{5}\right)^2 = 4\left(\frac{3}{20}\right)(y-k)$

$h^2 = 12p - 4pk$

$\left(\frac{7}{5}\right)^2 = 12\left(\frac{3}{20}\right) + 4\left(\frac{3}{20}\right)k$

$\frac{49}{25} = \frac{9}{5} + \frac{3}{5}k$

$\frac{3}{5}k = \frac{49}{25} - \frac{9}{5}$

$\frac{3}{5}k = \frac{49-45}{25}$

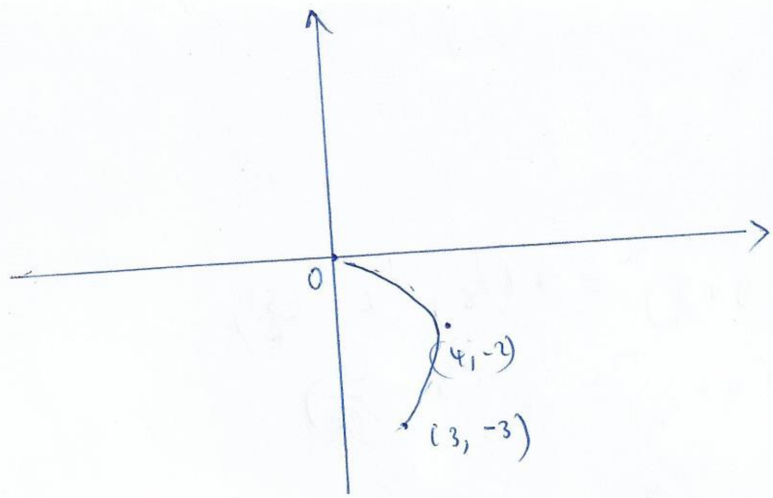
$\frac{3}{5}k = \frac{4}{25}$

$k = \frac{4}{15}$

eqn : $\left(x-\frac{7}{5}\right)^2 = \frac{3}{5}\left(y-\frac{4}{15}\right)$

6 T1

(b)



opens : LHS

$(y-k)^2 = -4p(x-h)$

$(k)^2 = -4p(-h)$

$k^2 = 4ph$ ---- (i) | (0,0)

$(-2-k)^2 = -4p(4-h)$ --- (ii) | (4,-2)

$(-3-k)^2 = -4p(3-h)$ --- (iii) | (3,-3)

$4+4k+k^2 = -16p+4ph$

$9+6k+k^2 = -12p+4ph$

$4+4k+\cancel{k^2} = -16p+\cancel{k^2}$

$9+6k+\cancel{k^2} = -12p+\cancel{k^2}$

$4k+16p = -4$

$6k+12p = -9$

$\cancel{2}k+8p = -2$

$\cancel{2}k+4p = -3$

$-$

$8p = 1 \qquad p = \frac{1}{8}$

$k = -2-4\left(\frac{1}{8}\right) = -\frac{5}{2}$

7 T 1

$$\left(\frac{5}{2}\right)^2 = 4\left(\frac{1}{8}\right)h$$

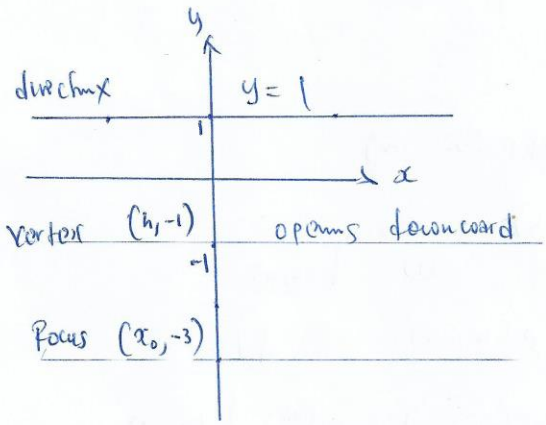
$$\frac{25}{4} = \frac{1}{2}h$$

$$h = \frac{25}{2}$$

eqn $\left(y+\frac{5}{2}\right)^2 = 4\left(\frac{1}{8}\right)\left(x-\frac{25}{2}\right)$

$$\left(y+\frac{5}{2}\right)^2 = \frac{1}{2}\left(x-\frac{25}{2}\right)$$

(c)



$4P = 8$

$$(x-h)^2 = -4P(y-k)$$

$$(x-h)^2 = -8(y-k)$$

$$(x-h)^2 = -8(y+1)$$

$$x^2 - 2xh + h^2 = -8y - 8$$

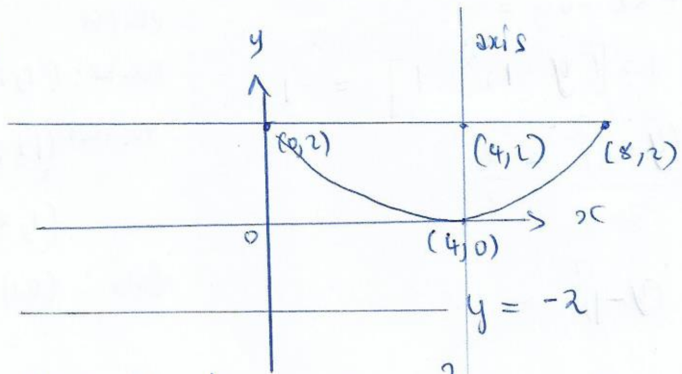
$$x^2 - 2xh + 8y + h^2 + 8 = 0$$

let $t = h$

$x^2 - 2xt + 8y + t^2 + 8 = 0$, $\forall t \in \mathbb{R}$ the graph opens downwards.

8 T1

(d)



opens upwards

$$(x-h)^2 = 4p(y-k)$$
$$(x-4)^2 = 4(2)y$$
$$x^2 - 8x + 16 = 8y.$$

(4) (a) $x^2 + 4x + y^2 = 12$ circle

$$(x+2)^2 - 4 + y^2 = 12$$
$$(x+2)^2 + y^2 = 16$$

center $(-2, 0)$

radius : 4 units

(b) $x^2 + 5y^2 + 4x = 1$ Ellipse

$$(x+2)^2 - 4 + 5y^2 = 1$$
$$\frac{(x+2)^2}{5} + \frac{5y^2}{5} = \frac{5}{5}$$
$$\frac{(x+2)^2}{5} + y^2 = 1$$

~~centre~~ $(-2, 0)$

Vertices: $(-2 \pm \sqrt{5}, 0)$

$(\pm 1, -2)$

focus : $(-4, 0)$ $(0, 0)$

9 T1

(c) $x^2 + 2y^2 - 2x - 4y = -1$

$(x-1)^2 - 1 + 2[(y-1)^2 - 1] = -1$

$\frac{(x-1)^2}{2} + \frac{2(y-1)^2}{2} = \frac{2}{2}$

$\frac{(x-1)^2}{2} + (y-1)^2 = 1$

ellipse

centre: $(1,1)$

vertices: $(1 \pm \sqrt{2}, 1)$

$(1,2), (1,0)$

foci: $(0,1)$, $(2,1)$

(d) $x^2 - y^2 - 2x + 4y = 4$

$(x-1)^2 - [(y-2)^2 - 4] - 1 = 4$

$(x-1)^2 - (y-2)^2 = 1$

Hyperbola

centre $(1,2)$

vertices $(0,2)$ $(2,2)$

foci $(1 \pm \sqrt{2}, 2)$

Asymptotes: $x-1 = \pm(y-2)$

(e) $y^2 - 4x^2 + 16x = 24$

$y^2 - 4(x^2 - 4x) = 24$

$y^2 - 4((x-2)^2 - 4) = 24$

$y^2 - 4(x-2)^2 + 16 = 24$

$y^2 - 4(x-2)^2 = 8$

$\frac{y^2}{8} - \frac{(x-2)^2}{2} = 1$

$a^2 = 8 \quad b^2 = 2$

$c^2 = 8 + 2$

$c^2 = 10$

Hyperbola

centre $(2,0)$

vertices $(2 \pm \sqrt{8}, 0)$

foci $(2 \pm \sqrt{10}, 0)$

Asymptotes:

$y = \pm 4(x-2)$

10 T1

(f) $2x^2 - y^2 + 6y = 3$

$2x^2 - (y^2 - 6y) = 3$

$2x^2 - \left((y-3)^2 - 9\right) = 3$

$2x^2 - (y-3)^2 + 9 = 3$

$2x^2 - (y-3)^2 = -6$

$$\frac{(y-3)^2}{6} - \frac{x^2}{3} = 1$$

$a^2 = 6 \quad b^2 = 3$

$c^2 = 9$

$c = \pm 3$

Hyperbola

centre $(0, 3)$

vertices $(0, 3 \pm \sqrt{6})$

foci $(0, 3 \pm 3)$

$y - 3 = \pm 2x$

(g) $x^2 - y^2 + 4x - 6y = 6$

$(x+2)^2 - (y^2 + 6y) - 4 = 6$

$(x+2)^2 - \left((y+3)^2 - 9\right) - 4 = 6$

$(x+2)^2 - (y+3)^2 + 5 = 6$

$(x+2)^2 - (y+3)^2 = 1$

$a^2 = 1 \quad b^2 = 1$

$c^2 = 2$

Hyperbola

- centre $(-2, -3)$

vertices $(-2 \pm 1, -3)$

foci $(-2 \pm \sqrt{2}, -3)$

Asymptotes

$x + 2 = \pm (y+3)$

(h) $2x^2 + 2y^2 - 28x + 12y = -144$

$2(x^2 - 14) + 2(y^2 + 6) = -144$

$2\left[(x-7)^2 - 49\right] + 2\left[(y+3)^2 - 9\right] = -144$

$2(x-7)^2 - 98 + 2(y+3)^2 - 18 = -144$

$$\frac{(y+3)^2}{14} - \frac{(x-7)^2}{14} = 1$$

$a^2 = 14 \quad b^2 = 14 \quad c^2 = 28$

Hyperbola

centre $(7, -3)$

vertices $(7, -3 \pm \sqrt{14})$

foci $(7, -3 \pm \sqrt{14})$

Asymptotes

$y + 3 = \pm (x - 7)$

(i) $y^2 - 4y - 8x - 12 = 0$

$(y-2)^2 - 4 - 8x - 12 = 0$

$(y-2)^2 = 8x + 16$

$(y-2)^2 = 4(2)(x+2)$

parabola
vertex $(-2, 2)$
Focus $(0, 2)$

(5) eccentricity

$e = 1$ parabola

$0 < e < 1$ ellipse

$e > 1$ Hyperbola

(a) Foci : $(0, \pm 3)$; Eccentricity : 0.5

$$e = \frac{c}{a}$$

$$a = \frac{3}{0.5} = 6$$

$$a^2 = b^2 + c^2$$

$$6^2 = b^2 + 3^2$$

$$b^2 = (6+3)(6-3)$$

$$b^2 = 27$$

Ellipse with major axis parallel to the y-axis

eqn : $\dfrac{y^2}{36} + \dfrac{x^2}{27} = 1$

(b) Vertices : $(0, \pm 70)$; Eccentricity

Conic : Ellipse

$$0.1 = \frac{c}{70}$$

$$c = 7$$

$$b^2 = a^2 - c^2 = 70^2 - 7^2 = 4851$$

$$\frac{y^2}{4900} + \frac{x^2}{4851} = 1$$

12 T1

(c) foci : $(\pm 8, 0)$ eccentricity : 0.2

Conic : ellipse

$e = \frac{c}{a} = 0.2 = \frac{8}{a}$ $\quad a = 40$

$b^2 = 40^2 - 8^2$
$= 1600 - 64$
$= 1536$

$\frac{x^2}{1600} + \frac{y^2}{1536} = 1$

(d) vertices $(\pm 10, 0)$; eccentricity : 0.24 : ellipse

$0.24 = \frac{c}{10}$

$c = 2.4$ $\quad b^2 = 100 - 5.76 = 94.24$

$\frac{x^2}{100} + \frac{y^2}{94.24} = 0$

(e) $e = 3$, $V(0, \pm 1)$ $\quad 3 = \frac{c}{1} \Rightarrow c = 3$

$b^2 = c^2 - a^2 = 9 - 1 = 8$

Hyperbola

$\frac{y^2}{1} - \frac{x^2}{8} = 1$

(f) $e = 2$, $V(\pm 2, 0)$ Hyperbola

$2 = \frac{c}{2} \Rightarrow c = 4$ $\quad b^2 = 16 - 4 = 12$

$\frac{x^2}{4} - \frac{y^2}{12} = 1$

(g) $e = 3$, $f(\pm 3, 0)$ hyperbola

$3 = \frac{3}{a} \Rightarrow a = 1$ $\quad b^2 = 9 - 1 = 8$

$x^2 - \frac{y^2}{8} = 1$

13 T1

(6) $D = B^2 - 4AC$ $D=0$: parabola $D<0$ Ellipse / circle
$D>0$ Hyperbola

(a) $(-24)^2 - 4(16)(6) > 0 \Rightarrow$ Hyperbola

(b) $(-4)^2 - 4(1)(-2) > 0 \Rightarrow$ Hyperbola

(c) $(-8)^2 - 4(13)(7) < 0 \Rightarrow$ Ellipse

(d) $(4)^2 - 4(2)(5) < 0 \Rightarrow$ Ellipse

(e) $(6)^2 - 4(-5) > 0 \Rightarrow$ Hyperbola

(f) $(-60)^2 - 4(36)(25) = 0 \Rightarrow$ parabola

(g) $(4)^2 - 4(1)(4) = 0 \Rightarrow$ parabola

(h) $(1)^2 - 4(1)(4) < 0 \Rightarrow$ Ellipse

(7) (a) $xy = 2$ Hyperbola

$\cot 2\theta = 0 \Rightarrow \theta = \frac{\pi}{4}$

$x = x'\cos\theta - y'\sin\theta = \frac{x'}{\sqrt{2}} - \frac{y'}{\sqrt{2}}$
$y = x'\sin\theta + y'\cos\theta$

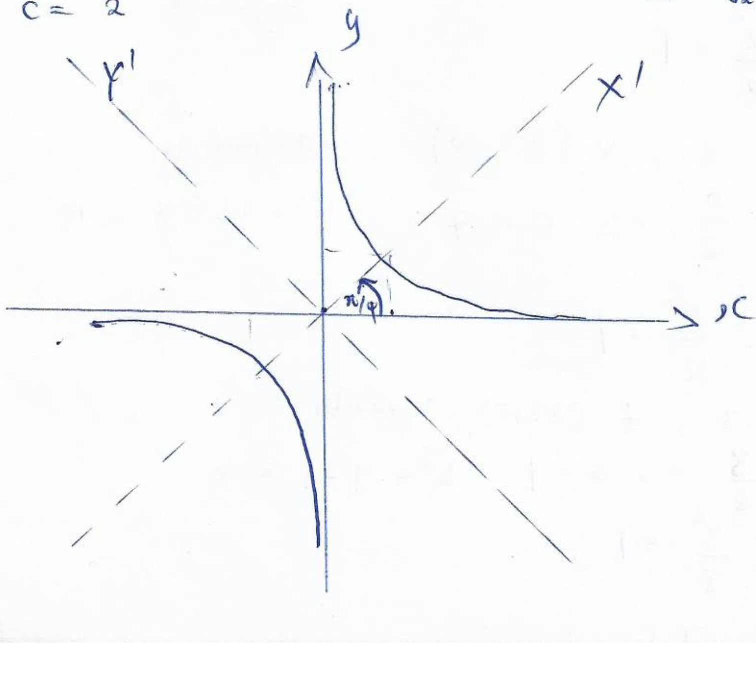
$$\Rightarrow \left(\frac{x'}{\sqrt{2}} - \frac{y'}{\sqrt{2}}\right)\left(\frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{2}}\right) = 2$$

$$\frac{(x')^2}{2} - \frac{(y')^2}{2} = 2$$

$c^2 = 2 + 2 = 4$
$c = 2$

$C' = (0,0)$ $V' = (\pm\sqrt{2}, 0)$
$F' = (\pm 2, 0)$

$C = (0,0)$ $V = (\pm 1, \pm 1)$
$F = \left(\pm\frac{2}{\sqrt{2}}, \pm\frac{2}{\sqrt{2}}\right)$



14 T1

(b) $x^2 + xy + y^2 = 1$

$D = B^2 - 4AC = (1)^2 - 4(1)(1) < 0 \Rightarrow$ conic ; ellipse

$\cot 2\theta = \frac{A-C}{B} = \frac{0}{1} = 0 \quad \Rightarrow \quad \theta = \frac{\pi}{4}$

$x = x'\cos\theta - y'\sin\theta, \quad y = x'\sin\theta + y'\cos\theta$

$x = \frac{x'}{\sqrt{2}} - \frac{y'}{\sqrt{2}}, \quad y = \frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{2}}$

$$\left(\frac{x'}{\sqrt{2}} - \frac{y'}{\sqrt{2}}\right)^2 + \left(\frac{x'}{\sqrt{2}} - \frac{y'}{\sqrt{2}}\right)\left(\frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{2}}\right) + \left(\frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{2}}\right)^2 = 1$$

$$\frac{(x')^2}{2} \cancel{- \frac{2x'y'}{2}} + \frac{(y')^2}{2} + \frac{(x')^2}{2} \cancel{- \frac{(y')^2}{2}} + \frac{(x')^2}{2} \cancel{+ \frac{2x'y'}{2}} + \frac{(y')^2}{2} = 1$$

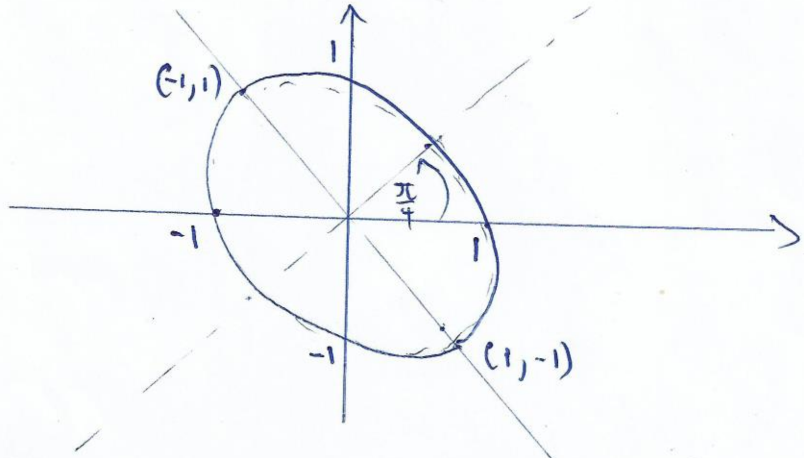
$$\frac{3(x')^2}{2} + \frac{(y')^2}{2} = 1$$

$$\frac{(x')^2}{\left(\frac{2}{3}\right)} + \frac{(y')^2}{2} = 1 \qquad a^2 = 2 \qquad b^2 = \frac{2}{3}$$

The major axis lies on y'-axis

$C' ; (0,0), \quad V' ; (0, \pm\sqrt{2}) \quad v' (\pm\sqrt{\tfrac{2}{3}}, 0)$

$C : (0,0) \quad V ; (\pm 1, \mp 1)$



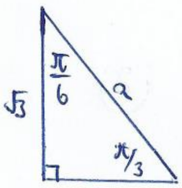
1S T1

(c) $3x^2 + 2\sqrt{3}xy + y^2 - 8x + 8\sqrt{3}y = 0$

$D = B^2 - 4AC = (2\sqrt{3})^2 - 4(3)(1) = 0 \Rightarrow$ Conic: parabola

$\cot 2\theta = \frac{A-C}{B} = \frac{3-1}{2\sqrt{3}} = \frac{1}{\sqrt{3}}$

$\tan 2\theta = \sqrt{3} \quad 2\theta = \frac{\pi}{3} \Rightarrow \theta = \frac{\pi}{6}$



$x = x'\cos(\frac{\pi}{6}) - y'\sin(\frac{\pi}{6})$, $y = x'\sin(\frac{\pi}{6}) + y'\cos(\frac{\pi}{6})$

$x = \frac{x'\sqrt{3}}{2} - \frac{y'}{2}$, $y = \frac{x'}{2} + \frac{y'\sqrt{3}}{2}$

$$3\left(\frac{x'\sqrt{3}}{2} - \frac{y'}{2}\right)^2 + 2\sqrt{3}\left(\frac{x'\sqrt{3}}{2} - \frac{y'}{2}\right)\left(\frac{x'}{2} + \frac{y'\sqrt{3}}{2}\right) + \left(\frac{x'}{2} + \frac{y'\sqrt{3}}{2}\right)^2$$
$$- 8\left(\frac{x'\sqrt{3}}{2} - \frac{y'}{2}\right) + 8\sqrt{3}\left(\frac{x'}{2} + \frac{y'\sqrt{3}}{2}\right) = 0$$

$$3\left(\frac{3(x')^2}{4} - \frac{2\sqrt{3}x'y'}{4} + \frac{(y')^2}{2}\right) + 2\sqrt{3}\left(\frac{(x')^2\sqrt{3}}{4} + \frac{3x'y'}{4} - \frac{x'y'}{4} - \frac{\sqrt{3}(y')^2}{4}\right)$$
$$\frac{(x')^2}{4} + \frac{2x'y'\sqrt{3}}{4} + \frac{3(y')^2}{4} - \frac{8\sqrt{3}x'}{2} + \frac{8y'}{2} + \frac{8\sqrt{3}x'}{2} + \frac{24y'}{2} = 0$$

$$\frac{9(x')^2}{4} - \cancel{\frac{6\sqrt{3}x'y'}{4}} + \cancel{\frac{3(y')^2}{4}} + \frac{6(x')^2}{4} + \cancel{\frac{4\sqrt{3}}{4}x'y'} - \cancel{\frac{6(y')^2}{4}} + \frac{(x')^2}{4}$$
$$+ \cancel{\frac{2x'y'\sqrt{3}}{4}} + \cancel{\frac{3(y')^2}{4}} - \cancel{\frac{8\sqrt{3}x'}{2}} + \frac{8y'}{2} + \cancel{\frac{8\sqrt{3}x'}{2}} + \frac{24}{2}y' = 0$$

$\frac{16(x')^2}{4} + 16y' = 0$ Axis of symmetry is y'-axis

$4(x')^2 + 16y' = 0$ $V'(0,0)$ $F'(0,-4)$

$4(x')^2 = -16y'$ directrix $y' = 4$

$(x')^2 = -4y'$

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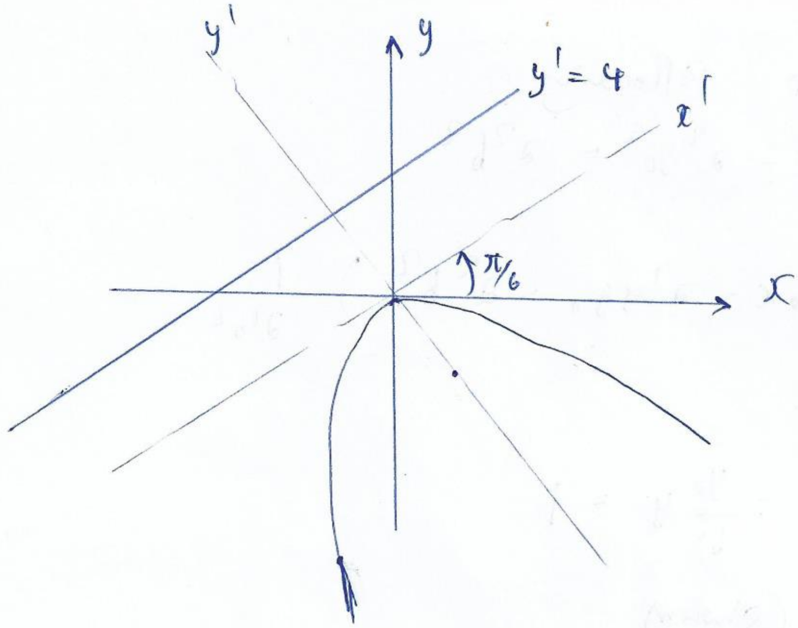

(d) $x^2 - \sqrt{3}xy + 2y^2 = 1$

$D = B^2 - 4AC = (-\sqrt{3})^2 - 4(1)(2) < 0 \Rightarrow$ ellipse

$\cot 2\theta = \frac{A-C}{B} = \frac{1-2}{(-\sqrt{3})} = \frac{1}{\sqrt{3}}$

$\theta = \frac{\pi}{6}$

$x = \frac{x'\sqrt{3}}{2} - \frac{y'}{2}, \quad y = \frac{x'}{2} + \frac{y'\sqrt{3}}{2}$

$$\left(\frac{x'\sqrt{3}}{2} - \frac{y'}{2}\right)^2 - \sqrt{3}\left(\frac{x'\sqrt{3}}{2} - \frac{y'}{2}\right)\left(\frac{x'}{2} + \frac{y'\sqrt{3}}{2}\right) + 2\left(\frac{x'}{2} + \frac{y'}{2}\sqrt{3}\right)^2 = 1$$

$$\frac{3(x')^2}{4} - \frac{2\sqrt{3}x'y'}{4} + \frac{(y')^2}{4} - \sqrt{3}\left(\frac{\sqrt{3}(x')^2}{4} + \frac{3}{4}x'y' - \frac{x'y'}{4} - \frac{\sqrt{3}(y')^2}{2}\right)$$

$$+ 2\left(\frac{(x')^2}{4} + \frac{2\sqrt{3}x'y'}{4} + \frac{3(y')^2}{4}\right) = 1$$

$$\cancel{\frac{3(x')^2}{4}} - \cancel{\frac{2\sqrt{3}x'y'}{4}} + \frac{(y')^2}{4} - \cancel{\frac{3(x')^2}{4}} - \frac{2\sqrt{3}x'y'}{4} + \frac{3}{4}(y')^2 + \frac{2(x')^2}{4} + \frac{4\cancel{\sqrt{3}}x'y'}{\cancel{4}} + \frac{6(y')^2}{4}$$

$$\frac{10(y')^2}{4} + \frac{2(x')^2}{4} = 1$$

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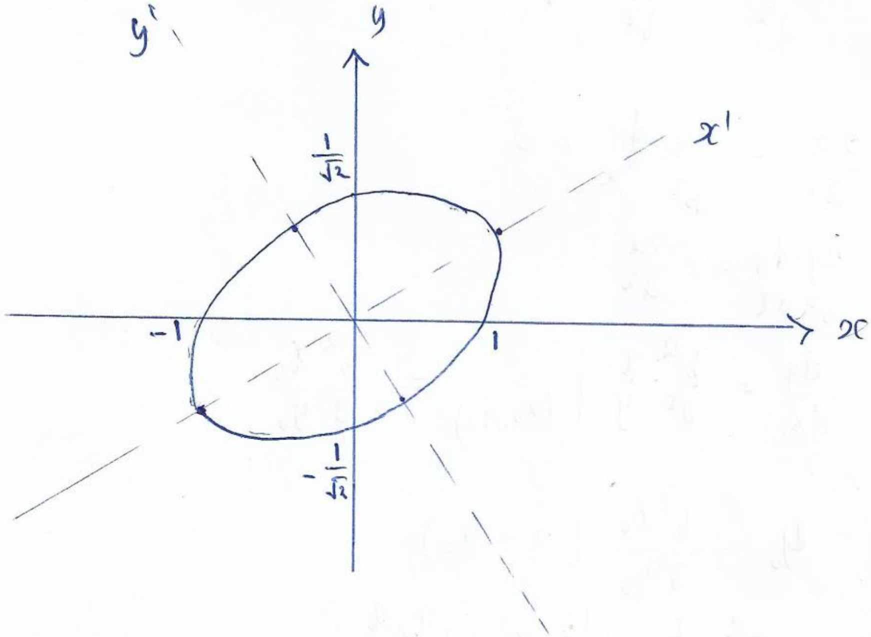
$$\frac{5(y')^2}{2} + \frac{(x')^2}{2} = 1$$

major axis is parallel to the x'- axis

$$\frac{(y')^2}{\left(\frac{2}{5}\right)} + \frac{(x')^2}{2} = 1 \qquad a^2 = 2 \quad b^2 = \frac{2}{5}$$



(c) $x^2 - 2xy + y^2 = 2 \qquad D = B^2 - 4AC = (-2)^2 - 4(1)(1) = 0$

Parabola

$$\cot 2\theta = \frac{A-C}{B} = \frac{1-1}{(-2)} = 0$$

$$\theta = \frac{\pi}{2}$$

$$x = \frac{x'}{\sqrt{2}} - \frac{y'}{\sqrt{2}}, \quad y = \frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{2}}$$

$$\left(\frac{x'}{\sqrt{2}} - \frac{y'}{\sqrt{2}}\right)^2 - 2\left(\frac{x'}{\sqrt{2}} - \frac{y'}{\sqrt{2}}\right)\left(\frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{2}}\right) + \left(\frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{2}}\right)^2 = 2$$

$$\frac{(x')^2}{2} - \frac{2(x'y')}{\cancel{2}} + \frac{(y')^2}{2} - 2\left(\frac{(x')^2}{2} - \frac{(y')^2}{2}\right) + \frac{(x')^2}{2} + \frac{2x'y'}{\cancel{2}} + \frac{(y')^2}{2} = 2$$

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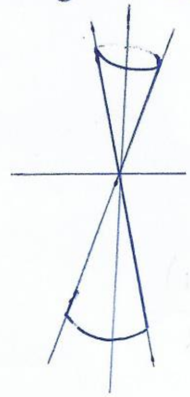
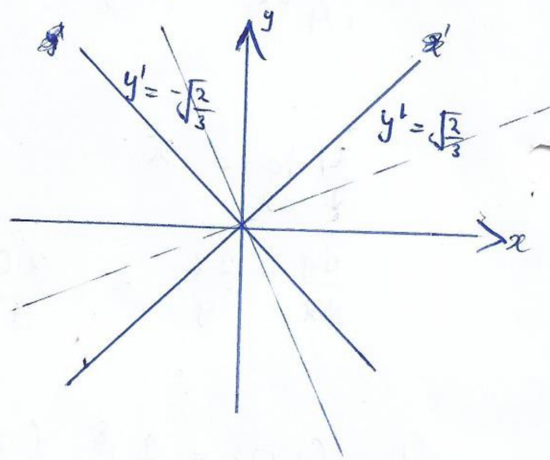
$$\frac{(y')^2}{2} + \frac{2(y')^2}{2} + \frac{(y')^2}{2} = 2$$

$$\frac{3(y')^2}{3} = \frac{2}{3}$$

$(y') = \pm\sqrt{\frac{2}{3}}$ Indegenarate case: the plane cuts the cone through the vertex





The rest of the questions the procedure is the same. In degenerate cases arise when the plane cutting the cones pass through the vertices at any angle of rotation. And the resulting graph is a straight line.

8. $\sqrt{x} + \sqrt{y} = 1$

$\sqrt{x} \geq 0 \Rightarrow x \geq 0$ Similarly $\sqrt{y} \geq 0 \Rightarrow y \geq 0$

$\sqrt{x} = 1 - \sqrt{y}$ $\sqrt{y} = 1 - \sqrt{x}$

thus follows that

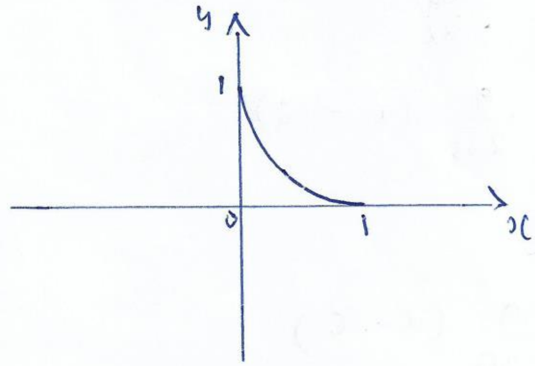
$1 - \sqrt{y} \geq 0$ $1 - \sqrt{x} \geq 0$

$-\sqrt{y} \geq -1$ $-\sqrt{x} \geq -1$

$\sqrt{y} \leq 1$ $\sqrt{x} \leq 1$

$y \leq 1$ $x \leq 1$

$R = \{0 \leq y \leq 1 \mid y \in \mathbb{R}\}$ $D = \{0 \leq x \leq 1 \mid x \in \mathbb{R}\}$



$\sqrt{y} = 1 - \sqrt{x}$

$y = (1 - \sqrt{x})^2 = 1 - 2\sqrt{x} + x$

$2\sqrt{x} = 1 + x - y$

$4x = (1 + x - y)^2$

$4x = 1 + x - y + x + x^2 - xy - y - xy + y^2$

$4x = 1 + 2x - 2y - 2xy + x^2 + y^2$

$x^2 + y^2 - 2x - 2y - 2xy + 1 = 0$

$x^2 - 2xy + y^2 - 2x - 2y + 1 = 0$

$D = B^2 - 4AC = (-2)^2 - 4(1)(1) = 0$

$\therefore$ $\sqrt{x} + \sqrt{y} = 1$ is a portion of a parabola

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(9)

$$r = \frac{ed}{1 \pm e\cos\theta} \qquad r = \frac{ed}{1 \pm e\sin\theta} \qquad d > 0$$

(a) $r = \dfrac{(1)(2)}{1 + \cos\theta}$

$r = \dfrac{2}{1 + \cos\theta}$

vertical directrix to the right of of the pole

$r = \dfrac{ed}{1 + e\cos\theta}$

(b) $e = 5$, $y = -6$

$r = \dfrac{(5)(6)}{1 - 5\sin\theta}$

$r = \dfrac{30}{1 - 5\sin\theta}$

Horizontal directrix below the pole

$r = \dfrac{ed}{1 - e\sin\theta}$

(c) $e = \frac{1}{2}$, $x = 1$

$r = \dfrac{(\frac{1}{2})(1)}{1 + \frac{1}{2}\cos\theta}$

$r = \dfrac{1}{2 + 1\cos\theta}$

(d) $e = \frac{1}{5}$, $y = -10$

$r = \dfrac{(\frac{1}{5})(10)}{1 - \frac{1}{5}\sin\theta}$

$r = \dfrac{2}{5 - \sin\theta}$

(e) $e = 1$, $y = 2$

$r = \dfrac{(1)(2)}{1 + \sin\theta}$

$r = \dfrac{2}{1 + \sin\theta}$

(f) $e = 2$, $x = 4$

$r = \dfrac{(2)(4)}{1 + 2\cos\theta}$

$r = \dfrac{8}{1 + 2\cos\theta}$

(g) $e = \frac{1}{4}$, $x = -2$

$r = \dfrac{(\frac{1}{4})(-2)}{1 - (\frac{1}{4})\cos\theta}$

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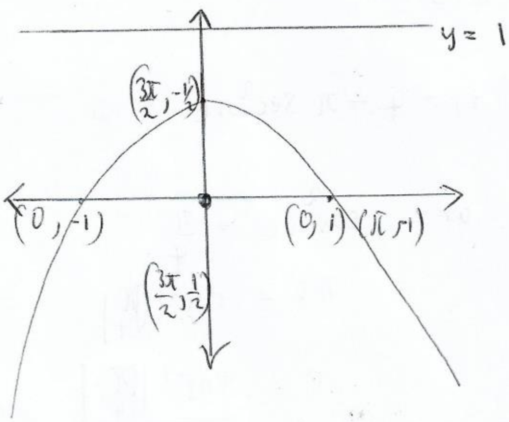
(h) $e = \frac{1}{3}$, $y = 6$

$r = \dfrac{(\frac{1}{3})(6)}{1 + (\frac{1}{3})\sin\theta}$

(10).

(a) $r = \dfrac{-1}{1 - \sin\theta}$

θ	0	$\pi/2$	π	$3\pi/2$	2π
r	-1	∞	-1	$-\frac{1}{2}$	-1

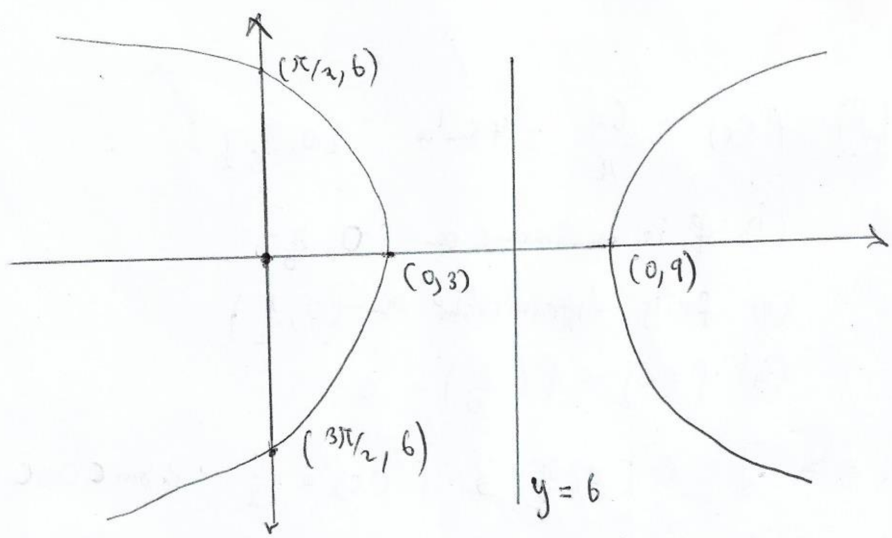


parabola

(b) $r = \dfrac{6}{1 + \cos\theta}$

θ	0	$\pi/2$	π	$3\pi/2$	2π
r	3	6	∞	6	3

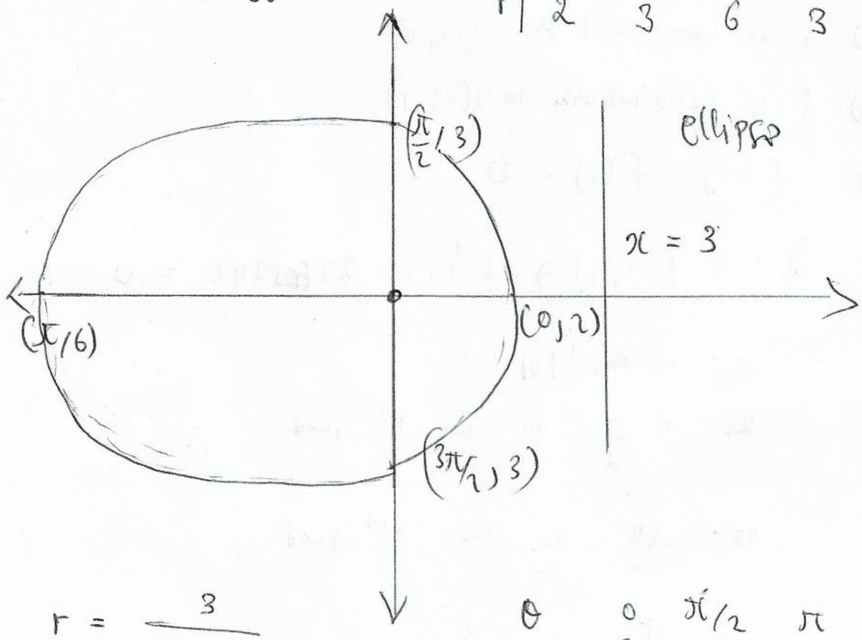
Hyperbola



22 T1

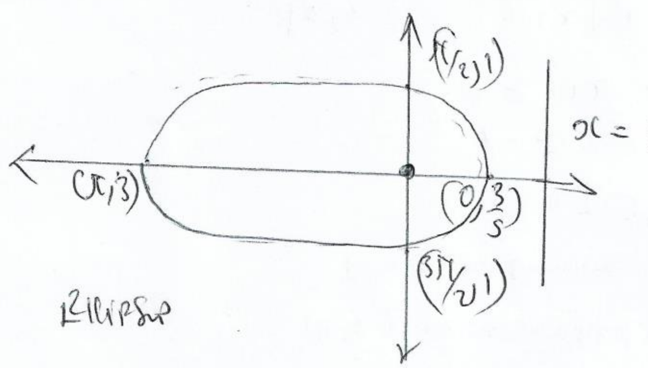
(c) $r = \dfrac{6}{2 + \cos\theta}$

θ	0	$\pi/2$	π	$3\pi/2$	2π
r	2	3	6	3	2



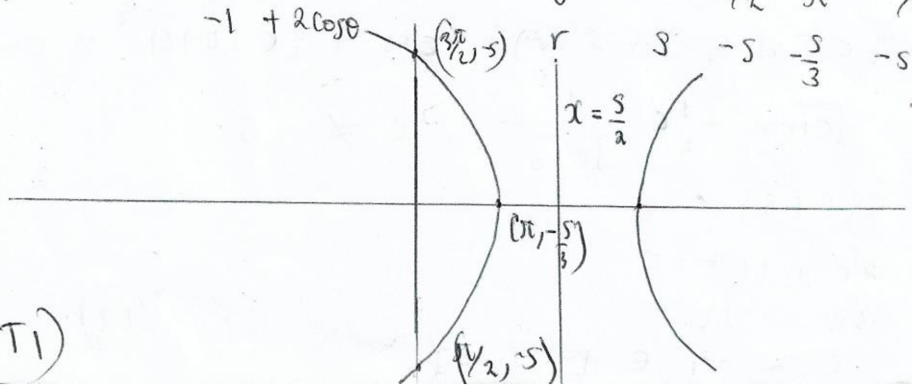
(d) $r = \dfrac{3}{3 + 2\cos\theta}$

θ	0	$\pi/2$	π	$3\pi/2$	2π
r	$\frac{3}{5}$	1	3	1	$\frac{3}{5}$



(e) $r = \dfrac{5}{-1 + 2\cos\theta}$

θ	0	$\pi/2$	π	$3\pi/2$	2π
r	5	-5	$-\frac{5}{3}$	-5	5

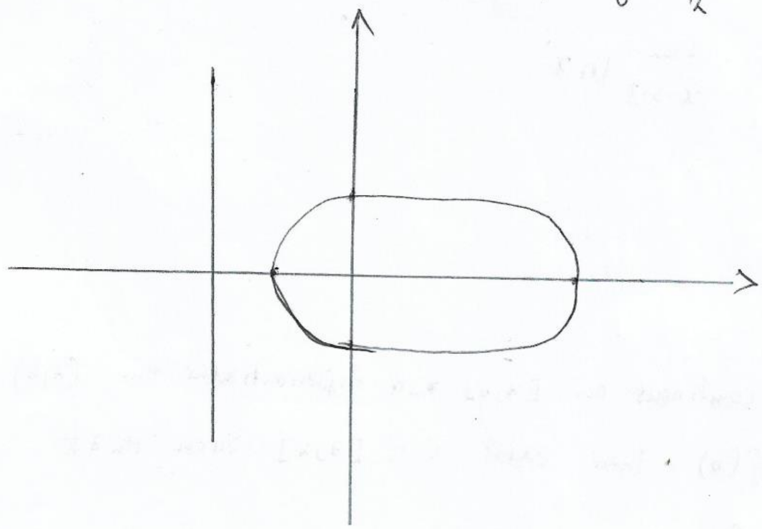


(23 T1)

(9) $r(3 - 2\cos\theta) = 6$

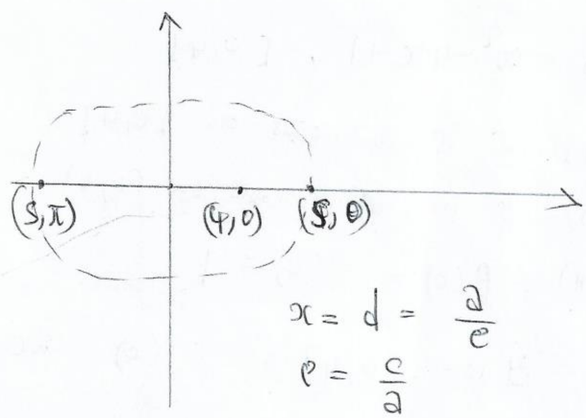
$$r = \frac{6}{3 - 2\cos\theta}$$

θ	0	$\pi/2$	π	$3\pi/2$	2π
r	6	2	$\frac{6}{5}$	2	6



(11)

(a)



$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

$$\frac{r^2\cos^2\theta}{25} + \frac{r^2\sin^2\theta}{9} = 1$$

$$r^2 = \frac{1}{\frac{\cos^2\theta}{25} + \frac{\sin^2\theta}{9}}$$

$$r^2 = \frac{225}{9\cos^2\theta + 25\sin^2\theta}$$

$$r = \frac{15}{\sqrt{9\cos^2\theta + 25\sin^2\theta}}$$

$x = d = \frac{a}{e}$

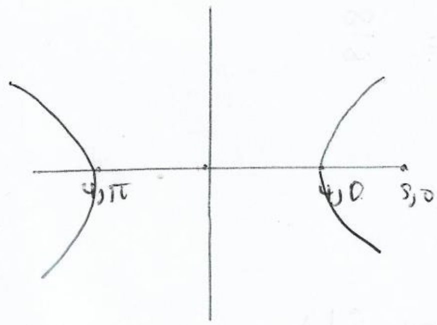
$e = \frac{c}{a}$

$e = \frac{4}{5}$ $d = \frac{25}{4}$

$$b^2 = a^2 - c^2$$
$$= 25 - 16$$
$$= 9$$

(2ω T.)

(b)



$$b^2 = c^2 - a^2$$
$$= 25 - 16$$
$$= 9$$

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

$$\frac{r^2\cos^2\theta}{16} - \frac{r^2\sin^2\theta}{9} = 1$$

$$r^2 = \frac{144}{9\cos^2\theta - 16\sin^2\theta}$$

$$r = \frac{12}{\pm\sqrt{9\cos^2\theta - 16\sin^2\theta}}$$

(c) $$\frac{x^2}{9} - \frac{y^2}{16} = 1$$

$$\frac{r^2\cos^2\theta}{9} - \frac{r^2\sin^2\theta}{16} = 1$$

$$r^2 = \frac{144}{16\cos^2\theta - 9\sin^2\theta}$$

$$r = \frac{12}{\pm\sqrt{16\cos^2\theta - 9\sin^2\theta}}$$

(d) $$\frac{x^2}{9} + y^2 = 1$$

$$\frac{r^2\cos^2\theta}{9} + r^2\sin^2\theta = 1$$

$$r^2 = \frac{9}{\cos^2\theta + 9\sin^2\theta}$$

$$r = \frac{3}{\pm\sqrt{\cos^2\theta + 9\sin^2\theta}}$$

(25 T1)

THE UNIVERSITY OF ZAMBIA
SCHOOL OF NATURAL SCIENCES
DEPARTMENT OF MATHEMATICS & STATISTICS

MAT 2100–Analytic Geometry and Calculus

Tutorial Sheet 2

April, 2018

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1. Evaluate the following limits:
   (a) $\lim\limits_{x\to 0} \dfrac{e^x - e^{-x}}{2\sin x}$ (c) $\lim\limits_{x\to 0^-} \dfrac{3\sin x}{\sqrt{x}}$ (d) $\lim\limits_{x\to 0} \dfrac{\sin x - \tan x}{x^2 \sin x}$
   (b) $\lim\limits_{x\to 0^+} \dfrac{7^{\sqrt{x}} - 1}{2^{\sqrt{x}} - 1}$

2. Give an $\epsilon - \delta$ proof of the following limits:
   (a) $\lim\limits_{x\to c} \sqrt{x} = \sqrt{c}$ for $c > 0$. (c) $\lim\limits_{x\to 0}(2x - 1) = -1$ (e) $\lim\limits_{x\to 5} \dfrac{1}{x-1} = \dfrac{1}{4}$
   (b) $\lim\limits_{x\to 5} \dfrac{1}{x} = \dfrac{1}{5}$. (d) $\lim\limits_{x\to 5} \dfrac{x^2 - 25}{x - 5} = 10$

3. Find the following limits:
   (a) $\lim\limits_{x\to \frac{\pi}{2}^-} (\tan x)^{\cos x}$ (d) $\lim\limits_{x\to\infty} x^{x}$ (g) $\lim\limits_{x\to 0}(x + e^{\frac{x}{3}})^{\frac{3}{x}}$
   (b) $\lim\limits_{x\to 0}(\cos x)^{\frac{1}{x}}$ (e) $\lim\limits_{x\to 0^+} \dfrac{\ln \sin^2 x}{\ln \tan x}$ (h) $\lim\limits_{x\to\infty} \dfrac{(\ln x)^2}{2^x}$
   (c) $\lim\limits_{x\to\infty} \left(1 + \dfrac{1}{x}\right)^x$ (f) $\lim\limits_{x\to \frac{\pi}{2}} (\tan x - \sec x)$ (i) $\lim\limits_{x\to 0^+} x^{\frac{1}{2}} \ln x$

4. Verify that the function satisfies the three hypotheses of Rolle's theorem on the given interval. Then find all numbers c that satisfy the conclusion of Rolle's theorem.
   (a) $f(x) = x^2 - 4x + 1, [0,4]$ (d) $f(x) = x\sqrt{x+6}, [-6,0]$
   (b) $f(x) = x^3 - 3x + 2x + 5, [0,2]$ (e) $f(x) = 4x - \tan \pi x, [-\frac{1}{4}, \frac{1}{4}]$
   (c) $f(x) = \sin 2\pi x, [-1,1]$ (f) $f(x) = \frac{6x}{\pi} - 4\sin^2 x, [0, \frac{\pi}{6}]$

5. Decide whether the Mean Value Theorem applies to the given function on the given interval. If it does, find all possible values of c.
   (a) $f(x) = |x|;\quad [1,2]$ (c) $f(t) = \dfrac{1}{t-1};\quad [0,2]$
   (b) $F(x) = \dfrac{x^3}{3};\quad [-2,2]$ (d) $h(x) = \dfrac{x}{x-3};\quad [0,2]$

6. Use the Mean value Theorem to show that $s = \frac{1}{t}$ decreases on any interval over which it is defined.

7. The Intermediate value Theorem states that if f is a continuous function on the interval $[a,b]$ and W is a number between $f(a)$ and $f(b)$, then there is at least one number c such between a and b such that $f(c) = W$.

Use the Intermediate Value Theorem to show that the equation $x - \cos x = 0$ has a solution between $x = 0$ and $x = \frac{\pi}{2}$.

8. Let

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ -x & \text{if } x \text{ is irrational.} \end{cases}$$

Sketch the graph of the function and decide where it is continuous.

9. Find equations for (i) the tangent lines and (ii) the normal lines to the hyperbola for the given values of x.

   (a) $\frac{x^2}{9} - y^2 = 1, \quad x = 6$

   (b) $\frac{y^2}{4} - \frac{x^2}{2} = 1, \quad x = 4$

10. Show that the equation of the tangent line to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_0, y_0) is $\frac{x_0}{a^2}x - \frac{y_0}{b^2}y = 1$.

11. Let C denote the circle whose equation is $(x-5)^2 + y^2$. Notice that the point $(8, -4)$ lies on the circle C. Find the equation of the line that is tangent to C at the point $(8, -4)$. Also, find the normal to the tangent at the same point.

12. The so called devil's curve is described by the equation

$$y^2(y^2 - 4) = x^2(x^2 - 5)$$

   (a) Compute the y-intercept of the curve.

   (b) Use implicit differentiation to find an expression for $\frac{dy}{dx}$ at the point (x, y).

   (c) Give an equation for the tangent line to the curve $(\sqrt{5}, 0)$.

MAT 2100 - Analytic Geometry and Calculus

Tutorial Sheet 2

(i) ~~Evaluate~~

(a) $\lim\limits_{x \to 0} \dfrac{e^{x} - e^{-x}}{2\sin x} = \dfrac{0}{0}$ (Indeterminate)

by L'Hôpital's rule we get

$\lim\limits_{x \to 0} \dfrac{e^{x} + e^{x}}{2\cos x} = \dfrac{2}{2} = 1$

(b) $\lim\limits_{x \to 0^{+}} \dfrac{7^{\sqrt{x}} - 1}{2^{\sqrt{x}} - 1}$

$= \lim\limits_{x \to 0^{+}} \dfrac{e^{\sqrt{x}\ln 7} - 1}{e^{\sqrt{x}\ln 2} - 1} = \dfrac{0}{0}$ (Indeterminate)

by L'Hôpital's rule we get

$= \lim\limits_{x \to 0^{+}} \dfrac{\cancel{\frac{1}{2}x^{-1/2}}\ln 7}{\cancel{\frac{1}{2}x^{-1/2}}\ln 2}$

$= \dfrac{\ln 7}{\ln 2}$

≈ 2.807 3 d.p

(c) $\lim\limits_{x \to 0^{-}} \dfrac{3\sin x}{\sqrt{x}} = \dfrac{0}{0}$ (Inderterminate)

by L'Hôpital's rule we get

$= \lim\limits_{x \to 0^{-}} \dfrac{3\cos x}{\frac{1}{2}x^{-1/2}}$

$$= \lim_{x \to 0^-} 6\sqrt{x}\cos x = 0$$

(d) $\lim_{x \to 0} \dfrac{\sin x - \tan x}{x^2 \sin x} = \dfrac{0}{0}$ (indeterminate)

$$= \lim_{x \to 0} \frac{\cancel{\sin x}(1 - \sec x)}{x^2 \cancel{\sin x}} = \frac{0}{0} \text{ (indeterminate)}$$

(L) $= \lim_{x \to 0} \dfrac{-\sec x \tan x}{2x} = \dfrac{0}{0}$ (indeterminate)

(L) $= \lim_{x \to 0} \left(\dfrac{-\sec^3 x - \sec x \tan^2 x}{2} \right)$

$$= -\frac{1}{2}$$

(2) (a) Let $\varepsilon > 0$ be given, we have determine $\delta > 0$ such that $0 < |x - c| < \delta$ and $|f(x) - L| < \varepsilon$.

that is $|\sqrt{x} - \sqrt{c}| < \varepsilon$ for $c > 0$

$$\Rightarrow |\sqrt{x} - \sqrt{c}| = \left| \frac{x - c}{\sqrt{x} + \sqrt{c}} \right| < \varepsilon$$

Since $c > 0$ we write $\left| \dfrac{x - c}{\sqrt{x} + \sqrt{c}} \right| < \varepsilon$

as $\dfrac{1}{\sqrt{x} + \sqrt{c}} |x - c| = \left| \dfrac{x - c}{\sqrt{x} + \sqrt{c}} \right|$

$$\sqrt{x} + \sqrt{c} \geq \sqrt{c}$$

$$x + c > c$$

$$\frac{1}{\sqrt{x}+\sqrt{c}} < \frac{1}{\sqrt{c}}$$

$$\Rightarrow \frac{1}{\sqrt{x}+\sqrt{c}}|x-c| < \frac{1}{\sqrt{c}}|x-c| < \varepsilon$$

$$\Rightarrow |x-c| < \sqrt{c}\,\varepsilon$$

Choosing $\delta \leq \sqrt{c}\,\varepsilon$, we have that

$$0 < |x-c| < \sqrt{c}\,\varepsilon = \delta \Rightarrow$$

$$|\sqrt{x}-\sqrt{c}| < \varepsilon = \frac{1}{\sqrt{c}} \text{ (proven)}$$

(b) $\lim\limits_{x \to 5} \frac{1}{x} = \frac{1}{5}$

given $\varepsilon > 0$, we need to find $\delta > 0 \ni$
$0 < |x-c| < \delta \Rightarrow |f(x) - L| < \varepsilon$

That is $0 < |x-5| < \delta \Rightarrow \left|\frac{1}{x} - \frac{1}{5}\right| < \varepsilon$

$$\left|\frac{5-x}{5x}\right| < \varepsilon$$

$$\left|\frac{1}{5x}\right| |x-5| < \varepsilon$$

$$\delta = \min\{5-x, 5+x\}$$

$\Rightarrow \quad \delta < 5\varepsilon \qquad$ because $\varepsilon < \frac{1}{5} \quad \varepsilon > 0$

$\delta > 0$

So $\quad 0 < |x - \varepsilon| < 5\varepsilon = \delta \quad \Rightarrow$

$$\left|\frac{1}{x} - \frac{1}{5}\right| < \varepsilon = \frac{\delta}{5}$$

(proven)

(c) $\lim\limits_{x \to 0} (2x - 1) = -1$

Let $\varepsilon > 0$ be given, we need to find $\delta > 0 \ni$

$0 < |x - 0| < \delta \quad \Rightarrow \quad |(2x-1) + 1| < \varepsilon$

This is the same as $|x| < \delta \Rightarrow |2x| < \varepsilon$

$\Rightarrow |2||x| < \varepsilon \quad \Rightarrow \quad |x| < \frac{\varepsilon}{2}$

Choosing $\delta = \frac{\varepsilon}{2}$, we have that

$0 < |x| < \delta = \frac{\varepsilon}{2} \quad \Rightarrow \quad |2x| < \varepsilon = 2\delta$

(proven).

FIZIE MATH LAB

(e) $\lim_{x \to 5} \frac{1}{x-1} = \frac{1}{4}$

Let $\varepsilon > 0$ be given, we need to find $\delta > 0 \ni$

$0 < |x-5| < \delta \Rightarrow \left|\frac{1}{x-1} - \frac{1}{4}\right| < \varepsilon$

$\Rightarrow \left|\frac{4-x+1}{x-1}\right| < \varepsilon$

$\left|\frac{5-x}{x-1}\right| < \varepsilon$

$\left|\frac{1}{x-1}\right| \left|\frac{x-5}{}\right| < \varepsilon$

$x \in (\delta - x, \ \delta + x)$

$\delta = \min\{\delta - x, \ \delta + x\}$

$\Rightarrow \quad -\varepsilon < \frac{x-5}{x-1} < \varepsilon$

$\left|\frac{1}{x-1} - \frac{1}{4}\right| \leq \left|\frac{1}{x-1}\right| + \left|\frac{1}{4}\right| < \varepsilon$

$\frac{1}{|x-1|} < \varepsilon \qquad \left|\frac{1}{4}\right| < \varepsilon$

(d) $\lim_{x \to 5} \dfrac{x^2 - 25}{x - 5} = 10$

Let $\varepsilon > 0$ be given, we need to find $\delta > 0$

$\ni \quad 0 < |x-5| < \delta \Rightarrow \left| \dfrac{x^2-25}{x-5} - 10 \right| < \varepsilon$

$$\left| \frac{x^2 - 25 - 10x + 50}{x-5} \right| < \varepsilon$$

$$\left| \frac{x^2 - 10x + 25}{x-5} \right| < \varepsilon$$

$$\left| \frac{\cancel{(x-5)}(x-5)}{\cancel{(x-5)}} \right| < \varepsilon$$

$$|x-5| < \varepsilon$$

Choosing $\delta = \varepsilon$, we have that

whenever $0 < |x-5| < \delta = \varepsilon$ we have that

$\left| \dfrac{x^2-25}{x-5} - 10 \right| < \varepsilon = \delta$.

(3) a) $\lim_{x \to \frac{\pi}{2}^-} (\tan x)^{\cos x} = \infty^0$ (indeterminate)

$= \lim_{x \to \frac{\pi}{2}^-} e^{\cos x \ln(\tan x)}$

$= \lim_{x \to \frac{\pi}{2}^-} e^{\left(\frac{\ln(\tan x)}{\frac{1}{\cos x}}\right)}$

$= e^{\lim_{x \to \frac{\pi}{2}^-} \left(\frac{\ln(\tan x)}{\frac{1}{\cos x}}\right)}$

$= e^{\lim_{x \to \frac{\pi}{2}^-} \left(\frac{\ln(\tan x)}{\sec x}\right)} = e^{\frac{\infty}{\infty}}$ (indeterminate)

$\overset{(L)}{=} e^{\lim_{x \to \frac{\pi}{2}^-} \left(\frac{\frac{\sec^2 x}{\tan x}}{\sec x \tan x}\right)}$

$= e^{\lim_{x \to \frac{\pi}{2}^-} \left(\frac{\cancel{\sec^2 x} \cot x}{\cancel{\sec x} \tan x}\right)}$

$= e^{\lim_{x \to \frac{\pi}{2}^-} \left(\frac{\csc x}{\tan x}\right)}$

$= e^{\frac{\lim_{x \to \frac{\pi}{2}^-} \csc x}{\lim_{x \to \frac{\pi}{2}^-} (\tan x)}}$ $\qquad \lim_{x \to \frac{\pi}{2}^-} \csc x = 1$

$= e^0$ $\qquad \lim_{x \to \frac{\pi}{2}^-} \tan x = \infty$

$= 1$

(b) $\lim_{x \to 0} (\cos x)^{\frac{1}{x}} = 1^{\infty}$ (indeterminate)

$$= \lim_{x \to 0} e^{\ln(\cos x)^{\frac{1}{x}}}$$

$$= \lim_{x \to 0} e^{\frac{1}{x} \ln \cos x}$$

$$= \lim_{x \to 0} e^{\frac{\ln \cos x}{x}}$$

$$= e^{\lim_{x \to 0} \frac{\ln \cos x}{x}}$$ $\because \lim_{x \to 0} \frac{\ln \cos x}{x} = \frac{0}{0}$ (indeterminate)

$$\stackrel{(L)}{=} e^{\lim_{x \to 0} (-\tan x)}$$

$$= e^{0}$$

$$= 1$$

(c) $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^{x} = 1^{\infty}$ (indeterminate)

$$= \lim_{x \to \infty} e^{x \ln\left(1 + \frac{1}{x}\right)}$$

$$= \lim_{x \to \infty} e^{\left[\frac{\ln\left(1 + \frac{1}{x}\right)}{\frac{1}{x}}\right]}$$

$$= e^{\lim_{x \to \infty} \left[\frac{\ln\left(1 + \frac{1}{x}\right)}{\frac{1}{x}}\right]} = e^{\frac{0}{0}}$$ (indeterminate)

(L)

$$= e^{\lim_{x \to \infty} \left[\frac{\left(\frac{1}{1+\frac{1}{x}}\right)\cancel{\left(-\frac{1}{x^2}\right)}}{\cancel{\left(-\frac{1}{x^2}\right)}} \right]}$$

$$= e^{\lim_{x \to \infty} \frac{1}{\left(1+\frac{1}{x}\right)}}$$

$$= e^{\frac{1}{\lim_{x \to \infty}\left(1+\frac{1}{x}\right)}}$$

$$= e^{\frac{1}{1+\lim_{x \to \infty}\left(\frac{1}{x}\right)}} \quad ; \lim_{x \to \infty} \frac{1}{x} = 0$$

$$= e^{1}$$

$$= e$$

(d) $\lim_{x \to \infty} x^x = \infty^{\infty}$ (indeterminate)

$$= \lim_{x \to \infty} e^{x \ln x}$$

$$= e^{\lim_{x \to \infty} x \ln x}$$

$$= e^{\lim_{x \to \infty} x \cdot \lim_{x \to \infty} \ln x}$$

$$= e^{\infty \cdot \infty}$$

$$= \tilde{\infty} \quad \text{(complex infinity)}$$

(e) $\lim_{x \to 0^+} \frac{\ln \sin^2 x}{\ln \tan x} = \frac{\infty}{\infty}$ (indeterminate)

(L)
$$= \lim_{x \to 0^+} \left(\frac{\frac{1}{\sin^2 x}(2 \cancel{\sin x} \cos x)}{\frac{1}{\tan x} \sec^2 x} \right)$$

$$= \lim_{x \to 0^+} \frac{2 \tan^2 x}{\sec^2 x}$$

$$= \lim_{x \to 0^+} \frac{2(\sec^2 x - 1)}{\sec^2 x}$$

$$= \lim_{x \to 0^+} (2 - 2\cos^2 x)$$

$$= 0$$

(f) $\lim_{x \to \frac{\pi}{2}} (\tan x - \sec x) = (\infty - \infty)$ (indeterminate)

$$= \lim_{x \to \frac{\pi}{2}} \left(\frac{\tan^2 x - \sec^2 x}{\tan x + \sec x} \right)$$

$$= \lim_{x \to \frac{\pi}{2}} \left(\frac{\sin x - 1}{\cos x} \right) = \frac{0}{0} \text{ (indeterminate)}$$

(L)
$$= \lim_{x \to \frac{\pi}{2}} \left(\frac{\cos x}{-\sin x} \right)$$

$$= \lim_{x \to \frac{\pi}{2}} \left(\frac{0}{1} \right) = 0$$

(c) $f(t) = \dfrac{1}{t-1}$ $[0,2]$

(i) Not continuous on $[0,2]$ because $t \neq 1$.

MVT not applicable

(d) $h(x) = \dfrac{x}{x-3}$ $[0,2]$

(i) continuous on $[0,2]$

(ii) differentiable on $(0,2)$

$$\Rightarrow h'(c) = \frac{f(2)-f(0)}{2}$$

$$\left.\frac{(x-3)-x}{(x-3)^2}\right|^{c} = -1$$

$$\frac{-3}{(c-3)^2} = -1$$

$$(c-3)^2 = 3$$

$$c^2-6c+9 = 3$$

$$c^2-6c+6 = 0$$

$$c = \frac{6 \pm \sqrt{6^2-4(6)}}{2}$$

$c \approx 4.73$ or $c \approx 1.27$

$c \approx 1.27 \in [0,2]$

(g) $\lim_{x \to 0} \left(x + e^{\frac{x}{3}} \right)^{\frac{3}{x}} = e^{\infty}$ (indeterminate)

$= \lim_{x \to 0} e^{\ln\left(x + e^{\frac{x}{3}}\right)^{\frac{3}{x}}}$

$= \lim_{x \to 0} e^{\frac{3}{x} \ln\left(x + e^{\frac{x}{3}}\right)}$

$= \lim_{x \to 0} e^{\frac{3 \ln\left(x + e^{\frac{x}{3}}\right)}{x}}$

$= e^{3 \lim_{x \to 0} \frac{\ln\left(x + e^{\frac{x}{3}}\right)}{x}} = e^{\frac{0}{0}}$ (indeterminate)

$= e^{3 \lim_{x \to 0} \frac{\left(\frac{1}{x + e^{x/3}}\right)\left(1 + \frac{1}{3} e^{\frac{x}{3}}\right)}{1}}$

$= e^{3 \lim_{x \to 0} \left(\frac{1 + \frac{1}{3} e^{\frac{x}{3}}}{x + e^{x/3}}\right)}$

$= e^{3\left(1 + \frac{1}{3}\right)}$

$= e^4$

(h) $\lim_{x \to \infty} \frac{(\ln x)^2}{2^x} = \frac{\infty}{\infty}$

$= \lim_{x \to \infty} \frac{(\ln x)^2}{e^{x \ln 2}}$

$\stackrel{(L)}{=} \lim_{x \to \infty} \frac{2(\ln x)\left(\frac{1}{x}\right)}{\ln 2 \, e^{x \ln 2}}$

$= \lim_{x \to \infty} \frac{2 \ln x}{x \ln 2 \, e^{x \ln 2}}$

$\stackrel{(L)}{=} \lim_{x \to \infty} \frac{2\left(\frac{1}{x}\right)}{\ln 2 \, e^{x \ln 2} + x(\ln 2)^2 e^{x \ln 2}}$

$= \lim_{x \to \infty} \frac{2}{x e^{x \ln 2}(\ln 2 + x(\ln 2)^2)}$

$= 0$

(1) $\lim_{x \to 0^+} x^{\frac{1}{2}} \ln x$

$= \lim_{x \to 0^+} \sqrt{x} \cdot \lim_{x \to 0^+} \ln x$

$= \infty$

Rolle's Theorem

Let f be continuous on $[a,b]$ and differentiable on (a,b). Then If $f(a) = f(b)$, There exist $c \in [a,b]$ such that $f'(c) = 0$.

(2) $f(x) = x^2 - 4x + 1$, $[0,4]$

(i) f is continuous on $[0,4]$

(ii) f is differentiable on $(0,4)$

(iii) $f(0) = f(4) = 1$

$\Rightarrow \exists\, c \in [0,4] \ni f'(c) = 2c - 4 = 0$

$\therefore c = 2 \in [0,4]$

(b) $f(x) = x^3 - 3x + 2x + 5$, $[0,2]$ is

(i) continuous on $[0,2]$

(ii) differentiable on $(0,2)$

(iii) $f(0) = 5$ $f(2) = 11$

RT not applicable ; property (III) fails.

(c) $f(x) = \sin 2\pi x \quad [-1, 1]$

(i) f is continuous on $[-1, 1]$

(ii) f is differentiable in $(-1, 1)$

(iii) $f(-1) = f(1) = 0$

$\Rightarrow \exists\, c \in [-1, 1] \ni f'(c) = 2\pi \cos 2\pi c = 0$

$\therefore \quad 2\pi c = \cos^{-1}|0|$

$2\pi c = \frac{\pi}{2}$ in the 1st quad

$2\pi c = \frac{3\pi}{2}$ in the 3rd quad

$c = \frac{1}{4}, \quad c = \frac{3}{4}$

(d) $f(x) = x\sqrt{x+6}, \quad [-6, 0]$

Domain $x + 6 \geq 0$

$x \geq -6$

$[-6, 0] \subset [-6, \infty)$

(i) f is continuous on $[-6, 0]$

(ii) f is differentiable on $(-6, 0)$

(iii) $f(-6) = f(0) = 0$

$\Rightarrow \exists\, c \in [-6, 0] \ni f'(c) = \sqrt{c+6} + \frac{1}{2}c\,(c+6)^{-\frac{1}{2}} = 0$

$\sqrt{c+6} = -\frac{1}{2}c\,\frac{1}{\sqrt{c+6}}, \quad c \neq -6$

$2(c+6) = -c$

$2c + 12 = -c$

$3c = -12$

$c = -4 \in [-6, 0]$

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(e) $f(x) = 4x - \tan \pi x \quad [-\frac{1}{4}, \frac{1}{4}]$

(i) f is continuous on $[-\frac{1}{4}, \frac{1}{4}]$

(ii) f is differentiable on $(-\frac{1}{4}, \frac{1}{4})$

(iii) $f(-\frac{1}{4}) = f(\frac{1}{4}) = 0$

$\Rightarrow \exists\, c \in [-\frac{1}{4}, \frac{1}{4}] \ni f'(c) = 4 - \pi \sec^2 \pi c = 0$

$\therefore \quad \sec^2(\pi c) = \frac{4}{\pi} \qquad$ or $\qquad \cos^2 \pi c = \frac{\pi}{4}$

$\sec(\pi c) = \pm\sqrt{\frac{4}{\pi}} \qquad \pi c = \cos^{-1}\left|\sqrt{\frac{\pi}{4}}\right|$

$\pi c = \sec^{-1}\left|\frac{2}{\sqrt{\pi}}\right| \qquad c = \frac{\cos^{-1}\left|\sqrt{\frac{\pi}{4}}\right|}{\pi}$

$c \approx 0.1533 \in [-\frac{1}{4}, \frac{1}{4}]$

(f) $f(x) = \frac{6x}{\pi} - 4\sin^2 x \quad [0, \frac{\pi}{6}]$

(i) f is continuous on $[0, \frac{\pi}{6}]$

(ii) f is differentiable on $(0, \frac{\pi}{6})$

(iii) $f(0) = f(\frac{\pi}{6}) = 0$

$\Rightarrow \exists\, c \in [0, \frac{\pi}{6}] \ni f'(c) = \frac{6}{\pi} - 8\sin c \cos c = 0$

$\therefore \quad \sin c \cos c = \frac{3}{4\pi}$

$\frac{1}{2}\sin 2c = \frac{3}{4\pi} \qquad c \approx 0.2489 \in [0, \frac{\pi}{6}]$

(5) Mean value theorem:

Let f be continuous on $[a,b]$ and differentiable on (a,b). There exist $c \in [a,b]$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

(a) $f(x) = |x| : [1,2]$

(i) $f(x) = x$, $[1,2]$ continuous

(ii) $f(x)$ differentiable on $(1,2)$

$\Rightarrow$

$$f'(c) = \frac{|c|}{c} = \frac{f(2) - f(1)}{(2)-(1)} = 1$$

$$c = \{1 \leq c \leq 2 \mid c \in \mathbb{R}\}$$

(b) $f(x) = \frac{x^3}{3}$ $[-2,2]$

(i) f is continuous on $[-2,2]$

(ii) f is differentiable on $(-2,2)$

$$\Rightarrow f'(c) = \frac{f(-2) + f(2)}{(+4)}$$

$$c^2 = \frac{+\frac{8}{3} + \frac{8}{3}}{+4}$$

$$c^2 = \frac{16}{12}$$

$$c = \pm \frac{4}{2\sqrt{3}}$$

$$c = \pm \frac{2}{\sqrt{3}} \in [-2,2]$$

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(6) $s = \frac{1}{t}$ is defined on $\{t \neq 0 \mid t \in \mathbb{R}\}$

(i) is continuous on $\{t \neq 0 \mid t \in \mathbb{R}\}$
(ii) is differentiable on $\{t \neq 0 \mid t \in \mathbb{R}\}$

$$s'(c) = \frac{s(b) - s(a)}{b - a}$$

$s'(t) = -\frac{1}{t^2}$, $\forall t \in \{t \neq 0 \mid t \in \mathbb{R}\}$ $s(t)$ is decreasing

because $s'(t) < 0 \Rightarrow \frac{s(b) - s(a)}{b - a} < 0$ $\forall a, b \in \mathbb{R}$,

$b - a \neq 0$.

(7) The intermediate value theorem

If f is a continuous function on the interval $[a, b]$ and W is a number between $f(a) = f(b)$, then there is at least one number c such between a and b such that $f(c) = W$

$x - \cos x = 0 \qquad [0, \frac{\pi}{2}]$

$f(a) \leq f(c) \leq f(b)$

(1) continuous on $[0, \frac{\pi}{2}]$

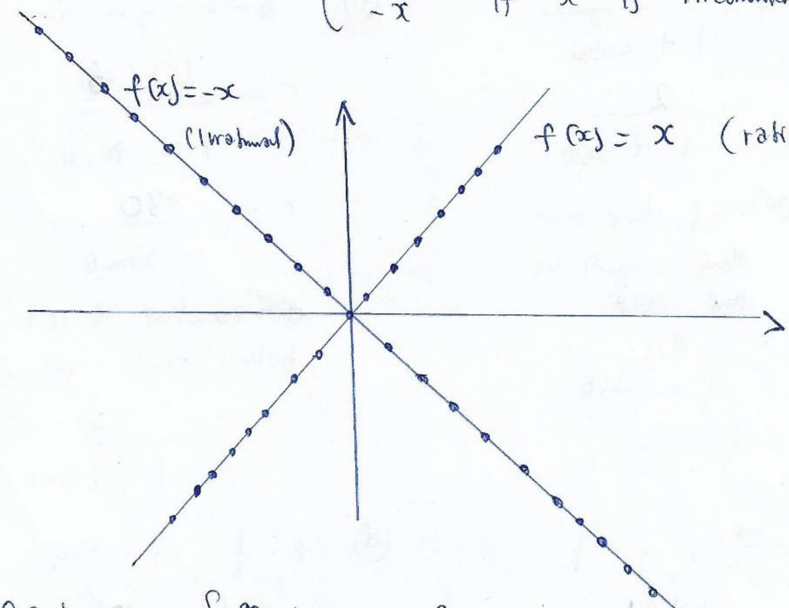
$f(0) = -1 \leq f(c) = 0 \leq f(\frac{\pi}{2}) = \frac{\pi}{2}$

since $f(c) = 0 \in [-1, \frac{\pi}{2}]$

$\Rightarrow \quad c \in [0, \frac{\pi}{2}]$

(8)

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ -x & \text{if } x \text{ is irrational} \end{cases}$$



$f(x) = x = \left\{ \frac{m}{n} ;\ m, n \in \mathbb{Z} \right\}$

This function is not continuous it has gaps of compliments of rationals and irrationals alternatively.

(9) (a) $\frac{x^2}{9} - \frac{y^2}{1} = 1, \quad x \in 6, \quad y = \pm\sqrt{3}$

(i) Tangents

$$\frac{2x}{9} - 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = \frac{2x}{9}$$

$$2(\pm\sqrt{3})\frac{dy}{dx} = \frac{4}{3}$$

$$\frac{dy}{dx} = \pm\frac{2}{3\sqrt{3}}$$

$$y - y_1 = \frac{dy}{dx}(x - x_1)$$

$$y - (\pm\sqrt{3}) = \pm\frac{2\sqrt{3}}{9}(x-6)$$

$$y + \sqrt{3} = -\frac{2\sqrt{3}}{9}(x-6) \quad \text{or} \quad y - \sqrt{3} = \frac{2\sqrt{3}}{9}(x-6)$$

(ii) Normals

$$y - y_1 = -\frac{1}{\frac{dy}{dx}}(x - x_1)$$

$$y - \sqrt{3} = \frac{9}{2\sqrt{3}}(x-6)$$

or

$$y + \sqrt{3} = -\frac{9}{2\sqrt{3}}(x-6)$$

(b) $\frac{y^2}{4} - \frac{x^2}{2} = 1, \; x = 4$

$$y^2 = 4\left(1 + \frac{x^2}{2}\right)$$

$$y = \pm\sqrt{4(1+8)}$$

$$y = \pm 12$$

(i) Tangents

$$y - y_1 = \frac{dy}{dx}(x - x_1)$$

$$\frac{\cancel{2}y}{\cancel{2}4}\left(\frac{dy}{dx}\right) - \frac{\cancel{2}x}{x} = 0$$

$$\frac{y}{2}\frac{dy}{dx} = x$$

$$\frac{dy}{dx} = \frac{2x}{y} = \frac{2(4)}{\pm 12}$$

$$y - (\pm 12) = \pm\frac{8}{12}(x - 4)$$

$$y + 12 = -\frac{2}{3}(x-4) \quad \text{or} \quad y - 12 = \frac{2}{3}(x-4)$$

(ii) Normals

$$y - y_1 = -\frac{1}{\frac{dy}{dx}}(x - x_1)$$

$$y - 12 = -\frac{3}{2}(x - 4) \quad \text{or} \quad y + 12 = \frac{3}{2}(x-4)$$

(10)

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad , \quad (x_0, y_0)$$

$$\frac{x_0^2}{a^2} - \frac{y_0^2}{b^2} = 1 \quad --- (1)$$

$$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\frac{2y}{b^2}\frac{dy}{dx} = \frac{2x}{a^2}$$

$$\frac{dy}{dx} = \frac{b^2}{a^2}\cdot\frac{x}{y}\bigg|_{(x_0, y_0)} = \frac{b^2 x_0}{a^2 y_0} .$$

$$y - y_0 = \frac{b^2 x_0}{a^2 y_0}(x - x_0)$$

$$a^2 y y_0 - a^2 y_0^2 = b^2 x_0 x - b^2 x_0^2$$

$$\left(\frac{b^2 x_0^2}{a^2 b^2 y_0 x_0} - \frac{a^2 y_0^2}{a^2 b^2 y_0 x_0} = \frac{b^2 x_0 x}{a^2 b^2 y_0 x_0} - \frac{a^2 y y_0}{a^2 b^2 y_0 x_0}\right)$$

$$\frac{x_0}{a^2 y_0} - \frac{y_0}{b^2 x_0} = \frac{x}{a^2 y_0} - \frac{y}{b^2 x_0}$$

$$b^2 x_0^2 - a^2 y_0^2 = b^2 x_0 x - a^2 y y_0 \quad ---- (2)$$

replacing (1) into (2)

we get the following

$$b^2x_0^2 - a^2y_0^2 = a^2b^2$$

$$\Rightarrow \left(b^2x_0x - a^2yy_0 = a^2b^2\right)\frac{1}{a^2b^2}$$

$$\frac{x_0}{a^2}x - \frac{y_0}{b^2}y = 1$$

(Shown)

(ii) $C(x,y) = (x-5)^2 + y^2$ $(8,-4)$

centre $(5,0)$

$$\frac{d}{dx}(x-5)^2 + \frac{d\,y^2}{dx} = 0$$

$$2(x-5) + 2y\frac{dy}{dx} = 0$$

$$\cancel{2}y\frac{dy}{dx} = -\cancel{2}(x-5)$$

$$\frac{dy}{dx} = \frac{-(8-5)}{(-4)} = \frac{3}{4}$$

$$y+4 = \frac{3}{4}(x-8)$$ Tangent

$$y+4 = -\frac{4}{3}(x-8)$$ normal.

12 $$y^2(y^2-4) = x^2(x^2-5)$$

(a) y-Intercept.

$$y^2(y^2-4) = 0$$

$$y = 0 \quad y = 2 \text{ or } y = -2$$

(b) $\frac{dy}{dx}$?

$$\frac{d}{dx}\left(y^2(y^2-4)\right) = \frac{d}{dx}\left(x^2(x^2-5)\right)$$

$$2y\frac{dy}{dx}y^2 + 2y\frac{dy}{dx}(y^2-4) = 2x\frac{d}{dx}x^2 + 2x\frac{d}{dx}(x^2-5)$$

$$2y^3\frac{dy}{dx} + 2y(y^2-4)\frac{dy}{dx} = 4x^2 + 4x^2$$

$$\frac{dy}{dx} = \frac{8x^2}{2y^3+2y^3-8y}$$

$$\frac{dy}{dx} = \frac{8x^2}{4y^3-8y}$$

$$\frac{dy}{dx} = \frac{x^2}{y^3-2y}$$

03

(b) $x^2 + xy + y^2 = 1$

$D = B^2 - 4AC = (1)^2 - 4(1)(1) < 0 \Rightarrow$ conic ; ellipse

$\cot 2\theta = \frac{A-C}{B} = \frac{0}{1} = 0 \quad \Rightarrow \quad \theta = \frac{\pi}{4}$

$x = x'\cos\theta - y'\sin\theta \ , \quad y = x'\sin\theta + y'\cos\theta$

$x = \frac{x'}{\sqrt{2}} - \frac{y'}{\sqrt{2}} \quad , \quad y = \frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{2}}$

$$\left(\frac{x'}{\sqrt{2}} - \frac{y'}{\sqrt{2}}\right)^2 + \left(\frac{x'}{\sqrt{2}} - \frac{y'}{\sqrt{2}}\right)\left(\frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{2}}\right) + \left(\frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{2}}\right)^2 = 1$$

$$\frac{(x')^2}{2} \cancel{- \frac{2x'y'}{2}} + \frac{(y')^2}{2} + \frac{(x')^2}{2} \cancel{- \frac{(y')^2}{2}} + \frac{(x')^2}{2} \cancel{+ \frac{2x'y'}{2}} + \frac{(y')^2}{2} = 1$$

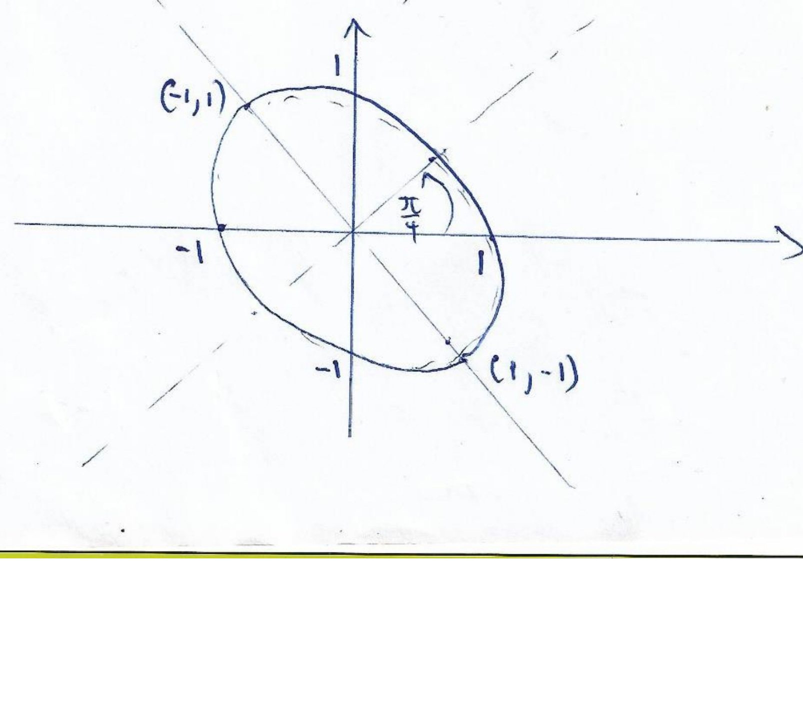
$$\frac{3(x')^2}{2} + \frac{(y')^2}{2} = 1$$

$$\frac{(x')^2}{\left(\frac{2}{3}\right)} + \frac{(y')^2}{2} = 1 \qquad a^2 = 2 \qquad b^2 = \frac{2}{3}$$

The major axis lies on y'-axis

$C' : (0,0) \ , \quad V' : (0, \pm\sqrt{2}) \qquad v' \left(\pm\sqrt{\tfrac{2}{3}}, 0\right)$

$C : (0,0) \qquad V : (\pm 1, \mp 1)$



1.5 T1

THE UNIVERSITY OF ZAMBIA
SCHOOL OF NATURAL SCIENCES
DEPARTMENT OF MATHEMATICS & STATISTICS
MAT 2100 –Analytic Geometry and Calculus
Tutorial Sheet 3

April, 2018

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1. Determine whether the following sequences converge or diverge. If it converges, find $\lim_{n\to\infty} a_n$:

   (a) $a_n = \dfrac{3n^2+2}{2n-1}$ (b) $a_n = \dfrac{\sqrt{3n^2+2}}{2n+1}$ (c) $a_n = \dfrac{e^{2n}}{4^n}$.

   integral test

2. Confirm that the integral test can be applied to the following series and use it to determine whether they converge or diverge.

   (a) $\sum_{n=1}^{\infty} \dfrac{1}{\sqrt{n+2}}$ (c) $\sum_{n=1}^{\infty} \dfrac{\ln n}{n^2}$ (e) $\sum_{n=1}^{\infty} \dfrac{n}{n^4+1}$

   (b) $\sum_{n=2}^{\infty} \dfrac{\ln n}{n^3}$ (d) $\sum_{n=2}^{\infty} \dfrac{1}{n\sqrt{\ln n}}$

3. Use the direct comparison to determine the convergence or divergence of the following series:

   (a) $\sum_{n=1}^{\infty} \dfrac{1}{n^2+1}$ (c) $\sum_{n=1}^{\infty} \dfrac{1}{2n-1}$ (e) $\sum_{n=1}^{\infty} \dfrac{3^n}{2^n-1}$

   (b) $\sum_{n=1}^{\infty} \dfrac{1}{3n^2+2}$ (d) $\sum_{n=2}^{\infty} \dfrac{1}{\sqrt{n}-1}$

   limit comparison

4. Use the limit comparison test to determine the convergence or divergence of the following series:

   (a) $\sum_{n=1}^{\infty} \sin\dfrac{1}{n}$ (c) $\sum_{n=1}^{\infty} \dfrac{1}{\sqrt{n^2+1}}$

   (b) $\sum_{n=1}^{\infty} \dfrac{5}{4^n+1}$ (d) $\sum_{n=1}^{\infty} \dfrac{2^n+1}{5^n+1}$

5. Determine whether the following series converge or diverge:

   (a) $\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n+1}$ (b) $\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{3n+2}$ (c) $\sum_{n=1}^{\infty} \dfrac{(-1)^n}{3^n}$ alternating.

FIZIE MATH LAB

(d) $\sum_{n=1}^{\infty} \frac{(-1)^n}{e^n}$ (e) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{\ln(n+1)}$

6. Use the ratio test to determine the convergence or divergence of the following series:

(a) $\sum_{n=1}^{\infty} \frac{1}{2^n}$ (c) $\sum_{n=1}^{\infty} n\left(\frac{6}{5}\right)^n$ (e) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\left(\frac{3}{2}\right)^n}{n^2}$ ratio

(b) $\sum_{n=1}^{\infty} \frac{n!}{3^n}$ (d) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+2)}{n(n+1)}$

7. Use the root test to determine the convergence or divergence of the following series:

(a) $\sum_{n=1}^{\infty} \left(\frac{2n}{n+1}\right)^n$ (c) $\sum_{n=0}^{\infty} e^{-3n}$

(b) $\sum_{n=1}^{\infty} \left(\frac{4n+3}{2n-1}\right)$ (d) $\sum_{n=2}^{\infty} \frac{n}{(\ln n)^n}$

8. Find the Maclaurin polynomial of degree n for the following functions

(a) $f(x) = e^{3x}$; $n = 4$ (c) $f(x) = \frac{x}{x+1}$; $n = 4$ (d) $f(x) = \sec x$; $n = 2$

(b) $f(x) = x^2 e^{-x}$; $n = 4$

9. Find the nth Taylor polynomial centred at c.

(a) $f(x) = \frac{2}{x}$; $n = 3, c = 1$

(b) $f(x) = \sqrt{x}$; $n = 3, c = 4$ taylor.

(c) $f(x) = x^2 \cos x, n = 2, c = \pi$

10. Find the radius of convergence of the following power series

(a) $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1}$ (c) $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{5^n}$

(b) $\sum_{n=0}^{\infty} (4x)^n$ (d) $\sum_{n=0}^{\infty} \frac{(2n)! x^{2n}}{n!}$

11. Find the interval of convergence of the following series (Be sure to check for convergence at the end points of the interval.)

(a) $\sum_{n=0}^{\infty} \frac{(x-3)^{n+1}}{(n+1)4^{n+1}}$ (b) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ (c) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$

TUTORIAL SHEET 3

(1) (a) $a_n = \dfrac{3n^2+2}{2n-1}$

Let $f(x) = a_n$

$\Rightarrow \lim_{x\to\infty} f(x) = \lim_{n\to\infty} a_n$

$= \lim_{x\to\infty} \dfrac{3x^2+2}{2x+1} = \lim_{x\to\infty} \dfrac{\left(3+\frac{2}{x^2}\right)\cancel{x^2}}{\left(\frac{2}{x}+\frac{1}{x^2}\right)\cancel{x^2}}$

$= 3\left(\frac{1}{0}\right) = \infty$

Thus $a_n = \dfrac{3n^2+2}{2n-1}$ diverges since $\lim_{n\to\infty} a_n = \infty$ is not unique, $\forall n \in \mathbb{Z}^{\geq 0}$ ✓

(b) $a_n = \dfrac{\sqrt{3n^2+2}}{2n+1}$

Let $f(x) = a_n \Rightarrow \lim_{x\to\infty} f(x) = \lim_{n\to\infty} a_n$

$= \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \dfrac{\sqrt{3x^2+2}}{2x+1}$

$= \lim_{x\to\infty} \dfrac{\cancel{\sqrt{x^2}}\left(\sqrt{3+\frac{2}{x^2}}\right)}{\cancel{x}\left(2+\frac{1}{x}\right)}$

$= \lim_{x\to\infty} \dfrac{\sqrt{3+\frac{2}{x^2}}}{2+\frac{1}{x}}$

$= \dfrac{\sqrt{3}}{2}$

Thus a_n converges to the $\lim_{n\to\infty} a_n = \dfrac{\sqrt{3}}{2}$ ✓

R =

(c) $\lim_{n \to \infty} \dfrac{e^{2n}}{4^n}$

Let $\lim_{x \to \infty} f(x) = \lim_{n \to \infty} a_n$

$$= \lim_{x \to \infty} \frac{e^{2x}}{4^x} = \lim_{x \to \infty} 4^{-x} e^{2x}$$

Since e^{2x} grows asymptotically faster than 4^x as $x \to \infty$, $\lim_{x \to \infty} 4^{-x} e^{2x} = \pm\infty$. Additionally $e^{2x} > 0$ and $4^x > 0$; $x \to \infty$ so

$$\lim_{x \to \infty} 4^{-x} e^{2x} = \infty$$

Hence $\lim_{n \to \infty} a_n = \infty$ implies $a_n = \dfrac{e^{2n}}{4^n}$

diverges.

2. Intergral test

If f is positive, continuous and decreasing for $x \geq 1$ and $a_n = f(n)$ then

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \int_1^{\infty} f(x)\,dx \quad \text{either}$$

both converge or both diverge

②

(a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+2}}$

Let $f(n) = \frac{1}{\sqrt{n+2}}$, $\forall n \geq -2$ f is continuous

which follows that f is continuous on $n \geq 1 \subset n \geq -2$.

(i) continuity is satisfied.

$f'(n) < 0$ that f is decreasing

$$f'(n) = \frac{d}{dn}(n+2)^{-1/2}$$

$$= -\frac{1}{2}(n+2)^{-3/2}$$

$$= -\frac{1}{2} \frac{1}{\sqrt{(n+2)^3}}$$

$f'(n) = -\frac{1}{2\sqrt{(n+2)^3}} < 0$ $f'(n) < 0, \forall n \geq -2 \Rightarrow f(n) < 0$

$\forall n \geq 1 \subset n \geq -2$

(ii) monotonic decreasing satisfied

(iii) observe that $f(n) > 0, \forall n \geq -2 \Rightarrow \forall n \geq 1 \subset n \geq -2$
$f(n) > 0$. positivity satisfied.

also to determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+2}}$ by limit comparison test we can compare the series to $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ a divergent P-series so $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+2}}$ diverges so integral test applys

2(a) $\int_1^{\infty} \frac{1}{\sqrt{n+2}} \, dn$

let $u^2 = n+2$

$2u \frac{du}{dn} = 1 \quad \Rightarrow \quad \int_{\sqrt{3}}^{\infty} \frac{1}{\cancel{u}} 2\cancel{u} \, du$

$2u \, du = dn$

$= 2\int_{\sqrt{3}}^{\infty} du$

$= 2[u]_{\sqrt{3}}^{\infty}$

$= 2\left[\sqrt{n+2}\right]_1^{\infty}$

$= 2\lim_{b \to \infty} \left[\sqrt{n+2}\right]_1^b$

$= \infty$

Hence by integral test the series diverges

(b) $\sum_{n=2}^{\infty} \frac{\ln n}{n^3}$

let $f(n) = \frac{\ln n}{n^3}$

(i) $f(n) > 0 \quad \forall \, n \geq 2$

(ii) $f(n)$ is continuous on $\forall n \geq 2$

(iii) $f'(n) < 0 \Rightarrow f(n)$ is monotonic decreasing

$f'(n) = \frac{d}{dn}\left(\frac{\ln n}{n^3}\right)$

~~this~~ implies integral test applies

$$= n^{-3}\frac{d}{dn}\ln n + \ln n \frac{d}{dx} n^{-3}$$

$$= \frac{n^{-3}}{n} + \ln n \cdot (-3n^{-4})$$

$$= n^{-4}(1-3\ln n)$$

$$= \frac{1-3\ln n}{n^4} < 0$$

$$\Rightarrow n^4\left(\frac{1-3\ln n}{n^4}\right) < 0 \cdot n^4$$

$$1-3\ln n < 0$$

$$e^{3\ln n} > e^1$$

$$n > e^{\frac{1}{3}}$$

Since $n \in \mathbb{N}^{\geq 2}$ or $\mathbb{Z}^{\geq 2}$

$$n > e^{\frac{1}{3}} \approx 2$$

$$n > 2$$

~~Determining convergence or divergence of $\sum_{n=2}^{\infty} \frac{\ln n}{n^3}$ we compute the limit of the the sequence of partial sums $\lim_{n\to\infty} \frac{\ln n}{n^3} =$~~

~~(L) $= \lim_{n\to\infty} \frac{\frac{d}{dn}\ln n}{\frac{d}{dn} n^3}$~~

$$= \lim_{n\to\infty} \frac{\frac{1}{n}}{3\frac{1}{n^2}} = \frac{1}{3}\lim_{n\to\infty}\frac{1}{n^2} = \lim_{n\to\infty}\frac{n^2}{3} = \infty$$

$$\Rightarrow \sum_{n=2}^{\infty} \frac{\ln n}{n^3} \text{ converges by comparison test}$$

Now we compute

$$\int_2^\infty \frac{\ln n}{n^3}\,dn$$

By Integration by parts

$$\int u\,dv = uv - \int v\,du$$

Let $u = \ln n \Rightarrow du = \frac{1}{n}\,dn$

$dv = n^{-3} \Rightarrow v = -\frac{1}{2n^2}$

$$\int_2^\infty \frac{\ln n}{n^3}\,dn = -\frac{\ln n}{2n^2} + \int \frac{1}{2n^2}\left(\frac{1}{n^3}\right)dn$$

$$= \frac{-\ln n}{2n^2} + \frac{1}{2}\int n^{-5}\,dn$$

$$= \frac{-\ln n}{2n^2} - \frac{1}{4n^4}$$

$$= \lim_{b\to\infty}\left[\frac{-\ln n}{2n^2} - \frac{1}{4n^4}\right]_2^\infty$$

$$= \lim_{b\to\infty}\left[\frac{-\ln n}{2n^2}\right]_2^\infty - \frac{1}{4}\lim_{b\to\infty}\left[\frac{1}{n^4}\right]_2^\infty$$

$$= -\frac{1}{2}\lim_{b\to\infty}\left[\frac{\ln n}{n^2}\right]_2^{\infty} - \frac{1}{4}\lim_{b\to\infty}\left[\frac{1}{n^4}\right]_2^{\infty}$$

$\left(\frac{\infty}{\infty}\right)$ indeterminate form

by L' Hôpitals

$$= -\frac{1}{2}\lim_{b\to\infty}\left[\frac{\frac{1}{n}}{2n}\right]_2^{\infty} + \frac{1}{64}$$

$$= -\frac{1}{4}\lim_{b\to\infty}\left[\frac{1}{n^2}\right]_2^{\infty} + \frac{1}{64} = \text{Real number}$$

~~Converges~~

Since both $\sum_{n=2}^{\infty}\frac{\ln n}{n^3}$ and $\int_2^{\infty}\frac{\ln n}{n^3}\,dn$ ~~diverges~~

It follows the series ~~diverges~~ Converges

(e) $\sum_{n=2}^{\infty}\frac{\ln n}{n^2}$

(i) $f(n) > 0 \quad \forall n \geq 2$

(ii) $f(n)$ is continuous $\forall n \geq 2$

(iii) If $f'(n) < 0 \Rightarrow f(n)$ is decreasing

$$f'(n) = \frac{d}{dn}\left(\frac{\ln n}{n^2}\right)$$

$$f'(n) = n^{-2}\frac{d}{dn}\ln n + \ln n\frac{d}{dn}n^{-2}$$

$= n^{-3} - 2\ln n \cdot n^{-3}$

$= n^{-3}(1 - 2\ln n) < 0$

$n^4 \cdot n^{-3}(1 - 2\ln n) < 0 \cdot n^4$

$n(1 - 2\ln n) < 0$

$n - 2n\ln n < 0$

$-2n\ln n < -n$

$2\ln n < 1$

$e^{\ln n} > e^{\frac{1}{2}}$

$n > \sqrt{e} \approx 2$

Since $n > 2$ & $n \geq 1$

$f(n)$ decreasing

So integral test applies

It follows either both $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ and

$\int_1^{\infty} \frac{\ln n}{n^2}\, dn$ converges or diverges.

$$\lim_{n\to\infty} \frac{\ln n}{n^2} = \frac{\lim_{n\to\infty} \frac{d}{dn} \ln n}{\lim_{n\to\infty} \frac{d}{dn} n^2}$$

$$= \frac{1}{2} \lim_{n\to\infty} \frac{1}{n^2} = 0$$

$\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ converges

$$\int_1^\infty \frac{\ln n}{n^2}\,dn$$

$$\int u\,dv = uv - \int v\,du$$

let $u = \ln n \Rightarrow du = \frac{1}{n}\,dn$

$dv = n^{-2}\,dn \Rightarrow v = -\frac{1}{n}$

$$\int_1^\infty \frac{\ln n}{n^2}\,dn = -\frac{\ln n}{n} + \int \frac{1}{n^2}\,dn$$

$$= \frac{-\ln n}{n} - \frac{1}{n}$$

$$= \lim_{b\to\infty}\left[-\frac{\ln n}{n} - \frac{1}{n}\right]_1^\infty$$

$$= \lim_{b\to\infty}\left[-\frac{\ln n}{n}\right]_1^\infty + \lim_{b\to\infty}\left[-\frac{1}{n}\right]_1^\infty$$

$\left(\frac{\infty}{\infty} \text{ indeterminate}\right)$

= by l'Hôpital's rule

$$= \lim_{b\to\infty}\left[-\frac{1}{n}\right]_1^\infty + \lim_{b\to\infty}\left[-\frac{1}{n}\right]_1^\infty$$

$$= 0$$

both $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ and $\int_1^\infty \frac{\ln n}{n^2}$ converges

So the series converges.

(d) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

Let $f(x) = \dfrac{1}{x\sqrt{\ln x}}$

(i) $f(x) > 0$ for $\forall\ x \geq 2$

(ii) $f(x)$ is continuous $\forall\ x \geq 2$

(iii) If $f'(x) < 0$ $f(x)$ is monotonic decreasing

$$f'(x) = \frac{-\frac{d}{dx}\left(x\sqrt{\ln x}\right)}{x^2 \ln x}$$

$$= \frac{-\left(\sqrt{\ln x} - \frac{1}{2\sqrt{(\ln x)^3}}\right)}{x^2 \ln x}$$

$$= \frac{-\left(2(\ln x)^2 - 1\right)}{2x^2(\ln x)^2} < 0$$

$$= -2(\ln x)^2 + 1 < 0$$

$$= \ln x > \frac{1}{\sqrt{2}}$$

$$= x > \frac{1}{e^{\sqrt{2}}} \approx 2$$

$\Rightarrow n = 2$ since $n \in \mathbb{N}^{\geq 2}$

$\Rightarrow n \geq 2$ $f(n)$ decreases

So integral test applies

$\Rightarrow$ Either both $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ and $\int_2^{\infty} \frac{1}{n\sqrt{\ln n}}$ converges or diverges.

by direct comparison test

$$a_n = \frac{1}{n\sqrt{\ln n}} \leq \frac{1}{n} = b_n$$

Since b_n diverges $\Rightarrow$ a_n ~~converges~~ diverges

Now $$\int_2^{\infty} \frac{1}{x\sqrt{\ln x}}\, dx$$

Let $p^2 = \ln x$

$$2p\frac{dp}{dx} = \frac{1}{x}$$

$$dx = 2px\, dp$$

$$= 2\int_{\sqrt{\ln 2}}^{\infty} dp$$

$$= 2[p]_1^{\infty}$$

$$= 2\lim_{b\to\infty}\left[\sqrt{\ln n}\right]_1^{\infty}$$

$$= 2\left(\sqrt{\ln b} - 1\right)$$

$$= \infty$$

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$\Rightarrow \int_1^{\infty} \frac{1}{n\sqrt{\ln n}} dn$ diverge which follows that the series diverges

(d) $\sum_{n=1}^{\infty} \frac{n}{n^4+1}$

(i) $f(n) > 0, \forall n \geq 1$

(ii) $f(n)$ is continuous $\forall n \geq 1$

(iii) $f'(n) < 0 \Rightarrow f(n)$ is monotonic decreasing

Let $f(n) = f(x)$

$$f'(x) = \frac{(x^4+1)\frac{dx}{dx} - x\frac{d}{dx}(x^4+1)}{(x^4+1)^2}$$

$$= \frac{x^4+1-4x^4}{(x^4+1)^2}$$

$$= \frac{1-3x^4}{(x^4+1)^2} < 0$$

$$= 1-3x^4 < 0$$

$$3x^4 > 1$$

$$x > \frac{1}{\sqrt[4]{3}} \approx 1$$

$\Rightarrow n > 1 \Rightarrow f'(n) < 0 \Rightarrow f(n)$ is decreasing

so integral test applys

therefore either both $\sum_{n=1}^{\infty} \frac{n}{n^4+1}$ and $\int_1^{\infty} \frac{n}{n^4+1}\, dn$ diverges or converges.

By ~~direct~~ comparison test

$$a_n = \frac{n}{n^4+1} < \frac{1}{n^3} = b_n$$

$a_n > 0$ and $b_n > 0$

$\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a convergent p-series

$$\lim_{n \to \infty} \left(\frac{a_n}{b_n}\right) = \lim_{n \to \infty} \left(\frac{1}{n^3} \cdot \frac{n^4+1}{n}\right)$$

$$= \lim_{n \to \infty} \left(1 + \frac{1}{n^4}\right)$$

$$= 1$$

We conclude by the limit comparison test that the series converges.

Now

$$\int_1^{\infty} \frac{x}{x^4+1}\, dx = \int_1^{\infty} \frac{x}{(x^2)^2+1}\, dx$$

Let $p = x^2$

$dp = dx\, 2x$

$dx = \frac{1}{2x}\, dp$

$$= \int_1^\infty \frac{\cancel{x}}{(p^2+1)} \cdot \frac{1}{2\cancel{x}}\, dp$$

$$= \frac{1}{2} \int_1^\infty \frac{1}{p^2+1}\, dp$$

$$= \frac{1}{2} \left[\arctan p\right]_1^\infty$$

$$= \frac{1}{2} \left[\arctan x^2\right]_1^\infty$$

$$= \frac{1}{2} \lim_{b \to \infty} \left[\arctan x^2\right]_1^\infty$$

$$= \frac{1}{2} \left[\arctan b - \arctan 1 \right]$$

$$= \frac{1}{2} \left(\frac{\pi}{2} - \frac{\pi}{4} \right)$$

$$= \frac{\pi}{8}$$

$\Rightarrow \int_1^\infty \frac{n}{n^4+1}\, dn$ converges so then

series $\sum_{n=1}^{\infty} \frac{n}{n^4+1}$ converges.

(3) Direct comparison test

let $0 \leq a_n \leq b_n \quad \forall n \in \mathbb{N}$ or $\mathbb{Z}$

(i) If $\sum b_n$ converges $\Rightarrow \sum a_n$ converges

(ii) if $\sum a_n$ diverges $\Rightarrow \sum b_n$ diverges

(2) $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

let $a_n = \frac{1}{n^2+1} \leq \frac{1}{n^2} = b_n$

$$n^4\left(\frac{1}{n^2+1}\right) \leq \left(\frac{1}{n^2}\right)(n^4)$$

$$\begin{array}{r|l} n^2+1 & \overline{\;n^2-1\;} \\ & n^4 \\ & -\;(n^4+n^2) \\ \hline & -n^2 \\ & -\;(-n^2-1) \\ \hline & 1 \end{array}$$

$$n^2-1+\frac{1}{n^2+1} \leq n^2$$

as $n \to \infty$, $\frac{1}{n^2+1} \to 0$ so $a_n \leq b_n$ is true

We know $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series) then

$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges

is

(b) $\sum_{n=1}^{\infty} \frac{1}{3n^2+2}$

Let $a_n = \frac{1}{3n^2+2} \le \frac{1}{n^2} = b_n$

$$\frac{3n^2}{3n^2+2} \le 3$$

$$1 - \frac{2}{3n^2+2} \le 3$$

as $n \to \infty$, $-\frac{2}{3n^2+2} \to \infty$

$\Rightarrow a_n \le b_n$ is true

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{3n^2+2}$ converges

√ 2

(c) $\sum_{n=1}^{\infty} \frac{1}{2n-1}$

Let $a_n = \frac{1}{2n-1} \ge \frac{1}{2n} = b_n$ $\quad a_n \ge b_n$

$\sum_{n=1}^{\infty} \frac{1}{2n}$ (harmonic diverges)

So $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ diverges

(d) $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$

$a_n = \frac{1}{\sqrt{n}-1} < \frac{1}{\sqrt{n}} = b_n$ not true

thus $\left(\frac{1}{\sqrt{n}-1}\right)^2 < \frac{1}{n} \quad \forall n > 2$

$a_n = \frac{1}{\sqrt{n}-1} > \frac{1}{\sqrt{n}} = b_n \quad \forall n > 2$

Since $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ diverges $\Rightarrow \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$ diverges

(e) $\sum_{n=1}^{\infty} \frac{3^n}{2^n - 1}$

$\sqrt[n]{\frac{3^n}{2^n-1}} > \frac{3}{2}$

$\frac{3}{\sqrt[n]{2^n-1}} > \frac{3}{2}$

$\frac{1}{\sqrt[n]{2^n-1}} > \frac{1}{2}$

$2 > \sqrt[n]{2^n-1}$

$2^n > 2^n - 1$

which is true

$$\sqrt[n]{\frac{3^n}{2^n-1}} > \frac{3}{2}$$

$$a_n = \frac{3^n}{2^n-1} > \left(\frac{3}{2}\right)^n = b_n$$

since $\sum_{n=2}^{\infty}\left(\frac{3}{2}\right)^n$ diverges $\Rightarrow \sum_{n=2}^{\infty}\frac{3^n}{2^n-1}$ diverges

(4)

$b_n > 0 \qquad a_n > 0$

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = L$$

(a) compare $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ to $\sum_{n=1}^{\infty}\frac{1}{n}$ (diverges harmonic)

$$\lim_{n\to\infty}\left(\frac{\sin\frac{1}{n}}{\frac{1}{n}}\right)$$

$$= \lim_{n\to\infty}\left(n \sin\frac{1}{n}\right)$$

(L) $\lim_{n\to\infty}\left(\sin\frac{1}{n} + n\left(\cos\left(\frac{1}{n}\right)\cdot\left(-\frac{1}{n^2}\right)\right)\right)$

$$\lim_{n\to\infty} \sin\frac{1}{n} = 0 \quad \text{and} \quad \lim_{n\to\infty} n = \infty$$

$$= \lim_{n\to\infty} n \sin\frac{1}{n}$$

$$= \lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{\frac{d}{dn}\sin\left(\frac{1}{n}\right)}{\frac{d}{d}\left(\frac{1}{n}\right)}$$

$$= \lim_{n\to\infty} \frac{\cancel{-\frac{1}{n^2}}\cos\left(\frac{1}{n}\right)}{\cancel{-\frac{1}{n^2}}}$$

$$= \lim_{n\to\infty} \cos\left(\frac{1}{n}\right)$$

$$= \cos\left(\lim_{n\to\infty} \frac{1}{n}\right)$$

$$= \cos(0)$$

$$= 1 \qquad \text{inconclusive}$$

So we compare $\sum_{n=1}^{\infty} \sin\frac{1}{n}$ to $\sum_{n=1}^{\infty} \sin n$ (diverges)

$$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\sin n} = -\infty \text{ to } \infty$$

so $\sum_{n=1}^{\infty} \sin n$ diverges

(b) $\sum_{n=1}^{\infty} \frac{5}{4^n+1}$ we compare to $\sum_{n=1}^{\infty} \frac{1}{4^n}$

(convergent geometric series)

$$= \lim_{n\to\infty} \left(\frac{\frac{5}{4^n+1}}{\frac{1}{4^n}} \right)$$

$$= \lim_{n\to\infty} \frac{5(4^n)}{4^n+1} \quad \left(\frac{\infty}{\infty}\right)$$

$$\overset{(L)}{=} \lim_{n\to\infty} \frac{5 \cdot \frac{d}{dn} 4^n}{\frac{d}{dn} 4^n + \frac{d}{dn}(1)}$$

$$= \lim_{n\to\infty} \frac{5 \cancel{\ln 4\, e^{n\ln 4}}}{\cancel{\ln 4\, e^{n\ln 4}}}$$

$$= 5$$

So the series converges

(c) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ we compare to $\sum_{n=1}^{\infty} \frac{1}{n}$ (divergent harmonic series)

$$\Rightarrow \lim_{n\to\infty} \left(\frac{n}{\sqrt{n^2+1}} \right)$$

$$= \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}\sqrt{1+\frac{1}{n^2}}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{1}} = 1$$

the series diverges

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(d) $\sum_{n=1}^{\infty} \frac{2^n+1}{5^n+1}$ we compare to $\sum_{n=1}^{\infty} \frac{2^n}{5^n}$

(convergent geometric series)

$$\lim_{n\to\infty} \frac{2^n 5^n + 5^n}{2^n 5^n + 2^n} = 1$$

$$\sum_{n=1}^{\infty} \frac{1+2^n}{1+5^n} \approx 1.80836$$

$$= \lim_{n\to\infty} \frac{\left(\frac{5}{2}\right)^n (2^n+1)}{5^n+1}$$

the leading term in the denominator of $\frac{\left(\frac{5}{2}\right)^n (2^n+1}{5^n+1}$ is 5^n. Dividing the numerator and denominator yields

$$= \lim_{n\to\infty} \frac{2^{-n}+1}{5^{-n}+1}$$

The expressions 2^{-n} and 5^{-n} both tend to zero as $n \to \infty$:

$$= \frac{1}{1}$$

$$= 1$$

So the series converges

5 (a) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}$ is an alternating series

(i) $\lim_{n \to \infty} \frac{1}{n+1} = 0$ $\qquad a_n = \frac{1}{n+1} \geq 0$

(ii) $a_{n+1} \leq a_n$ non-increasing

$$a_{n+1} = \frac{1}{n+2} \leq \frac{1}{n+1} = a_n \quad , \quad \forall n \geq 1$$

the series converges by alternating series test

$\lim_{n \to \infty} \frac{1}{n+1}$

Using the reciprocal rule,

$$\lim_{n \to \infty} \frac{1}{n+1} = \frac{1}{\lim_{n \to \infty} (n+1)}$$

Note that

$$\lim_{n \to \infty} (n+1) = \lim_{n \to \infty} n$$

$$\lim_{n \to \infty} \frac{1}{n+1} = \frac{1}{\lim_{n \to \infty} n}$$

$\therefore$ Since $\lim_{n \to \infty} n = \infty$

$$\Rightarrow \lim_{n \to \infty} \frac{1}{n+1} = 0$$

(b) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3n+2}$

$a_n = \frac{1}{3n+2} > 0 \qquad a_n > 0$

(i) $\lim_{n\to\infty} \frac{1}{3n+2}$

$= \frac{1}{\lim_{n\to\infty}(3n+2)}$

$= \frac{1}{3\lim_{n\to\infty} n}$

$= 0$

(ii) $a_{n+1} = \frac{1}{3n+3} \leq \frac{1}{3n+2} = a_n$

So the series converges

(c) $\sum_{n=1}^{\infty} \frac{(-1)^n}{3^n} \qquad a_n = \frac{1}{3^n} > 0$

(i) $\lim_{n\to\infty} \frac{1}{3^n} = \lim_{n\to\infty} 3^{-n}$

since $3 > 1$ and 3^n grows faster

$\lim_{n\to\infty} 3^{-n} = 0$

(ii) $a_{n+1} = \frac{1}{3^{n+1}} \leq \frac{1}{3^n} = a_n$

the series converges

(d) $\sum_{n=1}^{\infty} \frac{(-1)^n}{e^n}$

$a_n = \frac{1}{e^n} > 0$

(i) $\lim_{n\to\infty} \frac{1}{e^n} = \lim_{n\to\infty} e^{-n}$

$= e^{\lim_{n\to\infty}(-n)}$

$= e^{-\lim_{n\to\infty} n}$

$= 0$ since $\lim_{n\to\infty} n = \infty$

(ii) $a_{n+1} = \frac{1}{e^{n+1}} \leq \frac{1}{e^n} = a_n$

monotonic decreasing

$e^{n+1} \geq e^n$ $\qquad a > b$

since $n+1 \geq n$ $\qquad \frac{1}{a} < \frac{1}{b}$

the series converges

(e) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{\ln(n+1)}$

$a_n = \frac{n}{\ln(n+1)} > 0$

(i) $\lim_{n\to\infty} \frac{n}{\ln(n+1)} = \infty$

Since the polynomial n grows asymptotically faster than $\ln(n+1)$ as $n \to \infty$.
Additionally $n > 0$ and $\ln(n+1) > 0$ as n approaches ∞, the series diverges

6 (a) $\sum_{n=1}^{\infty} \frac{1}{2n}$

$$a_n = \frac{1}{2n} \qquad a_{n+1} = \frac{1}{2^{n+1}}$$

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{2n}{2^n + 1} \right|$$

The leading term in the denominator $\frac{2^n}{2^n+1}$ is 2^n

Divide the numerator and denominator by this

$$= \lim_{n \to \infty} \frac{1}{2^{-n} + 1}.$$

$$\left(\lim_{n \to \infty} 2^{-n} = 0 \right)$$

$$= \frac{1}{1}$$

$$= 1$$

ratio test is inconclusive

(b) $\sum_{n=1}^{\infty} \frac{n!}{3^n}$.

$$a_n = \frac{n!}{3^n} \qquad a_{n+1} = \frac{(n+1)!}{3^{n+1}}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{3^{n+1}} \cdot \frac{3^n}{n!}\right|$$

$$= \lim_{n\to\infty} \left|\frac{(n+1)!}{3\,n!}\right|$$

$$= \lim_{n\to\infty} \left|\frac{(n+1)\cancel{n!}}{3\cancel{n!}}\right|$$

$$= \frac{1}{3}\lim_{n\to\infty} (n+1)$$

$$= \infty$$

the series diverges

(c) $\sum_{n=1}^{\infty} n\left(\frac{6}{5}\right)^n$

$$a_n = n\left(\frac{6}{5}\right)^n \qquad a_{n+1} = (n+1)\left(\frac{6}{5}\right)^{n+1}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)\left(\frac{6}{5}\right)^{n+1}}{n\left(\frac{6}{5}\right)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n+1}{n}\cdot\left(\frac{6}{5}\right)\right|$$

$$= \frac{6}{5}\lim_{n\to\infty}\left|1+\frac{1}{n}\right|$$

$$= \frac{6}{5} > 1.$$

the series diverges

(d) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+2)}{n(n+1)}$

$a_n = \frac{(-1)^{n+1}(n+2)}{n(n+1)}$ $\quad a_{n+1} = \frac{(-1)^{n+2}(n+3)}{(n+1)(n+2)}$

$= \lim_{n\to\infty} \left| \frac{(-1)^{n+2}(n+3)}{\cancel{(n+1)}(n+2)} \cdot \frac{n\cancel{(n+1)}}{(-1)^{n+1}(n+2)} \right|$

$= \lim_{n\to\infty} \left| \frac{-n(n+3)}{(n+2)(n+2)} \right|$

$= \lim_{n\to\infty} \left| \frac{-n^2-3n}{n^2+4n+4} \right|$

$= \lim_{n\to\infty} \left| \frac{-1-\frac{3}{n}}{1+\frac{4}{n}+\frac{4}{n^2}} \right|$

$= \lim_{n\to\infty} |-1| = 1$

Thus Ratio test is inconclusive

(e) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\left(\frac{3}{2}\right)^n}{n^2}$

$$a_n = \frac{(-1)^{n-1}\left(\frac{3}{2}\right)^n}{n^2} \qquad a_{n+1} = \frac{(-1)^n\left(\frac{3}{2}\right)^{n+1}}{(n+1)^2}$$

$$\lim_{n\to\infty}\left|\frac{(-1)^n\left(\frac{3}{2}\right)^{n+1}}{(n+1)^2}\cdot\frac{n^2}{(-1)^{n-1}\left(\frac{3}{2}\right)^n}\right| \qquad \frac{3}{2}^{(n+1-n)}$$

$$= \lim_{n\to\infty}\left|\frac{(-1)\left(\frac{3}{2}\right)n^2}{(n+1)^2}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(-1)\,3n^2}{2(n+1)^2}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(-1)\,3n^2}{2n^2+4n+2}\right|$$

$$= \lim_{n\to\infty}\left|\frac{-3}{2+\frac{4}{n}+\frac{2}{n^2}}\right|$$

$$= \frac{3}{2} > 1$$

the series diverges

7. Root test

$\sum a_n$ converges absolutely if $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$

$\sum a_n$ diverges is $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$

or

$\lim_{n\to\infty} \sqrt[n]{a_n} = \infty$

(a) $\sum_{n=1}^{\infty} \left(\frac{2n}{n+1}\right)^n$

$a_n = \left(\frac{2n}{n+1}\right)^n \qquad a_{n+1} = \left(\frac{2n+2}{n+2}\right)^{n+1}$

$= \lim_{n\to\infty} \sqrt[n]{\left|\frac{2n}{n+1}\right|^n}$

$= \lim_{n\to\infty} \left|\frac{2n}{n+1}\right|$

$= \lim_{n\to\infty} \left|\frac{2}{1+\frac{1}{n}}\right|$

$= 2 > 1$

thus series diverges

(b) $\sum_{n=1}^{\infty} \left(\frac{4n+3}{2n-1}\right)$

the root test is ~~[illegible]~~ inconclusive since $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$

$\lim_{n\to\infty} \sqrt[n]{\left|\frac{4n+3}{2n-1}\right|}$

$= \lim_{n\to\infty} \left(\frac{4n+3}{2n-1}\right)^{1/n}$

$= \lim_{n\to\infty} e^{\ln\left(\frac{4n+3}{2n-1}\right)^{\frac{1}{n}}}$

$= \lim_{n\to\infty} e^{\frac{\ln\left(\frac{4n+3}{2n-1}\right)}{n}}$

$= e^{\lim_{n\to\infty} \frac{\ln\left(\frac{4n+3}{2n-1}\right)}{n}}$

$= e^{\frac{\ln\left(\lim_{n\to\infty} \frac{4n+3}{2n-1}\right)}{\lim_{n\to\infty} n}}$

$= e^{\frac{\ln\left(\lim_{n\to\infty} \frac{4+\frac{3}{n}}{2-\frac{1}{n}}\right)}{\lim_{n\to\infty} n}}$

$\frac{\ln 2}{n}$

$= e^{\frac{\ln 2}{\lim_{n\to\infty} n}}$ note $\lim_{n\to\infty} n = \infty$

$= e^{0}$ note $\frac{\ln 2}{\infty} = \ln 2\left(\frac{1}{\infty}\right) = 0$

$= 1$

(c) $\sum_{n=0}^{\infty} e^{-3n}$

$$= \lim_{n\to\infty} \sqrt[n]{|e^{-3n}|}$$

$$= \lim_{n\to\infty} |(e^{-3n})|^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \left|\frac{1}{e^{3}}\right| < 1$$

$\therefore \sum_{n=0}^{\infty} e^{-3n}$ converges absolutely so it converges

(d) $\sum_{n=2}^{\infty} \frac{n}{(\ln n)^n}$

$$= \lim_{n\to\infty} \sqrt[n]{\left|\frac{n}{(\ln n)^n}\right|}$$

$$= \lim_{n\to\infty} \left|\frac{n}{(\ln n)^n}\right|^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \left|\frac{n^{\frac{1}{n}}}{\ln n}\right|$$

By Quotient rule

$$= \frac{\lim_{n\to\infty} n^{\frac{1}{n}}}{\lim_{n\to\infty} \ln n}$$

$$= \frac{\lim_{n\to\infty} e^{\ln n^{\frac{1}{n}}}}{\lim_{n\to\infty} \ln n}$$

$$= \frac{\lim_{n\to\infty} e^{\frac{\ln n}{n}}}{\lim_{n\to\infty} \ln n}$$

$$= \frac{e^{\lim_{n\to\infty} \frac{\ln n}{n}}}{\lim_{n\to\infty} \ln n}$$

Since $\ln n$ grows asymptotically slower ~~than~~ the polynomial n as $n \to \infty$ $\lim_{n\to\infty} \frac{\ln n}{n} = 0$

$$= \frac{e^{0}}{\lim_{n\to\infty} \ln n}$$

$$= \frac{1}{\lim_{n\to\infty} \ln n} \qquad \lim_{n\to\infty} \ln n = \infty$$

$$= 0 < 1$$

So the series converges

(8) $f(x) = y$

$$P_n(x_0) = \sum_{n=0}^{n} \frac{f^{(n)}(0)}{n!}(x)^n \quad \text{maclaurin polynomial}$$

$$P_n(x_0) = \sum_{n=0}^{n} \frac{f^{(n)} x_0}{n!}(x-x_0)^n \quad \text{Taylor polynomial}$$

(a) $f(x) = e^{3x}$; $n = 4$

$$P_4(0) = \sum_{n=0}^{4} \frac{\frac{d}{dx} e^{3x}\Big|_{x_0=0}}{n!} x^n$$

$$= 1 + \frac{3e^{3x}\big|_{x_0=0}}{1!} . x + \frac{9e^{3x}\big|_{x_0=0} . x^2}{2!} + \frac{27e^{3x}\big|_{x_0=0} x^3}{3!}$$

$$+ \frac{81e^{3x}\big|_{x_0=0} . x^4}{4!} + \ldots$$

$$e^{3x} = 1 + 3x + \frac{9}{2!}x^2 + \frac{27}{3!}x^3 + \frac{81}{4!}x^4 + \ldots = \sum_{n=0}^{\infty} \frac{3^n x^n}{n!}$$

(b) $f(x) = x^2 e^{-x}$; $n = 4$

$$P_4(0) = \sum_{n=0}^{4} \frac{\frac{d}{dx} x^2 e^{-x}\big|_{x_0=0}}{n!} x^n \qquad \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

$$= 0 + \sum_{n=1}^{4} \frac{\frac{d}{dx}\frac{x^2}{e^x}\Big|_{x_0=0}}{n!} . x^n$$

$$= \frac{2xe^x - x^2e^x}{e^{2x}}\Big|^{x^0=0} x + \frac{1}{2!}\left[\frac{e^x(2-2x)e^{2x} - 2e^{2x}e^x(2x-x^2)}{e^{4x}}\right]^{x^0=0} x^2 +$$

$$\frac{1}{3!}\frac{\left(e^x(2-2x) + e^x(-2) - 6e^{3x}(2x-x^2) - 2e^{3x}(2-2x)\right)e^{4x}}{e^{8x}} x^3$$

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$$+\frac{1}{4!}\frac{d}{dx}\left(\frac{(-2xe^{x}-8xe^{3x}+6x^2e^{3x}-4e^{3x})e^{4x}}{e^{8x}}-\frac{4e^{4x}(2e^{x}-2xe^{x}-4xe^{3x}+2x^2e^{3x})}{e^{8x}}\right)\Bigg|_{x_0=0}\cdot x^4$$

$$= x^2 e^{-x} = \sum_{n=2}^{\infty}\frac{x^n(-1)^n}{(-2+n)!}$$

$$= x^2 - x^3 + \frac{x^4}{2} + \dots \doteq \sum_{n=2}^{4}\frac{x^n(-1)^n}{(-2+n)!}$$

(c) $f(x) = \frac{x}{x+1}$

$$P_n(0) = \sum_{n=0}^{4}\frac{f^{(n)}(0)}{n!}x^n$$

$$\frac{x}{1+x} = \frac{1}{2} + \sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^{1+n}(-1+x)^n$$

$$\frac{x}{1+x} = x - x^2 + x^3 - x^4 + \dots$$

$$= \sum_{n=0}^{4}\frac{f^{(n)}(0)\,x^n}{n!}$$

$$\frac{d^n y}{dx^n} = \frac{(-1)^{n-1}\,n!}{(x+1)^{n+1}}$$

$= \frac{1}{(x+1)^2}$ +

$$= \left[\frac{1}{(x+1)^2}x - \frac{1}{2!}\frac{2}{(x+1)^3}x^2 + \frac{1}{3!}\frac{6}{(x+1)^4}x^3 + \frac{1}{4!}\frac{24}{(x+1)^5}x^4 + \cdots\right]_{x_0=0}$$

$$= x - x^2 + x^3 - x^4 + \cdots$$

(d) $f(x) = \sec x$

$$P_n(x) = \sum_{n=0}^{4} \frac{f^{(n)}(0)}{n!} x^n$$

Expansion at $x_0 = 0$

$$\sec x = -2\sum_{n=1}^{\infty} (-1)^n q^{-1+2n} \quad \text{for } q = e^{jx}$$

$$= \frac{\sec 0}{0!}x^0 + \frac{\frac{d}{dx}\sec x\big|_{x_0=0}}{1!}.x + \frac{\frac{d^2}{dx^2}\sec x\big|_{x_0=0}}{2!}.x^2 + \cdots$$

$$= 1 + \frac{\sec x \tan x\big|_{x_0=0}}{1!}\cdot x + \frac{\sec^3 x + \tan^2 x \sec x\big|_{x_0=0}}{2!}.x^2$$

$$= 1 + \frac{x^2}{2} + \cdots$$

9. n^{th} Taylor polynomial

$$P_n(x) = \sum_{n=0}^{n} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

(a) $f(x) = \frac{2}{x}$; $n=3$, $x_0 = 1$

$$P_n(x) = \frac{2}{x}\Big|_{x_0=1}(x-x_0)^0 + \frac{1}{1!}\frac{d}{dx}\left(\frac{2}{x}\right)\Big|_{x_0=1}(x-x_0)^1 + \frac{1}{2!}\frac{d^2}{dx^2}\left(\frac{2}{x}\right)\Big|_{x_0=1}(x-x_0)^2$$

$$+ \frac{1}{3!}\frac{d^3}{dx^3}\left(\frac{2}{x}\right)\Big|_{x_0=1}(x-x_0)^3 + \dots$$

$$= 2 + \left(\frac{-2}{x_0^2}\right)(x-1) + \frac{1}{2}\left(\frac{4}{x_0^3}\right)(x-1)^2 + \frac{1}{6}\left(\frac{-12}{x_0^3}\right)(x-1)^3$$

$$= 2 - 2(x-1) + 2(x-1)^2 - \cancel{2}(x-1)^3 + \dots \quad \text{(open form)}$$

n^{th} derivative of $f(x) = \frac{2}{x}$

$$f'(x) = \frac{-2}{x^2}$$

$$f''(x) = \frac{4}{x^3}$$

$$f'''(x) = \frac{-12}{x^4}$$

$$f^n(x) = \frac{2(-1)^n n!}{x^n}$$

$$\Rightarrow P_n(x) = \sum_{n=0}^{3} \frac{2(-1)^n \cancel{n!}}{x_0^n \cancel{n!}} (x - x_0)^n \Big|_{x_0 = 1}$$

i.e $$f^n(x_0) = \frac{2(-1)^n n!}{x_0^n}$$

$$P_n(x) = \sum_{n=1}^{3} 2(-1)^n (x-1)^n$$

(b) $f(x) = \sqrt{x}$; $n = 3$, $c = 4$

nth derivative by successive differentiation

$f(x) = \sqrt{x}$ $\qquad$ 0th order

$f'(x) = \frac{1}{2} x^{-1/2}$ $\qquad$ 1st order

$f''(x) = -\frac{1}{4} x^{-3/2}$ $\qquad$ 2nd order

$f'''(x) = \frac{3}{8} x^{-\frac{5}{2}}$ $\qquad$ 3rd order

$f^n(x) = x^{(1-2n)\frac{1}{2}}$?

$$P_n(x) = \frac{\sqrt{x_0}\,(x-4)^0}{0!} + \frac{\frac{d}{dx}\sqrt{x_0}\,(x-4)}{1!} + \frac{\frac{d^2}{dx^3}\sqrt{x_0}\,(x-4)^2}{2!} + \frac{\frac{d^3}{dx^3}\sqrt{x_0}\,(x-4)^3}{3!} + \cdots \Bigg|_{x_0=4}$$

$$= 2 + \frac{1}{2}\sqrt{(4)}^{-1/2}(x-4) + \frac{1}{2!}\left(-\frac{1}{4}(4)^{-3/2}\right)(x-4)^2 + \frac{1}{3!}\left(\frac{3}{8}(4)^{-5/2}\right)(x-4)^3 + \cdots$$

$$= 2 + \frac{x-4}{4} - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3 + \cdots$$

$$= \sum_{n=0} 2^{1-2n}(-4+x)^n \binom{\frac{1}{2}}{n}$$

(c) $f(x) = x^2 \cos x$; $n = 2$, $x_0 = \pi$

n^{th} derivatives

$$f'(x) = 2x\cos x - x^2 \sin x \Big|_{x_0=\pi} = -2\pi$$

$$f''(x) = 2\cos x - 2x\sin x - 2x\sin x - x^2\cos x$$

$$= 2\cos x - 4x\sin x - x^2\cos x \Big|_{x_0=\pi} = -2+\pi^2$$

$$f'''(x) = -2\sin x - 4\sin x - 4x\cos x + x^2\sin x - 2x\cos x$$

$$= -6\sin x - 6x\cos x + x^2\sin x \Big|_{x_0=\pi} = -6\pi$$

$$P_n(x) = \sum_{n=0}^{2} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

$$= -\pi^2 - 2\pi(x-\pi) + \frac{(-2+\pi^2)}{2!}(x-\pi)^2$$

$$= -\pi^2 - 2\pi(x-\pi) + \frac{1}{2}(-2+\pi^2)(x-\pi)^2 + \cdots$$

$$= -\pi^2 - 2\pi(x-\pi) + \left(\frac{\pi^2}{2} - 1\right)(x-\pi)^2 + \cdots$$

(10) (a) $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1}$

Let $a_n = \frac{(-1)^n}{n+1}$

by ratio test we have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} \cdot x^{n+1}}{n+2} \cdot \frac{n+1}{x^n (-1)^n} \right|$$

$$= \lim_{n\to\infty} |x| \left|\frac{n+1}{n+2}\right|$$

$$= |x| \lim_{n\to\infty} \left|\frac{n+1}{n+2}\right| = 1$$

By the ratio test, the series converges when $|x| < 1$

Radius of Convergence : $R = \frac{1}{L} = 1$

(b) $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{5^n}$

$a_n = \frac{(-1)^n}{5^n}$

By ratio test $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

$$= \lim_{n\to\infty} \left|\frac{(-1)^{n+1}}{5^{n+1}} \cdot \frac{5^n}{(-1)^n}\right|$$

$$= \lim_{n\to\infty} \left|-5^{-1}\right|$$

$$= \lim_{n\to\infty} \left|-\frac{1}{5}\right| = \frac{1}{5}$$

$R = \frac{1}{\cancel{5}}$ $\sum_{n=1}^{\infty} \left(-\frac{1}{5}\right)^n x^n$ Converges when $|x| < 5$

$R = 5$

(b) $\sum_{n=0}^{\infty} (4x)^n$

$\sum_{n=0}^{\infty} 4^n x^n$

$a_n = 4^n$ $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{4^{n+1}}{4^n} \right| = 4$

$R = \frac{1}{4}$ Converges when $|x| < \frac{1}{4}$

(d) $\sum_{n=0}^{\infty} \frac{(2n)!\, x^{2n}}{n!}$ $R = \frac{1}{2}$

$a_n = \frac{(2n)!}{n!}$

$= \lim_{n \to \infty} \left| \frac{(2n+1)!}{(n+1)!} \cdot \frac{n!}{(2n)!} \right|$ Converges when $|x| < \frac{1}{2}$

$= \lim_{n \to \infty} \left| \frac{(2n+1)\cancel{(2n)!}}{\cancel{n!}(n+1)} \cdot \frac{\cancel{n!}}{\cancel{(2n)!}} \right|$

$= \lim_{n \to \infty} |2|$

$= 2$

when $x = 7$ the series becomes

$$\sum_{n=0}^{\infty} \frac{\cancel{4^{n+1}}}{(n+1)\cancel{4^{n+1}}} = \sum_{n=0}^{\infty} \frac{1}{n+1}$$

By intergral test

(i) $a_n > 0 \quad \forall n \geq 0$

(ii) a_n is continuous on $\forall n \geq 0$

(iii) $a_n = f(n) \quad f'(n) < 0 \quad f(n)$ is decreasing

$$f'(n) = \frac{-1}{(n+1)^2} < 0, \quad \forall n \geq 0$$

$$\int_0^{\infty} \frac{1}{n+1}\, dn$$

let $p = n+1$
$dp = dn$

$$= \int_1^{\infty} \frac{1}{p}\, dp$$

$$= \Big[\ln|p|\Big]_1^{\infty}$$

$$= \lim_{b \to \infty} \ln b - \ln(1)$$

$$= \infty$$

diverges

10(a)

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1}$$

let $a_n = \frac{(-1)^n}{n+1}$

by ratio test.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}\cdot x^{n+1}}{n+2}\cdot\frac{n+1}{x^n(-1)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n+1}{n+2}\right|$$

$$= |x| \lim_{n\to\infty}\left|\frac{n+1}{n+2}\right| = 1$$

the series converges because $|x|<1$

(b) $\sum_{n=0}^{\infty}(4x)^n$ $\quad \sum_{n=0}^{\infty} 4^n x^n$

$a_n = 4^n \quad \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{4^{n+1}}{4^n}\right| = 4.$

$R = \frac{1}{4}$ converges when $|x| < \frac{1}{4}$

(c) $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{5^n}$ $\quad a_n = \frac{(-1)^n}{5^n}$ By ratio test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{5^{n+1}}\right|\cdot\frac{5^n}{(-1)^n}$$

$$= \lim_{n\to\infty}\left|-5^{-1}\right| = \lim_{n\to\infty}\left|-\frac{1}{5}\right| = \frac{1}{5}.$$

$R = \frac{1}{\frac{1}{5}}$ $\sum_{n=1}^{\infty}\left(\frac{-1}{5}\right)^n x^n$ converges when $|x|<5$

$\underline{R=5}$.

(d) $\sum_{n=0}^{\infty} \frac{(2n)!\, x^{2n}}{n!}$ $\quad a_n = \frac{(2n)!}{n!}$

$= \lim_{n\to\infty} \left| \frac{(2n+1)!}{(n+1)!} \cdot \frac{n!}{2n!} \right|$

$= \lim_{n\to\infty} \left| \frac{(2n+1)(2n)!}{n!\,(n+1)} \cdot \frac{n!}{(2n)!} \right|$

$= \lim_{n\to\infty} |2| = 2.$

$R = \frac{1}{2}$ Converges when $|x| < \frac{1}{2}$

(11) (a) $\sum_{n=0}^{\infty} \frac{(x-3)^{n+1}}{(n+1)4^{n+1}}$

$= \lim_{n\to\infty} \left| \frac{(x-3)^{n+2}}{(n+2)4^{n+2}} \right| \cdot \frac{(n+1)\, 4^{n+1}}{(x-3)^{n+1}}$

$= \lim_{n\to\infty} \left| \frac{(x-3)(n+1)}{4(n+2)} \right| = \left| \frac{x-3}{4} \right| \lim_{n\to\infty} \left| \frac{n+1}{n+2} \right| = \left| \frac{x-3}{4} \right|$

the series by ratio test converges when $|x-3| < 4$

Interval of converges $|x-3| < 4 = -1 < x < 7$.

(b) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2n+1}}{2n+2} \cdot \frac{2n+1}{(-1)^n x^{2n+1}} \right|$

$= \lim_{n\to\infty} \left| \frac{(-1)x(2n+1)}{2n+2} \right|$ $\quad R = 1$

the series converges when $|x| < 1$

Interval $= -1 < x < 1$

11 (a) $\sum_{n=0}^{\infty} \frac{(x-3)^{n+1}}{(n+1)4^{n+1}}$

$$= \lim_{n\to\infty} \left| \frac{(x-3)^{n+2}}{(n+2)4^{n+2}} \cdot \frac{(n+1)\,4^{n+1}}{(x-3)^{n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(x-3)(n+1)}{4(n+2)} \right|$$

$$= \left| \frac{x-3}{4} \right| \lim_{n\to\infty} \left| \frac{n+1}{n+2} \right| = \left| \frac{x-3}{4} \right|$$

the series by ratio test converges when $|x-3| < 4$

Interval of convergence $|x-3| < 4 = -1 < x < 7$

(b) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$

$$= \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2n+2}}{2n+2} \cdot \frac{2n+1}{(-1)^n x^{2n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1)\, x\, (2n+1)}{2n+2} \right| \qquad R = 1$$

the series converges when $|x| < 1 = -1 < x < 1$

(c) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$

$$= \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2n+1}}{(n+1)!} \cdot \frac{n!}{(-1)^n x^{2n}} \right|$$

$$= \lim_{n\to\infty} |-x| \left| \frac{1}{n+1} \right| = 0$$

$R = \infty$

this series converges on $\forall x = -\infty < x < \infty$

Second part of testing for convergence at end points of Interval of convergence cont...

$\longrightarrow$

Checking for convergence at the end points of the interval of convergence.

We use

(i) Integral test

(ii) Alternating series test

(iii) Geometric series test

(iv) power series test

11 (a) $(-1, 7)$

when $x = -1$ the series is

$$\sum_{n=0}^{\infty} \frac{(-4)^{n+1}}{(n+1)4^{n+1}} = \sum_{n=0}^{\infty} (-1)^{n+1}\left(\frac{1}{n+1}\right)$$

by an alternating series test

$$a_n = \frac{1}{n+1}$$

$$a_{n+1} = \frac{1}{n+2}$$

(1) $\lim_{n \to \infty} \frac{1}{n+1} = 0$ ---- (satisfied)

(2) $a_{n+1} \leq a_n$

$$a_{n+1} = \frac{1}{n+2} \leq \frac{1}{n+1} = a_n \text{ ---- satisfied}$$

$$n+2 \geq n+1$$

So the series converges at $x = -1$

FIZIE MATH LAB

(c) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$

$= \lim_{n \to \infty} \left| \frac{(-1)^{n+1} x^{2n+1}}{(n+1)!} \right| \cdot \frac{n!}{(-1)^n x^{2n}}$

$= \lim_{n \to \infty} |-x| \left| \frac{1}{n+1} \right| = 0$

$R = \infty$

the series converges on

$|x| = -\infty < x < \infty$

THE UNIVERSITY OF ZAMBIA
SCHOOL OF NATURAL SCIENCES
DEPARTMENT OF MATHEMATICS & STATISTICS
MAT2100 Tutorial Sheet 4

May, 2018

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1. Evaluate the Riemann sum for $f(x) = x^2 + 1$ on the interval $[-1, 2]$ using equally spaced partition points $-1 < -0.5 < 0 < 0.5 < 1 < 1.5 < 2$, with the sample point $\overline{x}_i$ being the midpoint of the ith subinterval.

2. Evaluate the Riemann sum for
$$f(x) = (x+1)(x-2)(x-4) = x^3 - 5x^2 + 2x + 8$$
on the interval $[0, 5]$ using the partition P with points $0 < 1.1 < 2 < 3.2 < 4 < 5$ and the corresponding sample points $\overline{x}_1 = 0.5$, $\overline{x}_2 = 1.5$, $\overline{x}_3 = 2.5$, $\overline{x}_4 = 3.6$ and $\overline{x}_5 = 5$.

3. Use the definition to evaluate the following definite integrals
(a) $\int_{-2}^{3} (x+3)dx$ (b) $\int_{-1}^{3} (2x^2 - 8)dx$

4. Sketch the region corresponding to each definite integral. Then evaluate each integral using a geometric formula
(a) $\int_{1}^{3} 4dx$ (b) $\int_{0}^{3} (x+2)dx$ (c) $\int_{-2}^{2} \sqrt{4-x^2}dx$

5. Evaluate the following integrals using the limit definition:
(a) $\int_{-1}^{1} x^3 dx$ (b) $\int_{1}^{2} (x^2+1)dx$ (c) $\int_{1}^{2} 4x^2 dx$

6. Suppose f is integrable on $[a, b]$ and $m \le f(x) \le M$ for all $x \in [a, b]$, and $m > 0$, $M > 0$. Prove that
$$m(b-a) \le \int_a^b f(x)dx \le M(b-a)$$
Use this result to estimate $\int_0^1 \sqrt{1+x^4}dx$.

7. Evaluate the following derivatives:
(a) $\frac{d}{dx}\left[\int_1^x t^3 dt\right]$ (b) $\frac{d}{dx}\left[\int_2^x \frac{t^{3/2}}{\sqrt{t^2+17}}dt\right]$

(c) $\frac{d}{dx}\left[\int_x^4 \tan^2 t \cos t dt\right], \quad \frac{\pi}{2} < x < \frac{3\pi}{2}$

8. Find the average value of the function defined by $f(x) = x \sin x^2$ on the interval $[0, \sqrt{\pi}]$.

9. Find all values c that satisfy the mean value Theorem for integrals for $f(x) = \frac{1}{(x+1)^2}$ on $[0, 2]$.

10. The region $\mathfrak{R}$ under the curve $y = 1/x^2\sqrt{x^2-1}$, above the x-axis and between the lines $x = 2/\sqrt{3}$ and $x = 2$, is revolved around the y-axis. Find the volume of the resulting solid.

11. Find the volume of the solid generated when the region between the semicircle $y = 1 - \sqrt{1-x^2}$ and the line $y = 1$ is rotated around the x-axis.

12. Sketch and find the area of the region between the curve $y = x^3$ and the lines $y = -x$ and $y = 1$.

13. Find the area of the region bounded by the parabola $x = y^2 + 2$ and the line $y = x - 8$.

14. Find the area of the bounded region in the first quadrant between the curves $4y + 3x = 7$ and $y = x^{-2}$.

15. Find the length of the arc of $24xy = x^4 + 48$ from $x = 2$ to $x = 4$.

16. Find the length of the arc of $27y^2 = 4(x-2)^3$ from $(2, 0)$ to $(11, 6\sqrt{3})$.

17. Find the length of the arc of $x = 3y^{3/2} - 1$ from $y = 0$ to $y = 4$.

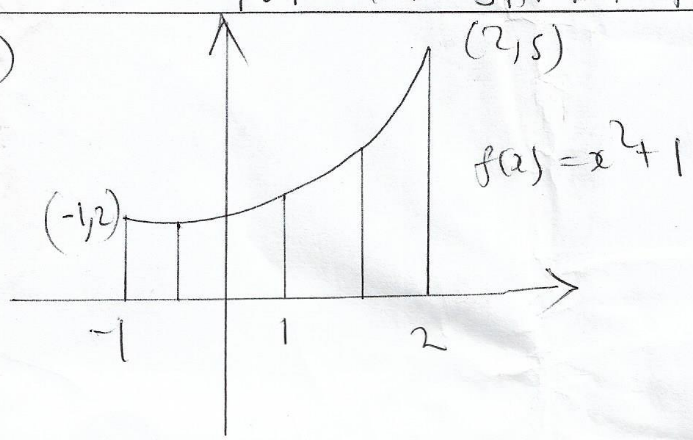
18. Find the centroid of the given planar region bounded by the following region:
   (a) $y = x^2, y = 0$ and $x = 1$
   (b) the semicircle $y = \sqrt{a^2 - x^2}$, and $y = 0$
   (c) $y = \sin x, y = 0$ from $x = 0$ to $x = \pi$

19. Find the volume generated by revolving the given region about the given axis.
   (a) $y = x^3$, under the line $y = 1$, and between $x = 0$ and $x = 1$; about the x-axis
   (b) $y = 2x$, above the x-axis, and between $x = 0$ and $x = 1$; about the y-axis

Done

TUTORIAL SHEET 4

(1)



$$\text{Riemann Sum} = \sum_{i=1}^{n} f(\bar{x}_i)\,\Delta x$$

$$= \sum_{i=1}^{6} \left((\bar{x}_i)^2+1\right)\frac{1}{2}$$

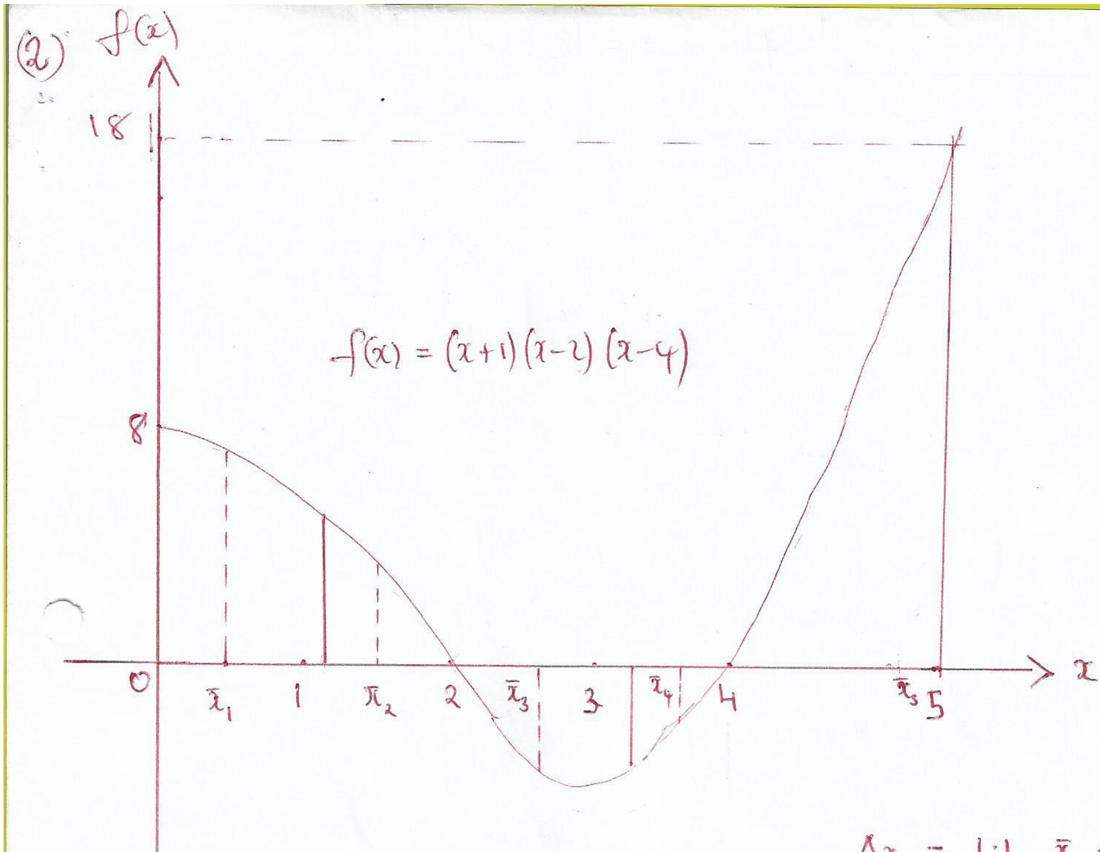
$\bar{x}_j = -\frac{1}{2} + \frac{1}{4}i \qquad \Delta x = \frac{1}{2}$

$\bar{x}_j = -0.75,\ -0.25,\ 0.25,\ 0.75,\ 1.25,\ 1.75$

$$\text{Sum} = 2\left((0.75)^2+1\right)0.5 + 2\left((0.25)^2+1\right)0.5 + \left((1.25)^2+1\right)0.5$$
$$+ \left((1.75)^2+1\right)0.5$$
$$= 1.0625 + 1.5625 + 1.28125 + 2.03125$$

(ANS): $= \underline{\underline{5.9375}}$

(2)



$$\lim_{n \to \infty} \sum_{j=1}^{n} f(\bar{x}_j)\, \Delta x$$

$\Delta x_1 = 1.1$, $\bar{x}_1 = 0.5$

$\Delta x_2 = 0.9$, $\bar{x}_2 = 1.5$

$\Delta x_3 = 1.2$, $\bar{x}_3 = 2.5$

$\Delta x_4 = 0.8$, $\bar{x}_4 = 3.6$

$\Delta x_5 = 1$, $\bar{x}_5 = 5$

neiman sum $= \sum_{j=1}^{n} f(\bar{x}_j)\, \Delta x_j$

$= \sum_{j=1}^{5} f(\bar{x}_j)\, \Delta x_j$

$= (0.5+1)(0.5-2)(0.5-4)(1.1) + (1.5+1)(1.5-2)(1.5-4)(0.9) +$
$(2.5+1)(2.5-2)(2.5-4)(1.2) + (3.6+1)(3.6-2)(3.6-4)(0.8)$
$+ 18$

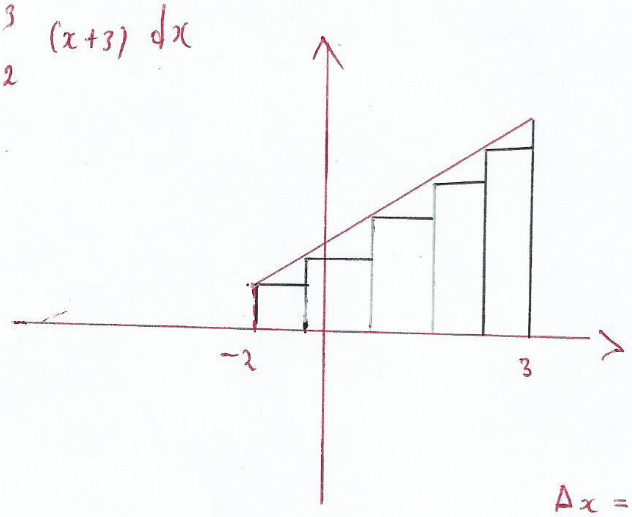
$= (1.5)(-1.5)(-3.5)(1.1) + (2.5)(-0.5)(-2.5)(0.9) + (3.5)(0.5)(-1.5)$
$+ (4.6)(1.6)(-0.4)(0.8) + 18 = \underline{\underline{24.4948}}$ (ANS) (2)

FIZIE MATH LAB

(3) Definition

$$\int_{x_0}^{x_n} f(x)\,dx = \lim_{n \to \infty} \sum_{j=1}^{n} f(c_j)\,\Delta x \; ; \quad c_j \in (x_0, x_n) \; , \; \Delta x = \frac{x_0 + x_n}{n}$$

(a) $\displaystyle\int_{-2}^{3} (x+3)\,dx$



$$\Delta x = \frac{5}{n}$$

Choosing x_0 the right end point we obtain

$$c_i = -2 + \frac{5}{n} i$$

$$= \lim_{n \to \infty} \sum_{j=1}^{n} f(c_j)\,\Delta x = \int_{x_0}^{x_n} f(x)\,dx$$

$$= \lim_{n \to \infty} \sum_{j=1}^{n} \left(-2 + \frac{5}{n} i + 3\right) \frac{5}{n}$$

$$= \lim_{n \to \infty} \sum_{j=1}^{n} \left(1 + \frac{5}{n} i\right) \frac{5}{n}$$

FIZIE MATH LAB

$$= \lim_{n\to\infty}\left[\sum_{j=1}^{n}\left(\frac{5}{n} + \left(\frac{5}{n}\right)^2 j\right)\right]$$

$$= \lim_{n\to\infty}\left[\sum_{j=1}^{n}\frac{5}{n} + \sum_{j=1}^{n}\left(\frac{5}{n}\right)^2 j\right]$$

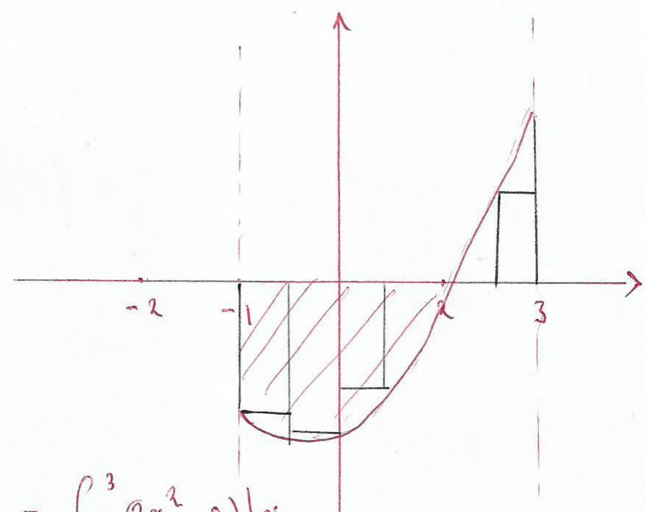
$$= \lim_{n\to\infty}\left[5 + \left(\frac{5}{n}\right)^2 \frac{1}{2}[n(n+1)]\right]$$

$$= 5 + \lim_{n\to\infty} \frac{25}{2}\left(\frac{n+1}{n}\right)$$

$$= 5 + \frac{25}{2}$$

$$= \underline{\frac{35}{2}} \quad \text{(ANS)}$$

(b) $\displaystyle\int_{-1}^{3} (2x^2 - 8)\, dx$



$$\lim_{n\to\infty}\sum_{j=1}^{n} f(c_i)\,\Delta x = \int_{-1}^{3} (2x^2 - 8)\,dx$$

$$\Delta x = \frac{4}{n}$$

Taking c_i as a mid point of a subinterval we have
$c_i = \frac{2}{n} j$

$\Delta x = \frac{4}{n}$

Choosing Right end point -1 we obtain

$c_i = -1 + \frac{4i}{n}$

$$\lim_{n\to\infty} \sum_{i=1}^{n} f\left(-1 + \frac{4}{n}i\right)\frac{4}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} \left[2\left(-1+\frac{4}{n}i\right)^2 - 8\right]\frac{4}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[2\left(1 - \frac{8}{n}i + \frac{16}{n^2}i^2\right) - 8\right]\frac{4}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[2 - \frac{16}{n}i + \frac{32}{n^2}i^2 - 8\right]\frac{4}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{128}{n^3}i^2 - \frac{64}{n^2}i - \frac{24}{n}\right]$$

$$= \lim_{n\to\infty} \left[\frac{128}{n^3}\sum_{i=1}^{n} i^2 - \frac{64}{n^2}\sum_{i=1}^{n} i - \sum_{i=1}^{n}\frac{24}{n}\right]$$

$$= \lim_{n\to\infty} \left[\frac{128}{n^2} \cdot \frac{n(n+1)(2n+1)}{6} - \frac{64}{n^2} \cdot \frac{n(n+1)}{2} - 24\right]$$

$$= \lim_{n\to\infty} \left[\frac{64}{3} \frac{(n+1)(2n+1)}{n^2} - \frac{32(n+1)}{n} - 24\right]$$

$$= \lim_{n\to\infty} \left[\frac{64}{3} \frac{2n^2+3n+1}{n^2} - 32\left(1+\frac{1}{n}\right) - 24\right]$$

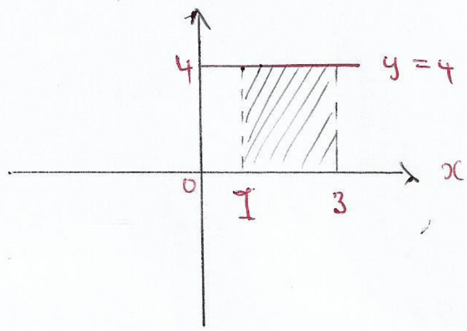
$$= \lim_{n\to\infty}\left[\frac{64}{3}\left(2+\frac{3}{n}+\frac{1}{n^2}\right)-32\left(1+\frac{1}{n}\right)-24\right]$$

$$= \lim_{n\to\infty}\left[\frac{128}{3}-32-24\right]$$

$$= \underline{\underline{\frac{-40}{3}}} \quad : (\text{ANS})$$

(4)

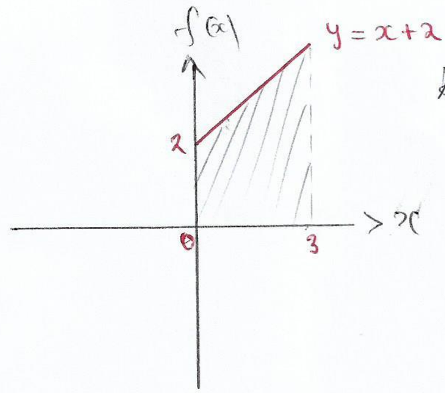
(a)



$Lb = (4)(2) = 8$ (ANS)

$$\int_1^3 4\,dx = 4x\Big|_1^3 = 8$$

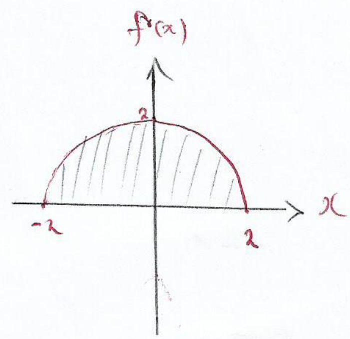
(b) $$\int_0^3 (x+2)\,dx = \frac{x^2}{2}+2x\Big|_0^3 = \frac{21}{2} = \frac{1}{2}(2+5)3 = \frac{21}{2}$$



ANS: $\frac{1}{2}(2+7)3 = \frac{21}{2}$

(c)

$$\int_{-2}^{2} \sqrt{4-x^2}\,dx$$



$4 - x^2 > 0$ $\quad = \frac{1}{2}\pi r^2$

$x^2 < 4$

$-2 < x < 2 \Rightarrow y > 0$ $\quad = 2\pi$ (ANS)

5

(a) $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$

$$\Delta x = \frac{b-a}{n}$$

$$x_i = a + i\Delta x$$

$$\int_{-1}^{1} x^3\,dx \qquad \Delta x = \frac{2}{n} \qquad x_i = -1 + \frac{2i}{n}$$

$$f(x_i) = (x_i)^3 = \left(-1 + \frac{2i}{n}\right)^3 = -1 + 6\frac{i}{n} - 12\frac{i^2}{n^2} + 8\frac{i^3}{n^3}$$

$$\sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n} \left(-1 + 6\frac{i}{n} - 12\frac{i^2}{n^2} + 8\frac{i^3}{n^3}\right)\frac{2}{n}$$

$$= \sum_{i=1}^{n} \left(-\frac{1}{n} + 6\frac{i}{n^2} - 12\frac{i^2}{n^3} + 8\frac{i^3}{n^4}\right)$$

$$= \sum_{i=1}^{n}\left(-\frac{1}{n}\right) + \sum_{i=1}^{n}\left(6\frac{1}{n^2}\right) - \sum_{i=1}^{n}\left(12\frac{i^2}{n^3}\right) + \sum_{i=1}^{n}\left(8\frac{i^3}{n^4}\right)$$

$$= -\frac{1}{8}\sum_{i=1}^{n}(1) + \frac{6}{n^2}\sum_{i=1}^{n}(i) - \frac{12}{n^3}\sum_{i=1}^{n}(i)^2 + \frac{8}{n^4}\sum_{i=1}^{n}(i)^3$$

$$= -\frac{1}{n}(n) + \frac{6}{n^2}\left(\frac{n(n+1)}{2}\right) - \frac{12}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) + \frac{8}{n^4}\left(\frac{n^2(n+1)^2}{4}\right)$$

$$\sum_{i=1}^{n} f(x_i)\,\Delta x = -\frac{1}{n}(n) + \frac{6}{n^2}\left(\frac{n(n+1)}{2}\right) - \frac{12}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) + \frac{8}{n^4}\left(\frac{n^2(n+1)^2}{4}\right)$$

$$= -1 + 3\left(\frac{n(n+1)}{n^2}\right) - 2\left(\frac{n(n+1)(2n+1)}{n^3}\right) + 2\left(\frac{n^2(n+1)^2}{n^4}\right)$$

8

$$\lim_{n\to\infty}\left(\frac{n(n+1)}{n^2}\right) = 1$$

$$\lim_{n\to\infty}\left(\frac{n(n+1)(2n+1)}{n^3}\right) = 2$$

$$\lim_{n\to\infty}\frac{n^2(n+1)^2}{n^4} = 1$$

$$\int_{-1}^{1} x^3\,dx = -1 + 3(1) - 2(2) + 2(1) = \underline{\underline{0}} \quad (\text{ANS})$$

(b) $\int_1^2 (x^2+1)\,dx$

$$\int_1^2 (x^2+1)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\,\Delta x$$

$$\Delta x = \frac{1}{n}$$

$$c_i = 1 + \frac{i}{n}$$

$$f(c_i) = \left(1+\frac{i}{n}\right)^2 + 1 = 1 + \frac{2}{n}i + \frac{i^2}{n^2} + 1$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left(2 + \frac{2i}{n} + \frac{i^2}{n^2}\right)\frac{1}{n}$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{2}{n} + \frac{2i}{n^2} + \frac{i^2}{n^3}\right)$$

$$= \lim_{n\to\infty}\left[\frac{2}{n}\sum_{i=1}^{n}(1) + \frac{2}{n^2}\sum_{i=1}^{n} i + \frac{1}{n^3}\sum_{i=1}^{n} i^2\right]$$

$$= \lim_{n\to\infty}\left[2 + \cancel{2}\frac{(n+1)}{\cancel{2}n} + \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)\right]$$

$$= \lim_{n\to\infty}\left[2 + \left(1+\frac{1}{n}\right) + \frac{2n^2+3n+1}{6n^2}\right]$$

$$= \lim_{n\to\infty}\left[2 + 1 + \frac{1}{n} + \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right]$$

$$= \lim_{n\to\infty}\left[2 + 1 + \frac{1}{3}\right]$$

$$= \frac{10}{3} \quad \text{(ANS)}$$

(c) $\int_1^2 4x^2\,dx$ $\qquad \Delta x = \frac{1}{n} \qquad c_i = 1 + \frac{i}{n}$

$$f(c_i) = 4\left(1+\frac{i}{n}\right)^2 = 4\left(1 + \frac{2i}{n} + \frac{i^2}{n^2}\right) = \left(4 + \frac{8i}{n} + \frac{4i^2}{n^2}\right)$$

$$\int_1^2 4x^2\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\,\Delta x$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left(4 + \frac{8i}{n} + \frac{4i^2}{n^2}\right)\frac{1}{n}$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{4}{n} + \frac{8i}{n^2} + \frac{4i^2}{n^3}\right)$$

$$= \lim_{n\to\infty}\left[\frac{4}{n}\sum_{i=1}^{n}(1) + \frac{8}{n^2}\sum_{i=1}^{n} i + \frac{4}{n^3}\sum_{i=1}^{n} i^2\right]$$

$$= \lim_{n\to\infty}\left[4 + \frac{8}{n^2}\left(\frac{n(n+1)}{2}\right) + \frac{4}{n^3}\,\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{n\to\infty}\left[4 + 4 + \frac{4}{3}\right] = \frac{28}{3}$$

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(6) Proof: Since f is integrable on (a,b) so it is continuous on $[a,b]$ $\forall x \in [a,b]$. The set of values of x is bounded i.e bounded below, $\exists\, m \in \mathbb{R}$ s.t $m \leq x$ and bounded above, $\exists\, M \in \mathbb{R}$ s.t $x \leq M$ $\forall x \in [a,b]$. We are given $m > 0$ and $M > 0$.

By extreme value theorem: A continuous function f defined on a closed, bounded interval $[a,b]$ attains both an absolute maximum (M) and an absolute minimum (m) on that interval

$$\Rightarrow \quad m \leq f(x) \leq M, \quad \forall x \in [a,b]$$

Suppose $g(x) \leq h(x)$ $\forall x \in [a,b]$ and $g(x)$ and $h(x)$ are integrable on $[a,b]$ then

$$\int_a^b g(x)\,dx \leq \int_a^b h(x)\,dx$$

$$\Rightarrow \int_a^b m\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b M\,dx$$

$$mx\Big|_a^b \leq \int_a^b f(x)\,dx \leq Mx\Big|_a^b$$

$$m(b) - m(a) \leq \int_a^b f(x)\,dx \leq M(b) - M(a)$$

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

(proven) (ABUS)

$$\int_0^1 \sqrt{1+x^4}\, dx$$

$f(x) = \sqrt{1+x^4}$ is

(i) Continuous on $[0,1]$, and

(ii) Integrable on $[0,1]$

the $\min([0,1]) = 0$ and $\max([0,1]) = 1$

$$m(b-a) \le \int_a^b f(x)\, dx \le M(b-a)$$

$$\phi \le \int_0^1 \sqrt{1+x^4}\, dx \le \sqrt{2}$$

so $\int_0^1 \sqrt{1+x^4}\, dx \in [\phi, \sqrt{2}]$

(7) (a) $\frac{d}{dx}\left[\int_1^x t^3\, dt\right] = t^3\Big|_1^x = x^3 - 1$

(b) $\frac{d}{dx}\left[\int_2^x \frac{t^{3/2}}{\sqrt{t^2+17}}\right] = \frac{t^{3/2}}{\sqrt{t^2+17}}\Bigg|_2^x$

$$= \frac{x^{3/2}}{\sqrt{x^2+17}} - \frac{2^{3/2}}{\sqrt{2^2+17}}$$

$$= \frac{x^{3/2}}{\sqrt{x^2+17}} - \frac{2\sqrt{2}}{\sqrt{21}}$$

(c) $\frac{d}{dx}\left[\int_x^4 \tan^2 t \cot t \, dt\right]$, $\frac{\pi}{2} < x < \frac{3\pi}{2}$

$= \tan^2 t \cot t \Big]_x^4$

$= \tan^2(4)\cot(4) - \tan^2 x \cot x$

$= \frac{\sin^2 4}{\cos^2 4} \frac{\cos 4}{\sin 4} - \tan^2 x \cot x$

$= \frac{\sin 4}{\cos 4} - \tan x$

$= \tan 4 - \tan x$ (ANS)

(8) average value $= \frac{1}{b-a}\int_a^b f(x)\,dx$

$= \frac{1}{\sqrt{\pi}}\int_0^{\sqrt{\pi}} x \sin x^2 \, dx$

Let $u = x^2$ when $x = 0$ $u = 0$

$du = 2x dx$ $x = \sqrt{\pi}$ $u = \pi$

$dx = \frac{1}{2x} du$

$= \frac{1}{\sqrt{\pi}}\int_0^{\pi} \cancel{x} \sin u \frac{1}{2\cancel{x}} du$

$= \frac{1}{2\sqrt{\pi}}\int_0^{\pi} \sin u \, du$

13

$f(x) = x\sin x^2;\ x \in [0, \sqrt{\pi}]$

$$\frac{1}{\sqrt{\pi}} \int_0^{\sqrt{\pi}} x \sin x^2 \, dx$$

Let $\sin x^2 = y$

$$\frac{dy}{dx} = (2\cos x)(2x) = 4x\cos x$$

$$x^2 = u$$
$$du = 2x\,dx$$
$$dx = \frac{du}{2x}$$

$$= \frac{1}{2\sqrt{\pi}} \int_0^{\pi} \sin y \, dy$$

$$= -\left(\frac{1}{2\sqrt{\pi}} \cos y\right)\Big|_0^{\pi}$$

$$\cos\pi - \cos 0$$

$$= -\left(\frac{1}{2\sqrt{\pi}}(-1-1)\right)$$

$$= -\left(-\frac{1}{\sqrt{\pi}}\right) \quad \text{(ANS)}$$

$$= \frac{1}{\sqrt{\pi}}$$

(14)

$f(x) = x \sin x^2; \quad x \in [0, \sqrt{\pi}]$

$$\frac{1}{\sqrt{\pi}} \int_0^{\sqrt{\pi}} x \sin x^2 \, dx$$

Let $\sin x^2 = u$

$$\frac{du}{dx} = (2\cos x)(2x) = 4x\cos x$$

$x^2 = u$

$du = 2x\,dx$

$dx = \frac{du}{2x}$

$$= \frac{1}{2\sqrt{\pi}} \int_0^{\pi} \sin u \, du$$

$$= -\left(\frac{1}{2\sqrt{\pi}} \cdot \cos u\right)\Big|_0^{\pi}$$

$\cos\pi - \cos 0$

$$= -\left(\frac{1}{2\sqrt{\pi}}\left(-1-1\right)\right)$$

$$= -\left(-\frac{1}{\sqrt{\pi}}\right) \quad (\text{ANS})$$

$$= \frac{1}{\sqrt{\pi}}$$

9)

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

$$\frac{1}{b-a}\int_a^b f(x)\,dx = f(c)$$

$$\frac{1}{2}\int_0^2 \frac{1}{(x+1)^2}\,dx = \frac{1}{(c+1)^2}$$

Let $u = x+1$

$du = dx$ when: $x = 0, u = 1$

$x = 2, u = 3$

$$\Rightarrow \quad \frac{1}{2}\int_1^3 u^{-2}\,du = \frac{1}{(c+1)^2}$$

$$\frac{1}{2}\left[-\frac{1}{u}\right]_1^3 = \frac{1}{(c+1)^2}$$

$$\frac{1}{2}\left[-\frac{1}{3}+1\right] = \frac{1}{(c+1)^2}$$

$$\frac{1}{3} = \frac{1}{(c+1)^2}$$

$$(c+1)^2 = 3$$

$$c^2 + 2c - 2 = 0$$

$$c = \frac{-2 \pm \sqrt{2^2 - 4(-2)}}{2}$$

$$c = -1 \pm \sqrt{3}$$

$$c = \underline{\underline{\sqrt{3}-1}} \approx 0.73 \in [0,2]$$ (ANS)

13

(11)

$y = 1 - \sqrt{1-x^2}$

$y = 1$

$\sqrt{1-x^2} \geq 0$ Min$(y) = 0$

$\sqrt{1-x^2} = 1-y$

$\sqrt{1-x^2} = -(y-1)$

$-(y-1) \geq 0$

$-y+1 \geq 0$

$y \leq 1$ range $(y) = \{y \mid 0 \leq y \leq 1\}$

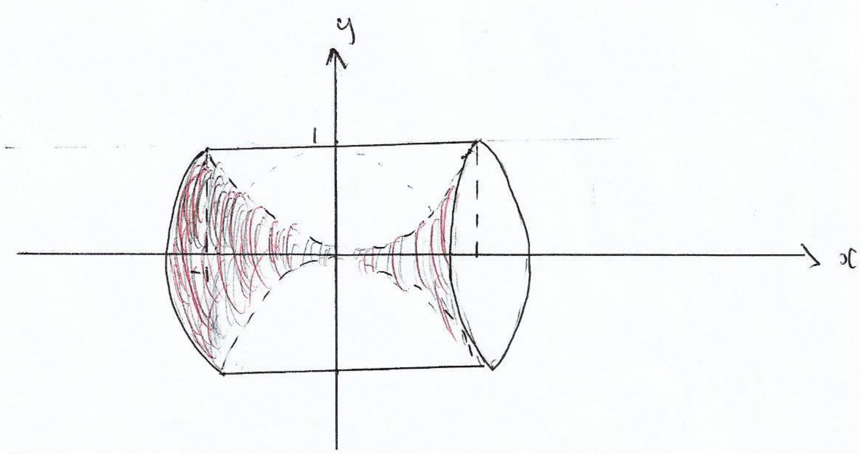
$(y-1)^2 = (-\sqrt{1-x^2})^2$

$1-x^2 \geq 0$

$(1+x)(1-x) \geq 0$ Domain $(y) = \{x \mid -1 \leq x < 1\}$

$x^2 + (y-1)^2 = 1$ $r^2 = 1$

centre $(0,1)$ radius $= \underline{1}$ unit



$$V = \pi \int_{-1}^{1} [R(x)]^2 - [r(x)]^2 \, dx$$

$$= \pi (1)^2 (2) - \pi \int_{-1}^{1} \left(1 - \sqrt{1-x^2}\right)^2 dx$$

$$= 2\pi - \pi \int_{-1}^{1} \left(2 - x^2 - 2\sqrt{1-x^2}\right) dx$$

$$= 2\pi - 4(\pi) + \frac{2\pi}{3} + 2\pi \int_{-1}^{1} \sqrt{1-x^2} \, dx$$

$$= \frac{-4\pi}{3} + 2\pi \int_{-1}^{1} \sqrt{1-x^2} \, dx$$

Considering $\int_{-1}^{1} \sqrt{1-x^2} \, dx$

Let $x = \cos\theta$ $\quad dx = -\sin\theta \, d\theta$

when $x = 1$ $\quad \theta = \cos^{-1}(1)$

$x = -1$ $\quad \theta = \cos^{-1}(-1)$

$$= \int_{\cos^{-1}|-1|}^{\cos^{-1}|1|} \sin\theta(-\sin\theta) \, d\theta$$

$$= \int_{\pi}^{0} -\sin^2\theta \, d\theta$$

$$= -\int_{0}^{\pi} -\sin^2\theta \, d\theta$$

$$= \int_{0}^{\pi} \sin^2\theta \, d\theta$$

$$A = \frac{1}{2} + \int_0^1 dx - \int_0^1 x^3\, dx$$

$$= \frac{1}{2} + x\Big|_0^1 - \frac{x^4}{4}\Big|_0^1$$

$$= \frac{1}{2} + 1 - \frac{1}{4}$$

$$= \underline{\underline{\frac{5}{4}\ \text{units}^2}} \quad \text{(ANS)}$$

15. $x = y^2 + 2 \qquad y = x - 8$

$x = (x-8)^2 + 2$

$x = x^2 - 16x + 64 + 2$

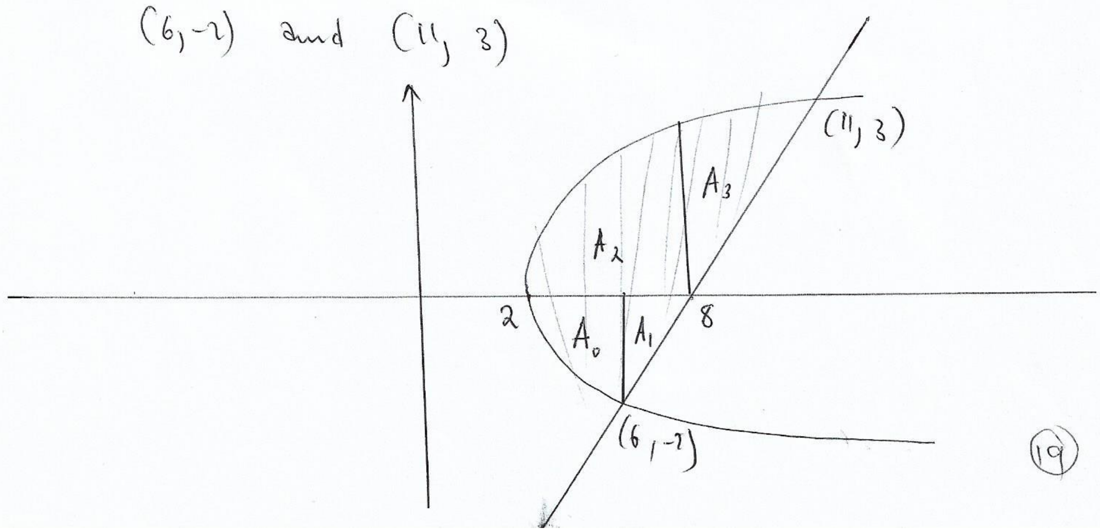
$x^2 - 17x + 66 = 0$

$(x-6)(x-11) = 0$

$x = 6$ or 11

Points of intersection

$(6,-2)$ and $(11, 3)$



$A = A_0 + A_1 + A_2 + A_3$

$A = \cancel{2} + \int_2^6 \sqrt{x-2}\,dx + \int_2^8 \sqrt{x-2}\,dx + \int_8^{11} \sqrt{x-2} - (x-8)\,dx$

$A = \cancel{2} + \frac{16}{3} + \cancel{\frac{2\sqrt{6}^3}{3}} + 18 - \cancel{\frac{2\sqrt{6}^3}{3}} + 88 - 64 - \frac{11^2}{2} + \frac{8^2}{2}$

$= \cancel{2}\,\cancel{84} + \frac{16}{3} - \frac{11^2}{2} + \frac{8^2}{2}$

$= \frac{85}{6}$ unit^2

$\approx \underline{14.83 \text{ unit}^2\text{s}}$ ANS

14. $4y + 3x = 7$

$y = x^{-2}$

points of intersection

$\frac{4}{x^4} + 3x = 7$

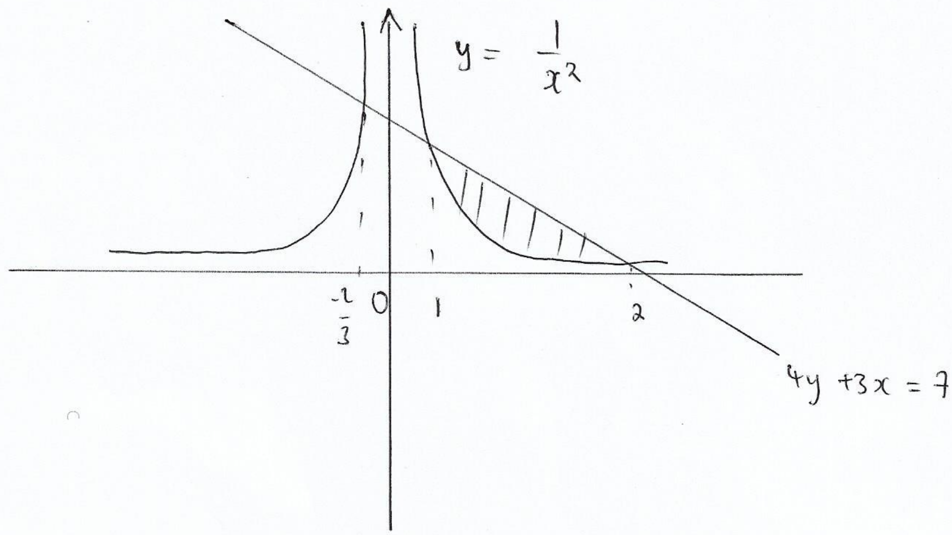
$4 + 3x^3 = 7x^2$

$3x^3 - 7x^2 + 4 = 0$

$$\begin{array}{rrrr|l} 3 & -7 & 0 & 4 & 2 \\ + & 6 & -2 & -4 & \\ \hline 3 & -1 & -2 & 0 & \end{array}$$

$(3x^2 - x - 2)(x-2) = 0$

$x = 1$ or $-\frac{2}{3}$ or 2



$$A = \int_1^2 \left(\frac{7}{4} - \frac{3}{4}x - \frac{1}{x^2}\right) dx$$

$$A = \int_1^2 \left(\frac{7}{4} - \frac{3}{4}x - x^{-2}\right) dx$$

$$A = \left[\frac{7}{4}x - \frac{3x^2}{8} + \frac{1}{x}\right]_1^2$$

$$A = \left(\frac{7}{2} - \frac{3}{2} + \frac{1}{2}\right) - \left(\frac{7}{4} - \frac{3}{8} + 1\right)$$

$$A = \frac{5}{2} - \frac{19}{8}$$

$$A = \frac{1}{8} \text{ unit}^2$$

ANS

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15.

$$L = \int_a^b \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx$$

$$24xy = x^4 + 48 \; ; \; [2,4]$$

$$24x\frac{dy}{dx} + 24y\frac{dx}{dx} = 4x^3$$

$$24x\frac{dy}{dx} = 4x^3 - 24y$$

$$\frac{dy}{dx} = \frac{x^3 - 6y}{6x}$$

$$\frac{dy}{dx} = \frac{x^3 - 6\left(\frac{x^3}{24} + \frac{2}{x}\right)}{6x}$$

$$\frac{dy}{dx} = \frac{x^3}{6} - \frac{1}{x}\left(\frac{x^3}{24} + \frac{2}{x}\right)$$

$$= \frac{x^2}{8} - \frac{2}{x^2}$$

$$= \frac{x^4 - 16}{8x^2}$$

$$\left[\frac{dy}{dx}\right]^2 = \left[\frac{x^4 - 16}{8x^2}\right]^2$$

$$= \frac{x^8 - 32x^4 + 256}{(8x^2)^2}$$

$$\Rightarrow \quad L = \int_2^4 \sqrt{1 + \frac{x^8 - 32x^4 + 256}{(8x^2)^2}}$$

$$= \int_2^4 \sqrt{\frac{x^8 + 32x^4 + 256}{64x^4}}$$

$$= \int_2^4 \sqrt{\frac{(x^4+16)^2}{64x^4}}$$

$$= \int_2^4 \frac{x^4 + 16}{8x^2}$$

$$= \int_2^4 \left(\frac{1}{8}x^2 + 2x^{-2}\right) dx$$

$$= \left[\frac{1}{8} \cdot \frac{x^3}{8} - \frac{2x^{-1}}{1}\right]_2^4$$

$$= \left[\frac{8}{3} - \frac{1}{2}\right] - \left[\frac{8}{24} - 1\right]$$

$$= \frac{17}{6} \text{ units}$$

(Ans) $\approx 2.8\bar{3}$ units

(23)

16. $27y^2 = 4(x-2)^3 \quad ; \quad (2,0), (11, 6\sqrt{3})$

$$27(2)y\frac{dy}{dx} = 4\frac{d}{dx}(x-2)^3$$

$$54y\frac{dy}{dx} = 12(x-2)^2$$

$$\frac{dy}{dx} = \frac{2}{9}\frac{(x-2)^2}{y}$$

$$y^2 = \frac{4(x-2)^3}{27}$$

$$y = \frac{2\sqrt{(x-2)^3}}{3\sqrt{3}}$$

$$\frac{\cancel{2}\sqrt{(x-2)^3}}{3\sqrt{3}}\frac{dy}{dx} = \frac{\cancel{2}}{9}(x-2)^2$$

$$\frac{dy}{dx} = \frac{\sqrt{3}}{3}(x-2)^{1/2}$$

$\Rightarrow$

$$L = \int_2^{11}\sqrt{1+\left[\frac{\sqrt{3}}{3}(x-2)^{1/2}\right]^2}$$

$$= \int_2^{11}\sqrt{1+\frac{3}{9}(x-2)}$$

(24)

$$= \int_3^{11} \sqrt{\frac{1+x}{3}}\,dx$$

Let $u = \frac{1+x}{3}$

$3u = 1 + x$

$3\,du = dx$

$$= 3\int_1^4 u^{1/2}\,du$$

$$= 3\left[\frac{2}{3}u^{3/2}\right]_1^4$$

$$= \frac{2}{3}(8) - \frac{2}{3}$$

$$= \frac{14}{3} \text{ units}$$

Ans: ≈ 4.67 units

17. $x = 3y^{3/2} - 1$

$$\frac{dx}{dy} = \left(\frac{3}{2}\right)(3)\,y^{1/2}$$

$$\frac{dx}{dy} = \frac{9}{2}y^{1/2}$$

$$L = \int_0^4 \sqrt{1 + \left[\frac{dx}{dy}\right]^2}\,dy$$

$$= \int_0^4 \sqrt{1 + \left[\frac{9}{2} y^{1/2}\right]^2}$$

$$= \int_0^4 \sqrt{1 + \frac{81}{4} y}$$

Let $u = 1 + \frac{81}{4} y$

when $y = 0$, $u = 1$

$y = 4$, $u = 82$

$$du = \frac{81}{4} dy$$

$$= \frac{4}{81} \int_1^{82} u^{1/2} \, du$$

$$= \frac{4}{81} \cdot \frac{2}{3} u^{\frac{3}{2}} \Big|_1^{82}$$

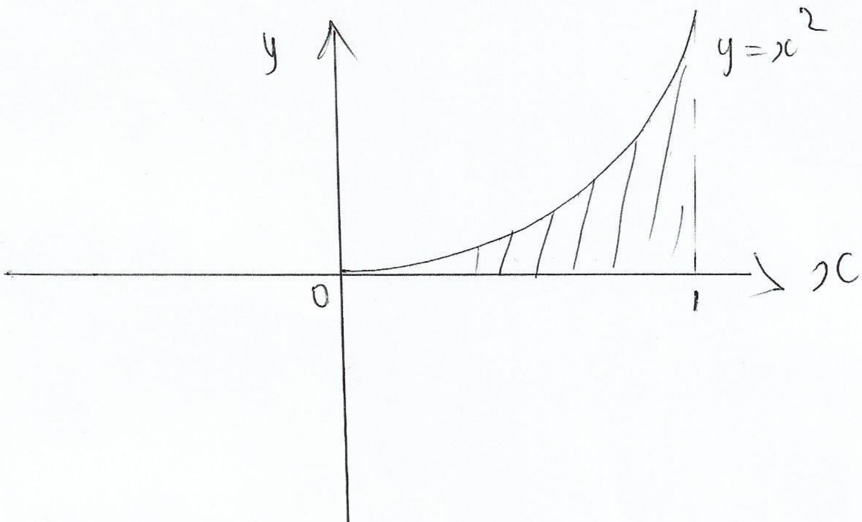
$$= \frac{4}{81} \frac{(2)\left((82)^2 - 1\right)}{3}$$

ANS: ≈ 24.4 units

(26)

18.

a) $y = x^2$, $y = 0$, $x = 1$



$$\bar{x} = \frac{\int_a^b x f(x)\,dx}{\int_a^b f(x)\,dx}$$

$$\bar{y} = \frac{\int_a^b \frac{1}{2}(f(x))^2\,dx}{\int_a^b f(x)\,dx}$$

$$\bar{x} = \frac{\int_0^1 x(x^2)\,dx}{\int_0^1 x^2\,dx}$$

$$= \frac{\int_0^1 x^3\,dx}{\int_0^1 x^2\,dx}$$

$$= \frac{x^4}{4}\Big|^1 \Big/ \frac{x^3}{3}\Big|^1 = \frac{3}{4}$$

FIZIE MATH LAB

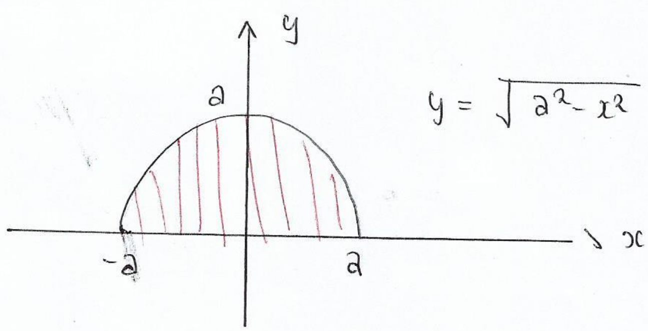
$$\bar{y} = \frac{\int_0^1 \frac{1}{2}(x^2)^2 \, dx}{\int_0^1 x^2 \, dx}$$

$$\bar{y} = \frac{\left.\frac{x^5}{10}\right|^1}{\left.\frac{x^3}{3}\right|^1}$$

$$\bar{y} = \frac{\frac{1}{10}}{\frac{1}{3}} = \frac{3}{10}$$

$$\text{centroid} = \left(\frac{3}{4}, \frac{3}{10}\right) \quad \text{(ANS)}$$

(b)



$$\bar{x} = \frac{\int_a^b x f(x) \, dx}{\int_a^b f(x) \, dx}, \quad \bar{y} = \frac{\int_a^b \frac{1}{2}(f(x))^2 \, dx}{\int_a^b f(x)}$$

$$\bar{x} = \frac{\int_{-a}^{a} x\sqrt{a^2 - x^2} \, dx}{\int_{-a}^{a} \sqrt{a^2 - x^2} \, dx}$$

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Let $u^2 = a^2 - x^2$

$2u\,du = -2x\,dx$

when $x = a \quad u = 0$

$x = -a \quad u = 0$

$$= \frac{\int_0^0 \frac{x\,u(-u)}{x}\,du}{\int_{-a}^{a} \sqrt{a^2-x^2}\,dx}$$

$$= \frac{0}{\frac{a^2}{2}\pi}$$

$$= 0$$

$$\bar{y} = \frac{\int_a^b \frac{1}{2}(f(x))^2\,dx}{\int_a^b f(x)\,dx}$$

$$= \frac{\frac{1}{2}\int_{-a}^{a} (a^2-x^2)\,dx}{\int_{-a}^{a} \sqrt{a^2-x^2}\,dx}$$

$$= \frac{\frac{1}{2}\left(a^2x - \frac{x^3}{3}\right)\Big|_{-a}^{a}}{\int_{-a}^{a} \sqrt{a^2-x^2}}$$

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$$= \frac{\frac{2a^3}{3}}{\int_{-a}^{a} \sqrt{a^2 - x^2}\, dx}$$

Let $x = a\cos\theta$

$dx = -a \sin a \, d\theta$

$$\Rightarrow \int_{-a}^{a} \sqrt{a^2 - x^2}\, dx \equiv \int_{\cos^{-1}|-1|}^{\cos^{-1}|1|} a\sin\theta\,(-a\sin\theta)\,d\theta$$

$$= a^2 \int_0^{\pi} \sin^2\theta \, d\theta$$

$$= a^2 \int_0^{\pi} \frac{1 - \cos 2\theta}{2}\, d\theta$$

$$= \frac{a^2}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi}$$

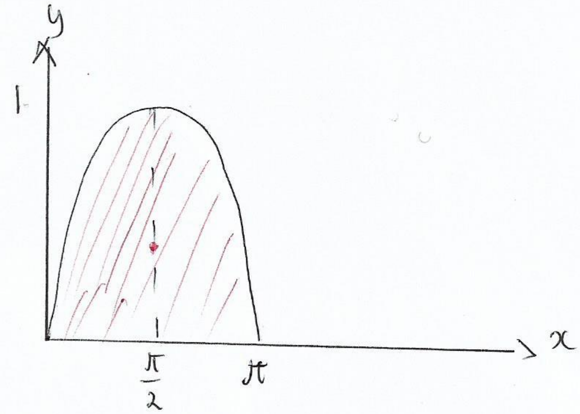
$$= \frac{a^2 \pi}{2}$$

$$\Rightarrow \bar{y} = \frac{\frac{2a^3}{3}}{\frac{a^2\pi}{2}}$$

$$\bar{y} = \frac{4a}{3\pi} \quad \forall a \in R$$

Centroid $\left(0, \frac{4a}{3\pi}\right)$ (ANS)

(c) $y = \sin x$, $y = 0$, $x = 0$ $x = \pi$



$$\bar{x} = \frac{\int_0^{\pi} x \sin x \, dx}{\int_0^{\pi} \sin x \, dx}$$

$$= \frac{-x \cos x \Big|_0^{\pi} + \int_0^{\pi} \cos x \, dx}{-\cos x \Big|_0^{\pi}}$$

$$= \frac{\pi + \sin x \Big|_0^{\pi}}{2}$$

$$= \frac{\pi}{2}$$

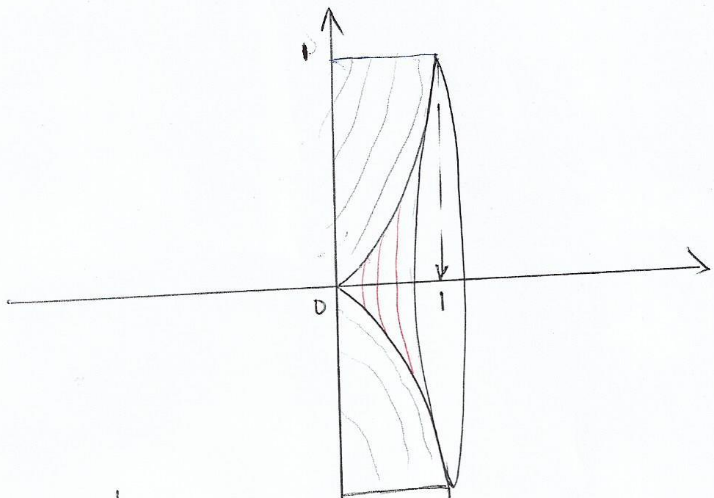
$$\bar{y} = \frac{\frac{1}{2} \int_0^{\pi} \sin^2 x \quad dx}{2}$$

$$\bar{y} = \frac{\frac{1}{4} \int_0^{\pi} (1 - \cos 2x) \, dx}{2} = \frac{\pi}{8}$$

Centroid : $\left(\frac{\pi}{2}, \frac{\pi}{8}\right)$

(19)

(a)



$$V = \pi \int_0^1 (x^3)^2 \, dx$$

$$= \pi \int_0^1 x^6 \, dx$$

$$= \left.\frac{\pi x^7}{7}\right|_0^1$$

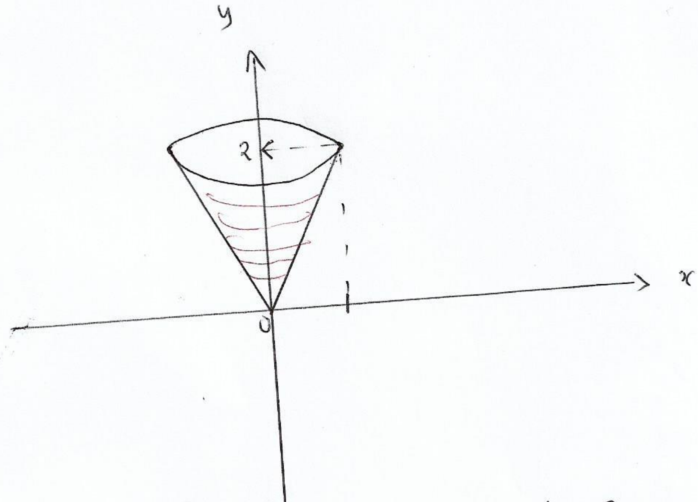
$$= \frac{\pi}{7} \text{ unit}^3 \quad \text{(ANS)}$$

$V_R = \pi - \frac{\pi}{7}$

$V_R = \frac{6}{7}\pi$

b

$y = 2x$



Volume of the cone $= \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi (1)^2(2)$

$= \frac{2}{3}\pi$ units^3

Or

$V = \pi \int_0^2 \left(\frac{y}{2}\right)^2 dy$

$= \frac{\pi}{4} \int_0^2 y^2\, dy$

$= \frac{\pi}{4} \left.\frac{y^3}{3}\right|_0^2$

$= \frac{\pi}{\cancel{4}} \left(\frac{\cancel{8}^{2}}{3}\right)$

$= \frac{2\pi}{3}$ units^3 (ANS)

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THE UNIVERSITY OF ZAMBIA
SCHOOL OF NATURAL SCIENCES
DEPARTMENT OF MATHEMATICS & STATISTICS
MAT2100 Tutorial Sheet 5

June, 2018

1. Evaluate the following antiderivatives:
   (a) $\int \frac{1}{\sqrt{x-1}}\,dx$
   (b) $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$
   (c) $\int (4-2t^2)^7 t\,dt$
   (d) $\int x^2\sqrt[3]{x^3+5}\,dx$
   (e) $\int \frac{x}{\sqrt{1+5x^2}}\,dx$
   (f) $\int x\sqrt{ax+b}\,dx$
   (g) $\int \frac{\cos 3x}{\sin^2 3x}\,dx$
   (h) $\int \tan^2 x\sec^4 x\,dx$
   (i) $\int x^2\csc^2 x^3\,dx$
   (j) $\int (\tan^2 x+\cot^2 x)^2\,dx$
   (k) $\int \sqrt{e^{2x}-1}\,dx$
   (l) $\int \sqrt{2^{2x}-1}\,dx$

2. Evaluate the following integrals:
   (a) $\int \frac{dx}{4+9x^2}$
   (b) $\int \frac{dx}{\sqrt{25-x^2}}$
   (c) $\int \frac{dx}{\sqrt{25-16x^2}}$
   (d) $\int \frac{dx}{(x-3)\sqrt{x^2-6x+8}}$
   (e) $\int \frac{dx}{\sqrt{3-2x^2}}$
   (f) $\int \frac{dx}{\sqrt{2+7x^2}}$
   (g) $\int \frac{dx}{x\sqrt{9x^2-16}}$
   (h) $\int \frac{dx}{x\sqrt{3x^2-2}}$
   (i) $\int \frac{dx}{(1+x)\sqrt{x}}$
   (j) $\int \frac{2x\,dx}{\sqrt{6x-x^2}}$
   (k) $\int \frac{x\,dx}{\sqrt{x^2+8x+20}}$
   (l) $\int \frac{x^3\,dx}{\sqrt{x^2-2x+4}}$
   (m) $\int \frac{x\,dx}{4-x^4}$
   (n) $\int \frac{e^x\,dx}{4+e^{2x}}$
   (o) $\int \frac{\cos x\,dx}{5+\sin^2 x}$

3. Find the indicated antiderivative.
   (a) $\int x^2 e^x\,dx$
   (b) $\int \cos(\ln x)\,dx$
   (c) $\int e^{ax}\cos bx\,dx$
   (d) $\int (\ln x)^2\,dx$
   (e) $\int x^2\tan^{-1} x\,dx$
   (f) $\int \frac{\ln x}{x^2}\,dx$
   (g) $\int x\sqrt{1+x}\,dx$
   (h) $\int (\sin^{-1} x)^2$
   (i) $\int x^n\ln x\,dx$
   (j) $\int e^{\sqrt{x}}\,dx$
   (k) $\int \ln(x^2+1)\,dx$
   (l) $\int_0^{2\pi} x\sin nx\,dx$
   (m) $\int x\pi^x\,dx$
   (n) $\int x^2 2^{-x}\,dx$
   (o) $\int x2^{\sqrt{2x}\,dx}$
   (p) $\int e^t\sec^3(e^t-1)\,dt$
   (q) $\int \frac{\csc^3\sqrt{\theta}}{\sqrt{\theta}}\,d\theta$
   (r) $\int \frac{\sqrt{x}}{\sqrt{1-x}}\,dx$
   (s) $\int x\sin^{-1}(x^2)\,dx$
   (t) $\int x^2\tan^{-1} x\,dx$
   (u) $\int \sin^2 x\cos x\ln(\sin x)\,dx$

FIZIE MATH LAB

4. Find the reduction formula for the following:

(a) $\int \cos^n x\,dx$ (e) $\int x^n \sin ax\,dx$ (i) $\int \tan^n x\,dx$ (m) $\int \frac{x^n\,dx}{\sqrt{(1+x^2)^n}}$

(b) $\int \sin^n x\,dx$ (f) $\int \frac{x^2\,dx}{(a^2+x^2)^n}$ (j) $\int x^n e^{x^2}\,dx$ (n) $\int \sin^n x \cos^m x\,dx$

(c) $\int \sec^n x\,dx$ (g) $\int x^m (\ln x)^n\,dx$ (k) $\int x^n \cos \pi x$ (o) $\int x^n \sin^{-1} x\,dx$

(d) $\int x^n e^{ax}\,dx$ (h) $\int (\ln x)^n\,dx$ (l) $\int x(\ln x)^n\,dx$ (p) $\int (x^2+a^2)^n\,dx$

5. Use the substitution $z = \tan\theta/2$ to evaluate the following:

(a) $\int \frac{dx}{1-\sin x}$ (f) $\int \frac{dt}{1+3\cos t}$ (k) $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{d\theta}{1+\sin\theta+\cos\theta}$

(b) $\int \frac{d\theta}{5+4\sin\theta}$ (g) $\int_0^{\frac{\pi}{2}} \frac{d\theta}{2+\cos\theta}$ (l) $\int \frac{d\theta}{\sin\theta-\cos\theta}$

(c) $\int \frac{dx}{1+2\sin x}$ (h) $\int \frac{dx}{2\sin x+\cos x}$ (m) $\int \frac{dx}{1+\cos x}$

(d) $\int \frac{d\theta}{3-4\sin\theta+2\cos\theta}$ (i) $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{dx}{3-2\sin x}$ (n) $\int \frac{dx}{1+\sin x}$

(e) $\int \frac{dx}{3-4\sin x}$ (j) $\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \frac{\cos\theta\,d\theta}{\sin\theta\cos\theta+\sin\theta}$ (p) $\int \frac{\cos x}{\sin x-1}\,dx$

6. Evaluate the following:

(a) $\int \frac{x^4-4x^2+x+1}{x^2-4}\,dx$ (g) $\int \frac{x^2\,dx}{(x-1)(x^2+4)^2}$ (m) $\int \frac{dx}{\sqrt{\sqrt{x}+1}}$

(b) $\int \frac{2x^2-1}{(x-1)(x-2)(x-3)}\,dx$ (h) $\int \frac{x^3+1}{x(x^2+x+1)^2}\,dx$ (n) $\int \frac{dx}{x\sqrt{1+3x}}$

(c) $\int \frac{x^2-4}{x^3-3x^2-x+3}\,dx$ (i) $\int \frac{dx}{1+e^x}$ (o) $\int \frac{dx}{1+\sqrt[4]{x-1}}$

(d) $\int \frac{x^2\,dx}{(x-1)(x^2+4x+5)}$ (j) $\int \sqrt{1+e^x}\,dx$ (p) $\int \frac{dx}{\sqrt[3]{x}-x}$

(e) $\int \frac{dx}{(x^2+1)(x^2+4)}$ (k) $\int \frac{x^{\frac{2}{3}}}{x+1}\,dx$ (q) $\int \frac{dx}{\sqrt[3]{x}+\sqrt{x}}$

(f) $\int \frac{dx}{x(x^2+1)^2}$ (l) $\int \frac{dx}{\sin x} + \cos x$ (r) $\int \frac{dx}{\sqrt[3]{x}-\sqrt{x}}$

(1)

(a) $\int \frac{1}{\sqrt{x-1}}\, dx$

Let $u^2 = x-1$

$2u\,du = dx$

$= 2\int \frac{1}{\cancel{u}} \cancel{u}\, du$

$= 2\int du$

$= 2u + 2c$

$= 2\sqrt{x+1} + 2c$

$= 2\sqrt{x+1} + C \quad ; \quad C = 2c$

(b) $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\, dx$

Let $u^2 = x$

$du\, 2u = dx$

$= \int \frac{\cos u}{\cancel{u}}\, 2\cancel{u}\, du$

$= 2\int \cos u\, du$

$= 2\sin u + 2c$

$= 2\sin\sqrt{x} + C \quad ; \quad C = 2c$

(c) $\int (4-2t^2)^7 t\, dt$

Let $u = 4-2t^2$

$du = -4t\, dt$

$dt = -\frac{1}{4t}\, du$

$$\int u^7 \cancel{t} \left(-\frac{1}{4\cancel{t}}\right) du$$

$$= -\frac{1}{4}\int u^7 \, du$$

$$= -\frac{1}{4}\left[\frac{u^8}{8} + c\right]$$

$$= -\frac{1}{4}\left[\frac{(4-2t^2)^8}{8} + c\right]$$

$$= -\frac{(4-2t^2)^8}{32} - \frac{1}{4}c$$

$$= -\frac{(4-2t^2)^8}{32} + c \; ; \; c = -\frac{1}{4}c$$

(d) $\int x^2 \sqrt[3]{x^3+5}\, dx$

Let $u = x^3 + 5$

$du = 3x^2\, dx \qquad dx = \frac{1}{3x^2}\, du$

$$= \int \cancel{x^2}\, \sqrt[3]{u} \left(\frac{1}{3\cancel{x^2}}\right) du$$

$$= \frac{1}{3}\int u^{1/3}\, du$$

$$= \frac{1}{\cancel{3}}\left[\frac{\cancel{3}u^{\frac{4}{3}}}{4}\right] + \frac{1}{3}c$$

$$= \frac{1}{4}\sqrt[3]{(x^3+5)^4} + c \; ; \; c = \frac{1}{3}c$$

(e) $\int \frac{x}{\sqrt{1+5x^2}}\,dx$

Let $u^2 = 1+5x^2 \Rightarrow 2u\,du = 10x\,dx$

$x\,dx = \frac{1}{5}u\,du$

$$\Rightarrow \int \frac{x}{\sqrt{1+5x^2}}\,dx = \frac{1}{5}\int \frac{1}{\cancel{u}}(\cancel{u}\,du)$$

$$= \frac{1}{5}\int du$$

$$= \frac{1}{5}u + \frac{1}{5}c$$

$$= \frac{1}{5}u + C \quad ; \quad C = \frac{1}{5}c$$

$$= \frac{1}{5}\sqrt{1+5x^2} + C$$

(f) $\int x\sqrt{ax+b}\,dx$

let $u = \sqrt{ax+b}$, $u^2 = ax+b$ $\quad 2u\,du = a\,dx$ $\quad x = \frac{u^2-b}{a}$

$$\Rightarrow \int x\sqrt{ax+b}\,dx = \int \left(\frac{u^2-b}{a}\right) u \left(\frac{2u}{a}\,du\right)$$

$$= \frac{2}{a^2}\int (u^2-b)(u^2)\,du$$

$$= \frac{2}{a^2}\int u^4 - bu^2\,du$$

$$= \frac{2}{a^2}\left(\frac{u^5}{5} - \frac{bu^3}{3}\right) + C$$

$$= \frac{2(ax+b)^{5/2}}{a^2} - \frac{2b(ax+b)^{3/2}}{3a^2} + C$$

(g) $\displaystyle\int \frac{\cos 3x}{\sin^2 3x}\,dx$

Let $u = \sin 3x$

$du = 3\cos 3x\,dx$

$dx = \dfrac{du}{3\cos 3x}$

$= \displaystyle\int \frac{\cancel{\cos 3x}}{u^2}\left(\frac{du}{3\cancel{\cos 3x}}\right)$

$= \dfrac{1}{3}\displaystyle\int u^{-2}\,du$

$= \dfrac{1}{3}\left(-\dfrac{1}{u}\right) + C$

$= -\dfrac{1}{3\sin 3x} + C$

$= -\dfrac{1}{3}\csc 3x + C$

(h) $\displaystyle\int \tan^2 x \sec^4 x\,dx = \int \tan^2 x \sec^2 x \sec^2 x\,dx$

$1 + \tan^2 x = \sec^2 x$

$\Rightarrow \displaystyle\int \tan^2 x \sec^2 x \sec^2 x\,dx = \int \tan^2 x \sec^2 x\,(1+\tan^2 x)\,dx$

$= \displaystyle\int \tan^2 x \sec^2 x\,dx + \int \tan^4 x \sec^2 x\,dx$

Let $u = \tan x$, $du = \sec^2 x\,dx$

$= \displaystyle\int u^2 \cancel{\sec^2 x}\left(\frac{du}{\cancel{\sec^2 x}}\right) + \int u^4 \cancel{\sec^2 x}\left(\frac{du}{\cancel{\sec^2 x}}\right)$

$= \displaystyle\int u^2\,du + \int u^4\,du = \frac{u^3}{3} + \frac{u^5}{5} + C$

$= \dfrac{\tan^3 x}{3} + \dfrac{\tan^5 x}{5} + C$

(i) $\int x^2 \csc^2 x^3 \, dx$

Let $u = x^3 \implies du = 3x^2 \, dx$

$$\implies \int x^2 \csc^2 x^3 \, dx = \frac{1}{3}\int \cancel{x^2} \csc^2 u \cdot \left(\frac{1}{\cancel{x^2}}\right) du$$

$$= \frac{1}{3}\int \csc^2 u \, du$$

$$= -\frac{1}{3}\cot u + c$$

$$= -\frac{1}{3}\cot x^3 + c$$

(j) $\int (\tan^2 x + \cot^2 x)^2 \, dx$

$$= \int (\tan^4 x + 2\tan^2 x \cot^2 x + \cot^4 x) \, dx$$

$$= \int \tan^4 x \, dx + 2\int \tan^2 x \cot^2 x \, dx + \int \cot^4 x \, dx$$

$$= \int \tan^4 x \, dx + \int \cot^4 x \, dx + 2\int dx$$

$$= \int \tan^2 x (\sec^2 x - 1) \, dx + \int \cot^2 x (\csc^2 x - 1) \, dx + 2\int dx$$

$$= \int \tan^2 x \sec^2 x \, dx - \int \tan^2 x \, dx + \int \cot^2 x \csc^2 x \, dx - \int \cot^2 x + 2\int dx$$

$$= \frac{\tan^3 x}{3} - \int (\sec^2 x - 1) \, dx - \frac{\cot^3 x}{3} - \int \cot^2 x \, dx + 2\int dx$$

$$= \frac{\tan^3 x}{3} - \frac{\cot^3 x}{3} + \cot x - \tan x + 4x + C$$

$\sqrt{e^{2x}-1} \geq 0 \quad \forall x \geq 0$ and $\tan^{-1}\sqrt{e^{2x}-1} < \frac{\pi}{2}$

$\Rightarrow e^{2x} - 1 \geq 0$

$\ln e^{2x} \geq \ln 1$

$2x \geq 0$

$\forall x \geq 0, \quad \int \sqrt{e^{2x}-1}\, dx = \sqrt{e^{2x}-1} - \tan^{-1}\sqrt{e^{2x}-1} + c$

(i) $\int \sqrt{2^{2x}-1}\, dx$

$\int \sqrt{e^{\ln 2^{2x}}-1}\, dx$

$\int \sqrt{e^{2x\ln 2}-1}\, dx \qquad$ Let $u = e^{2x\ln 2}$

$du = 2\ln 2\, e^{2x\ln 2}\, dx$

$du = 2\ln 2\, u\, dx$

$= \int \frac{\sqrt{u-1}}{2\ln 2\, u}\, du$

$= \frac{1}{2\ln 2} \int \frac{\sqrt{u-1}}{u}\, du$

Let $v^2 = u - 1$

$2v \frac{dv}{du} = 1$

$= \frac{1}{\ln 2} \int \frac{v^2}{v^2+1}\, dv$

Let $v = \tan\theta$

$\theta = \tan^{-1}\sqrt{e^{2x}-1}$

$dv = \sec^2\theta \, d\theta$

$$= \frac{1}{\ln 2}\int \frac{\tan^2\theta}{\cancel{\sec^2\theta}} \cancel{\sec^2\theta}\, d\theta$$

$$= \frac{1}{\ln 2}\int \tan^2\theta \, d\theta$$

$$= \frac{1}{\ln 2}\int (\sec^2\theta - 1)\, d\theta$$

$$= \frac{1}{\ln 2}(\tan\theta - \theta)$$

$$= \frac{1}{\ln 2}\left(\tan(\tan^{-1}\sqrt{2^{2x}-1} - \tan^{-1}\sqrt{2^{2x}-1}\right) + c$$

$$= \frac{1}{\ln 2}\left(\sqrt{2^{2x}-1} - \tan^{-1}\sqrt{2^{2x}-1}\right) + c, \quad \forall x \geq 0$$

and $\tan^{-1}\sqrt{2^{2x}-1} < \frac{\pi}{2}$

(2)

(a) $\int \frac{dx}{4+9x^2}$

$\int \frac{dx}{4+(3x)^2}$

Let $3x = 2\tan\theta \qquad \theta = \mathrm{Tan}^{-1}\left(\frac{3x}{2}\right)$

$\Rightarrow \quad 3\,dx = 2\sec^2\theta \cdot d\theta$

$\int \frac{dx}{4+9x^2} \equiv \int \frac{\frac{2}{3}\sec^2\theta}{4+4\tan^2\theta}\,d\theta$

$= \frac{1}{4}\int \frac{\frac{2}{3}\cancel{\sec^2\theta}}{\cancel{\sec^2\theta}}\,d\theta$

$= \frac{1}{6}\int d\theta$

$= \frac{1}{6}\theta + C$

$= \frac{1}{6}\mathrm{Tan}^{-1}\left(\frac{3x}{2}\right) + C$

(b) $\int \frac{dx}{\sqrt{25-x^2}}$

Let $x = 5\sin\theta \quad , \quad \theta = \sin^{-1}\left(\frac{x}{5}\right)$

$dx = 5\cos\theta\,d\theta$

$= \int \frac{5\cos\theta}{\sqrt{25-25\sin^2\theta}}\,d\theta$

$= 5\int \frac{\cancel{\cos\theta}}{5\cancel{\cos\theta}}\,d\theta$

$= \theta + C \quad = \sin^{-1}\left(\frac{x}{5}\right) + C \ , \quad |x| \leq 5$

(c) $\displaystyle\int \frac{dx}{\sqrt{25-16x^2}}$

Let $4x = 5\sin\theta$

$4\,dx = 5\cos\theta\, d\theta$

$= \frac{5}{4}\displaystyle\int \frac{\cos\theta}{\sqrt{25-25\sin^2\theta}}\, d\theta$

$= \frac{\cancel{5}}{4}\displaystyle\int \frac{\cancel{\cos\theta}}{\cancel{5}\,\cancel{\cos\theta}}\, d\theta$

$= \frac{1}{4}\theta + C$

$= \frac{1}{4}\sin^{-1}\left(\frac{4x}{5}\right) + C \quad , \quad |x| \le \frac{5}{4}$

or

$= -\frac{1}{4}\cos^{-1}\left(\frac{4x}{5}\right) + C \quad , \quad |x| \le \frac{5}{4}$

(d) $\displaystyle\int \frac{dx}{(x-3)\sqrt{x^2-6x+8}}$

Consider the quad expr : x^2-6x+8

By completing the square we have

$x^2-6x+8 = (x-3)^2 - 1$

So. $= \displaystyle\int \frac{dx}{(x-3)\sqrt{(x-3)^2-1}}$

Let $u = x-3 \Rightarrow du = dx$

$= \displaystyle\int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1}(u) + C = \sec^{-1}(x-3) + C$

(e)

$$\int \frac{dx}{\sqrt{3 - 2x^2}}$$

Let $x\sqrt{2} = \sqrt{3}\sin\theta \qquad \theta = \sin^{-1}\left(\frac{x\sqrt{6}}{3}\right)$

$\sqrt{2}\,dx = \sqrt{3}\cos\theta\,d\theta$

$$= \frac{\sqrt{3}}{\sqrt{2}}\int \frac{\cos\theta}{\sqrt{3 - 3\sin^2\theta}}\,d\theta$$

$$= \frac{\cancel{\sqrt{3}}}{\sqrt{2}}\int \frac{\cos\theta}{\cancel{\sqrt{3}}\sqrt{1-\sin^2\theta}}\,d\theta$$

$$= \frac{1}{\sqrt{2}}\theta + C$$

$$= \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{x\sqrt{6}}{3}\right) + C, \quad |x| \leq \frac{3}{\sqrt{6}}$$

(f)

$$\int \frac{dx}{\sqrt{2 + 7x^2}}$$

Let $x\sqrt{7} = \sqrt{2}\sinh\theta \qquad \theta = \sinh^{-1}\left(\frac{x\sqrt{7}}{\sqrt{2}}\right)$

$dx\sqrt{7} = \sqrt{2}\cosh\theta\,d\theta$

$$= \frac{\sqrt{2}}{\sqrt{7}}\int \frac{\cosh\theta}{\sqrt{2 + 2\sinh^2\theta}}\,d\theta$$

$$= \frac{1}{\sqrt{7}}\int \frac{\cancel{\cosh\theta}}{\cancel{\cosh\theta}}\,d\theta$$

$$= \frac{1}{\sqrt{7}}\sinh^{-1}\left(\frac{x\sqrt{14}}{2}\right) + C$$

$$\sinh^{-1} z = \ln\left(z + z\sqrt{z^2-1}\right)$$

$\Rightarrow$

$$\frac{1}{\sqrt{7}} \sinh^{-1}\left(\frac{x\sqrt{14}}{2}\right) + C =$$

$$\frac{\sqrt{7}}{7} \ln\left(\frac{x\sqrt{14}}{2} + \frac{\sqrt{14x^2+4}}{2}\right)$$

or

$$\int \frac{dx}{\sqrt{7x^2+2}}$$

Let $x\sqrt{7} = \sqrt{2}\tan u \qquad u = \tan^{-1}\left(\frac{x\sqrt{7}}{\sqrt{2}}\right)$

$$dx\sqrt{7} = \sqrt{2}\sec^2 u \, du$$

$$= \int \frac{\frac{\sqrt{2}}{\sqrt{7}}\sec^2 u \, du}{\sqrt{2\tan^2 u + 2}}$$

$$= \frac{\cancel{\sqrt{2}}}{\sqrt{7}} \int \frac{\sec^2 u}{\cancel{\sqrt{2}}\sec u} du$$

$$= \frac{1}{\sqrt{7}} \int \sec u \, du$$

$$= \frac{1}{\sqrt{7}} \int \frac{\sec^2 u + \sec u \tan u}{\sec u + \tan u} du$$

FIZIE MATH LAB

Let $s = \tan u + \sec u$

$ds = \sec^2 u + \sec u \tan u$

$= \frac{1}{\sqrt{7}} \int \frac{1}{s}\, ds$

$= \frac{1}{\sqrt{7}} \ln|s|$

$= \frac{1}{\sqrt{7}} \ln\left| \tan u + \sec u \right|$

$= \frac{1}{\sqrt{7}} \ln\left| \tan\left(\tan^{-1}\left(\frac{x\sqrt{7}}{\sqrt{2}}\right)\right) + \sec\left(\tan^{-1}\left(\frac{x\sqrt{7}}{\sqrt{2}}\right)\right) \right|$

Note $\sec\left(\tan^{-1}(z)\right) = \sqrt{z^2+1}$

$\tan\left(\tan^{-1}(z)\right) = z$

$= \frac{1}{\sqrt{7}} \ln\left| \frac{x\sqrt{7}}{\sqrt{2}} + \sqrt{\left(\frac{x\sqrt{7}}{\sqrt{2}}\right)^2 + 1} \right|$

(g) $\displaystyle\int \frac{dx}{x\sqrt{9x^2-16}}$

Let $3x = 4\sec\theta$, $\theta = \sec^{-1}\left(\frac{3x}{4}\right)$

$3\,dx = 4\sec\theta\tan\theta\,d\theta$

$$\Rightarrow \left(\frac{3}{4}\cdot\frac{4}{3}\right)\int \frac{\sec\theta\tan\theta}{\sec\theta\sqrt{16\sec^2\theta-16}}\,d\theta$$

$$= \frac{1}{4}\int \frac{\tan\theta}{\tan\theta}\,d\theta$$

$$= \frac{1}{4}\int d\theta$$

$$= \frac{1}{4}\theta + c$$

$$= \frac{1}{4}\sec^{-1}\left(\frac{3x}{4}\right) + c$$

(h) $\displaystyle\int \frac{dx}{x\sqrt{3x^2-2}}$

Let $\sqrt{3}x = \sqrt{2}\sec\theta$, $x = \frac{\sqrt{2}}{\sqrt{3}}\sec\theta$

$\theta = \sec^{-1}\left(\frac{\sqrt{3}x}{\sqrt{2}}\right)$

$\sqrt{3}\,dx = \sqrt{2}\sec\theta\tan\theta\,d\theta$

$$= \int \frac{\cancel{\frac{\sqrt{2}}{\sqrt{3}}\sec\theta}\tan\theta}{\cancel{\frac{\sqrt{2}}{\sqrt{3}}\sec\theta}\sqrt{2\sec^2\theta-2}}\,d\theta$$

$$= \frac{1}{\sqrt{2}}\int \frac{\cancel{\tan\theta}}{\cancel{\tan\theta}}\,d\theta$$

$$= \frac{1}{\sqrt{2}}\theta + C = \frac{1}{\sqrt{2}}\sec^{-1}\left(\frac{\sqrt{3}}{\sqrt{2}}x\right) + C$$

(i) $\int \frac{dx}{(1+x)\sqrt{x}}$

Let $u^2 = x$, $\sqrt{x} = u$

$\Rightarrow 2u\,du = dx$

$= \int \frac{2\cancel{u}}{\cancel{u}(1+u^2)}\, du$

$= 2\int \frac{1}{u^2+1}\, du$

Let $u = \tan\theta$ $\qquad \theta = \tan^{-1} u = \tan^{-1}\sqrt{x}$

$du = \sec^2\theta\, d\theta$

$= 2\int \frac{1}{\cancel{\tan^2\theta + 1}} \cancel{\sec^2\theta}\, d\theta$

$= 2\int d\theta$

$= 2\theta + c$

$= 2\tan^{-1}\sqrt{x} + c$

(j) $\int \frac{2x}{\sqrt{6x - x^2}}\, dx$

$\int \frac{2x}{\sqrt{-(x-3)^2+9}}\, dx$

∴ By completing the square of $6x - x^2$

$= -(x-3)^2 + 9$

$$= \int \frac{2x}{\sqrt{9-(x-3)^2}}\,dx$$

Let $x-3 = 3\sin\theta$ $\qquad \frac{x-3}{3} = \sin\theta$

$dx = 3\cos\theta\, d\theta$

$$= 2\int \frac{3\sin\theta + 3}{\sqrt{9 - 9\sin^2\theta}}\,dx$$

$$= 2\int \left(\frac{3\sin\theta + 3}{\cancel{3}\,\cancel{\cos\theta}}\right) \cancel{3}\,\cancel{\cos\theta}\, d\theta$$

$$= 6\int (\sin\theta + 1)\, d\theta$$

$$= 6(-\cos\theta + \theta) + c$$

$$= -6\cos\left(\sin^{-1}\left(\frac{x-3}{3}\right)\right) + 6\sin^{-1}\left(\frac{x-3}{3}\right) + C$$

$$= -6\left[\sqrt{1 - \sin^2\left(\sin^{-1}\left(\frac{x-3}{3}\right)\right)}\right] + 6\sin^{-1}\left(\frac{x-3}{3}\right) + c$$

$$= -6\sqrt{1-\left(\frac{x-3}{3}\right)^2} + 6\sin^{-1}\left(\frac{x-3}{3}\right) + c$$

(k) $\displaystyle\int \frac{x\,dx}{\sqrt{x^2+8x+20}}$

$$= \int \frac{x\,dx}{\sqrt{(x+4)^2+4}}$$

Let $x+4 = 2\sinh\theta$

$dx = 2\cosh\theta\, d\theta$

$\frac{x+4}{2} = \sinh\theta$

$$= \int \left(\frac{2\sinh\theta - 4}{\sqrt{4\sinh^2\theta + 4}}\, 2\cosh\theta\right) d\theta$$

$$= \int \frac{2\sinh\theta - 4}{\cancel{2\cosh\theta}}\, d\theta\; \cancel{2\cosh\theta}$$

$$= \int (2\sinh\theta - 4)\, d\theta$$

$$= 2\cosh\left(\sinh^{-1}\left(\frac{x+4}{2}\right)\right) - 4\sinh^{-1}\left(\frac{x+4}{2}\right) + c$$

$$= 2\sqrt{1+\sinh^2\left(\sinh^{-1}\left(\frac{x+4}{2}\right)\right)} - 4\sinh^{-1}\left(\frac{x+4}{2}\right) + c$$

$$= 2\sqrt{1+\left(\frac{x+4}{2}\right)^2} - 4\sinh^{-1}\left(\frac{x+4}{2}\right) + c$$

$$= 2\sqrt{1+\left(\frac{x+4}{2}\right)^2} - 4\ln\left|\sqrt{1+\left(\frac{x+4}{2}\right)^2} + \left(\frac{x+4}{2}\right)\right| + c$$

Or

$$\int \frac{x}{\sqrt{(x+4)^2+4}}$$

let $\quad x+4 = 2\tan\theta$

$\quad dx = 2\sec^2\theta \, d\theta$

$$= \int \frac{2\tan\theta - 4}{\sqrt{4\tan^2\theta + 4}} \, 2\sec^2\theta \, d\theta$$

$$= \int (2\tan\theta - 4)\sec\theta \, d\theta$$

$$= 2\sec\theta - 4\ln|\sec\theta + \tan\theta| + c$$

$$= 2\sec\left(\tan^{-1}\left(\frac{x+4}{2}\right)\right) - 4\ln\left|\sec\left(\tan^{-1}\left(\frac{x+4}{2}\right)\right) + \frac{x+4}{2}\right|$$

Notice $\quad \sec(\tan^{-1} z) = \sqrt{z^2+1}$

$$= 2\sqrt{1+\left(\frac{x+4}{2}\right)^2} - 4\ln\left|\sqrt{1+\left(\frac{x+4}{2}\right)^2} + \frac{x+4}{2}\right|$$

(L) $\displaystyle\int \frac{x^3}{\sqrt{x^2-2x+4}}\,dx$

complete the : $x^2-2x+4 = (x-1)^2+3$

$$= \int \frac{x^3}{\sqrt{(x-1)^2+3}}\,dx$$

let $x-1 = \sqrt{3}\tan\theta \qquad \theta = \tan^{-1}\left(\frac{x-1}{\sqrt{3}}\right)$

$dx = \sqrt{3}\sec^2\theta\,d\theta$

$$= \int \frac{(1+\sqrt{3}\tan\theta)^3}{\cancel{\sqrt{3\tan^2\theta+3}}}\left(\cancel{\sqrt{3}\sec^2\theta}\right)d\theta$$

$$= \int \left(1 + 3\sqrt{3}\tan\theta + 9\tan^2\theta + 3\sqrt{3}\tan^3\theta\right)d\theta$$

$$= \theta - 3\sqrt{3}\ln|\cos\theta| + \int (9\tan^2\theta + 3\sqrt{3}\tan^3\theta)\,d\theta$$

$$= \theta - 3\sqrt{3}\ln|\cos\theta| + 9\int(\sec^2\theta-1)\,d\theta + 3\sqrt{3}\int \tan^2\theta\tan\theta\,d\theta$$

$$= \theta - 3\sqrt{3}\ln|\cos\theta| + 9(\tan\theta-\theta) + 3\sqrt{3}\int(\sec^2\theta-1)\tan\theta\,d\theta$$

$$= \theta - 3\sqrt{3}\ln|\cos\theta| + 9(\tan\theta-\theta) + 3\sqrt{3}\int(\sec^2\theta\tan\theta - \tan\theta)\,d\theta$$

$$= \frac{3\sqrt{3}}{2}\tan^2\theta + 9\tan\theta - 8\theta + c$$

$$= \frac{3\sqrt{3}}{2}\tan^2\left(\tan^{-1}\left(\frac{x-1}{\sqrt{3}}\right)\right) + 9\tan\left(\tan^{-1}\left(\frac{x-1}{\sqrt{3}}\right)\right) - 8\tan^{-1}\left(\frac{x-1}{\sqrt{3}}\right) + c$$

$$= \frac{3\sqrt{3}}{2}\left(\frac{x-1}{\sqrt{3}}\right)^2 + 9\left(\frac{x-1}{\sqrt{3}}\right) - 8\tan^{-1}\left(\frac{x-1}{\sqrt{3}}\right) + c$$

$$= \frac{\sqrt{3}\,(x-1)^2}{2} + 3(x-1) - 8\tan^{-1}\left(\frac{x-1}{\sqrt{3}}\right) + c$$

(M) $\displaystyle\int \frac{x}{4-x^4}\,dx$

Let $x^2 = 2\sin\theta \qquad \theta = \sin^{-1}\left(\frac{x^2}{2}\right)$

$2x\,dx = 2\cos\theta\,d\theta$

$dx = \frac{\cos\theta}{x}\,d\theta$

$= \displaystyle\int \frac{\cancel{x}}{4-4\sin^2\theta}\cdot\frac{\cos\theta}{\cancel{x}}\cdot d\theta$

$= \displaystyle\int \frac{\cos\theta}{4\cos^2\theta}\,d\theta$

$= \displaystyle\frac{1}{4}\int \sec\theta\,d\theta$

$= \displaystyle\frac{1}{4}\int \frac{\sec^2\theta + \sec\theta\tan\theta}{\sec\theta + \tan\theta}\,d\theta$

Let $s = \sec\theta + \tan\theta$

$ds = (\sec^2\theta + \sec\theta\tan\theta)\,d\theta$

$= \displaystyle\frac{1}{4}\int \frac{1}{s}\,ds$

$= \frac{1}{4}\ln|s| + c$

$= \frac{1}{4}\ln|\sec\theta + \tan\theta| + c$

$= \frac{1}{4}\ln\left|\sec\left(\sin^{-1}\left(\frac{x^2}{2}\right)\right) + \tan\left(\sin^{-1}\left(\frac{x^2}{2}\right)\right)\right| + c$

Notice $\sec(\sin^{-1}p) = \frac{1}{\sqrt{1-p^2}}$, $\tan(\sin^{-1}p) = \frac{p}{\sqrt{1-p^2}}$

$= \frac{1}{4}\ln\left|\frac{2}{\sqrt{4-x^4}} + \frac{x^2}{\sqrt{4-x^4}}\right| + c$

$= \frac{1}{4}\ln\left|\frac{2+x^2}{\sqrt{4-x^4}}\right| + c$

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(n) $\int \frac{e^x}{4 + e^{2x}}\, dx$

let $u = e^x$

$du = e^x\, dx$

$du = u\, dx$

$= \int \frac{\cancel{u}}{4 + u^2} \left(\frac{1}{\cancel{u}}\right) du$

$= \int \frac{1}{4+u^2}\, du$

let $u = 2\tan\theta$, $\theta = \tan^{-1}\left(\frac{u}{2}\right)$

$du = 2\sec^2\theta\, d\theta$

$= \int \frac{2\sec^2\theta}{4 + 4\tan^2\theta}\, d\theta$

$= \frac{1}{2} \int d\theta$

$= \frac{1}{2}\theta + C$

$= \frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right) + C$

$= \frac{1}{2}\tan^{-1}\left(\frac{e^x}{2}\right) + C$

(c) $\int \frac{\cos x}{5 + \sin^2 x} \, dx$

Let $u = \sin x$

$du = \cos x \, dx$

$= \int \frac{\cancel{\cos x}}{5 + u^2} \left(\frac{1}{\cancel{\cos x}}\right) du$

$= \int \frac{1}{5 + u^2} \, du$

Let $u = \sqrt{5} \tan \theta$ $\quad \theta = \tan^{-1}\left(\frac{u}{\sqrt{5}}\right)$

$du = \sqrt{5} \sec^2 \theta \, d\theta$

$= \int \frac{\sqrt{5} \sec^2 \theta}{5 + 5\tan^2 \theta} \, d\theta$

$= \frac{1}{5} \int \frac{\sqrt{5} \cancel{\sec^2 \theta}}{\cancel{\sec^2 \theta}} \, d\theta$

$= \frac{\sqrt{5}}{5} \theta + C$

$= \frac{\sqrt{5}}{5} \tan^{-1}\left(\frac{u}{\sqrt{5}}\right) + C$

$= \frac{\sqrt{5}}{5} \tan^{-1}\left(\frac{\sin x}{\sqrt{5}}\right) + C$

3.

(a) $\int x^2 e^x \, dx$

$$\int u_1 dv_1 = u_1 v_1 - \int v_1 du_1$$

Let $u_1 = x^2 \Rightarrow du_1 = 2x \, dx$ and

$$\int dv_1 = \int e^x dx \Rightarrow v_1 = e^x$$

$$= \int x^2 e^x \, dx = x^2 e^x - 2\int x e^x \, dx \text{ --- (i)}$$

picking $\int xe^x \, dx$ to integrate independently we obtain

$$\int x e^x$$

$$\int u_2 \, dv_2 = u_2 v_2 - \int v_2 \, du_2$$

Let $u_2 = x \Rightarrow du_2 = dx$ and

$$\int dv_2 = \int e^x dx \Rightarrow v_2 = e^x$$

$$= \int xe^x = xe^x - \int e^x dx = xe^x - e^x \text{ --- (ii)}$$

Substituting (ii) into (i) we get

$$\int x^2 e^x dx = x^2 e^x - 2(xe^x - e^x) + c$$

$$= x^2 e^x - 2xe^x + 2e^x + c$$

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(b) $\int \cos(\ln x)\, dx$

Let $u_1 = \cos(\ln x) \Rightarrow du_1 = \frac{-\sin(\ln x)}{x} dx$

$\int dv_1 = \int dx \Rightarrow v_1 = x$

$\int \cos(\ln x)\, dx = x\cos(\ln x) + \int \sin(\ln x)\, dx$ ---- (i)

picking $\int \sin(\ln x)\, dx$ to integrate independently we get

$\int \sin(\ln x)\, dx \equiv \int u_2\, dv_2$

$u = \sin(\ln x) \Rightarrow du = \frac{\cos(\ln x)}{x}$

$\int dv = \int dx \Rightarrow v = x$

$\int \sin(\ln x) dx = x\sin(\ln x) - \int \cos(\ln x) dx$ ---- (ii)

Substituting (ii) into (i) we get

$\int \cos(\ln x)\, dx = x\cos(\ln x) + x\sin(\ln x) - \int \cos(\ln x)\, dx$

$2\int \cos(\ln x) dx = x\cos(\ln x) + x\sin(\ln x)$

$\int \cos(\ln x)\, dx = \frac{1}{2}x\cos(\ln x) + \frac{1}{2}x\sin(\ln x) + c$

(c) $\int e^{ax} \cos bx \, dx$

Let: $u_1 = e^{ax} \Rightarrow du_1 = ae^{ax}\,dx$ and

$\int dv_1 = \int \cos bx \, dx \Rightarrow v_1 = \frac{1}{b} \sin bx$

$$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{b} \sin bx - \frac{a}{b} \int e^{ax} \sin bx \, dx \quad \text{---(i)}$$

$\int e^{ax} \sin bx \, dx$ gives the following

Let $u_2 = e^{ax} \Rightarrow du_2 = a e^{ax} dx$ and

$\int dv_2 = \int \sin bx \, dx \Rightarrow v_2 = -\frac{1}{b} \cos bx$

$$\int e^{ax} \sin bx \, dx = -\frac{e^{ax}}{b} \cos bx + \frac{a}{b} \int e^{ax} \cos bx \, dx \quad \text{---(ii)}$$

$$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{b} \sin bx - \frac{a}{b}\left(-\frac{e^{ax}}{b} \cos bx + \frac{a}{b} \int e^{ax} \cos bx \, dx \right)$$

$$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{b} \sin bx + \frac{a e^{ax}}{b^2} \cos bx - \frac{a^2}{b^2} \int e^{ax} \cos bx \, dx$$

$$\frac{b^2 + a^2}{b^2} \int e^{ax} \cos bx = \frac{e^{ax}}{b^2} \sin bx + \frac{a e^{ax}}{b^2} \cos bx$$

$$\int e^{ax} \cos bx = \frac{e^{ax} \sin bx + a e^{ax} \cos bx}{b^2 + a^2} + c$$

(d) $\int (\ln x)^2 \, dx$

Let $u_1 = (\ln x)^2 \Rightarrow du_1 = \frac{2\ln x}{x} dx$

$\int dv_1 = \int dx \Rightarrow v_1 = x$

$\int (\ln x)^2 dx = x(\ln x)^2 - 2\int \ln x \, dx \; \ldots\ldots (i)$

$\int \ln x \, dx$

Let $u_2 = \ln x \Rightarrow du_2 = \frac{1}{x} dx$

$\int dv_2 = \int dx \Rightarrow v_2 = x$

$\int \ln x \, dx = x \ln x - \int dx = x\ln x - x \; \ldots (ii)$

Substituting (ii) into (i)

$\int (\ln x)^2 dx = x(\ln x)^2 - 2(x\ln x - x) + C$

$= x(\ln x)^2 - 2x\ln x + 2x + C$

(e) $\int x^2 \tan^{-1} x \, dx$

$u = \tan^{-1} x \Rightarrow du = \frac{dx}{1+x^2}$

$\int dv = \int x^2 dx \Rightarrow v = \frac{x^3}{3}$

$\int x^2 \tan^{-1} x \, dx = \frac{x^3 \tan^{-1} x}{3} - \frac{1}{3}\int \frac{x^3}{1+x^2} dx$

Simplify $\dfrac{x^3}{1+x^2}$

$$\begin{array}{r|l} & x \\ \hline 1+x^2 & x^3 \\ & -\ (x^3+x) \\ \hline & -x \end{array}$$

$$\frac{x^3}{1+x^2} = x - \frac{x}{(1+x^2)}$$

$$= \frac{x^3 \operatorname{Tan}^{-1}x}{3} - \frac{1}{3}\int\left(x - \frac{x}{(1+x^2)}\right)dx$$

$$= \frac{x^3 \operatorname{Tan}^{-1}x}{3} - \frac{1}{3}\int x\,dx + \frac{1}{3}\int \frac{x}{1+x^2}\,dx$$

For $\int \frac{x}{1+x^2}\,dx$, let $x = \operatorname{Tan}\theta$

$dx = \sec^2\theta\, d\theta$

$$= \int \frac{\operatorname{Tan}\theta}{\cancel{\sec^2\theta}} \cancel{\sec^2\theta}\, d\theta$$

$$= \int \operatorname{Tan}\theta\, d\theta$$

$$= -\ln|\cos\theta| = \ln|\sec\theta| = \ln|\sqrt{x^2+1}|$$

Now

$$\int x^2 \tan^{-1}x\, dx = \frac{x^3 \operatorname{Tan}^{-1}x}{3} - \frac{x^2}{6} - \frac{1}{3}\ln|\cos(\operatorname{Tan}^{-1}x)| + c$$

(f) $\int \frac{\ln x}{x^2}\, dx$

Let $u = \ln x \Rightarrow du = \frac{1}{x}\, dx$

$\int dv = \int \frac{1}{x^2}\, dx \Rightarrow v = -\frac{1}{x}$

$$\int \frac{\ln x}{x^2}\, dx = \frac{-\ln x}{x} + \int x^{-2}\, dx$$

$$= \frac{-\ln x}{x} - \frac{1}{x} + C$$

(g) $\int x\sqrt{1+x}\, dx$

Let $u^2 = 1+x$, $x = u^2 - 1$

$2u\, du = dx$

$$= 2\int (u^2-1)\, u \cdot u\, du$$

$$= 2\int (u^4 - u^2)\, du$$

$$= 2\left(\frac{u^5}{5} - \frac{u^3}{3}\right) + c$$

$$= 2\left(\frac{\sqrt{(1+x)^5}}{5} - \frac{\sqrt{(1+x)^3}}{3}\right) + C$$

(h)

$\int (\sin^{-1}x)^2 \, dx$

Let $u = (\sin^{-1}x)^2 \Rightarrow du = \dfrac{2\sin^{-1}(x)}{\sqrt{1-x^2}}\,dx$

$\int dv = \int dx \Rightarrow v = x$

$\int (\sin^{-1}(x))^2 dx = x(\sin^{-1}x)^2 - 2\int \dfrac{x\sin^{-1}(x)}{\sqrt{1-x^2}}\,dx$

for $\int \dfrac{x\sin^{-1}x}{\sqrt{1-x^2}}\,dx$ Let $x = \sin\theta$
$dx = \cos\theta \, d\theta$

$\Rightarrow -2\int \dfrac{\sin\theta \sin^{-1}(\sin\theta)}{\cancel{\cos\theta}} \cancel{\cos\theta}\, d\theta$

$= -2\int \theta \sin\theta \, d\theta$

by futher Integration by parts we obtain

$= -2(-\theta\cos\theta + \sin\theta)$

$= 2\theta\cos\theta - 2\sin\theta$

$= 2\sin^{-1}(x)\cos(\sin^{-1}x) - 2\sin(\sin^{-1}(x))$

$= 2\sin^{-1}(x)\sqrt{1-x^2} - 2x$

Now

$\int (\sin^{-1}x)^2 dx = x(\sin^{-1}(x))^2 + 2\sin^{-1}(x)(\sqrt{1-x^2}) - 2x + c$

(i)

$\int x^n \ln x \, dx$

Let $u = \ln x \Rightarrow du = \frac{1}{x} dx$

$\int dv = \int x^n dx \Rightarrow v = \frac{x^{n+1}}{n+1}$

$$\int x^n \ln x \, dx = \frac{x^{n+1}}{n+1} \ln x - \frac{1}{n+1} \int x^n \, dx$$

$$= \frac{x^{n+1}}{n+1} \ln x - \frac{x^{n+1}}{(n+1)^2} + C$$

(j)

$\int e^{\sqrt{x}} dx$

Let $u = \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}} dx$

$dx = 2u\,du$

$\Rightarrow \int e^{\sqrt{x}} dx = 2\int u e^u du$

$\int f dg = fg - \int g df$

Let $f = u \Rightarrow df = du$

$\int dg = \int e^u \Rightarrow g = e^u$

$2\int u e^u du = 2(ue^u - \int e^u du)$

$= 2ue^u - e^u + C$

$\therefore \int e^{\sqrt{x}} dx = 2\sqrt{x} e^{\sqrt{x}} - e^{\sqrt{x}} + C$

$= 2 e^{\sqrt{x}}(\sqrt{x} - 1) + C$

(1K) $\int \ln(x^2+1)\,dx$

Let $u = \ln(x^2+1) \Rightarrow$

$du = \dfrac{2x}{x^2+1}\,dx$

$\int dv = \int dx \Rightarrow v = x$

$$\int \ln(x^2+1)\,dx = x\ln(x^2+1) - 2\int \frac{x^2}{x^2+1}\,dx$$

by long division on $\int \dfrac{x^2}{x^2+1}$ we get

$$= x\ln(x^2+1) - 2\int \left(1 - \frac{1}{x^2+1}\right) dx$$

$$= x\ln(x^2+1) - 2x + 2\,\mathrm{Tan}^{-1}x + C$$

(L) $\int_0^{2\pi} x \sin nx\,dx$

Let $u = x \Rightarrow du = dx$

$\int dv = \int \sin nx\,dx \Rightarrow v = -\dfrac{1}{n}\cos nx$

$$\int_0^{2\pi} x\sin nx\,dx = \frac{-x}{n}\cos nx\Big|_0^{2\pi} + \frac{1}{n}\int_0^{2\pi} \cos nx\,dx$$

$$= \frac{-2\pi}{n} + \frac{1}{n^2}\sin nx\Big|_0^{2\pi}$$

$$= -\frac{2\pi}{n}$$

(m) $\int x\pi^x \, dx$

Let $u = x \Rightarrow du = dx$

$\int dv = \int \pi^x \, dx$

$\int dv = \int e^{x\ln\pi} \, dx$

$V = \frac{1}{\ln\pi} e^{x\ln\pi}$

$$\int x\pi^x \, dx = \frac{x e^{x\ln\pi}}{\ln\pi} - \frac{1}{\ln\pi}\int e^{x\ln\pi} \, dx$$

$$= \frac{x\pi^x}{\ln\pi} - \frac{\pi^x}{(\ln\pi)^2} + c$$

$$= \frac{\pi^x(x\ln\pi - 1)}{(\ln\pi)^2} + c$$

(n) $\int x^2 2^{-x} \, dx$

let $u = x^2 \Rightarrow du = 2x \, dx$

$\int dv = \int 2^{-x} \, dx = \int e^{-x\ln 2} \, dx$

$V = \frac{-e^{-x\ln 2}}{\ln 2}$

$$\int x^2 2^{-x} \, dx = -\frac{x^2 e^{-x\ln 2}}{\ln 2} + \frac{2}{\ln 2}\int \left(\frac{x e^{-x\ln 2}}{1}\right) dx$$

for $\int x e^{-x\ln 2} \, dx$

let $u = x \Rightarrow du = dx$, $\int dv = \int e^{-x\ln 2} \Rightarrow v = -\frac{e^{-x\ln 2}}{\ln 2}$

$$\int x\, e^{-x\ln 2}\,dx = \frac{-x e^{-x\ln 2}}{\ln 2} + \frac{1}{\ln 2}\int e^{-x\ln 2}\,dx$$

$$= \frac{-x e^{-x\ln 2}}{\ln 2} + \frac{e^{-x\ln 2}}{(\ln 2)^2}$$

$$\int x^2 2^{-x}\,dx = \frac{-x^2 e^{-x\ln 2}}{\ln 2} + \frac{2}{\ln 2}\left(\frac{-x e^{-x\ln 2}}{\ln 2} + \frac{e^{-x\ln 2}}{(\ln 2)^2}\right) + c$$

$$= \frac{-x^2 e^{-x\ln 2}}{\ln 2} - \frac{2x e^{-x\ln 2}}{(\ln 2)^2} + \frac{e^{-x\ln 2}}{(\ln 2)^3} + c$$

$$= \frac{-x^2 2^{-x}}{\ln 2} - \frac{2x\, 2^{-x}}{(\ln 2)^2} + \frac{2^{-x}}{(\ln 2)^3} + c$$

(b) $\int x\, 2^{\sqrt{2}x}\,dx$

Let $u = x \Rightarrow du = dx$

$\int dv = \int 2^{x\sqrt{2}}\,dx \Rightarrow$

$v = \int e^{x\sqrt{2}\ln 2} = \frac{1}{\sqrt{2}\ln 2} e^{x\sqrt{2}\ln 2}$

$$\int x\, 2^{x\sqrt{2}}\,dx = \frac{x e^{x\sqrt{2}\ln 2}}{\sqrt{2}\ln 2} - \int \frac{1}{\sqrt{2}\ln 2} e^{x\sqrt{2}\ln 2}\,dx$$

$$= \frac{x\, 2^{\sqrt{2}x}}{\sqrt{2}\ln 2} - \frac{2^{\sqrt{2}x}}{(\sqrt{2}\ln 2)^2} + c$$

(p) $\int e^t \sec^3(e^t - 1)\, dt$

Let $u = e^t - 1$

$du = e^t\, dt$

$du = (u+1)\, dt$

$dt = \dfrac{du}{u+1}$

$= \int \cancel{(u+1)} \sec^3 u \dfrac{du}{\cancel{(u+1)}}$

$= \int \sec^3 u\, du$

Using the reduction formula

$\therefore \int \sec^n u\, du = \int \sec^2 u \sec^{n-2} du$

Let $f = \sec^{n-2} u \Rightarrow df = (n-2)\sec^{n-3} u\, (\sec u \tan u)\, du$

and $\int \sec^2 u\, du = \int dg \Rightarrow g = \tan u$

$\int \sec^n u\, du = \sec^{n-2} u \tan u - (n-2)\int \sec^{n-3}(\sec u \tan^2 u)\, du$

$\int \sec^n u\, du = \sec^{n-2} u \tan u - (n-2)\int \sec^{n-2} u (\sec^2 u - 1)\, du$

$\int \sec^n u\, du = \sec^{n-2} u \tan u - (n-2)\int (\sec^n u - \sec^{n-2} u)\, du$

$\int \sec^n u\, du = \sec^{n-2} u \tan u - (n-2)\int \sec^n u + (n-2)\int \sec^{n-2} u\, du$

$(n-1)\int \sec^n u\, du = \sec^{n-2} u \tan u + (n-2)\int \sec^{n-2} u\, du$

$\int \sec^n u\, du = \dfrac{1}{n-1}\sec^{n-2} u \tan u + \dfrac{n-2}{n-1}\int \sec^{n-2} u\, du$

When $n = 3$

$$\int \sec^3 u\,du = \frac{1}{3-1}\sec^{3-2}u\,\tan u + \frac{3-2}{3-1}\int \sec^{3-2}u\,du$$

$$= \frac{1}{2}\sec u\tan u + \frac{1}{2}\int \sec u\,du$$

$$= \frac{1}{2}\sec u\tan u + \frac{1}{2}\ln|\sec u + \tan u| + C$$

$$= \frac{1}{2}\sec(e^t-1)\tan(e^t-1) + \frac{1}{2}\ln|\sec(e^t-1)+\tan(e^t-1)| + C$$

(9) $\int \frac{\csc^3\sqrt{\theta}}{\sqrt{\theta}}\,d\theta$

Let $u = \sqrt{\theta} \Rightarrow 2\sqrt{\theta}\,du = d\theta$

$$\int \frac{\csc^3 u}{\cancel{\sqrt{\theta}}}\,2\cancel{\sqrt{\theta}}\,du$$

$$2\int \csc^3 u\,du$$

Using the reduction formular

$$\int \csc^n x\,dx = \int \csc^2 x\,\csc^{n-2}(x)\,dx$$

Let $u = \csc^{n-2}(x) \Rightarrow du = -(n-2)\csc^{n-3}(x)(\csc x\cot x)$

$du = -(n-2)\csc^{n-2}(x)\cot x\,dx$

$$\int \csc^2 x\,dx = \int dv$$

$$v = -\cot x$$

$$\int \csc^n x\,dx = -\csc^{n-2}(x)\cot x - (n-2)\int \csc^{n-2}(x)\cot^2(x)\,dx$$

$$\int \csc^n x\,dx = -\csc^{n-2}(x)\cot x - (n-2)\int \csc^{n-2}(x)(\csc^2 x - 1)\,dx$$

$$= -\csc^{n-2}(x)\cot x - (n-2)\int (\csc^n x - \csc^{n-2} x)\,dx$$

$$= -\csc^{n-2}(x)\cot x - (n-2)\int \csc^n x\,dx + (n-2)\int \csc^{n-2}(x)\,dx$$

$$(n-1)\int \csc^n x\,dx = -\csc^{n-2}(x)\cot x + (n-2)\int \csc^{n-2}(x)\,dx$$

$$\int \csc^n x\,dx = -\frac{1}{(n-1)}\csc^{n-2}(x)\cot x + \frac{n-2}{n-1}\int \csc^{n-2}(x)\,dx$$

When $n = 3$

$$2\int \csc^3 u\,du = 2\left(-\frac{1}{2}\csc u\cot u + \frac{1}{2}\int \csc(u)\,du\right)$$

$$2\int \csc^3 u\,du = -\csc u\cot u - \ln|\csc u + \cot u| + C$$

Now

$$\int \frac{\csc^3\sqrt{\theta}}{\sqrt{\theta}}\,d\theta = -\csc\sqrt{3}\cot\sqrt{3} - \ln|\csc\sqrt{3} + \cot\sqrt{3}| + C$$

(r) $\int \frac{\sqrt{x}}{\sqrt{1-x}}\,dx$

Let $\sqrt{x} = \sin\theta \qquad \theta = \sin^{-1}\sqrt{x}$

$\frac{1}{2}x^{-1/2}\,dx = \cos\theta\,d\theta$

$dx = 2\sqrt{x}\cos\theta\,d\theta$

$= 2\int \frac{\sqrt{x}}{\sqrt{1-\sin^2\theta}}\sqrt{x}\cos\theta\,d\theta$

$= 2\int \frac{\sin^2\theta}{\cancel{\cos\theta}}\cancel{\cos\theta}\,d\theta$

$= +2\int \sin^2\theta\,d\theta$

$= +\cancel{2}\int \frac{1-\cos 2\theta}{\cancel{2}}\,d\theta$

$= \int(\cos 2\theta + 1)\,d\theta$

$= -\frac{1}{2}\sin 2\theta + \theta + C$

$= -\frac{1}{2}\sin 2\theta + \theta + C$

$= -\cos\theta\sin\theta + \theta + C$

$= -\cos(\sin^{-1}\sqrt{x})\sin(\sin^{-1}\sqrt{x}) - \sin^{-1}\sqrt{x} + C$

$= \sqrt{x}\sqrt{1-\sqrt{x}^2} + \sin^{-1}\sqrt{x} + C$

$= \sin^{-1}\sqrt{x} - \sqrt{x(1-x)} + C$

(5) $\int x \sin^{-1}(x^2)\, dx =$

Let $u = x^2 \Rightarrow du = 2x\, dx$

$= \frac{1}{2}\int \sin^{-1} u \, du$

For the integrand $\sin^{-1}(u)$ integrate by parts

$\int f\, dg = fg - \int g\, df$ where

$f = \sin^{-1}(u)$, $dg = du$, $df = \frac{1}{\sqrt{1-u^2}} du$, $g = u$

$= \frac{1}{2} u \sin^{-1} u - \frac{1}{2}\int \frac{u}{\sqrt{1-u^2}} du$

For the integrand $\frac{u}{\sqrt{1-u^2}}$, substitute $s = 1-u^2$

$ds = -2u\, du$

$= \frac{1}{2} u \sin^{-1}(u) + \frac{1}{4}\int \frac{1}{\sqrt{s}} ds$

$= \frac{1}{4} u \sin^{-1}(u) + \frac{\sqrt{s}}{2} + \frac{1}{4} u \sin^{-1}(u) + C$

$= \frac{\sqrt{1-u^2}}{2} + \frac{1}{2} u \sin^{-1} u + C$

$= \frac{\sqrt{1-x^4}}{4} + \frac{1}{2} x^2 \sin^{-1}(x^2) + C$

$= \frac{1}{2}\left(\sqrt{1-x^4} + x^2 \sin^{-1}(x^2)\right) + C$

(t) $\int x^2 \tan^{-1}x \, dx = \frac{x^3 \tan^{-1}x}{3} - \frac{x^2}{6} - \frac{1}{3}\ln|\cos(\tan^{-1}x)| + c$

NB: $\cos(\tan^{-1}x) = \frac{1}{\sqrt{1+x^2}}$

$$\int x^2 \tan^{-1}x \, dx = \frac{x^3 \tan^{-1}x}{3} - \frac{x^2}{6} + \frac{1}{3}\ln|\sqrt{1+x^2}| + c$$

(e) is the same as (t)

(4) $\int \sin^2(x)\cos(x)\ln(\sin x)\,dx = \frac{1}{9}\sin^2(x)\left(3\ln(\sin x) - 1\right) + c$

Let $u = \sin x$ $\quad du = \cos x \, dx$

$= \int u^2 \ln u \, du$

Integrating by parts we get

$\int f \, dg = fg - \int g \, df$

Let $f = \ln u \Rightarrow df = \frac{1}{u} du$

$\int dg = u^2 du \Rightarrow g = \frac{u^3}{3}$;

$$= \frac{1}{3}u^3 \ln u - \frac{1}{3}\int u^2 \, du$$

$$= \frac{1}{3}u^3 \ln(u) - \frac{u^3}{9} + C$$

$$= \frac{1}{3}\sin^3(x)\left(3\ln(\sin x) - 1\right) + C$$

(4)

(a) $\int \cos^n x \, dx \equiv \int \cos x \cos^{n-1} x \, dx$

Let $u = \cos^{n-1} x \Rightarrow du = (n-1)\cos^{n-2} x \, (-\sin x) \, dx$

$\int dv = \int \cos x \, dx \Rightarrow v = \sin x$

$\int \cos^n x \, dx = \sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} x \sin^2 x \, dx$

$\int \cos^n x \, dx = \cos^{n-1}(x) \sin x + (n-1)\int (\cos^{n-2}(x) - \cos^n(x)) dx$

$\int \cos^n x \, dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2}(x) \, dx - (n-1)\int \cos^n(x) dx$

$n\int \cos^n x \, dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2}(x) \, dx$

$\int \cos^n x \, dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2}(x) \, dx$

(b) $\int \sin^n x \, dx \equiv \int \sin x \sin^{n-1}(x) \, dx$

Let $u = \sin^{n-1}(x) \Rightarrow du = (n-1)\sin^{n-2}(x)\cos x \, dx$

$\int dv = \int \sin x \, dx \Rightarrow v = -\cos x$

$\int \sin^n x \, dx = -\sin^{n-1}(x) \cos x + (n-1)\int \sin^{n-2}(x) \cos^2 x \, dx$

$\int \sin^n x \, dx = -\sin^{n-1}(x) \cos x + (n-1)\int \sin^{n-2}(x) (1 - \sin^2 x) \, dx$

$n\int \sin^n x \, dx = -\sin^{n-1}(x) \cos x + (n-1)\int \sin^{n-2}(x) \, dx$

$\int \sin^n x \, dx = -\frac{1}{n}\sin^{n-1}(x) \cos x + \frac{n-1}{n}\int \sin^{n-2}(x) \, dx$

(c) $\int \sec^n x \, dx \equiv \int \sec^2 x \sec^{n-2} x \, dx$

Let $u = \sec^{n-2} x \Rightarrow du = (n-2)\sec^{n-3}(x)\sec x \tan x \, dx$

$\int dv = \int \sec^2 x \, dx \Rightarrow v = \tan x$

$$\int \sec^n x \, dx = \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x \tan^2 x \, dx$$

$$\int \sec^n x \, dx = \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x \,(\sec^2 x - 1)\, dx$$

$$\int \sec^n x \, dx = \sec^{n-2} x \tan x - (n-2)\int \sec^n x \, dx + (n-2)\int \sec^{n-2}(x)\, dx$$

$$\int \sec^n x \, dx = \sec^{n-2} x \tan x + (2-n)\int \sec^n x \, dx + (n-2)\int \sec^{n-2}(x)\, dx$$

$$\Rightarrow (n-1)\int \sec^n x \, dx = \sec^{n-2} x \tan x + (n-2)\int \sec^{n-2}(x)\, dx$$

$$\int \sec^n x \, dx = \frac{1}{n-1}\sec^{n-2} x \tan x + \frac{n-2}{n-1}\int \sec^{n-2}(x)\, dx$$

(d) $\int x^n e^{ax} \, dx$

Let $u = x^n \Rightarrow du = n x^{n-1} \, dx$

$\int dv = \int e^{ax} dx \Rightarrow v = \frac{1}{a} e^{ax}$

$$\Rightarrow \int x^n e^{ax} dx = \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1} e^{ax} \, dx$$

(e) $\int x^n \sin ax \, dx$

Let $u = x^n \Rightarrow du = n x^{n-1} dx$

$\int dv = \int \sin ax \, dx \Rightarrow v = -\frac{1}{a}\cos ax$

$$\int x^n \sin ax \, dx = -\frac{x^n \cos ax}{a} + \frac{n}{a}\int x^{n-1} \cos ax \, dx$$

(f) $\int \frac{x^2}{(a^2+x^2)^n}\,dx$

Let $u = (a^2+x^2)^{-n} \Rightarrow du = \frac{-2nx}{(a^2+x^2)^{n+1}}\,dx$

$\int dv = \int x^2\,dx \Rightarrow v = \frac{x^3}{3}$

$$\int \frac{x^2}{(a^2+x^2)^n}\,dx = \frac{x^3}{3(a^2+x^2)^n} + \frac{2n}{3}\int \frac{x^4}{(a^2+x^2)^{n+1}}\,dx$$

(g) $\int x^m (\ln x)^n\,dx$

Let $u = (\ln x)^n \Rightarrow du = \frac{n(\ln x)^{n-1}}{x}$

$\int dv = \int x^m\,dx \Rightarrow v = \frac{x^{m+1}}{m+1}$

$$\int x^m (\ln x)^n\,dx = \frac{x^{m+1}}{m+1}(\ln x)^n - \frac{n}{m+1}\int x^m (\ln x)^{n-1}\,dx$$

(h) $\int (\ln x)^n\,dx$

Let $u = (\ln x)^n \Rightarrow du = \frac{n(\ln x)^{n-1}}{x}$

$\int dv = \int dx \Rightarrow v = x$

$$\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx$$

(i) $\int Tan^n x\, dx$,

$= \int Tan^{n-2}x\, Tan^2 x\, dx$

$= \int Tan^{n-2}x\,(sec^2 x - 1)\, dx$

$= \int Tan^{n-2}x\, sec^2 x\, dx - \int Tan^{n-2}x\, dx$

$= \frac{1}{n-1} Tan^{n-1}(x) - \int Tan^{n-2}x\, dx$

$$\int Tan^n x\, dx = \frac{1}{n-1} Tan^{n-1}(x) - \int Tan^{n-2}(x)\, dx$$

(j) $\int x^n e^{x^2} dx$

Let $u = x^n \Rightarrow du = nx^{n-1} dx$

$\int dv = \int e^{x^2} dx \Rightarrow v = \int e^{x^2} dx$

$\Gamma(m) = \int x^{m-1} e^{-x} dx$

Let $y = x^2$

$dy = 2x\, dx$

$= \int e^{x^2} x^{n-1} (x\, dx)$

$= \frac{1}{2} \int (\sqrt{y})^{n-1} e^y\, dy$

$= \frac{1}{2} \int y^{\frac{1}{2}n - \frac{1}{2}} e^y\, dy$

$$= \frac{1}{2}\int (-y)^{\frac{n}{2}-\frac{1}{2}} e^{y}\, dy$$

$$= \frac{1}{2}\int (-1)^{\frac{n}{2}-\frac{1}{2}} (y)^{\left(\frac{n}{2}+\frac{1}{2}\right)-1} e^{y}\, dy$$

$$= \frac{(-1)^{\frac{n}{2}-\frac{1}{2}}}{2}\int y^{\left(\frac{n}{2}+\frac{1}{2}\right)-1} e^{y}\, dy$$

$$= \frac{(-1)^{\frac{n}{2}-\frac{1}{2}}}{2}\,\Gamma\left(\frac{n}{2}+\frac{1}{2},\ -x^{2}\right) + C$$

gama function.

k) $\int x^{n}\cos\pi x\, dx$

Let $u = x^{n} \Rightarrow du = \frac{nx^{n-1}}{n-1}\, dx$

$\int dv = \int \cos\pi x\, dx \Rightarrow v = \frac{1}{\pi}\sin\pi x$

$$\int x^{n}\cos\pi x\, dx = -\frac{n}{n-1}\int \frac{x^{n-1}\sin\pi x}{\pi} + \frac{x^{n}\sin\pi x}{\pi}$$

$$= \frac{x^{n}\sin\pi x}{\pi} - \frac{n}{\pi(n-1)}\int \frac{x^{n-1}\sin\pi x}{\pi}$$

for $\int x^{n-1} \sin \pi x$

let $u = x^{n-1} \Rightarrow du = (n-1) x^{n-2} dx$

$\int dv = \int \sin \pi x \, dx \Rightarrow v = -\frac{\cos \pi x}{\pi}$

$\int x^{n-1} \sin \pi x = -x^{n-1} \cos \pi x + \int \frac{n-1}{\pi} x^{n-2} \cos \pi x$

the formular is

$$\int x^n \cos \pi x \, dx = \frac{x^n \sin \pi x}{\pi} - \frac{n}{\pi (n-1)} \int x^{n-1} \sin \pi x \, dx$$

(1) $\int x (\ln x)^n \, dx$

let $u = (\ln x)^n \Rightarrow du = \frac{n (\ln x)^{n-1}}{x} dx$

$\int dv = \int x \, dx \Rightarrow v = \frac{x^2}{2}$

$$\int x (\ln x)^n dx = \frac{x^2 (\ln x)^n}{2} - \frac{n}{2} \int x (\ln x)^{n-1} dx$$

(m) $$\int \frac{x^n\,dx}{\sqrt{(1+x^2)^n}}$$

Let $u^2 = 1+x^2$; $x = \sqrt{u^2-1}$, $u = \sqrt{1+x^2}$

$$2u\,du = 2x\,dx$$

$$dx = \frac{u}{x}\,du$$

$$= \int \frac{\sqrt{(u^2-1)^n}}{u^n} \frac{u}{\sqrt{(u^2-1)}}\,du$$

$$= \int u^{1-n} \sqrt{(u^2-1)}^{\,n-1}\,du$$

$$\int f\,dg = fg - \int g\,df$$

Let $f = (u^2-1)^{\frac{n-1}{2}} \Rightarrow df = \frac{n-1}{2}(u^2-1)^{\frac{n-3}{2}} 2u\,du$

$$\int dg = \int u^{1-n}\,du \Rightarrow g = \frac{u^{2-n}}{2-n}$$

$$\Rightarrow \int u^{1-n}\sqrt{(u^2-1)}^{\,n-1}du = \frac{u^{2-n}\sqrt{(u^2-1)}^{\,n-1}}{2-n} - \int \frac{u^{2-n}}{2-n}(n-1)u\sqrt{(u^2-1)}^{\,n-3}_{u}$$

$$= \frac{u^{2-n}\sqrt{(u^2-1)}^{\,n-1}}{2-n} - \left(\frac{n-1}{2-n}\right)\int u^{3-n}\sqrt{(u^2-1)}^{\,n-3}\,du$$

$$= \frac{\sqrt{(1+x^2)}^{\,2-n}\,x^{n-1}}{2-n} - \left(\frac{n-1}{2-n}\right)\int \sqrt{(1+x^2)}^{\,3-n}\,x^{n-3}\left(\frac{x\,dx}{u}\right)$$

$$= \frac{x^{n-1}}{(2-n)\sqrt{(1+x^2)}^{n-2}} - \left(\frac{n-1}{2-n}\right)\int \frac{x^{n-2}}{\sqrt{(1+x^2)}^{n-2}}\,dx$$

$$= \frac{1}{2-n}\,\frac{x^{n-1}}{\sqrt{(1+x^2)}^{n-2}} - \left(\frac{n-1}{2-n}\right)\int \frac{x^{n-2}}{\sqrt{(1+x^2)}^{n-2}}\,dx$$

(n) $\int \sin^n x \cos^m x \, dx$

$u = \sin^n x$, $du = n \sin^{n-1} x \cos x \, dx$

$\int dv = \int \cos^m x \, dx$, $V = \frac{1}{m}\cos^{m-1} x \sin x + \frac{m+1}{m}\int \cos^{m-2} x \, dx$

$$\int \sin^n x \cos^m x \, dx = \frac{1}{m}\cos^{m-1} x \sin^{n+1} x + \frac{m+1}{m}\sin^n x \int \cos^{m-2} x \, dx$$

$$- \int \left(\frac{n}{m}\cos^m x \sin^n x + \frac{m+1}{m}\, n \sin^{n-1} x \cos x \int \cos^{m-2} x\right) dx$$

$$= \frac{1}{m}\cos^{m-1} x \sin^{n+1} x + \frac{m+1}{m}\sin^n x \int \cos^{m-2} x \, dx - \frac{n}{m}\int \cos^m x \sin^n x \, dx$$

$$- \frac{n(m+1)}{m}\int \sin^{n-1} x \cos x \, dx \iint \cos^{m-2} x \, dx \, dx$$

$$= \frac{1}{m}\cos^{m-1} x \sin^{n+1} x + \frac{m+1}{m}\sin^n x \int \cos^{m-2} x \, dx - \frac{n}{m}\int \cos^m x \sin^n x \, dx$$

$$- \frac{\cancel{n}(m+1)}{m}\,\frac{\sin^2 x}{\cancel{n}} \iint \cos^{m-2} x \, dx \, dx$$

$$= \frac{1}{m+n}\cos^{m-1} x \sin^{n+1} x + \frac{m+1}{m+n}\sin^n x \int \left(\cos^{m-2} x - \int \cos^{m-2} x\right) dx$$

(o) $\int x^n \sin^{-1}x \, dx$

Let $u = x^n \Rightarrow du = n x^{n-1} dx$

$\int dv = \int \sin^{-1}x \, dx \Rightarrow v =$

Let $\int f \, dg = fg - \int g \, df$

$f = \sin^{-1}x \Rightarrow df = \dfrac{1}{\sqrt{1-x^2}} dx$

$\int dg = \int dx \Rightarrow g = x$

$$v = x\sin^{-1}x - \int \frac{x}{\sqrt{1-x^2}} dx$$

$p^2 = 1 - x^2$

$2p\,dp = -2x \, dx$

$$v = x\sin^{-1}x + \int \frac{\cancel{x}}{\cancel{p}} \frac{\cancel{x}\cancel{p}}{\cancel{x}} dp$$

$$v = x\sin^{-1}x + p$$

$$v = x\sin^{-1}x + \sqrt{1-x^2}$$

$$\Rightarrow \int x^n \sin^{-1}x \, dx = x^n\left(x\sin^{-1}x + \sqrt{1-x^2}\right) - \int \left(x\sin^{-1}x + \sqrt{1-x^2}\right) n x^{n-1} dx$$

$$= x^{n+1}\sin^{-1}x + x^n\sqrt{1-x^2} - n\int x^n \sin^{-1}x - n\int x^{n-1}\sqrt{1-x^2} \, dx$$

$$= \frac{x^{n+1}\sin^{-1}x}{n+1} + \frac{x^n\sqrt{1-x^2}}{n+1} - \frac{n}{n+1}\int x^{n-1}\sqrt{1-x^2} \, dx$$

(p) $\int (x^2 + a^2)^n dx$

(5)

$$z = \tan\frac{\theta}{2}$$

$$\cos\theta = \frac{1-z^2}{1+z^2}$$

$$\sin\theta = \frac{2z}{1+z^2}$$

$$d\theta = \frac{2\,dz}{z^2+1}$$

(2) $\displaystyle\int \frac{dx}{1-\sin x}$

$$= \int \frac{1}{1-\frac{2z}{1+z^2}}\left(\frac{2}{z^2+1}\right)dz$$

$$= \int \frac{\cancel{1+z^2}}{1+z^2-2z}\,\frac{2}{\cancel{(z^2+1)}}\,dz$$

$$= \int \frac{2}{(z-1)^2}\,dz$$

for $\displaystyle\frac{2}{(z-1)^2} = \frac{A}{z-1} + \frac{B}{(z-1)^2}$

$$= \frac{A(z-1)+B}{(z-1)^2}$$

$$= \frac{Az-A+B}{(z-1)^2}$$

$A = 0$

$B = 2$

(50)

$$= \int \frac{2}{(z-1)^2}\, dz$$

$$= 2\int (z-1)^{-2}\, dz$$

$$= -2\,(z-1)^{-1} + c$$

$$= \frac{-2}{z-1} + c \qquad \text{or} \qquad \int \frac{1}{1-\sin x}\,dx = \frac{2\cos^2\frac{x}{2}}{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}$$

$$= \frac{-2}{\frac{\sin x}{1+\cos x} - 1} + c$$

$$= \frac{-2\,(1+\cos x)}{\sin x - 1 - \cos x} + c$$

$$= \frac{-2 - 2\cos x}{\sin x - \cos x - 1} + c$$

$$= \frac{2 + 2\cos x}{1 - \sin x + \cos x} + c$$

(b) $$\int \frac{d\theta}{5 + 4\sin\theta}$$

$$= \int \frac{1}{5 + 4\left(\frac{2z}{1+z^2}\right)} \cdot \frac{2\,dz}{1+z^2}$$

$$= \int \frac{\cancel{1+z^2}}{5 + 5z^2 + 8z} \cdot \frac{2}{\cancel{1+z^2}} \cdot dz$$

$$= \int \frac{2}{5z^2 + 8z + 5}\, dz$$

completing the square for

$$5z^2+8z+5 = 5\left(z^2+\frac{8}{5}z+1\right)$$

$$= 5\left(z^2+\left(\frac{4}{5}\right)^2-\left(\frac{4}{5}\right)^2+1\right)$$

$$= 5\left(\left(z+\frac{4}{5}\right)^2-\frac{16}{25}+1\right)$$

$$= 5\left(z+\frac{4}{5}\right)^2+\frac{9}{5}$$

so $$\int \frac{1}{5z^2+8z+5}\,dz = \int \frac{1}{5\left(z+\frac{4}{5}\right)^2+\frac{9}{5}}\,dz$$

Let $u = z+\frac{4}{5}$

$du = dz$

$$= \int \frac{1}{5u^2+\frac{9}{5}}\,du$$

$$= \int \frac{5}{(5u)^2+9}\,du$$

Let $5u = 3\tan y$

$5\,du = 3\sec^2 y$

$$= \cancel{5}\int \frac{1}{9\tan^2+9}\,\frac{3}{\cancel{5}}\sec^2 y\,dy$$

$$= \frac{3}{9}\int \frac{1}{\sec^2 y}\sec^2 y\,dy$$

$$= \frac{1}{3}\int dy$$

$$= \frac{1}{3} y + C$$

$$= \frac{1}{3}\tan^{-1}\left(\frac{5y}{3}\right) + C$$

$$= \frac{1}{3}\tan^{-1}\left(\frac{5}{3}z + \frac{4}{3}\right) + C$$

$$= \frac{1}{3}\tan^{-1}\left(\frac{5}{3}\tan\frac{\theta}{2} + \frac{4}{3}\right) + C$$

(c) $\displaystyle\int \frac{dx}{1 + 2\sin x}$

$$= \int \frac{1}{1 + 2\left(\frac{2z}{1+z^2}\right)}\left(\frac{2dz}{1+z^2}\right)$$

$$= \int \frac{\cancel{(1+z^2)}}{1+z^2+4z}\,\frac{2\,dz}{\cancel{(1+z^2)}}$$

$$= \int \frac{2}{z^2+4z+1}\,dz$$

$$= \int \frac{2}{(z+2)^2-3}\,dz$$

Let $z + 2 = \sqrt{3}\cosh\theta$, $\theta = \cosh^{-1}\left(\frac{z+2}{\sqrt{3}}\right)$

$dz = \sqrt{3}\sinh\theta\, d\theta$

$$= \int \frac{2}{3\cosh^2\theta - 3} \sqrt{3}\sinh\theta \, d\theta$$

$$= \int \frac{2\sqrt{3}\cancel{\sinh\theta}}{3(\cancel{\sinh\theta})} d\theta$$

$$= \frac{2\sqrt{3}}{3}\theta + c$$

$$= \frac{2\sqrt{3}}{3}\cos^{-1}\left(\frac{z+2}{\sqrt{3}}\right) + c \qquad \frac{z+2}{\sqrt{3}} = p$$

$$= \frac{2\sqrt{3}}{3}\ln\left|p + \sqrt{p^2-1}\right| + c$$

$$= \frac{2\sqrt{3}}{3}\ln\left|\frac{z+2}{\sqrt{3}} + \sqrt{\left(\frac{z+2}{\sqrt{3}}\right)^2 - 1}\right| + c$$

$$= \frac{2\sqrt{3}}{3}\ln\left|\frac{\tan\frac{\theta}{2}+2}{\sqrt{3}} + \sqrt{\left(\frac{\tan\frac{\theta}{2}+2}{\sqrt{3}}\right)^2 - 1}\right| + c$$

(d) $\int \frac{d\theta}{3-4\sin\theta+2\cos\theta}$

$$= \int \frac{1}{3-4\left(\frac{2u}{1+u^2}\right)+2\left(\frac{1-u^2}{1+u^2}\right)} \frac{2\,du}{1+u^2}$$

$$= \int \frac{\cancel{1+u^2}}{3(1+u^2)-8u+2(1-u^2)} \left(\frac{2\,du}{\cancel{1+u^2}}\right)$$

$$= \int \frac{2}{3+3u^2-8u+2-2u^2}\,du$$

$$= \int \frac{2}{u^2-8u+5}\,du$$

$$= \int \frac{2}{(u-4)^2-11}\,du$$

Let $u-4 = \sqrt{11}\cosh\theta$

$\theta = \cosh^{-1}\left(\frac{u-4}{\sqrt{11}}\right)$

$$= \frac{2\sqrt{11}}{11} \ln\left| \frac{\tan\frac{\theta}{11}+2}{\sqrt{11}} + \sqrt{\left(\frac{\tan\frac{\theta}{11}+2}{\sqrt{11}}\right)^2-1}\,\right| + C$$

(e) $\int \frac{dx}{3-4\sin x}$

$$= \int \frac{1}{3-4\left(\frac{2u}{1+u^2}\right)} \frac{2\,du}{1+u^2}$$

$$= \int \frac{\cancel{1+u^2}}{3(1+u^2)-8u} \cdot \frac{2}{\cancel{1+u^2}} \cdot du$$

$$= \int \frac{2}{3+3u^2-8u}\,du$$

$$= \int \frac{2}{3u^2-8u+3}\,du$$

$$= \int \frac{2}{3\left(u-\frac{4}{3}\right)^2-\frac{13}{3}}\,du = \frac{2}{3}\int \frac{1}{\left(u-\frac{4}{3}\right)^2-\frac{13}{9}}\,du$$

Let $u-\frac{4}{3} = \frac{\sqrt{13}}{3}\cosh\theta \qquad \theta = \cosh^{-1}\left(\frac{3u-4}{\sqrt{13}}\right)$

$$= \frac{2\sqrt{13}}{6}\ln\left|\frac{3u-4}{\sqrt{13}} + \sqrt{\left(\frac{3u-4}{\sqrt{13}}\right)^2-1}\right| + C$$

$$= \frac{2\sqrt{13}}{6}\ln\left|\frac{3\tan\frac{\theta}{2}-4}{\sqrt{13}} + \sqrt{\left(\frac{3\tan\frac{\theta}{2}-4}{\sqrt{13}}\right)^2-1}\right| + C$$

(f) $\int \frac{dt}{1+3\cos t}$

$$= \int \frac{1}{1+\frac{3(1-u^2)}{1+u^2}} \cdot \frac{2\,du}{1+u^2}$$

$$= \int \frac{\cancel{1+u^2}}{1+u^2+3-3u^2} \cdot \frac{2\,du}{\cancel{1+u^2}}$$

$$= \int \frac{2}{-2u^2+4}\,du$$

$$= \int \frac{1}{2-u^2}\,du$$

let $u = \sqrt{2}\sin\theta \qquad \theta = \sin^{-1}\left(\frac{u}{\sqrt{2}}\right)$

$du = \sqrt{2}\cos\theta\,d\theta$

$$= \frac{1}{2}\int \frac{1}{\cos^2\theta}\sqrt{2}\cos\theta\,d\theta$$

$$= \frac{\sqrt{2}}{2}\int \sec\theta\,d\theta$$

$$= \frac{\sqrt{2}}{2}\ln|\sec\theta+\tan\theta| + C$$

$$= \frac{\sqrt{2}}{2}\ln\left|\frac{1}{\sqrt{1-\left(\frac{u}{\sqrt{2}}\right)^2}} + \frac{\frac{u}{\sqrt{2}}}{\sqrt{1-\left(\frac{u}{\sqrt{2}}\right)^2}}\right| + C$$

$$= \frac{\sqrt{2}}{2} \ln \left| \frac{1 + \frac{u}{\sqrt{2}}}{\sqrt{1 - \left(\frac{u}{\sqrt{2}}\right)^2}} \right| + C$$

$$= \frac{\sqrt{2}}{2} \ln \left| \frac{\sqrt{2} + u}{\sqrt{2 - u^2}} \right| + C$$

$$= \frac{\sqrt{2}}{2} \ln \left| \frac{\sqrt{2} + \tan^2 \frac{\theta}{2}}{\sqrt{2 - \tan^2 \frac{\theta}{2}}} \right| + C$$

(g) $\int_0^{\frac{\pi}{2}} \frac{d\theta}{2 + \cos\theta}$

$$\cos\theta = \frac{1-z^2}{1+z^2}$$

$$d\theta = \frac{2\,dz}{z^2+1}$$

$$= \int_0^1 \frac{1}{2 + \frac{1-z^2}{1+z^2}} \cdot \frac{2\,dz}{z^2+1}$$

$$= \int_0^1 \frac{\cancel{(z^2+1)}}{\cancel{2}(1+z^2) + 1 - z^2} \cdot \frac{\cancel{2}}{\cancel{(z^2+1)}}\,dz$$

$$= 2\int_0^1 \frac{dz}{2 + 2z^2 + 1 - z^2}$$

$$= 2\int_0^1 \frac{dz}{3 + z^2}$$

$$= 2\int_0^1 \frac{dz}{3+z^2}$$

let $z = \sqrt{3}\,\text{Tan}\,\theta \Rightarrow dz = \sqrt{3}\sec^2\theta\, d\theta$

$$= 2\int_0^1 \frac{\sqrt{3}\sec^2\theta}{3+3\,\text{Tan}^2\theta}\, d\theta$$

$$= 2\int_0^1 \frac{\sqrt{3}}{3}\, d\theta$$

$$= \frac{2\sqrt{3}}{3}\,\theta \Big|_0^1$$

$$= \frac{2\sqrt{3}}{3}$$

(h) $\displaystyle\int \frac{dx}{2\sin x + \cos x}$

$$= \int \frac{1}{2\left(\frac{2z}{z^2+1}\right) + \frac{1-z^2}{1+z^2}} \cdot \frac{2dz}{z^2+1}$$

$$= \int \frac{2}{4z+1-z^2}\, dz$$

$$= \int \frac{2}{4z + 1 - z^2}\, dz$$

$$= 2 \int \frac{1}{-z^2 + 4z + 1}\, dz$$

$$-(z^2 - 4z + 1)$$

$$-(z^2 + (-2)^2 - (-2)^2 + 1)$$

$$-[(z-2)^2 - 4 + 1]$$

$$-(z-2)^2 + 3$$

$$= 2 \int \frac{1}{3 - (z-2)^2}\, dz$$

Let $z - 2 = \sqrt{3} \sin\theta$

$dz = \sqrt{3} \cos\theta \, d\theta$

$$= 2 \int \frac{\sqrt{3} \cos\theta}{3 - 3\sin^2\theta}\, d\theta$$

$$= \frac{2\sqrt{3}}{3} \int \sec\theta \, d\theta$$

$$= \frac{2\sqrt{3}}{3} \ln\left| \sec\theta + \tan\theta \right| + C$$

$$= \frac{2\sqrt{3}}{3} \ln\left| \sec\left(\sin^{-1} \frac{z-2}{\sqrt{3}}\right) + \tan\left(\sin^{-1} \frac{z-2}{\sqrt{3}}\right) \right| + c$$

$$= \frac{2\sqrt{3}}{3} \ln\left| \frac{1}{\sqrt{1 - \left(\frac{z-2}{\sqrt{3}}\right)^2}} + \frac{\frac{z-2}{\sqrt{3}}}{\sqrt{1 - \left(\frac{z-2}{\sqrt{3}}\right)^2}} \right| + c \ \ldots$$

(6)

(a) $\int \frac{x^4 - 4x^2 + x + 1}{x^2 - 4}\, dx$

$$
\begin{array}{r|l}
 & x^2 \\
x^2-4 & x^4-4x^2+x+1 \\
 & -\ (x^4-4x^2) \\
\hline
 & x+1
\end{array}
$$

$= \int \frac{x^4 - 4x^2 + x + 1}{x^2 - 4}\, dx = \int \left(x^2 + \frac{x+1}{x^2-4} \right) dx$

for $\frac{x+1}{x^2-4} = \frac{x+1}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}$

By Cover-up method

$A = \left. \frac{x+1}{x+2} \right|_{x=2} = \frac{3}{4}$

$B = \left. \frac{x+1}{x-2} \right|_{x=-2} = \frac{-1}{-4} = \frac{1}{4}$

so

$= \int x^2\, dx + \frac{3}{4} \int \frac{1}{x-2}\, dx + \frac{1}{4} \int \frac{1}{x+2} + c$

$= \frac{x^3}{3} + \frac{3}{4} \ln|x-2| + \frac{1}{4} \ln|x+2| + c$

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(b) $$\int \frac{2x^2-1}{(x-1)(x-2)(x-3)}\,dx$$

$$= \int \frac{2x^2-1}{(x-1)(x-2)(x-3)}\,dx = A\int \frac{1}{(x-1)}\,dx + B\int \frac{1}{(x-2)}\,dx + C\int \frac{1}{(x-3)}\,dx$$

$$= A\ln|x-1| + B\ln|x-2| + C\ln|x-3| + C$$

But by cover-up method we have that

$$A = \left.\frac{2x^2-1}{(x-2)(x-3)}\right|_{x=1}$$

$$A = \frac{1}{(-1)(-2)} = \frac{1}{2}$$

$$B = \left.\frac{2x^2-1}{(x-1)(x-3)}\right|_{x=2} = \frac{7}{(1)(-1)}$$

$$B = -7$$

$$C = \left.\frac{2x^2-1}{(x-1)(x-2)}\right|_{x=3} = \frac{17}{(2)(1)}$$

$$C = \frac{17}{2}$$

So the integral

$$= \frac{1}{2}\ln|x-1| - 7\ln|x-2| + \frac{17}{2}\ln|x-3| + C$$ 62

(1) $\int \frac{dx}{1+e^x}$

Let $u = e^x$ $\qquad \frac{1}{u(1+u)} = \frac{A}{u} + \frac{B}{1+u}$

$du = e^x\,dx$

$du = u\,dx$

$dx = \frac{1}{u}\,du$

$\Rightarrow \int \frac{dx}{1+e^x} = \int \frac{1}{u(1+u)}\,du$

$= A\int \frac{1}{u}\,du + B\int \frac{1}{1+u}\,du$

$A = 1 \qquad B = -1$

$= \int \frac{1}{u}\,du - \int \frac{1}{1+u}\,du$

$= \ln|u| - \ln|1+u| + C$

$= \ln e^x - \ln|1+e^x| + C$

$= x + \ln\left|\frac{1}{1+e^x}\right| + C$

(j) $\int \sqrt{1+e^{x}}\, dx$ (see 1-K)

(k) $\int \frac{x^{\frac{2}{3}}}{x+1}\, dx$

Let $u = x+1 \Rightarrow du = dx$

$$\int \frac{x^{\frac{2}{3}}}{x+1}\, dx = \int \frac{(u-1)^{2/3}}{u}\, du = \int \frac{\sqrt[3]{(u-1)^2}}{u}\, du$$

Let $p^3 = u+1$

$dp\,(3p^2) = du$

$$= \int \frac{p^2}{p^3-1}\, 3p^2\, dp = 3\int \frac{p^4}{p^3-1}\, dp$$

by long division

$$\begin{array}{r} p \\ p^3-1 \,\big)\, p^4 \\ -\ \ p^4 - p \\ \hline p \end{array}$$

$$\Rightarrow 3\int \frac{p^4}{p^3-1}\, dp = 3\int \left(p + \frac{p}{p^3-1}\right) dp$$

Difference of 2 cubes

$$\begin{aligned}(A-B)^3 &= (A-B)(A^2-2AB+B^2) \\ &= A^3 - 2A^2B + AB^2 - A^2B + 2AB^2 - B^3 \\ &= A^3 - B^3 - 3A^2B + 3AB^2 \\ &= A^3 - B^3 - 3AB(A-B)\end{aligned}$$

$$\Rightarrow A^3 - B^3 = (A-B)^3 + 3AB(A-B)$$

$$A^3 - B^3 = (A-B)\left((A-B)^2 + 3AB\right)$$
$$= (A-B)\left(A^2 - 2AB + B^2 + 3AB\right)$$
$$= (A-B)\left(A^2 + AB + B^2\right)$$

So $$3\int\left(p + \frac{p}{p^3-1}\right)dp = \frac{3p^2}{2} + 3\int\frac{p}{(p-1)(p^2+p+1)}dp$$

$$\frac{p}{(p-1)(p^2+p+1)} \equiv \frac{A}{p-1} + \frac{Bp + C}{p^2+p+1}$$
$$\equiv \frac{Ap^2 + Ap + A + Bp^2 - Bp + Cp - C}{(p-1)(p^2+p+1)}$$
$$\equiv \frac{Ap^2 + Bp^2 + Ap - Bp + Cp + A - C}{(p-1)(p^2+p+1)}$$

$A + B = 0$ $\qquad A = \frac{1}{3}, \quad B = -\frac{1}{3}, \quad C = \frac{1}{3}$
$A - B + C = 1$
$A - C = 0$

$$= \cancel{3}\int\frac{1}{\cancel{3}}\left(\frac{1}{p-1}\right)dp + \cancel{3}\int\frac{1}{\cancel{3}}\left(\frac{-p+1}{p^2+p+1}\right)dp + \frac{3p^2}{2}$$
$$= \ln|p-1| + \frac{3p^2}{2} + \int\frac{1-p}{p^2+p+1}\,dp$$

completing the square

$$p^2 + p + 1$$

$$p^2 + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1$$

$$\left(p + \frac{1}{2}\right)^2 + \frac{3}{4}$$

$$= \frac{3p^2}{2} + \ln|p-1| + \int \frac{p - p}{\left(p+\frac{1}{2}\right)^2 + \frac{3}{4}}\, dp$$

let $p + \frac{1}{2} = \frac{\sqrt{3}}{2} \tan\theta$, $\theta = \tan^{-1}\left(\frac{2p+1}{\sqrt{3}}\right)$

$$dp = \frac{\sqrt{3}}{2} \sec^2\theta \, d\theta$$

$$= \frac{3p^2}{2} + \ln|p-1| + \int \frac{1 - \left(\frac{\sqrt{3}}{2}\tan\theta - \frac{1}{2}\right)}{\frac{3}{4}\tan^2\theta + \frac{3}{4}}\, d\theta \, \frac{\sqrt{3}}{2}\sec^2\theta$$

$$= \frac{3p^2}{2} + \ln|p-1| + \int \frac{\left(\frac{3}{2} - \frac{\sqrt{3}}{2}\tan\theta\right)\frac{\sqrt{3}}{2}\cancel{\sec^2\theta}}{\frac{3}{4}\cancel{\sec^2\theta}}\, d\theta$$

$$= \frac{3p^2}{2} + \ln|p-1| + \frac{4}{3}\cdot\frac{\sqrt{3}}{2}\int \frac{3}{2} - \frac{\sqrt{3}}{2}\tan\theta \, d\theta$$

$$= \frac{3}{2}p^2 + \ln|p-1| + \frac{\sqrt{3}}{3}\int 3 - \sqrt{3}\tan\theta \, d\theta$$

$$= \frac{3}{2}p^2 + \ln|p-1| + \frac{\sqrt{3}}{\cancel{3}}\cancel{3}\theta + \frac{\cancel{\sqrt{3}}^2}{\cancel{3}}\ln|\cos\theta| + C$$

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$$= \frac{3}{2}p^2 + \ln|p-1| + \ln|\cos\theta| + \sqrt{3}\,\theta + c$$

$$= \frac{3}{2}p^2 + \ln|p-1| + \ln\left|\cos\left(\tan^{-1}\frac{2p+1}{\sqrt{3}}\right)\right| + 2p+1 + c$$

$$= \frac{3}{2}\sqrt[3]{(u+1)^2} + \ln\left|\sqrt[3]{u+1}-1\right| + \ln\left|\cos\left(\tan^{-1}\frac{2\sqrt[3]{u+1}+1}{\sqrt{3}}\right)\right| +$$
$$2\sqrt[3]{u+1} + 1 + c$$

$$= \frac{3}{2}\sqrt[3]{(x+2)^2} + \ln\left|\sqrt[3]{x+2}-1\right| + \ln\left|\cos\tan^{-1}\left(\frac{2\sqrt[3]{x+2}+1}{\sqrt{3}}\right)\right| +$$
$$2\sqrt[3]{x+2} + 1 + c$$

(m) $\int \frac{dx}{\sqrt{\sqrt{x}+1}}$

let $u^2 = x \Rightarrow$

$2u\,du = dx$

$$\int \frac{dx}{\sqrt{\sqrt{x}+1}} = \int \frac{2u\,du}{\sqrt{u+1}} = 2\int \frac{u}{\sqrt{u+1}}\,du$$

let $p^2 = u+1 \Rightarrow 2p\,dp = du$

$$= 2\int \frac{p^2-1}{\cancel{p}}\,2\cancel{p}\,dp$$

$$= 4\int p^2 - 1\,dp$$

$$= \frac{4p^3}{3} - 4p + C$$

$$= \frac{4\sqrt{(u+1)^3}}{3} - 4\sqrt{u+1} + C$$

$$= \frac{4\sqrt{(\sqrt{x}+1)^3}}{3} - 4\sqrt{\sqrt{x}+1} + C$$

(9) $\int \frac{dx}{\sqrt[3]{x} + \sqrt{x}}$

$$= \int \frac{dx}{\sqrt{x}(x+1)}$$

let $y^2 = x$

$2y\,dy = dx$

$$= \int \frac{2\cancel{y}\,dy}{\cancel{y}(y^2+1)}$$

$$= 2\int \frac{1}{y^2+1}\,dy$$

$$= 2\,\mathrm{Tan}^{-1} y + C$$

$$= 2\,\mathrm{Tan}^{-1}\sqrt{x} + C$$

(r) $\int \frac{dx}{3\sqrt{x}-\sqrt{x}}$

$= \int \frac{dx}{\sqrt{x}(x-1)}$

Let $u^2 = x$

$2u\,du = dx$

$= \int \frac{2\cancel{u}\,du}{\cancel{u}(u^2-1)}$

$= 2\int \frac{1}{u^2-1}\,du$

$= 2\int \frac{1}{u^2-1}\,du = 2\int \frac{A}{u+1}\,du + 2\int \frac{B}{u-1}\,du$

$= 2A\ln|u+1| + 2B\ln|u-1| + C$

$\frac{1}{u^2-1} = \frac{A}{u+1} + \frac{B}{u-1}$

$A = \frac{1}{u-1}\Big|_{x=-1} = -\frac{1}{2}$, $B = \frac{1}{u+1}\Big|_{u=1} = \frac{1}{2}$

$= -\ln|u+1| + \ln|u-1| + C$

$= \ln\left|\frac{u-1}{u+1}\right| + C$

$= \ln\left|\frac{\sqrt{x}-1}{\sqrt{x}+1}\right| + C$

THE UNIVERSITY OF ZAMBIA
SCHOOL OF NATURAL SCIENCES
DEPARTMENT OF MATHEMATICS & STATISTICS
MAT2100 Tutorial Sheet 6

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1. For the following vectors find; $(i)\mathbf{A}\cdot\mathbf{B}$, $||\mathbf{A}||, ||\mathbf{B}||$; (ii) the cosine of the angle between $\mathbf{A}$ and $\mathbf{B}$; (iii) the scalar component of $\mathbf{B}$ in the direction of $\mathbf{A}$; (iv) the vector $\text{proj}_{\mathbf{A}}\mathbf{B}$.
   (a) $\mathbf{A} = 2i - 4j + \sqrt{5}k, \mathbf{B} = -2i + 4j - \sqrt{5}k$
   (b) $\mathbf{A} = (3/5)i + (4/5)k, \mathbf{B} = 5i + 12j$
   (c) $\mathbf{A} = 10i + 11j - 2k, \mathbf{B} = 3j + 4k$
   (d) $\mathbf{A} = 2i + 10j - 11k, \mathbf{B} = 2i + 2j + k$
   (e) $\mathbf{A} = -2i + 7j, \mathbf{B} = k$
   (f) $\mathbf{A} = \frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{3}}j + \frac{1}{\sqrt{6}}k, \mathbf{B} = \frac{1}{\sqrt{2}}J - k$
   (g) $\mathbf{A} = -i + j, \mathbf{B} = \sqrt{2}i + \sqrt{3}j + 2k$
   (h) $A = -5i + j, B = 2i + \sqrt{17}j + l0k$

2. Find the length of the indicated portion of the curve.
   (a) $r(t) = (4\cos t)i + (4\sin t)j + 3tk, \quad 0 \le t \le \pi/2$
   (b) $r(t) = (\cos t + t\sin t)i + (\sin t - t\cos t)j, \quad \pi/2 \le t \le \pi$
   (c) $r(t) = (e^t\cos t)i + (e^t\sin t)j + e^tk, \quad -\ln 4 \le t \le 0$
   (h) $r(t) = (1 + 2t)i + (1 + 3t)j + (6 - 6t)k, \quad -1 \le t \le 0$

3. Find the unit vectors that are tangent and normal to the curve at the given point (four vectors in all). Then sketch the vectors and curve together.
   (a) $y = x^2, (2, 4)$
   (b) $x^2 + 2y^2 = 6, (2, 1)$
   (c) $y = \tan^{-1} x, (1, \pi/4)$
   (d) $3x^2 + 8xy + 2y^2 - 3 = 0, (1, 0)$
   (e) $x^2 - 6xy + 8y^2 - 2x - 1 = 0, (1, 1)$
   (f) $y = \int_0^x \sqrt{3 + t^4}\, dt, (0, 0)$
   (g) $y = \int_e^x \ln(\ln t)\, dt(e, 0)$

4. Verify that $\mathbf{A} \times \mathbf{B}$ is perpendicular to both A and B, when $\mathbf{A} = (1, 3, -1)$ and $\mathbf{B} = (2, 0, 1)$.

5. Find a vector $\mathbf{N}$ that is perpendicular to the plane of the three points $P(1, -1, 4), Q(2, 0, 1)$ and $R(0, 2, 3)$.

6. Find the area of ΔPQR of problem 5.

7. Find the distance d from the origin to the plane of problem 5.

8. Find the of the parallelepiped formed by the vectors $\vec{PQ}$ and $\vec{PR}$ problem 5 and the vector $\vec{PS}$, where $S = (3, 5, 7)$.

9. If $A = (2, -3, 1)$ is normal to one pane $\mathfrak{P}_1$. and $B = (-1, 4, -2)$ is normal to another plane $\mathfrak{P}_2$, show that $\mathfrak{P}_1$ and $\mathfrak{P}_2$ intersect and find a vector parallel to the line $\mathfrak{L}$ in which they intersect.

10. Find the vector representation, the parametric equations,and the rectangular equations for the line through the points $P(1, -2, 5)$ and $Q(3, 4, 6)$.

11. Find the points at which the line of problem 10 cuts the coordinate planes.

12. Write equations for the line through the point $(1, 2, -6)$ and is parallel to the vector $(4, 1, 3)$.

13. Write parametric equations for the line through $(-1, 4, 2)$ and is parallel to the line

$$\frac{x-2}{4} = \frac{y}{5} = \frac{z+2}{3}.$$

14. Show that the lines $x = 1 + t$, $y = 2t$, $z = 1 + 3t$ and $x = 3s$, $y = 2s$, $z = 2 + s$ intersect,and find their point of intersection.

15. By the methods of calculus, find the point $P_1(x, y, z)$ on the line $x = 3 + t, y = 2 + t, z = 1 + t$ that is closest to the point $P_0(1, 2, 1)$, and verify that P_0P_1 is perpendicular to the lines.

16. Find the equation of the plane through the points $P(1, 3, 5)$, $Q(-1, 2, 4)$ and $R(4, 4, 0)$.

17. Find the equation of the plane through the points $(3, 2, -1)$, $(1, -1, 3)$ and $R(3, -24, 4)$.

18. Find the cosine of the angle θ between the planes $4x + 4y - 2z = 9$ and $2x + y + z = -3$.

19. Find parametric equations for the line $\mathfrak{L}$ that is the intersection of the planes in problem 16.

20. Find an equation of the plane containing the point $P(1, 3, 1)$ and the line $\mathfrak{L} : x = t, y = t, z = t + 2$.

21. Show that the line $\frac{x-3}{5} = \frac{y+1}{5} = \frac{z+4}{7}$ lies in the plane $3x + 4y - 5zz = 25$.

22. Show that the line $\mathfrak{L}$ of intersection of the planes $x + y - z = 0$ and $x - y - 5z + 7 = 0$ is parallel to the line $\mathfrak{M}$ determined by $\frac{x+3}{3} = \frac{y-1}{-2} = \frac{z-5}{1}$.

23. Find an equation of the plane $P_0(2, -1, -1)$ and $P_1(1, 2, 3)$ and perpendicular to the plane $2x + 3y - 5z - 6 = 0$.

24. Find $\mathbf{T}, \mathbf{N}$, and κ for the following plane curves:
   (a) $r(t) = ti + (\ln \cos t)j, -\pi/2 < t < \pi/2$
   (b) $r(t) = (\ln \sec t)i + tj, -\pi/2 < t < \pi/2$
   (c) $r(t) = (2t + 3)i + (5 - t2)j$
   (h) $r(t) = (\cos t + t \sin t)i + (\sin t - t \cos t)j, t > 0$

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Q 1 (a) $A = 2i - 4j + \sqrt{5}k$, $B = -2i + 4j - \sqrt{5}k$

(i) $A \cdot B = (2)(-2) + (-4)(4) + (\sqrt{5})(-\sqrt{5}) = -25$

$\|A\| = \sqrt{2^2 + (-4)^2 + (\sqrt{5})^2} = 5$

$\|B\| = \sqrt{(-2)^2 + 4^2 + (-\sqrt{5})^2} = 5$

(ii)

$$\cos\theta = \frac{A \cdot B}{\|A\|\|B\|} = -1$$

$$= \frac{-25}{5^2} = -1$$

(iii) $\text{Comp}_A B = \|B\| \cos\theta = -5$

(iv) $\text{proj}_A B = \text{Com}_A B \cdot \frac{A}{\|A\|} = -5 \cdot \frac{(-2i + 4j + \sqrt{5}k)}{5}$

$= -(2i - 4j + \sqrt{5}k) = -2i + 4j - \sqrt{5}k$

(b) $A = \frac{3}{5}i + \frac{4}{5}k$, $B = 5i + 12j$

(i) $A \cdot B = \left(\frac{3}{5}\right)(5) + 0 = 3$

$\|A\| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = 1$

$\|B\| = \sqrt{5^2 + 12^2} = 13$

(ii) $$\cos\theta = \frac{A \cdot B}{\|A\|\|B\|} = \frac{3}{13}$$

(iii) $\text{Comp}_A B = \|B\| \cos\theta = \frac{A \cdot B}{\|A\|} = 3$

(iv) $\text{proj}_A B = \text{Comp}_A B \cdot \frac{A}{\|A\|} = \frac{A \cdot B}{\|A\|^2} A = \frac{3 \cdot \left(\frac{3}{5}i + \frac{4}{5}k\right)}{1}$

$= \frac{9}{5}i + \frac{12}{5}k$

(c) $A = 10i + 11j - 2k$, $B = 3j + 4k$

(i) $A \cdot B = (11)(3) + (-2)(4) = 25$

$\|A\| = \sqrt{10^2 + 11^2 + (-2)^2} = \sqrt{225} = 15$

$\|B\| = \sqrt{3^2 + 4^2} = 5$

(ii) $\cos\theta = \dfrac{A \cdot B}{\|A\|\,\|B\|} = \dfrac{25}{(5)(15)} = \dfrac{1}{3}$

(iii) $\text{Comp}_A B = \|B\| \cos\theta = \dfrac{A \cdot B}{\|A\|} = \dfrac{5}{3}$

(iv) $\text{Proj}_A B = \text{Comp}_A B \cdot \dfrac{A}{\|A\|} = \dfrac{A \cdot B}{\|A\|^2} A = \dfrac{5}{3}\left(\dfrac{1}{15}(10i + 11j - 2k)\right)$

$= \dfrac{10}{9}i + \dfrac{11}{9}j - \dfrac{2k}{9}$

(d) $A = 2i + 10j - 11k$, $B = 2i + 2j + k$

(i) $A \cdot B = (2)(2) + (10)(2) + (-11)(1) = 13$

$\|A\| = \sqrt{2^2 + 10^2 + (-11)^2} = 15$

$\|B\| = \sqrt{2^2 + 2^2 + 1} = 3$

(ii) $\cos\theta = \dfrac{A \cdot B}{\|A\|\,\|B\|} = \dfrac{13}{(3)(15)} = \dfrac{13}{45}$

(iii) $\text{Comp}_A B = \|B\| \cos\theta = \dfrac{A \cdot B}{\|A\|} = \dfrac{13}{15}$

(iv) $\text{Proj}_A B = \text{Comp}_A B \cdot \dfrac{A}{\|A\|} = \dfrac{A \cdot B}{\|A\|^2} A = \dfrac{13}{15} \cdot \dfrac{1}{15}(2i + 10j - 11k)$

$= \dfrac{26}{225}i + \dfrac{130}{225}j - \dfrac{143}{225}k$

(e) $A = -2i + 7j \ , \quad B = k$

(i) $A \cdot B = 0$

$\|A\| = \sqrt{4+49} = \sqrt{53}$

$\|B\| = \sqrt{1} = 1$

(ii) $\cos\theta = \dfrac{A \cdot B}{\|A\|\|B\|} = 0$

(iii) $\text{Comp}_A B = \|B\| \cos\theta = 0$

(iv) $\text{proj}_A B = \text{Comp}_A B \cdot \dfrac{A}{\|A\|} = 0$ (zero vector)

(f) $A = \dfrac{1}{\sqrt{2}} i + \dfrac{1}{\sqrt{3}} j + \dfrac{1}{\sqrt{6}} k \ , \quad B = \dfrac{1}{\sqrt{2}} j - k$

(i) $A \cdot B = \left(\dfrac{1}{\sqrt{3}}\right)\left(\dfrac{1}{\sqrt{2}}\right) + \left(\dfrac{1}{\sqrt{6}}\right)(-1) = 0$

$\|A\| = \sqrt{\left(\dfrac{1}{\sqrt{2}}\right)^2 + \left(\dfrac{1}{\sqrt{3}}\right)^2 + \left(\dfrac{1}{\sqrt{6}}\right)^2} = 1$

? $\|B\| = \sqrt{\left(\dfrac{1}{\sqrt{2}}\right)^2 + (1)^2} = \sqrt{\dfrac{3}{2}} = \dfrac{\sqrt{6}}{2}$

(ii) $\cos\theta = \dfrac{A \cdot B}{\|A\|\|B\|} = 0$

(iii) $\text{Comp}_A B = \|B\| \cos\theta = 0$

(iv) $\text{proj}_A B = \text{Comp}_A B \cdot \dfrac{A}{\|A\|} = 0$ (zero vector)

(g) $A = -i + j$, $B = \sqrt{2}\,i + \sqrt{3}\,j + 2k$

(i) $A \cdot B = -\sqrt{2} + \sqrt{3} = \sqrt{3} - \sqrt{2}$

$\|A\| = \sqrt{1+1} = \sqrt{2}$

$\|B\| = \sqrt{2+3+4} = 3$

(ii) $\cos\theta = \dfrac{A \cdot B}{\|A\|\|B\|} = \dfrac{\sqrt{3}-\sqrt{2}}{3\sqrt{2}} = \dfrac{\sqrt{6}-2}{6}$

(iii) $\text{Comp}_A B = \|B\| \cos\theta = \dfrac{\sqrt{6}-2}{2}$

(iv) $\text{Proj}_A B = \text{Comp}_A B \cdot \dfrac{A}{\|A\|} = \dfrac{\sqrt{6}-2}{2}(-i + j)$

$= -\left(\dfrac{\sqrt{6}-2}{2}\right) i + \dfrac{\sqrt{6}-2}{2} j$

$= \dfrac{2-\sqrt{6}}{2} i + \dfrac{\sqrt{6}-2}{2} j$

(h) $A = -5i + j$, $B = 2i + \sqrt{17}\,j + 10k$

(i) $A \cdot B = -10 + \sqrt{17}$

$\|A\| = \sqrt{25 + 1} = \sqrt{26}$

$\|B\| = \sqrt{4 + 17 + 100} = 11$

(ii) $\cos\theta = \dfrac{\sqrt{17}-10}{11\sqrt{26}}$

(iii) $\text{Comp}_A B = \|B\| \cos\theta = \dfrac{\sqrt{17}-10}{\sqrt{26}}$

(iv) $\text{Proj}_A B = \text{Comp}_A B \cdot \dfrac{A}{\|A\|} = \dfrac{\sqrt{17}-10}{26}(-5i + j)$

(4) $= \dfrac{5\sqrt{17}-50}{26} i + \dfrac{\sqrt{17}-10}{26} j$

2.

Arc length of space curves is given by

let $r(t) = f(t)\,i + g(t)\,j + h(t)\,k$

$$S = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\,dt$$

(a) $r(t) = (4\cos t)\,i + (4\sin t)\,j + 3t\,k \quad , \quad 0 \le t \le \frac{\pi}{2}$

$$s = \int_0^{\pi/2} \sqrt{(-4\sin t)^2 + (4\cos t)^2 + (3)^2}\,dt$$

$$S = \int_0^{\pi/2} \sqrt{16\sin^2 t + 16\cos^2 t + 9}\,dt$$

$$S = \int_0^{\pi/2} \sqrt{25}\,dt$$

$$S = \int_0^{\pi/2} 5\,dt = 5t \Big|_0^{\frac{\pi}{2}} = \frac{5\pi}{2}$$

(b) $r(t) = (\cos t + t\sin t)\,i + (\sin t - t\cos t)\,j \; ; \; \frac{\pi}{2} \le t \le \pi$

$$S = \int_{\frac{\pi}{2}}^{\pi} \sqrt{(\cancel{-\sin t} + \cancel{\sin t} + t\cos t)^2 + (\cancel{\cos t} - \cancel{\cos t} - t\sin t)^2}\,dt$$

$$S = \int_{\frac{\pi}{2}}^{\pi} \sqrt{t^2\cos^2 t + t^2\sin^2 t}\,dt$$

$$S = \int_{\pi/2}^{\pi} t\,dt = \frac{t^2}{2}\Big|_{\frac{\pi}{2}}^{\pi} = \frac{\pi^2}{2} - \frac{\pi^2}{8} = \frac{3}{8}\pi^2$$

(c) $r(t) = (e^t\cos t)\,i + (e^t\sin t)\,j + e^t k \quad , \quad -\ln 4 \le t \le 0$

$$\cancel{S = \int_{-\ln 4}^{0} \sqrt{(e^t\cos t - e^t\sin t)^2}\,dt}$$

$$s = \int_{-\ln 4}^{0} \sqrt{\left[\frac{d}{dt}(e^t \cos t)\right]^2 + \left[\frac{d}{dt} e^t \sin t\right]^2 + \left[\frac{d}{dt} e^t\right]^2}\, dt$$

$$s = \int_{-\ln 4}^{0} \sqrt{[e^t \cos t - e^t \sin t]^2 + [e^t \sin t + e^t \cos t]^2 + e^{2t}}\, dt$$

$$s = \int_{-\ln 4}^{0} \sqrt{e^{2t}\cos^2 t - \cancel{2e^{2t}\cos t \sin t} + e^{2t}\sin^2 t + e^{2t}\sin^2 t + \cancel{2e^{2t}\cos t \sin t} + e^{2t}\cos^2 t + e^{2t}}$$

$$s = \int_{-\ln 4}^{0} \sqrt{2e^{2t}\cos^2 t + 2e^{2t}\sin^2 t + e^{2t}}\, dt$$

$$s = \int_{-\ln 4}^{0} \sqrt{3e^{2t}}\, dt$$

$$s = \int_{-\ln 4}^{0} \sqrt{3}\, e^t\, dt = \sqrt{3}\, e^t \Big|_{-\ln 4}^{0}$$

$$s = \sqrt{3} - \sqrt{3}\, e^{-\ln 4} = \sqrt{3} - \sqrt{3}\left(\frac{1}{4}\right) = \frac{3\sqrt{3}}{4}$$

(d) $r(t) = (1+2t)i + (1+3t)j + (6-6t)k, \quad -1 \le t \le 0$

$$s = \int_{-1}^{0} \sqrt{(2)^2 + (3)^2 + (-6)^2}\, dt$$

$$s = \int_{-1}^{0} \sqrt{49}\, dt = \int_{-1}^{0} 7\, dt = 7t \Big|_{-1}^{0}$$

$s = 7$

(6)

3 (a) $y = x^2$; $(2, 4)$

let $x = t \Rightarrow y = t^2$, $t \in \mathbb{R}$

$\Rightarrow r(t) = ti + t^2 j$, $t = 2$ (vector equation)

$r'(t) = i + 2tj = i + 4j$ (Tangent vector)

$$T(t) = \frac{r'(t)}{\|r'(t)\|} = \frac{i + 4j}{\sqrt{17}} = \frac{1}{\sqrt{17}} i + \frac{4}{\sqrt{17}} j$$

(unit tangent vector)

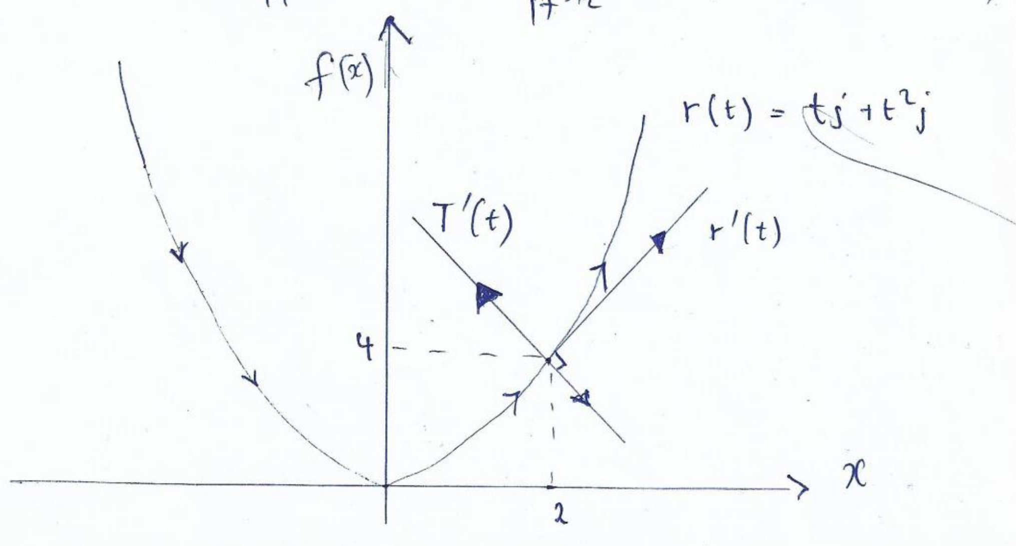
$$T(t) = \frac{1}{\sqrt{1+4t^2}} i + \frac{2t}{\sqrt{1+4t^2}} j$$

$$T'(t) = \frac{-8t\left(\frac{1}{2}\right)(1+4t^2)^{-1/2}}{\left(\sqrt{1+4t^2}\right)^2} i + \frac{2\sqrt{1+4t^2} - 8t\left(\frac{1}{2}\right)(1+4t^2)^{-1/2} 2t}{\left(\sqrt{1+4t^2}\right)^2} j$$

$$= \frac{-4t}{(1+4t^2)^{3/2}} i + \frac{2(1+4t^2)^{1/2} - 8t^2(1+4t^2)^{-1/2}}{1+4t^2} j$$

$$N(t) = \frac{-8}{\sqrt{68}} i + \frac{2}{\sqrt{68}} j$$

$$= \frac{-8}{17^{3/2}} i + \frac{2}{17^{3/2}} j$$ (unit normal vector)



$$T'(t) = \frac{d}{dt}\left(\frac{-\sqrt{6}\sin t}{\sqrt{6-3\cos^2 t}}\right)i + \frac{d}{dt}\left(\frac{\sqrt{3}\cos t}{\sqrt{6-3\cos^2 t}}\right)j$$

$$= \frac{-\sqrt{6}\cos t\sqrt{6-3\cos^2 t} + \sqrt{6}\sin t\left(\frac{1}{2}\right)\left(6-3\cos^2 t\right)^{-1/2}\left(+6\cos t\sin t\right)}{6-3\cos^2 t}\, i$$

$$+ \frac{\left(-\sqrt{3}\sin t\sqrt{6-3\cos^2 t} - \sqrt{3}\cos t\,\frac{1}{2}\left(6-3\cos^2 t\right)^{-1/2}\left(6\cos t\sin t\right)\right)}{6-3\cos^2 t}\, j$$

$$= \frac{-\cancel{\sqrt{6}}\left(\frac{2}{\cancel{\sqrt{6}}}\right)2 + \frac{\sqrt{6}}{2}\left(\frac{1}{\sqrt{3}}\right)(4)^{-1/2}\left(6\left(\frac{2}{\sqrt{6}}\right)\left(\frac{1}{\sqrt{3}}\right)\right)}{4}\, i$$

$$+ \frac{\left(-\cancel{\sqrt{3}}\left(\frac{1}{\cancel{\sqrt{3}}}\right)2\right) - \frac{\sqrt{3}\,(\cancel{2})}{\sqrt{6}}\left(\frac{1}{\cancel{2}}\right)(4)^{-1/2}\left(6\left(\frac{2}{\sqrt{6}}\right)\left(\frac{1}{\sqrt{3}}\right)\right)}{}\, j$$

$$= \frac{-4 + \frac{\sqrt{2}}{\cancel{24}_{4}}\left(\frac{\cancel{12}^{3}}{\sqrt{18}}\right)}{4}\, i + \frac{(-2) - \frac{\sqrt{2}}{\cancel{24}_{4}}\left(\frac{\cancel{12}^{3}}{\sqrt{18}}\right)}{4}\, j$$

$$= \frac{-3}{4}\, i + \frac{(-3)}{4}\, j$$

$$= -\frac{3}{4}\, i - \frac{3}{4}\, j$$ (unit normal vector

$$N(t) = -\frac{1}{\sqrt{2}}\, i - \frac{1}{\sqrt{2}}\, j$$

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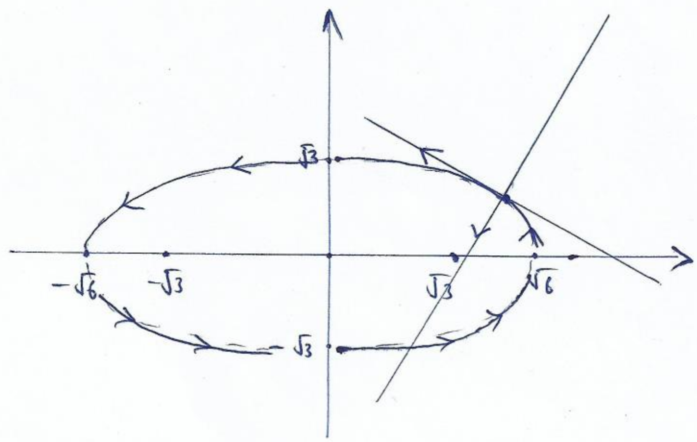

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(e) $y = \tan^{-1} x \quad (1, \pi/4)$

Let $\quad x = t \quad y = \tan^{-1} t \quad t \in \mathbb{R}$

$r(t) = t\,i + (\tan^{-1} t)\,j$ (vector equation)

$$r'(t) = i + \frac{j}{t^2+1} \quad \text{(Tangent vector)}$$

$$T(t) = \frac{r'(t)}{\|r'(t)\|} = \frac{i + \frac{1}{2} j}{\sqrt{1+\frac{1}{4}}}$$

$$= \left(\frac{\sqrt{5}}{2}\right)^{-1} i + \left(\frac{\sqrt{5}}{2}\right)^{-1} \frac{1}{2} j$$

$$= \frac{2\sqrt{5}}{5} i + \frac{\sqrt{5}}{5} j \quad \text{(unit tangent vector)}$$

$$T'(t) = \frac{d}{dt}\, i\left(\sqrt{1+(t^2+1)^2}\right)^{-1} + \frac{d}{dt} \frac{1}{(t^2+1)}\left(\sqrt{1+(t^2+1)^2}\right)^{-1}$$

$$= \frac{-\cancel{2}\,2t\,(t^2+1)}{\cancel{2}} \left(1+(t^2+1)^2\right)^{-3/2} i$$

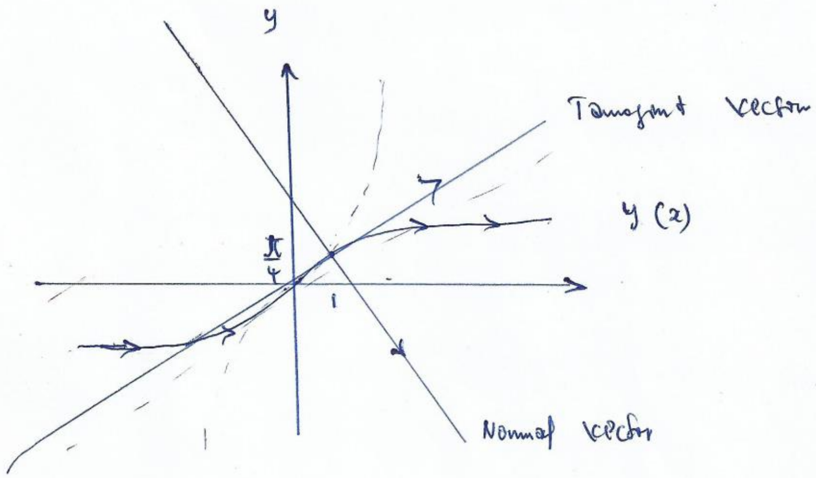
$$+ (-1)\left[(t^2+1)\left(1+(t^2+1)^2\right)^{1/2}\right]^{-2} \Big(2t\left(1+(t^2+1)^2\right)^{1/2}$$

$$+ (t^2+1)\left(\frac{1}{2}\right)\left(1+(t^2+1)^2\right)\left(2(t^2+1)2t\right) \Big)\, j$$

(11)

$$= (-1)\left(2\sqrt{5}\right)^{-2}\left(\left(\sqrt{10}\right)+40\right)j - \frac{4}{\sqrt{5^3}}\,j$$

$$= \frac{4}{5\sqrt{5}}\,i - \frac{\left(\sqrt{10}+40\right)}{20}\,j$$



$$1+\frac{\mu^2}{\alpha}+\frac{\Omega^2}{\beta} = 1+\frac{\frac{\sqrt{85}+7}{2\sqrt{85}}}{\frac{5\sqrt{85}-85}{2\sqrt{85}}}+\frac{\frac{\sqrt{85}-7}{2\sqrt{85}}}{\frac{5\sqrt{85}+85}{2\sqrt{85}}}$$

$$= 1+\frac{\sqrt{85}+7}{5\sqrt{85}-85}+\frac{\sqrt{85}-7}{5\sqrt{85}+85}$$

$$= \frac{2125-7225+1020+\cancel{130\sqrt{85}}+1020-\cancel{120\sqrt{85}}}{25(85)-(85)^2}$$

$$= \frac{-306\cancel{0}}{-510\cancel{0}}$$

$$= \frac{3}{5}$$

$$\Rightarrow \quad \beta\left(y'-\frac{\Omega}{\beta}\right)^2+\alpha\left(x'-\frac{\mu}{\alpha}\right)^2=\frac{3}{5}$$

$$\frac{5\beta\left(y'-\frac{\Omega}{\beta}\right)^2}{3}+\frac{5\alpha\left(x'-\frac{\mu}{\alpha}\right)^2}{3}=1$$

Let $y'(t)=\frac{\Omega}{\beta}+\frac{3}{5\beta}\cosh t \quad x'(t)=\frac{\mu}{\alpha}+\frac{3}{5\alpha}\sinh t$

$r(t) = x'(t)\,i + y'(t)\,j$

$$\Rightarrow (x')^2\left(\frac{1}{2}+\frac{7}{2\sqrt{85}}-\frac{18}{\sqrt{85}}+4-\frac{28}{\sqrt{85}}\right)+(y')^2\left(\frac{1}{2}-\frac{7}{2\sqrt{85}}+\frac{18}{\sqrt{85}}+4+\frac{28}{\sqrt{85}}\right)$$

$$-2x'\cos\theta+2y'\sin\theta-1=0$$

$$\cos\theta=\sqrt{\frac{1+\lambda}{2}} \qquad \sin\theta=\sqrt{\frac{1-\lambda}{2}}$$

$$=\sqrt{\frac{\sqrt{85}+7}{2\sqrt{85}}} \qquad =\sqrt{\frac{\sqrt{85}-7}{2\sqrt{85}}}$$

$$\Rightarrow (x')^2\left(\frac{\sqrt{85}+7-36+4\sqrt{85}-56}{2\sqrt{85}}\right)+(y')^2\left(\frac{\sqrt{85}-7+36+4\sqrt{85}+56}{2\sqrt{85}}\right)$$

$$-2x'\sqrt{\frac{\sqrt{85}+7}{2\sqrt{85}}}+2y'\sqrt{\frac{\sqrt{85}-7}{2\sqrt{85}}}-1=0$$

$$\Rightarrow (x')^2\left(\frac{5\sqrt{85}-85}{2\sqrt{85}}\right)+(y')^2\left(\frac{5\sqrt{85}+85}{2\sqrt{85}}\right)-2x'\sqrt{\frac{\sqrt{85}+7}{2\sqrt{85}}}+2y'\sqrt{\frac{\sqrt{85}-7}{2\sqrt{85}}}$$

Completing the square pairwise

$$\left(\frac{5\sqrt{85}-85}{2\sqrt{85}}\right)\left[(x')^2-\left(\frac{2\sqrt{\frac{\sqrt{85}+7}{2\sqrt{85}}}}{\frac{5\sqrt{85}-85}{2\sqrt{85}}}\right)x'\right]+\left(\frac{5\sqrt{85}+85}{2\sqrt{85}}\right)\left[(y')^2\right.$$

$$\left.-\left(\frac{2\sqrt{\frac{\sqrt{85}-7}{2\sqrt{85}}}}{\frac{5\sqrt{85}+85}{2\sqrt{85}}}\right)\right]-1=0$$

let $\alpha = \dfrac{5\sqrt{85} - 85}{2\sqrt{85}} < 0$

$\beta = \dfrac{5\sqrt{85} + 85}{2\sqrt{85}} > 0$

$\mu = \sqrt{\dfrac{\sqrt{85} + 7}{2\sqrt{85}}} > 0$

$\Omega = \sqrt{\dfrac{\sqrt{85} - 7}{2\sqrt{85}}} > 0$

$$\Rightarrow \alpha\left[(x')^2 - \frac{2\mu}{\alpha}x'\right] + \beta\left[(y')^2 - \frac{2\Omega}{\beta}(y')\right] = 1$$

$$\alpha\left[\left(x' - \frac{\mu}{\alpha}\right)^2 - \frac{\mu^2}{\alpha^2}\right] + \beta\left[\left(y' - \frac{\Omega}{\beta}\right)^2 - \frac{\Omega^2}{\beta^2}\right] = 1$$

$$\alpha\left(x' - \frac{\mu}{\alpha}\right)^2 - \frac{\mu^2}{\alpha} + \beta\left(y' - \frac{\Omega}{\beta}\right)^2 - \frac{\Omega^2}{\beta} = 1$$

$$\alpha\left(x' - \frac{\mu}{\alpha}\right)^2 + \beta\left(y' - \frac{\Omega}{\beta}\right)^2 = 1 + \frac{\mu^2}{\alpha} + \frac{\Omega^2}{\beta}$$

$$\Rightarrow \beta\left(y' - \frac{\Omega}{\beta}\right)^2 + \alpha\left(x' - \frac{\mu}{\alpha}\right)^2 = 1 + \frac{\mu^2}{\alpha} + \frac{\Omega^2}{\beta}$$

3 (d)

$$3x^2 + 8xy + 2y^2 - 3 = 0 \quad , (1, 0)$$

$$b^2 - 4ac = 8^2 - 4(3)(-3) > 0$$

$\Rightarrow$ This conic section is an hyperbola.

$$\tan 2\theta = \frac{b}{a-c} = 8$$

$$\cos 2\theta = \frac{|a-c|}{\sqrt{b^2 + (a-c)^2}} = \lambda \qquad \sin 2\theta = \frac{|b|}{\sqrt{b^2 + (a-c)^2}}$$

$$\sin\theta = \sqrt{\frac{1-\lambda}{2}} \qquad \cos\theta = \sqrt{\frac{1+\lambda}{2}}$$

$$\Rightarrow \cos 2\theta = \frac{1}{\sqrt{65}} \qquad \sin 2\theta = \frac{8}{\sqrt{65}}$$

$$x = x'\cos\theta - y'\sin\theta$$
$$y = x'\sin\theta + y'\cos\theta$$

$$\Rightarrow 3(x'\cos\theta - y'\sin\theta)^2 + 8(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta) + 2(x'\sin\theta + y'\cos\theta)^2 - 3 = 0$$

$$3(x')^2\cos^2\theta - 6x'y'\sin\theta\cos\theta + 3(y')^2\sin^2\theta + 8(x')^2\sin\theta\cos\theta + 8x'y'\cos 2\theta - 8(y')^2\sin\theta\cos\theta + 2(x')^2\sin^2\theta + 2x'y'\sin\theta\cos\theta + (y')^2\cos^2\theta - 3 = 0$$

$$\Rightarrow (x')^2\left(4\sin 2\theta + \frac{1}{2}\cos 2\theta + \frac{5}{2}\right) + (x'y')(8\cos 2\theta - \sin 2\theta) + (y')^2\left(-4\sin 2\theta + \frac{1}{2}\cos 2\theta + \frac{5}{2}\right) - 3 = 0$$

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$$\Rightarrow (x')^2\left(\frac{32}{\sqrt{65}}+\frac{1}{2\sqrt{65}}+\frac{5}{2}\right)+(y')^2\left(\frac{-32}{\sqrt{65}}+\frac{1}{2\sqrt{65}}+\frac{5}{2}\right)-3=0$$

$$(x')^2\left(\frac{65+3\sqrt{65}}{6\sqrt{65}}\right)-(y')^2\left(\frac{63-5\sqrt{65}}{6\sqrt{65}}\right)=1$$

$$\frac{(x')^2}{\left(\frac{6\sqrt{65}}{65+3\sqrt{65}}\right)}-\frac{(y')^2}{\left(\frac{6\sqrt{65}}{63-5\sqrt{65}}\right)}=1$$

$$a^2=\frac{6\sqrt{65}}{65+3\sqrt{65}}\simeq 0.542 \qquad b^2=\frac{6\sqrt{65}}{63-5\sqrt{65}}\simeq 2.132$$

let $\cosh t = \frac{x'}{a}$ and $\sinh t = \frac{y'}{b}$

$x'(t) = a\cosh t \qquad y'(t) = b\sinh t \qquad t\in\mathbb{R}$

$r(t) = x'(t)\,i + y'(t)\,j \qquad \begin{aligned} x' &= x\cos\theta - y\sin\theta \\ y' &= -x\sin\theta + y\cos\theta \end{aligned}$

Note that $x'(t)$ and $y'(t)$ are not derivatives but variables in $x'y'$ coordinate system.

$\Rightarrow r(t) = (a\cosh t)\,i + (b\sinh t)\,j$

$r'(t) = a\sinh t\, i + b\cosh t\, j$

$$T(t) = \frac{r'(t)}{\|r'(t)\|}$$

$$T(t) = \frac{a\sinh t\, i + b\cosh t\, j}{\sqrt{a^2\sinh^2 t + b^2\cosh^2 t}}$$

$$T(t) \approx 0.358i + 1.440j$$

$$T'(t) = \left[\frac{a\sinh t}{\sqrt{a^2\sinh^2 t + b^2\cosh^2 t}}\right]' i +$$

$$\left[\frac{b\cosh t}{\sqrt{a^2\sinh^2 t + b^2\cosh^2 t}}\right]' j$$

$$i'(t) = \left(\frac{a\cosh t}{\sqrt{a^2\sinh^2 t + b^2\cosh^2 t}} + \frac{a\sinh t\left(-\frac{1}{2}\right)\left(2a^2\sinh t\cosh t + 2b^2\sinh\cosh\right)}{\sqrt{\left(a^2\sinh^2 t + b^2\cosh^2 t\right)^3}}\right)$$

$$+j\left(\frac{b\sinh t}{\sqrt{a^2\sinh^2 t + b^2\cosh^2 t}} + \frac{a\cosh t\left(-\frac{1}{2}\right)\left(2a^2\sinh t\cosh t + 2b^2\sinh t\cosh\right)}{\sqrt{\left(a^2\sinh^2 t + b^2\cosh^2 t\right)^3}}\right)$$

$$T'(t) = \left[\frac{1}{\left(a^2\sinh^2 t + b^2\cosh^2 t\right)^{3/2}}\left(a^2b\sinh^3 t + b^3\cosh^2 t\sinh t\right.\right.$$

$$\left.\left. - a^3\sinh t\cosh^2 t - ab^2\sinh t\cosh^2 t\right)\right] i + D' j$$

$$N(t) = \frac{T'(t)}{\|T'(t)\|}$$

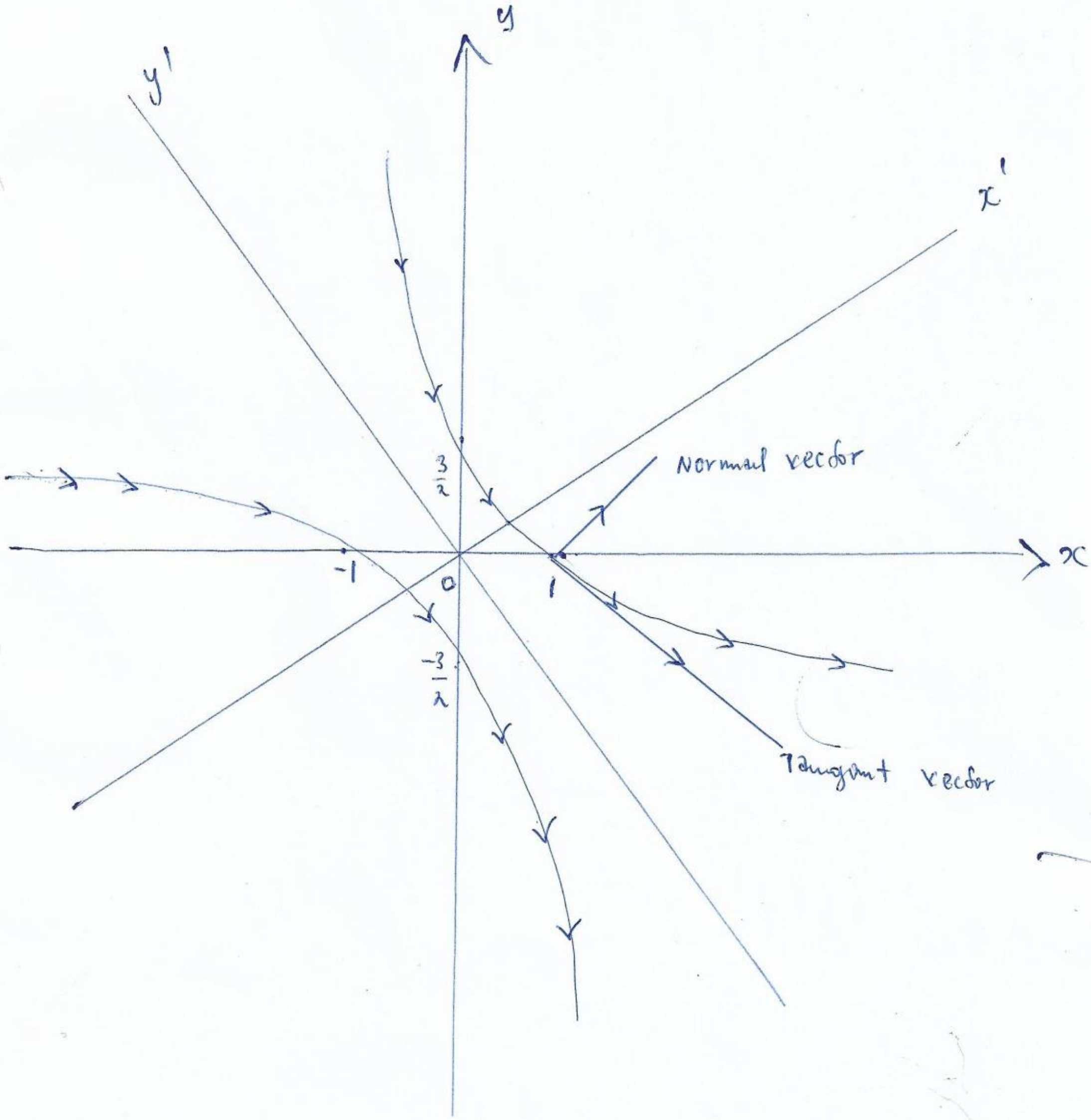
(18)

$$3x^2 + 8xy + 2y^2 - 3 = 0$$

$$3x^2 = 3$$

$$x = \pm 1, \quad y = 0$$

$$y = \pm \frac{\sqrt{6}}{2}, \quad x = 0$$



(e) $x^2 - 6xy + 8y^2 - 2x - 1 = 0$, $(1,1)$

$$b^2 - 4ac = (-6)^2 - 4(8) > 0$$

$\Rightarrow$ The curve is the Hyperbola

$$\tan 2\theta = \frac{b}{a-c} = \frac{-6}{1-8} = \frac{6}{7}$$

$$x = x'\cos\theta - y'\sin\theta \quad , \quad y = x'\sin\theta - y'\cos\theta$$

$$(x'\cos\theta - y'\sin\theta)^2 - 6(x'\cos\theta + y'\sin\theta)(x'\sin\theta - y'\cos\theta)$$
$$+ 8(x'\sin\theta + y'\cos\theta)^2 - 2(x'\cos\theta - y'\sin\theta) - 1 = 0$$

$$\Rightarrow (x')^2\cos^2\theta - 2x'y'\cos\theta\sin\theta + (y')^2\sin^2\theta - 6((x')^2\sin\theta\cos\theta - x'y'\cos^2\theta + x'y'\sin^2\theta)$$
$$+ 6(y')^2\sin\theta\cos\theta + 8(x')^2\sin^2\theta + 16x'y'\sin\theta\cos\theta + 8(y')^2\cos^2\theta$$
$$- 2x'\cos\theta + 2y'\sin\theta - 1 = 0$$

$$\Rightarrow (x')^2\left(\frac{1+\cos 2\theta}{2} - 3\sin 2\theta + \frac{8 - 8\cos 2\theta}{2}\right) + x'y'(-\sin 2\theta + 6\cos 2\theta + 8\sin 2\theta)$$
$$(y')^2\left(\frac{1-\cos 2\theta}{2} + 3\sin 2\theta + \frac{8 + 8\cos 2\theta}{2}\right) - 2x'\cos\theta + 2y'\sin\theta - 1 = 0$$

$$\cos 2\theta = \frac{|a-c|}{\sqrt{b^2 + (a-c)^2}} \equiv \lambda \quad \sin 2\theta = \frac{|b|}{\sqrt{b^2 + (a-c)^2}}$$

$$= \frac{7}{\sqrt{85}} \qquad\qquad = \frac{6}{\sqrt{85}}$$

(f) $y = \int_0^x \sqrt{3+t^4}\, dt$; $(0,0)$

$$y = \int_0^x \sqrt{3+t^4}\, dt = \sqrt{3}\int_0^x \left(1+\frac{t^4}{\sqrt{3}}\right)^{1/2} dt$$

By $(1+b)^k = \sum_{n=0}^{\infty} \binom{k}{n} b^n$ or $(1+b)^k = \sum_{n=0}^{\infty} \frac{k^{(n)}}{n!} b^n$

we have that

$$y = \sqrt{3}\int_0^x \sum_{n=0}^{\infty} \frac{(1/2)^{(n)}}{n!} \frac{t^{4n}}{(\sqrt{3})^n}\, dt$$

$$y = \sum_{n=0}^{\infty} (\sqrt{3})^{1-n} \frac{(1/2)^{(n)}}{n!} \frac{t^{4n+1}}{4n+1}\Big|_0^x$$

$$y = \sum_{n=0}^{\infty} (\sqrt{3})^{1-n} \frac{(1/2)^{(n)}}{n!} \frac{x^{4n+1}}{4n+1}$$

Let $x = t$

$$r(t) = t\hat{i} + \left[\sum_{n=0}^{\infty} 3^{\frac{1}{2}-\frac{n}{2}} \frac{(1/2)^{(n)}}{n!} \frac{x^{4n+1}}{4n+1}\right]\hat{j}$$

(4) $(A\times B)\cdot A = (A\times B)\cdot B = 0 \Rightarrow A\times B \perp A \wedge B$

$A = (1, 3, -1) \quad B = (2, 0, 1)$

$$A\times B = \begin{vmatrix} i & j & k \\ 1 & 3 & -1 \\ 2 & 0 & 1 \end{vmatrix} = i\begin{vmatrix} 3 & -1 \\ 0 & 1 \end{vmatrix} - j\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} + k\begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix}$$

$$= 3i - 3j - 6k$$

$$= (3, -3, -6)$$

This

$$(A\times B)\cdot A = (3, -3, -6)(1, 3, -1)$$

$$= 3 - 9 + 6$$

$$= 0 \Rightarrow A\times B \perp A$$

and $(A\times B)\cdot B = (3, -3, -6)(2, 0, 1)$

$$= 6 - 6$$

$$= 0 \Rightarrow A\times B \perp B$$

$\therefore (A\times B) \perp$ to both $A \wedge B$

(5) $P(1, -1, 4),\ Q(2, 0, 1),\ R, (0, 3, 3)$

$\vec{PQ}\times\vec{PR} = N$ (Normal vector to the plane of 3 points

$\vec{PQ} = \vec{OQ} - \vec{OP} = (1, 1, -3),\quad \vec{PR} = \vec{OR} - \vec{OP} = (-1, 3, -1)$

$$N = \begin{vmatrix} i & j & k \\ 1 & 1 & -3 \\ -1 & 3 & -1 \end{vmatrix} = i\begin{vmatrix} 1 & -3 \\ 3 & -1 \end{vmatrix} - j\begin{vmatrix} 1 & -3 \\ -1 & -1 \end{vmatrix} + k\begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix}$$

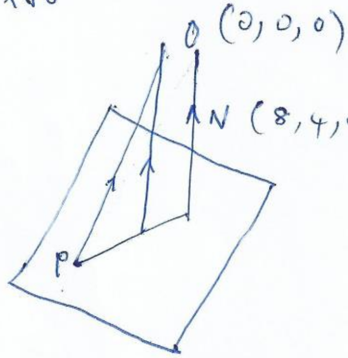
$$= 8i + 4j + 4k$$

(22)

(6) ΔPQR

$$A = \frac{1}{2}\|\overrightarrow{PQ}\|\|\overrightarrow{PR}\|\sin\theta$$
$$= \frac{1}{2}\|\overrightarrow{PQ}\times\overrightarrow{PR}\|$$
$$= \frac{1}{2}\|N\|$$
$$= \frac{1}{2}\sqrt{8^2+4^2+4^2}$$
$$= \frac{1}{2}\sqrt{96}$$
$$= 2\sqrt{6}$$

(7)



$P(1,-1,4)$

$$d = \frac{|a_0x_0 + b_1x_1 + c_2x_2 + D|}{\sqrt{a_0^2 + b_1^2 + c_2}}$$

$$a_0(x-x_0) + b_1(y-y_1) + c_2(z-z_1) = 0$$
$$8(x-1) + 4(y+1) + 4(z-4) = 0$$
$$8x - 8 + 4y + 4 + 4z - 16 = 0$$
$$8x + 4y + 4z - 20$$

$$d = \frac{|-20|}{\sqrt{96}} = \frac{20}{\sqrt{96}} = \frac{20}{4\sqrt{6}} = \frac{5}{\sqrt{6}} = \frac{5\sqrt{6}}{6}$$

or

$$d = \left| \operatorname{Comp}_N \vec{OP} \right| = \left| \frac{\vec{OP} \cdot N}{\|N\|} \right|$$

$$= \frac{|-8+4-16|}{\sqrt{8^2+4^2+4^2}}$$

$$= \frac{20}{\sqrt{96}}$$

(8) $\vec{PQ} \times \vec{PR} = (8, 4, 4)$

$\vec{PS} = \vec{OS} - \vec{OP}$

$= (2, 6, 3)$

$$V = \left| \left(\vec{PQ} \times \vec{PR} \right) \cdot \vec{PS} \right| = \left| \begin{pmatrix} 8 \\ 4 \\ 4 \end{pmatrix} \begin{pmatrix} 2 \\ 6 \\ 3 \end{pmatrix} \right| = 16 + 24 + 12 = 52$$

(9) If B_1 and B_2 intersect then following is true

$A_1 \cdot B_2 = \|A_1\| \|B_2\| \cos\theta$, for $0 \le \theta \le \pi$

$A = (2, -3, 1) \quad B = (-1, 4, -2)$

$$\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \begin{pmatrix} -1 \\ 4 \\ -2 \end{pmatrix} = \sqrt{2^2+3^2+1^2} \sqrt{1^2+4^2+2^2} \cos\theta$$

$$-2-12-2 = \sqrt{14}\sqrt{21} \cos\theta$$

$$\cos\theta = \frac{-16}{\sqrt{294}}$$

$0 < \cos^{-1} \left| \frac{-16}{\sqrt{294}} \right| < \pi$ so B_1 and B_2 intersect

(2c)

The planes B_1 and B_2 intersect at
$\theta = 21.1°$

$\therefore$ vector parallel to the line L of intersection is
given by $A \times B = L$

$$A \times B = \begin{vmatrix} i & j & K \\ 2 & -3 & 1 \\ -1 & 4 & 2 \end{vmatrix}$$

$$= i \begin{vmatrix} -3 & 1 \\ 4 & 2 \end{vmatrix} - j \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} + K \begin{vmatrix} 2 & -3 \\ -1 & 4 \end{vmatrix}$$

$$= -2i - 3j + 5K \quad = (-2, -3, 5)$$

(10) $r(t) = r_0 + tv \qquad v = \overrightarrow{PQ}$

$r(t) = (1, -2, 5) + t(2, 6, 1) \quad v = (2, 6, 1)$

$r(t) = (1 + 2t, -2 + 6t, 5 + t)$ vector representation

$x = 1 + 2t$, $y = -2 + 6t$, $z = 5 + t$ parametric equations

Let $t = z - 5$

$$x = 1 + 2(z - 5)$$
$$x = 1 + 2z - 10$$
$$x = 2z - 9 \quad \text{--- (i)}$$
$$y = -2 + 6(z - 5)$$
$$y = -2 + 6z - 30$$
$$y = 6z - 32 \quad \text{--- (ii)}$$

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Rectangular equations

$$x = 2z - 9$$
$$y = 6z - 32$$

(ii) point of interception are

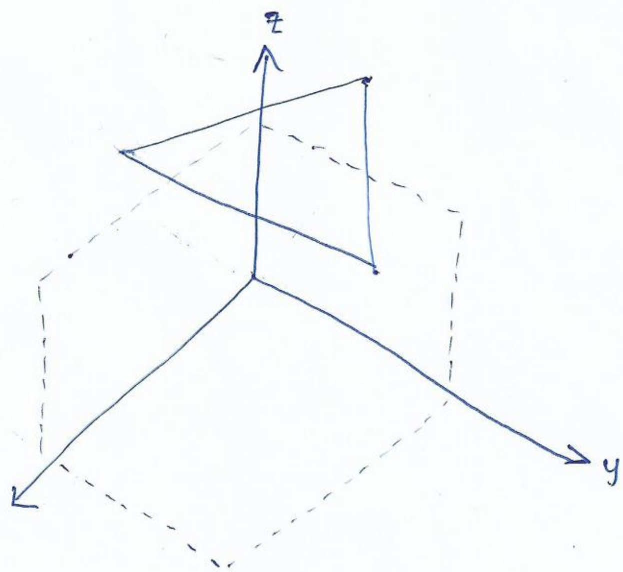
$(x,0,0)$ $(0,y,0)$ $(0,0,z)$ on axes

point of interception on the coordinate planes are $(x,y,0)$, $(0,y,z)$, $(x,0,z)$

$z = 0$ $(-9, -32, 0)$

$x = 0$ $(0, -5, \frac{9}{2})$

$y = 0$ $(\frac{14}{3}, 0, \frac{16}{3})$



(12) $P_0(1, 2, -6)$ $V(4, 1, 3)$

$x = 1 + 4t$, $y = 2 + t$, $z = -6 + 3t$

(13) $P_0(-1, 4, 2)$ $\parallel$ $\dfrac{x-2}{4} = \dfrac{y}{5} = \dfrac{z+2}{3}$

$V(4, 5, 3)$

$x = -1 + 4t$, $y = 4 + 5t$, $z = 2 + 3t$

(14)

$x = 1 + t$	$x = 3s$
$y = 2t$	$y = 2s$
$z = 1 + 3t$	$z = 2 + s$

$1 + t = 3s$ (i)

$2t = 2s$ (ii)

$1 + 3t = 2 + s$ (iii)

Coordinates : $\left(\frac{3}{2}, 1, \frac{5}{2}\right)$

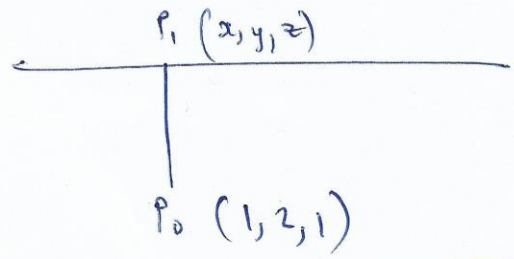
$t = s$

$t = s = \frac{1}{2}$

(27)

15 $x = 3+t$, $y = 2+t$, $z = 1+t$

$P_1(x,y,z)$ is on the line parallel to a vector $(1,1,1)$ and a point $P(3,2,1)$



$$d(P_1,P_0) = \sqrt{(x-1)^2+(y-2)^2+(z-1)^2}$$

$$= \sqrt{(3+t-1)^2+(2+t-2)^2+(1+t-1)^2}$$

$$= \sqrt{(2+t)^2+t^2+t^2}$$

$$= \sqrt{t^2+4t+4+t^2+t^2}$$

$$= \sqrt{3t^2+4t+4}$$

we now get the minimum value of $d(P_1,P_0)$

$$\frac{d}{dt}\left(\sqrt{3t^2+4t+4}\right) = 0$$

$$= \frac{1}{2}(3t^2+4t+4)^{-1/2}(6t+4) =$$

$$\Rightarrow t = -\frac{2}{3}$$

Thus $P_1 = \left(\frac{7}{3}, \frac{4}{3}, \frac{1}{3}\right)$

$\overrightarrow{P_0P_1} = \left(\frac{-4}{3}, \frac{2}{3}, \frac{2}{3}\right)$ $\quad$ $\overrightarrow{P_1P} = \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$

$$\overrightarrow{P_0P_1} \cdot \overrightarrow{P_1P} = \left(\frac{-4}{3}, \frac{2}{3}, \frac{2}{3}\right) \cdot \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$$

$$= \frac{(-4)(2)}{(3)(3)} + \frac{(2)(2)}{(3)(3)} + \frac{(2)(2)}{(3)(3)}$$

$$= 0$$

This shows that $\overrightarrow{P_0P_1} \perp \overrightarrow{P_1P}$.

(16) $P(1,3,5)$, $Q(-1,2,4)$ $R(4,4,0)$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\vec{PQ} = (-1,2,4) - (1,3,5) = (-2,-1,-1)$$

$$\vec{PR} = (4,4,0) - (1,3,5) = (3,1,-5)$$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} i & j & k \\ -2 & -1 & -1 \\ 3 & 1 & -5 \end{vmatrix} = i \begin{vmatrix} -1 & -1 \\ 1 & -5 \end{vmatrix} - j \begin{vmatrix} -2 & -1 \\ 3 & -5 \end{vmatrix} + k \begin{vmatrix} -2 & -1 \\ 3 & 1 \end{vmatrix}$$

$$= 6i - 13j + k$$

plane : $n_0(x - x_0) + n_1(y - y_1) + n_2(z - z_2) = 0$

$$6(x-1) - 13(y-3) + z - 5 = 0$$

$$6x - 6 - 13y + 39 + z - 5 = 0$$

$$6x - 13y + z + 18 = 0$$

(17) $P(3,2,-1)$, $Q(1,-1,3)$ $R(3,-24,4)$

$\vec{PQ} \times \vec{PR}$ (Normal vector)

$\vec{PQ} = (-2,-3,4)$ $\vec{PR} = (2,-23,1)$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} i & j & k \\ -2 & -3 & 4 \\ 2 & -23 & 1 \end{vmatrix}$$

$$= i \begin{vmatrix} -3 & 4 \\ -23 & 1 \end{vmatrix} - j \begin{vmatrix} -2 & 4 \\ 2 & 1 \end{vmatrix} + k \begin{vmatrix} -2 & -3 \\ 2 & -23 \end{vmatrix}$$

$$= 69i + 10j + 40k$$

$$69(x-3)+10(y-2)-(z+1)=0$$

$$69x-207+10y-20-z-1=0$$

$$69x+10y+z-228=0$$

(18) $4x+4y-2z=9$ has a normal vector $V_1(4,4,-2)$ and

$2x+y+z=-3$ has a normal vector $V_2(2,1,1)$

$$\cos\theta = \frac{V_1 \cdot V_2}{\|V_1\|\|V_2\|}$$

$$= \frac{\begin{pmatrix}4\\4\\-2\end{pmatrix}\begin{pmatrix}2\\1\\1\end{pmatrix}}{\left(\sqrt{4^2+4^2+2^2}\right)\left(\sqrt{2^2+2}\right)}$$

$$= \frac{8+4-2}{6\sqrt{6}}$$

$$= \frac{5}{3\sqrt{6}}$$

(19) $4x + 4y - 2z = 9$ --(1)

$2x + y + z = -3$ --(2)

Let $z = -3 - 2x - y$ --(3)

$2x = -3 - y - z$ --(4)

Replacing (3) into (2) and (4) into (1) we get the following

$8x + 6y = 9$ and

$2y - 4z = 15$ as the rectangular equations of the line L intersecting the planes at $\theta = \cos^{-1}\left(\frac{5}{3\sqrt{6}}\right) \approx 0.822$ rad (or $47.1°$)

Let $x = \frac{9}{8} - \frac{6}{8}y$ and $z = \frac{15}{4} + \frac{2}{4}y$

Let $y = t$ (parameter)

$\Rightarrow x = \frac{9}{8} - \frac{6}{8}t, \quad y = t, \quad z = \frac{15}{4} + \frac{2}{4}t$

are the parametric equns of the line L.

(20) $P(1,3,1)$ let $Q = (0,0,2)$

and $a = \overrightarrow{PQ} = (-1,-3,1)$

$b = (1,1,1)$

Normal vector $n = a \times b = \begin{vmatrix} i & j & k \\ -1 & -3 & 1 \\ 1 & 1 & 1 \end{vmatrix}$

$= i\begin{vmatrix} -3 & 1 \\ 1 & 1 \end{vmatrix} - j\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} + k\begin{vmatrix} -1 & -3 \\ 1 & 1 \end{vmatrix}$

$= -4i + 2j + 2k$

$n = (-4, 2, 2)$

plane : $n_0(x-x_0) + n_1(y-y_0) + n_2(z-z_0) = 0$

$-4(x-1) + 2(y-3) + 2(z-1) = 0$

$-4x + 4 + 2(y-3) + 2z - 2 = 0$

$-4x + 2y + 2z - 4 = 0$

(21) If $\dfrac{x-3}{5} = \dfrac{y+1}{5} = \dfrac{z+4}{7}$ lies in the plane

$\Rightarrow$ all points on the line satisfies the equation of the plane $3x + 4y - 5z = 25$

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point $P_0(3,-1,-4)$ lie on the line parallel to the vector $V = (5,5,7)$.

(i) $P_0(3,-1,-4)$ satisfies the equation

$$3x+4y-5z = 25$$

$$\Rightarrow 3(3)+4(-1)-5(-4)$$

$$= 25$$

(ii) the line is perpendicular to the normal to the plane. $n = (3,4,-5)$

$$\Rightarrow n \cdot V = 0$$

$$= \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 5 \\ 7 \end{pmatrix}$$

$$= 15+20-35$$

$$= 0$$

$\therefore$ The line lies in the given plane.

(22) $x+y-z=0$ $\qquad$ $m: \dfrac{x+3}{3} = \dfrac{y-1}{-2} = \dfrac{z-5}{1}$

$x-y-5z+7=0$

L: Intersection line.

If $L \parallel m \Rightarrow V_L = t\,V_m \quad , t \in \mathbb{R}$

Firstly we got rectangular equations

$z = x+y$

$x - y - 5(x+y) + 7 = 0$

$x - y - 5x - 5y + 7 = 0$

$-4x - 6y + 7 = 0$

$-4x = -7 + 6y$

$x = \frac{7}{4} - \frac{3}{2}y$

$x = z - y$

$z - y - y - 5z + 7 = 0$

$-4z - 2y + 7 = 0$

$-4z = -7 + 2y$

$z = \frac{7}{4} - \frac{2}{4}y$

Let $y = t$

$\Rightarrow \quad x = \frac{7}{4} - \frac{3}{2}t, \quad y = t, \quad z = \frac{7}{4} - \frac{1}{2}t$

$$\Rightarrow \quad L: \quad \frac{x - \frac{7}{4}}{-\frac{3}{2}} = y = \frac{z - \frac{7}{4}}{-\frac{1}{2}}$$

$V_L = \left(-\frac{3}{2}, 1, -\frac{1}{2}\right) \qquad V_m = (3, -2, 1)$

$V_L = -\frac{1}{2}(V_m) = -\frac{1}{2}(3, -2, 1)$

$\Rightarrow \quad L \parallel M$

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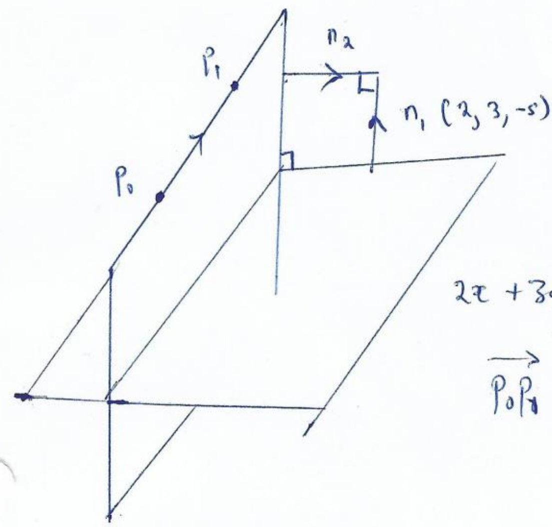
(23)



$$2x + 3y - 5z - 6 = 0$$

$$\overrightarrow{P_0P_1} = \overrightarrow{OP_1} - \overrightarrow{OP_0}$$
$$= (-1, 3, 4)$$

$\overrightarrow{P_0P_1} \times n_1 = n_2$

$$n_2 = \begin{vmatrix} i & j & k \\ -1 & 3 & 4 \\ 2 & 3 & -5 \end{vmatrix}$$

$$= i \begin{vmatrix} 3 & 4 \\ 3 & -5 \end{vmatrix} - j \begin{vmatrix} -1 & 4 \\ 2 & -5 \end{vmatrix} + k \begin{vmatrix} -1 & 3 \\ 2 & 3 \end{vmatrix}$$

$$= -27i + 3j - 9k$$

Choosing $P_0\ (2, -1, -1)$ we get

$$-27(x-2) + 3(y+1) - 9(z+1) = 0$$
$$-27x + 54 + 3y + 3 - 9z - 9 = 0$$
$$-27 + 3y - 9z + 48 = 0$$

(24)

(a) $r(t) = t\,i + (\ln \cos t)\,j \qquad -\pi/2 < t < \pi/2$

$$r'(t) = i + \left(\frac{-\sin t}{\cos t}\right) j$$

$$r'(t) = i - (\tan t)\,j$$

$$T(t) = \frac{r'(t)}{\|r'(t)\|} = \frac{i - \tan t\, j}{\sqrt{1 + \tan^2 t}}$$

$$= \frac{i - \tan t\, j}{\sec t}$$

$$= (\cos t)\, i - (\sin t)\, j$$

$$N(t) = \frac{T'(t)}{\|T'(t)\|} = \frac{(-\sin t)\, i - (\cos t)\, j}{\sqrt{\sin^2 t + \cos^2 t}}$$

$$= -\sin t\, i - \cos t\, j$$

$$K = \frac{\|T'(t)\|}{\|r'(t)\|} = \frac{\sqrt{\sin^2 t + \cos^2 t}}{\sqrt{1 + \tan^2 t}}$$

$$= \frac{1}{\sec t}$$

$$= \cos t$$

(b) $r(t) = (\ln \sec t)\mathbf{i} + t\mathbf{j}$, $-\pi/2 < t < \pi/2$

$r'(t) = (\tan t)\mathbf{i} + \mathbf{j}$

$$T(t) = \frac{r'(t)}{\|r'(t)\|}$$

$$= \frac{(\tan t)\mathbf{i} + \mathbf{j}}{\sqrt{\tan^2 t + 1}}$$

$$= \sin t\,\mathbf{i} + \cos t\,\mathbf{j}$$

$$T'(t) = \cos t\,\mathbf{i} - \sin t\,\mathbf{j}$$

$$N(t) = \frac{T'(t)}{\|T'(t)\|}$$

$$= \frac{\cos t\,\mathbf{i} - \sin t\,\mathbf{j}}{\sqrt{\cos^2 t + \sin^2 t}}$$

$$= \cos t\,\mathbf{i} - \sin t\,\mathbf{j}$$

$$K = \frac{\|T'(t)\|}{\|r'(t)\|} = \frac{\sqrt{\cos^2 t + \sin^2 t}}{\sqrt{\tan^2 t + 1}} = \cos t$$

(36)

(c) $r(t) = (2t+3)i + (5-2t)j$

$r'(t) = 2i - 2j$

$$T(t) = \frac{r'(t)}{\|r'(t)\|} = \frac{2i-2j}{\sqrt{2^2+2^2}} = \frac{\sqrt{2}}{2}i - \frac{\sqrt{2}}{2}j$$

~~has no~~

The normal is the zero vector. No unit normal vector.

$$K = \frac{\|T'(t)\|}{\|r'(t)\|} = 0$$

(d) $r(t) = (\cos t + t\sin t)i + (\sin t - t\cos t)j, \ t>0$

$r'(t) = (\cancel{-\sin t} + \cancel{\sin t} + t\cos t)i + (\cancel{\cos t} - \cancel{\cos t} + t\sin t)j$

$= (t\cos t)i + (t\sin t)j$

$$T(t) = \frac{r'(t)}{\|r'(t)\|} = \frac{(t\cos t)i + (t\sin t)j}{\sqrt{t^2\cos^2 t + t^2\sin^2 t}}$$

$$= \cos t\, i + \sin t\, j$$

$T'(t) = -\sin t\, i + \cos t\, j$

(37)

$$N(t) = \frac{T'(t)}{\|T'(t)\|}$$

$$= \frac{-\sin t\, i + \cos t\, j}{\sqrt{\sin^2 t + \cos^2 t}}$$

$$= -\sin t\, i + \cos t\, j$$

$$K = \frac{\|T'(t)\|}{\|r'(t)\|}$$

$$K = \frac{\sqrt{\sin^2 t + \cos^2 t}}{\sqrt{t^2\cos^2 t + t^2 \sin^2 t}}$$

$$K = \frac{1}{t}, \quad t \neq 0, \; t > 0$$

(38)

THE UNIVERSITY OF ZAMBIA
SCHOOL OF NATURAL SCIENCES
DEPARTMENT OF MATHEMATICS & STATISTICS
MAT2100 Tutorial Sheet 7

September, 2018

1. Find the domain of definition of the function
   (a) $f(x,y) = \frac{x+2y}{\sqrt{4-x^2-y^2}}$
   (b) $f(x,y) = \ln(16 - x^2 - y^2) + \ln(x^2 + y^2 - 1)$
   (c) $f(x,y) = e^x \ln(xy)$

2. Find the limit for the following, if it exists
   (a) $\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^2}$
   (b) $\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^4}$
   (c) $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$
   (d) $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$
   (e) $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$
   (f) $\lim_{(x,y)\to(0,0)} \frac{x}{x^2+y^2}$
   (g) $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^3+y^3}$
   (h) $\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y}$
   (i) $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+y^4}$

3. Determine whether the function

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

is continuous at the origin.

4. Suppose f is a function of two variables x and y. Show that if f is differentiable at (x_0, y_0) then it is continuous at (x_0, y_0).

5. Show that $f_x(0,0)$ and $f_y(0,0)$ both exists, but the function is not differentiable at $(0,0)$, where f is defined as

$$f(x,y) = \begin{cases} \frac{-3xy}{x^2+y^2} & \text{if } f(x,y) \neq (0,0) \\ 0 & \text{if } f(x,y) = (0,0) \end{cases}$$

6. Find the partial derivatives f_x, f_y and f_z
   (a) $f(x,y) = 4x^3 - 3x^2y^2 + 2x + 3y$
   (b) $f(x,y) = \frac{x^5}{\ln y}$
   (c) $f(x,y) = \sqrt{x^2+y^2}$
   (d) $f(x,y) = \frac{x+y}{xy-1}$
   (e) $f(x,y) = \tan^{-1}(\frac{y}{x})$
   (f) $f(x,y,z) = xy^2z^3$
   (g) $f(x,y,z) = \ln x - \ln y - ze^{zy}$
   (h) $f(x,y) = (x^2-1)(y+2)$
   (i) $f(x,y) = \frac{x+y}{xy-1}$
   (j) $f(x,y) = e^{xy} \ln y$
   (k) $f(x,y) = \cos^2(3x - y^2)$
   (l) $f(x,y) = \log_y x$

7. If z is implicitly defined as a function of x and y by $x^2 + y^2 + z^2 = 1$, show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z - \frac{1}{z}$

8. If z is implicitly defined as a function of x and y by $x^2 + y^2 - z^2 = 3$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

9. If $z = \ln\sqrt{x^2 + y^2}$, show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 1$

10. Let $z = u^3v^5$, where $u = x + y$ and $v = x - y$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

11. Verify Euler's theorem for the function
    (a) $f(x, y) = \sqrt{x^2 + y^2}$
    (b) $f(x, y) = xy^2 + x^2y - y^3$
    (c) $f(x, y, z) = 3xz^2 - 2xyz + y^2z$

12. If $z = \ln(x^2 + y^2)$, $x = e^{-t}$, $y = e^t$, find $\frac{dz}{dt}$.

13. Find all the local maxima, local minima, and saddle points of the following functions:
    (a) $f(x, y) = 2xy - 5x^2 - 2y^2 + 4x + 4y - 4$
    (b) $f(x, y) = 2xy - x^2 - 2y^2 + 3x + 4$
    (c) $f(x, y) = 2x^3 + 2y^3 - 9x^2 + 3y^2 - 12y$
    (d) $f(x, y) = \dfrac{1}{x} + xy + \dfrac{1}{y}$
    (e) $f(x, y) = y\sin x$
    (f) $f(x, y) = e^{2x}\cos y$

14. Find the absolute maximum and minimum values of the function $x^2 - 2xy + 2y$ on the rectangle $D = \{(x, y) : 0 \le x \le 3,\ 0 \le y \le 2\}$.

15. If $f(x, y) = e^x y^2 - x^3 \ln y$, verify that f_{xxy}, f_{xyx} and f_{yxx} all are equal.

16. Find the shortest distance from the point $(1, 0, -2)$ to the plane $x + 2y + z = 4$.

17. A rectangular box without a lid is to be made from $12m^2$ of cardboard. Find the maximum volume of such a box.

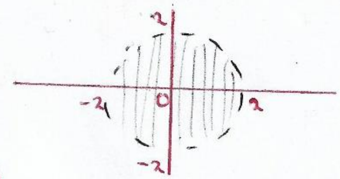
18. (a) If $z = f(x, y) = x^2 + 3xy - y^2$, find the differential dz.
    (b) If x changes from 2 to 2.05 and y changes from 3 to 2.96, compare the values of $\triangle z$ and dz.

19. The base and height of a right circular cone are measured as $10cm$ and $25cm$, respectively, with a possible error in the measurement of as much as $0.1cm$ each. Use differentials to estimate the maximum error in the calculated volume of the cone.

B. A

TUTORIAL SHEET 7

1. (a) $f(x,y) = \dfrac{x+2y}{\sqrt{4-x^2-y^2}}$



$\sqrt{4-x^2-y^2} \neq 0$

$4-x^2-y^2 > 0 \Rightarrow 4 > x^2+y^2$

The doman is defined in the interior $x^2+y^2<4$ of the circle (open disk) centred at $(0,0)$ with radius 2 units. $D = \{x^2+y^2<4 \mid (x,y) \in \mathbb{R}^2\}$

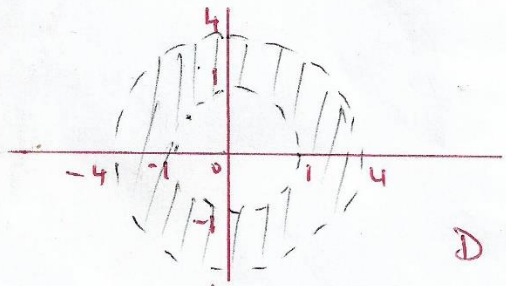
(b) $f(x,y) = \ln(16-x^2-y^2) + \ln(x^2+y^2-1)$

$16-x^2-y^2 > 0$ and $x^2+y^2-1 > 0$

$16 > x^2+y^2 \cap x^2+y^2 > 1$

$1 < x^2+y^2 < 16$

The doman is exculisively defined between open disks with radius 1 unit and 4 units centred at $(0,0)$

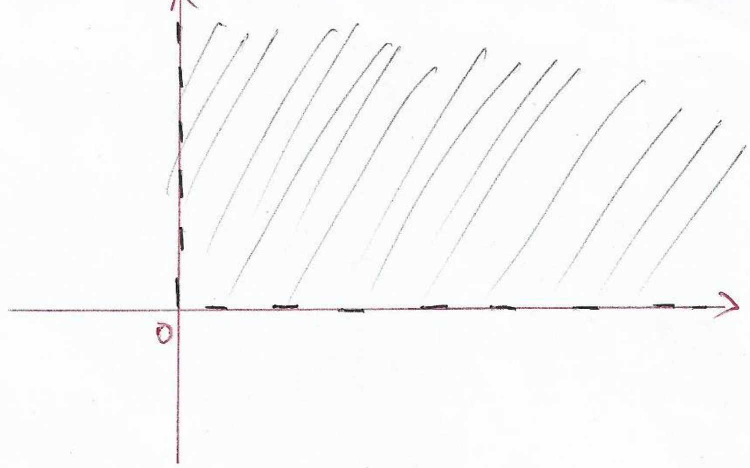


$D = \{x^2+y^2 \in (1,16) \mid (x,y) \in \mathbb{R}^2\}$

(c) $f(x,y) = e^x \ln(xy)$

$e^x \in \mathbb{R} \; \forall x \in \mathbb{R}$ but $\ln(xy) \in \mathbb{R} \; \forall xy > 0$

so $D = \{xy > 0 \mid (x,y) \in \mathbb{R}^2\}$



Domain is defined on the positive quadrant (first quadrant) bounded by $x > 0 \cap y > 0$ exclusively.

2. (a) $\lim\limits_{(x,y)\to(0,0)} \dfrac{2xy^2}{x^2+y^2}$

$\lim\limits_{(0,y)\to(0,0)} \dfrac{0}{y^2} = 0$, $\lim\limits_{(x,0)\to(0,0)} \dfrac{0}{x^2} = 0$

$\lim\limits_{(x,x)\to(0,0)} \dfrac{\cancel{2}x^3}{2x^2} = \lim\limits_{x\to 0} x = 0$,

From these 3 paths we begin to suspect $\lim\limits_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^2} = 0$

So. Given $\varepsilon > 0$, we need to find $\delta > 0$

Such that $\left| \dfrac{2xy^2}{x^2+y^2} - 0 \right| < \varepsilon$ whenever

$0 < \sqrt{(x-0)^2 + (y-0)^2} < \delta$.

$\Rightarrow \left| \dfrac{2xy^2}{x^2+y^2} - 0 \right| = \left| \dfrac{2xy^2}{x^2+y^2} \right| = \dfrac{2|x|y^2}{x^2+y^2} < \varepsilon$

Since $x^2 + y^2 \geq y^2$

$\Rightarrow \dfrac{2|x|y^2}{x^2+y^2} < \dfrac{2|x|\cancel{y^2}}{\cancel{y^2}} = 2|x| < \varepsilon$

(2)

Notice that $x^2 + y^2 > x^2$

$\sqrt{x^2+y^2} > x$

$\Rightarrow \frac{1}{2}\sqrt{x^2+y^2} > x$

Choosing $\delta < \frac{\varepsilon}{2}$

Therefore $\left|\frac{2xy^2}{x^2+y^2} - 0\right| < \varepsilon = 2\delta$ whenever $0 < \sqrt{x^2+y^2} < \delta = \frac{\varepsilon}{2}$

(b) $\lim\limits_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^4}$

$\lim\limits_{(0,y)\to(0,0)} \frac{0}{y^4} = 0, \quad y \neq 0 \qquad \lim\limits_{(x,0)\to(0,0)} \frac{0}{x^2} = 0, x \neq 0$

$\lim\limits_{(x,x)\to(0,0)} \frac{2x^3}{x^2+x^4} = \lim\limits_{(x,x)} \frac{2x}{1+x^2} = 0$

$\lim\limits_{(y^2,y)\to(0,0)} \frac{2y^4}{y^4+y^4}$

The limit does not exist because it is not the same for all paths.

(3)

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2+y^2}$$

$$\lim_{(0,y)\to(0,0)} \frac{-y^2}{y^2} = -1, \quad \lim_{(x,0)\to(0,0)} \frac{x^2}{x^2} = 1$$

$\therefore$ The limit does not exist

(d) $$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$$

$$\lim_{(0,y)\to(0,0)} \frac{0}{\sqrt{y^2}} = 0, \; y\neq 0, \quad \lim_{(x,0)\to} \frac{0}{\sqrt{x^2}} = 0, \; x\neq 0$$

$$\lim_{(x,x)\to(0,0)} \frac{x^2}{\sqrt{2x^2}} = \lim_{x\to 0} \frac{x}{\sqrt{2}} = 0$$

we begin to suspect the limit to be zero so by definition we have that

$\forall \varepsilon > 0 \; \exists \delta > 0 \ni \left|\frac{xy}{\sqrt{x^2+y^2}} - 0\right| < \varepsilon$ whenever $0 < \sqrt{(x-0)^2+(y-0)^2} < \delta$ $\quad 0 < \sqrt{x^2+y^2} < \delta$

$$\Rightarrow \left|\frac{xy}{\sqrt{x^2+y^2}} - 0\right| = \left|\frac{xy}{\sqrt{x^2+y^2}}\right| = \frac{|xy|}{\sqrt{x^2+y^2}} \leq \frac{|x||y|}{\sqrt{x^2}} \leq \frac{\cancel{|x|}|y|}{\cancel{|x|}}$$

$$\leq |y| < \varepsilon \quad . \quad y^2 \leq x^2+y^2$$

$$\Rightarrow \sqrt{y^2} < \sqrt{x^2+y^2} \Rightarrow |y| \leq \sqrt{x^2+y^2} < \varepsilon$$

Choosing $\delta = \varepsilon$

$$\Rightarrow \left|\frac{xy}{\sqrt{x^2+y^2}} - 0\right| < \varepsilon = \delta \text{ whenever } 0 < \sqrt{x^2+y^2} < \delta = \varepsilon$$

(e) $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$

$\lim_{(0,y)\to(0,0)} \frac{0}{y^2} = 0, \; y \neq 0$

$\lim_{(x,0)\to(0,0)} \frac{x^2}{x^2} = 1$

Limit does not exist

(f) $\lim_{(x,y)\to(0,0)} \frac{x}{x^2+y^2}$

$\lim_{(0,y)\to(0,0)} \frac{0}{y^2} = 0, \; y \neq 0$

$\lim_{(x,0)\to(0,0)} \frac{x}{x^2} = \lim_{x\to 0} \frac{1}{x}, \; x \neq 0$

Limit does not exist.

(g) $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^3+y^3}$

$\lim_{(0,y)\to(0,0)} \frac{0}{y^3} = 0$

$\lim_{(x,0)\to(0,0)} \frac{0}{x^3} = 0$

$\lim_{(y,y)\to(0,0)} \frac{y^4}{2y^3} = \frac{1}{2}\lim_{y\to 0} y = 0$

$\lim_{(x,x)\to(0,0)} \frac{x^4}{2x^3} = \frac{1}{2}\lim_{x\to 0} x = 0$

So we begin to suspect the limit to be $= 0$

So. Let $\varepsilon > 0$ be given, we have to find $\delta > 0$ such that

$$\left| \frac{x^2y^2}{x^3+y^3} - 0 \right| < \varepsilon$$

whenever

$$0 < \sqrt{x^2+y^2} < \delta$$

$$\left| \frac{x^2y^2}{x^3+y^3} - 0 \right| < g(x,y) < \varepsilon$$

$$\left| \frac{x^2y^2}{x^3+y^3} \right| \cancel{\leq} \frac{x^2y^2}{|x^3+y^3|} < g(x,y) < \varepsilon$$

$|x^3+y^3| \leq |x^3| + |y^3|$ by triangular inequality

$$|x^3+y^3| \leq |x^3| + |y^3| < |x^3|$$

(5)

$$\Rightarrow \frac{x^2y^2}{|x^3+y^3|} < \frac{x^2y^2}{|x^3|} < \varepsilon$$

$$< \frac{x^2y^2}{|x^2|} < \varepsilon$$

$$\frac{x^2y^2}{|x^2|} = y^2 < \varepsilon$$

$$\therefore \quad g(x,y) = y^2$$

$$\lim_{(x,y)\to(0,0)} y^2 = 0 \quad \Rightarrow \lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^3+y^3}$$

by the squeeze theorem.

(3)

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

$f(x,y)$ is continuous at (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b).$$

Thus $\displaystyle \lim_{(x,y)\to(0,0)} \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases} = \lim_{(x,y)} 0 = 0, \quad (x,y) = 0.$

But $\displaystyle \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$; for $y = x$ $\displaystyle \lim_{(x,x)\to(0,0)} \frac{x^2}{2x^2} = \frac{1}{2}$

Thus $f(x,y)$ is not continuous at $(0,0)$.

(6)

(4) Let f be a function $f: \mathbb{R}^2 \to \mathbb{R}$ that is $f(x,y) = z$, we have to show that if f is differentiable at (x_0, y_0) then it is continuous at (x_0, y_0). Suppose f is differentiable at (x_0, y_0) then the following is true

$$f_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

and

$$f_y(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}.$$

Since f is differentiable it follows consequently that the following is true

$$\lim_{(x,y) \to (x_0, y_0)} f(x,y) = f(x_0, y_0)$$ thus

f is continuous at (x_0, y_0).

(5) $$f(x,y) = \begin{cases} \frac{-3xy}{x^2 + y^2}, & \text{if } (x,y) \neq 0 \\ 0, & \text{if } (x,y) = 0 \end{cases}$$

$$f_x(x,y) = \frac{-3y(x^2 + y^2) + 3xy(2x)}{(x^2 + y^2)^2}$$

$$f_x(x,y) = \frac{-3yx^2 - 3y^3 + 6x^2 y}{(x^2 + y^2)^2}$$

$$f_x(x,y) = \frac{3yx^2 - 3y^3}{(x^2 + y^2)^2}$$

(7)

$$f_y(x,y) = \frac{-3x\,(x^2+y^2) + 3xy\,(2y)}{(x^2+y^2)^2}$$

$$f_y(x,y) = \frac{-3x^3 - 3xy^2 + 6xy^2}{(x^2+y^2)^2}$$

$$f_y(x,y) = \frac{3xy^2 - 3x^3}{(x^2+y^2)^2}$$

$$\lim_{(x,y)\to(0,0)} f_x(0,0) = \lim_{(x,y)\to(0,0)} \frac{3x^2y - 3y^3}{(x^2+y^2)^2}$$

$$= \lim_{(0,y)\to(0,0)} \frac{-3y^3}{y^4} = \lim_{y\to 0} \frac{-3}{y} =$$

$$= \lim_{(x,0)\to(0,0)} \frac{0}{x^4} = 0$$

$$\lim_{(x,y)\to(0,0)} f_y(0,0) = \lim_{(x,y)\to(0,0)} \frac{3xy^2 - 3x^3}{(x^2+y^2)^2}$$

$$= \lim_{(0,y)\to(0,0)} \frac{0}{y^4} = 0$$

$$= \lim_{(x,0)\to(0,0)} \frac{-3x^3}{x^4} = \lim_{x\to 0} \frac{-3}{x^4} =$$

$$f_y(x,y) = \frac{-3x\ (x^2+y^2) + 3xy\ (2y)}{(x^2+y^2)^2}$$

$$f_y(x,y) = \frac{-3x^3 - 3xy^2 + 6xy^2}{(x^2+y^2)^2}$$

$$f_y(x,y) = \frac{3xy^2 - 3x^3}{(x^2+y^2)^2}$$

$$\lim_{(x,y)\to(0,0)} f_x(0,0) = \lim_{(x,y)\to(0,0)} \frac{3x^2y - 3y^3}{(x^2+y^2)^2}$$

$$= \lim_{(0,y)\to(0,0)} \frac{-3y^3}{y^4} = \lim_{y\to 0} \frac{-3}{y} =$$

$$= \lim_{(x,0)\to(0,0)} \frac{0}{x^4} = 0$$

$$\lim_{(x,y)\to(0,0)} f_y(0,0) = \lim_{(x,y)\to(0,0)} \frac{3xy^2 - 3x^3}{(x^2+y^2)^2}$$

$$= \lim_{(0,y)\to(0,0)} \frac{0}{y^4} = 0$$

$$= \lim_{(x,0)\to(0,0)} \frac{-3x^3}{x^4} = \lim_{x\to 0} \frac{-3}{x^4} =$$

f is not differentiable since

$$f(x,y) = \begin{cases} \frac{-3xy}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

is not continuous at $(0,0)$

because

$$\lim_{(x,y)\to(0,0)} 0 = 0 \neq \lim_{(x,y)\to(0,0)} \frac{-3xy}{x^2+y^2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{-3xy}{x^2+y^2} = -3 \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

observe that $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ does not exist

refer to question (8).

(6) (a) $f(x,y) = 4x^3 - 3x^2y^2 + 2x + 3y$

$f_x = 12x^2 - 6xy^2 + 2$ and

$f_y = -6x^2y + 3$

(b) $f(x,y) = \frac{x^5}{\ln y}$

$f_x = \frac{5x^4}{\ln y}$ and

$$f_y = \frac{\partial}{\partial y} x^5 (\ln y)^{-1}$$

$$= \frac{-x^5 (\ln y)^{-2}}{y}$$

$$= \frac{-x^5}{y(\ln y)^2}$$

(c) $f(x,y) = \sqrt{x^2+y^2}$

$$f_x = \frac{1}{\cancel{2}} (x^2+y^2)^{-1/2} \cancel{2}x$$

$$f_x = \frac{x}{\sqrt{x^2+y^2}}$$ and

$$f_y = \frac{1}{\cancel{2}} (x^2+y^2)^{-1/2} (\cancel{2}y)$$

$$f_y = \frac{y}{\sqrt{x^2+y^2}}$$

(d) $f(x,y) = \dfrac{x+y}{xy-1}$

$$f_x = \frac{(xy-1) - (x+y)(y)}{(xy-1)^2}$$

$$f_x = \frac{\cancel{xy} - \cancel{xy} - y^2}{(xy-1)^2}$$

$$f_x = \frac{-y^2}{(xy-1)^2}$$

(10)

and

$$f_y = \frac{(xy-1) - (x+y)(x)}{(xy-1)^2}$$

$$f_y = \frac{-x^2}{(xy-1)^2}$$

(c) $f(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$

$$f_x = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right)$$

$$f_x = \frac{\cancel{x^2}}{x^2+y^2}\left(\frac{-y}{\cancel{x^2}}\right)$$

$$= \frac{-y}{(x^2+y^2)}$$

and

$$f_y = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right)$$

$$f_y = \frac{x^2}{x^2+y^2}\left(\frac{1}{x}\right)$$

$$= \frac{x}{x^2+y^2}$$

(11)

ans) f_{xyz} =

(f) $f(x,y,z) = xy^2z^3$

$f_x = y^2z^3,$

$f_y = 2xyz^3$ and

$f_z = 3xy^2z^2$

(g) $f(x,y,z) = \ln x - \ln y - ze^{zy}$

$f_x = \frac{1}{x},$

$f_y = -\frac{1}{y} - z^2e^{zy}$ and

$f_z = -z\frac{\partial}{\partial z}e^{zy} + e^{zy}\frac{\partial}{\partial z}(-z)$

$= -zye^{zy} - e^{zy}$

$= -e^{zy}(zy+1)$

(h) $f(x,y) = (x^2-1)(y+2)$

$f_x = (x^2-1)\frac{\partial}{\partial x}(y+2) + (y+2)\frac{\partial}{\partial x}(x^2-1)$

$f_x = 2x(y+2)$ and

$f_y = x^2-1$

(i) is the same as (j)

(j) $f(x,y) = e^{xy}\ln y$

$f_x = ye^{xy}\ln y$ and

$f_y = e^{xy}\frac{\partial}{\partial y}\ln y + \ln y\frac{\partial}{\partial y}e^{xy}$

$= \frac{1}{y}e^{xy} + x\ln y\, e^{xy}$

$= e^{xy}\left(\frac{1}{y} + x\ln y\right)$

(12)

(k) $f(x,y) = \cos^2(3x - y^2)$

$f_x = 2\cos(3x - y^2)(-\sin(3x - y^2))\,3$

$= -6\cos(3x-y^2)\sin(3x-y^2)$

$= -3\sin(6x - 2y^2)$

$f_y = 2\cos(3x - y^2)(-\sin(3x-y^2))(-2y)$

$= 4y\cos(3x-y^2)(\sin(3x-y^2))$

$= 2y\sin(6x - 2y^2)$

(l) $f(x,y) = \log_y x$

Let $p = \log_y x$

$\ln x = \ln y^p$

$p = \frac{\ln x}{\ln y}$

$\Rightarrow f(x,y) = \frac{\ln x}{\ln y}$

$f_x = \frac{\partial}{\partial x}\left(\frac{\ln x}{\ln y}\right)$

$= \frac{1}{\ln y}\frac{\partial}{\partial x}(\ln x)$

$= \frac{1}{\ln y}\left(\frac{1}{x}\right)$

$= \frac{1}{x}(\ln y)^{-1}$

$f_y = \frac{\partial}{\partial y}\ln x(\ln y)^{-1}$

$= \ln x\frac{\partial}{\partial y}(\ln y)^{-1}$

$= \frac{-(\ln x)(\ln y)^{-2}}{y}$

$= \frac{-\ln x}{y(\ln y)^2}$

(13)

(7). Given $x^2 + y^2 + z^2 = 1$ we have to show

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z - \frac{1}{z}$$

$$\frac{\partial z}{\partial x} = \frac{-\frac{\partial}{\partial x}(x^2+y^2+z^2-1)}{\frac{\partial}{\partial z}(x^2+y^2+z^2-1)}$$

$$= \frac{-\cancel{2}x}{\cancel{2}z} = -\frac{x}{z}$$

$$\frac{\partial z}{\partial y} = \frac{-\frac{\partial}{\partial x}(x^2+y^2+z^2-1)}{\frac{\partial}{\partial z}(x^2+y^2+z^2-1)}$$

$$= \frac{-2y}{2z} = \frac{-y}{z}$$

$$\Rightarrow x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = x\left(\frac{-x}{z}\right) + y\left(\frac{-y}{z}\right)$$

$$= \frac{-x^2}{z} + \frac{(-y)^2}{z}$$

$$= -\frac{(x^2+y^2)}{}$$

Notice that $x^2 + y^2 = 1 - z^2$

$$= -\frac{(1-z^2)}{z}$$

$$= \frac{z^2 - 1}{z}$$

$$= z - \frac{1}{z}$$

(hence shown)

(8) let $f(x,y,z) = x^2+y^2-z^2-3$

$$\frac{\partial z}{\partial x} = \frac{-\frac{\partial}{\partial x}(x^2+y^2-z^2-3)}{\frac{\partial}{\partial z}(x^2+y^2-z^2-3)} = \frac{+(\cancel{2}x)}{(\cancel{2}z)} = \frac{x}{z}$$

$$\frac{\partial z}{\partial y} = \frac{-\frac{\partial}{\partial y}(x^2+y^2-z^2-3)}{\frac{\partial}{\partial z}(x^2+y^2-z^2-3)} = \frac{+(\cancel{2}y)}{+\cancel{2}z} = \frac{y}{z}$$

(9) $z = \ln\sqrt{x^2+y^2}$ we have to show that

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 1$$

$$\frac{\partial z}{\partial x} = \frac{1}{\sqrt{x^2+y^2}}\left(\frac{1}{\cancel{2}}(x^2+y^2)^{-1/2}(\cancel{2}x)\right)$$

$$= \frac{x}{x^2+y^2}$$

$$\frac{\partial z}{\partial x} = \frac{1}{\sqrt{x^2+y^2}}\left(\frac{1}{\cancel{2}}(x^2+y^2)^{-1/2}(\cancel{2}y)\right)$$

$$= \frac{y}{x^2+y^2}$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = x\left(\frac{x}{x^2+y^2}\right) + y\left(\frac{y}{x^2+y^2}\right)$$

$$= \frac{x^2+y^2}{x^2+y^2}$$

$$= 1 \qquad \text{(shown)}$$

(10)

(10) $z = u^3 v^5$ where $u = x+y$ $v = x-y$

$z = (x+y)^3 (x-y)^5$

$$\frac{\partial z}{\partial x} = (x+y)^3 \frac{\partial}{\partial x}(x-y)^5 + (x-y)^5 \frac{\partial}{\partial x}(x+y)^3$$

$$= 5(x+y)^3 (x-y)^4 + 3(x+y)^2 (x-y)^5$$

$$= 5(x^2-y^2)^3 (x-y) + 3(x^2-y^2)^2 (x-y)^3$$

$$\frac{\partial z}{\partial y} = (x+y)^3 \frac{\partial}{\partial y}(x-y)^5 + (x-y)^5 \frac{\partial}{\partial y}(x+y)^3$$

$$= -5(x+y)^3 (x-y)^4 + 3(x-y)^5 (x+y)^2$$

$$= -5(x^2-y^2)^3 (x-y) + 3(x^2-y^2)^2 (x-y)^3$$

(11) Euler's theorem for homogeneous functions

If $f(x,y)$ is an homogeneous function of degree k then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = k f(x,y)$$

(a) $f(x,y) = \sqrt{x^2+y^2}$

f is homogeneous if it satisfies

$$f(\alpha x, \alpha y) = \alpha^k f(x,y)$$

$$\Rightarrow f(\alpha x, \alpha y) = \sqrt{(\alpha x)^2 + (\alpha y)^2}$$

(16)

$= \sqrt{a^2(x^2+y^2)}$

$= a\sqrt{x^2+y^2}$

$= a f(x,y) \qquad K=1$

$\Rightarrow$ f is an homogeneous function of degree 1.

$$\frac{\partial f}{\partial x} = \frac{1}{2}(x^2+y^2)^{-1/2}(2x) = \frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{\partial f}{\partial y} = \frac{1}{2}(x^2+y^2)^{-1/2}(2y) = \frac{y}{\sqrt{x^2+y^2}}$$

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial x} = \frac{x^2}{\sqrt{x^2+y^2}} + \frac{y^2}{\sqrt{x^2+y^2}}$$

$$= \frac{(x^2+y^2)^1\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}\sqrt{x^2+y^2}}$$

$$= \frac{\cancel{(x^2+y^2)}(\sqrt{x^2+y^2}}{\cancel{x^2+y^2}}$$

$$= \sqrt{x^2+y^2}$$

$$= f(x,y)$$

(verified)

(17)

(b) $f(x,y) = xy^2 + x^2y - y^3$

$f(ax, ay) = (ax)(ay)^2 + (ax)^2(ay) - (ay)^3$

$= a^3xy^2 + a^3x^2y - a^3y^3$

$= a^3(xy^2 + x^2y - y^3)$

$= a^3(f(x,y)) \qquad k = 3$

$\frac{\partial f}{\partial x} = y^2 + 2xy \qquad \frac{\partial f}{\partial y} = 2xy + x^2 - 3y^2$

$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = x(y^2 + 2xy) + y(2xy + x^2 - 3y^2)$

$= xy^2 + 2x^2y + 2xy^2 + x^2y - 3y^3$

$= 3xy^2 + 3x^2y - 3y^3$

$= 3(xy^2 + x^2y - y^3)$

$= 3f(x,y)$

(verified)

(c) $f(x,y,z) = 3xz^2 - 2xyz + y^2z$

$f(ax, ay, az) = 3(ax)(az)^2 - 2(ax)(ay)(az) + (ay)^2(az)$

$= 3a^3xz^2 - 2a^3xyz + a^3y^2z$

$= a^3(3xz^2 - 2xyz + y^2z)$

$= a^3(f(x,y,z))$

$k = 3$

(18)

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = k\, f(x,y,z)$$

$$\frac{\partial f}{\partial x} = 3z^2 - 2yz \qquad \frac{\partial f}{\partial y} = -2xz + 2yz$$

$$\frac{\partial f}{\partial z} = 6xz - 2xy + y^2$$

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = x(3z^2 - 2yz) + y(-2xz + 2yz) + z(6xz - 2xy + y^2)$$

$$= 3xz^2 - 2xyz - 2xyz + 2y^2z + 6xz^2 - 2xyz + zy^2$$

$$= 9xz^2 - 6xyz + 3y^2z$$

$$= 3(3xz^2 - 2xyz + y^2z)$$

$$= 3\, f(x,y,z)$$

verified

(12) $z = \ln(x^2 + y^2)$, $x = e^{-t}$ $y = e^{t}$

$$z = \ln\left(e^{-2t} + e^{2t}\right)$$

$$z = \ln(2\cosh 2t)$$

$$\frac{dz}{dt} = \frac{1}{2\cosh 2t}\,\frac{d}{dt}\,2\cosh 2t$$

(19)

$$= \frac{1}{2\cosh 2t} 4\sinh 2t$$

$$= \frac{4\sinh 2t}{2\cosh 2t}$$

$$= \frac{2(e^{2t} - e^{-2t})}{e^{2t} + e^{2t}}$$

(13) (a) $f(x,y) = 2xy - 5x^2 - 2y^2 + 4x + 4y - 4$

$f_x = 2y - 10x + 4$

$f_{xx} = -10$

$f_y = 2x - 4y + 4$

$f_{yy} = -4$

$f_{xy} = 2$

$$D = g(x,y) = f_{xx} f_{yy} - (f_{xy})^2$$

$$= (-10)(-4) - (2)^2$$

$$= 36$$

Critical points

$$2y - 10x + 4 = 0$$
$$2x - 4y + 4 = 0$$

$$2y - 10x = -4$$
$$-2y + x = -2$$
$$-9x = -6$$
$$x = 2/3$$

$$-2y = -2 - \frac{2}{3}$$
$$-2y = -\frac{8}{3}$$
$$y = \frac{4}{3}$$

Local max is $(2/3, 4/3, 0)$

Since $D > 0$ and $f_{xx} < 0$ f has a local maximum $f(2/3, 4/3)$ at $(2/3, 4/3, f(2/3, 4/3))$

(b) $f(x,y) = 2xy - x^2 - 2y^2 + 3x + 4$

$$f_x = 2y - 2x + 3 \qquad f_y = 2x - 4y$$
$$f_{xx} = -2 \qquad f_{yy} = -4$$
$$f_{xy} = 2$$

$$D = (-2)(-4) - (2)^2 = 4 < 0$$

Critical point

$$2y - 2x + 3 = 0$$
$$2x - 4y = 0$$

$$-2x + 2y = -3$$
$$x = 2y$$
$$-4y + 2y = -3$$
$$y = \frac{3}{2}$$
$$x = \frac{3}{[illegible]}$$

(21)

Since $D < 0$ and $f_{xx} < 0$ $f(3/2, 3/2)$ is the local maximum value and thus Local maximum point is

$$\left(3/2,\ 3/4,\ f(3/2, 3/2)\right) = \left(3/2,\ 3/2,\ 17/2\right)$$

(O) $f(x,y) = 2x^3 + 2y^3 - 9x^2 + 3y^2 - 12y$

$f_x = 6x^2 - 18x$, $f_{xx} = 18x - 18$

$f_y = 6y^2 + 6y - 12$, $f_{yy} = 18y + 6$

$f_{xy} = 18$

Critical point(s)

$6x^2 - 18x = 0 \Rightarrow 6x(x-3) = 0$

$6y^2 + 6y - 12 = 0 \Rightarrow (y-1)(y+2) = 0$

$x = 0$, $x = 3$, $y = 1$, $y = -2$

$(0,1)$, $(0,-2)$, $(3,1)$, $(3,-2)$

at $(0,1)$

$D = f_{xx} f_{yy} - (f_{xy})^2$

$D = (-18)(6) - (18)^2 < 0$

f has a saddle point at $(0,1)$

$(0, 1, f(0,1))$

(22)

at $(0,-2)$

$$D = (-18)(-30) - (18)^2 > 0$$

Since $f_{xx}(0,-2) < 0$ f has a local maximum at $(0, -2, f(0,-2))$

at $(3,1)$

$$D = (36)(24) - (18)^2 > 0 \quad , \quad f_{xx}(3,1) > 0$$

$\Rightarrow$ f has a local minimum at $(3, 1, f(3,1))$

at $(3,-2)$

$D = (36)(-30) - (18)^2 < 0 \Rightarrow$ f has a saddle point at $(3, -2, f(3,-2))$.

(D) $f(x,y) = \frac{1}{x} + xy + \frac{1}{y}$

$$f_x = -\frac{1}{x^2} + y \quad , \quad f_{xx} = \frac{2}{x^3}$$

$$f_y = x - \frac{1}{y^2} \quad , \quad f_{yy} = \frac{2}{y^3}$$

$$f_{xy} = 1$$

$\therefore$ Critical points

$$-\frac{1}{x^2} + y = 0 \quad , \quad x - \frac{1}{y^2} = 0$$

$$-1 + x^2 y = 0 \qquad xy^2 - 1 = 0$$

$-1 + x^2y = 0$ $\qquad y = \frac{1}{x^2}$ $\qquad -y^4 + y = 0$

$-1 + xy^2 = 0$ $\qquad -y(y^3 - 1) = 0$

$-1 + x\left(\frac{1}{x^2}\right)^2 = 0$ $\qquad y \neq 0, \ y = 1$

$-1 + \frac{1}{x^3} = 0$ $\qquad x \neq 0$

$-x^3 + 1 = 0$

$x^3 - 1 = 0$

$x = 1, \ y = 1$

critical point $(1, 1, f(1,1)) = (1, 1, 3)$

$D = (2)(2) - (1)^2 > 0$

since $f_{xx}(1,1) > 0 \Rightarrow (1, 1, f(1,1))$ is the local minimum point of f.

(e) $f(x,y) = y \sin x$

$f_x = y \cos x$, $\quad f_{xx} = -y \sin x$

$f_y = \sin x$, $\quad f_{yy} = 0$

$f_{xy} = \cos x$

Critical point(s)

$y \cos x = 0$ $\qquad y = 0$

$\sin x = 0$

for $\cos x = 0$ $\quad x = \frac{1}{2} n\pi$ where n is odd

for $\sin x = 0$ $\quad x = n\pi$ where n is even

$$D = -(\cos x)^2 < 0 \quad \forall \quad x = \frac{1}{2} n\pi \quad , n \in \mathbb{Z}^{O}$$
$$x = n\pi \quad , n \in \mathbb{Z}^{E}$$

So f has a saddle point at $\left(\frac{1}{2}n\pi, 0, f\left(\frac{1}{2}n\pi, 0\right)\right)$ and $\left(n\pi, 0, f(n\pi, 0)\right)$ $(A\pi, 0, 0)$

(f) $f(x,y) = e^{2x}\cos y$

$f_x(x,y) = 2e^{2x}\cos y \quad , f_{xx}(x,y) = 4e^{2x}\cos y$

$f_y(x,y) = -e^{2x}\sin y \quad , f_{yy}(x,y) = -e^{2x}\cos y$

$f_{xy}(x,y) = -2e^{2x}\sin y$

Critical point(s)

$2e^{2x}\cos y = 0 \quad , \quad -e^{2x}\sin y = 0$

$\Rightarrow \quad x \in \mathbb{R} \quad , \; y = \frac{1}{2}n\pi$ when n is odd intenger for $\cos y$ and $y = n\pi$ where n is even for $\sin y$

$$D = -\left(-2e^{2x}\sin y\right)^2 < 0$$

So f has no saddle point at this point because f has no local extrema.

(25)

(14) $f(x,y) = x^2 - 2xy + 2y$ $\quad D = \{ (x,y) : 0 \leq x \leq 3, 0 \leq y \leq 2 \}$

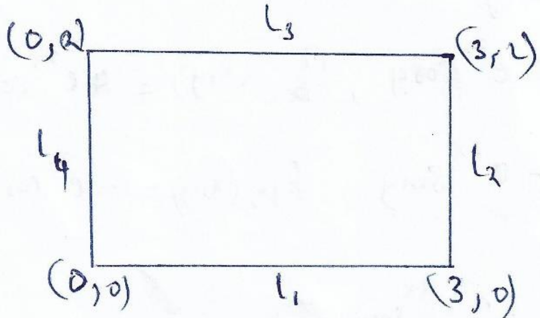
$f_x = 2x - 2y$, $f_y = -2x + 2$

Critical points

$2x - 2y = 0$

$-2x + 2 = 0 \Rightarrow x = 1, \ y = 1$

point $(1, 1, 1)$



$f_{xx} = 2 \quad f_{yy} = 0 \quad f_{xy} = -2$

$D = (2)(0) - (2)^2 < 0 \Rightarrow (1,1,1)$ is a saddle point

Along L_1, $y = 0 \Rightarrow f(x,0) = x^2$, $0 \leq x \leq 3$

$f(x,0)$ is increasing on $[0,3] \Rightarrow \min f(x,0) = f(0,0) = 0$

and $\max f(x,0) = f(3,0) = 9$

Along L_2, $x = 3 \Rightarrow f(3,y) = 9 - 4y$, $[0,2]$

is decreasing. $\max f(3,y) = f(3,0) = 9$

$\min f(3,y) = f(3,2) = 1$

Along L_3, $y = 2 \Rightarrow f(x,2) = x^2 - 4x + 4$, $[0,3]$
is decreasing; $\min f(x,2) = f(2,2) = 0$ and
$\max f(x,2) = f(0,2) = 4$

Along l_4, $x = 0 \Rightarrow f(0,y) = 2y$, $[0,2]$
is increasing, $\min f(0,y) = f(0,0) = 0$ and
$\max f(0,y) = f(0,2) = 4$

$\therefore$ the abs $\max f(x,y) = 9$ and abs $\min f(x,y) = 0$

(15) $f(x,y) = e^x y^2 - x^3 \ln y$

$$f_x = e^x y^2 - 3x^2 \ln y$$

$$f_{xx} = e^x y^2 - 6x \ln y$$

$$f_{xxy} = 2e^x y - \frac{6x}{y} \quad \text{- - - - (i)}$$

$$f_{xy} = 2e^x y - \frac{3x^2}{y}$$

$$f_{xyx} = 2e^x y - \frac{6x}{y} \quad \text{- - - - (ii)}$$

$$f_y = 2e^x y - \frac{x^3}{y}$$

$$f_{yx} = 2e^x y - \frac{3x^2}{y}$$

$$f_{yxx} = 2e^x y - \frac{6x}{y} \quad \text{- - - (iii)}$$

Notice that (i) = (ii) = (iii) (verified)

(16) the shortest distance from $(1, 0, -2)$ to $x + 2y + z - 4 = 0$ is the perpendicular distance.

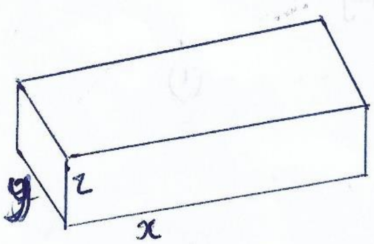
$$D = \frac{|a_0 x_0 + b_0 y_0 + c_0 z_0 + d|}{\sqrt{a_0^2 + b_0^2 + c_0^2}}$$

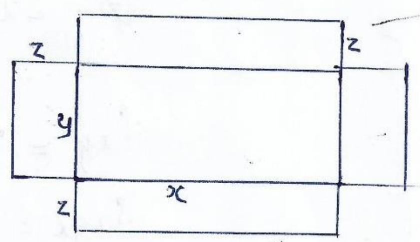
$$D = \frac{|(1)(1) + (2)(0) + (-2)(1) - 4|}{\sqrt{1^2 + 2^2 + 1^2}}$$

$$D = \frac{|-5|}{\sqrt{6}}$$

$$D = \frac{5\sqrt{6}}{6}$$

(17)





$$A = 2zy + (2z + y)x$$

$$\Rightarrow \quad 2zy + 2zx + xy = 12$$

let $f(x, y, z) = 2zy + 2zx + xy - 12$

$V = xyz$

(18) (a) $Z = f(x, y) = x^2 - 3xy - y^2$

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$\frac{\partial z}{\partial x} = 2x - 3y, \quad \frac{\partial z}{\partial y} = -3x - 2y$$

$$dz = (2x - 3y)\,dx + (-3x - 2y)\,dy$$

(b) $dz = (2(2) - 3(3))(0.05) + (-3(2) - 2(3))(0.04)$ $\quad dx = 0.05$

$dy = -0.04$

$$dz = (-5)(0.05) + (-12)(-0.04)$$

$$= -0.25 + 0.48$$

$$= 0.23$$

$$\Delta z = f(x+\Delta x,\ y+\Delta x) - f(x,y)$$

$$= f(2.05,\ 2.96) - f(2,3)$$

$$= (2.05)^2 - 3(2.05)(2.96) - (2.96)^2$$

$$- \left((2)^2 - 3(2)(3) - (3)^2\right)$$

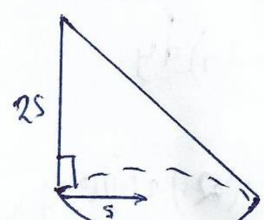
$$= -22.7631 - \left(2^2 - 3(2)(3) - 3^2\right)$$

$$= 23 - 22.7631$$

$$= 0.2369$$

$$\Rightarrow \quad dz \approx \Delta z$$

(19)



$\Delta = 0.1$

$$V = \frac{1}{3}\pi r^2 h$$

$$V = f(r,h) = \frac{1}{3}\pi r^2 h$$

$$dz = \frac{\partial z}{\partial r}\,dr + \frac{\partial z}{\partial h}\,dh$$

$$\frac{dz}{dr} = \frac{2}{3}\pi r h \qquad \frac{dz}{dh} = \frac{1}{3}\pi r^2$$

$$dv = \left(\frac{2}{3}\pi r h\right) dr + \left(\frac{1}{3}\pi r^2\right) dh$$

$$= \left(\frac{2}{3}\pi (5)(25)\right)(0.1) + \left(\frac{1}{3}\right)(\pi)(5)^2(0.1)$$

$$= \frac{250}{3}(0.1)\pi + \frac{25}{3}\pi(0.1)$$

$$= \frac{25}{3}\pi + \frac{2.5}{3}\pi$$

$$= \frac{27.5}{3}\pi$$

$$= \frac{55}{6}\pi \approx 28.8\,cm^3$$

Since $dv \approx \Delta v$ thus maximum error is

$e > \frac{55\pi}{6}\,cm^3$

(31)

$$2x = \frac{9-4y+2z}{2}$$

$$4x + 4y - 2z = 9 \quad (1)$$

$$2x + y + z = -3 \quad (2)$$

$$z = -3 - 2x - y \quad (3)$$

$$4x + 4y - 2(-3 - 2x - y) = 9$$

$$4x + 4y + 6 + 4x + 2y = 9$$

$$8x + 6y = 3$$

$$x = \frac{3}{8} - \frac{6y}{8}$$

$$\boxed{z = -3 - \frac{9 + 4y + 2z}{2} - y \qquad 2\cancel{z} = -6 - 9 + 4y + 2\cancel{z} - 2y \qquad y = t}$$

$$2x = -3 - y - z \quad (4)$$

$$2(-3 - y - z) + 4y - 2z = 9$$

$$-6 - 2y - 2z + 4y - 2z = 9$$

$$4z + 2y = 15$$

$$z = -\frac{15}{4} + \frac{y}{4}$$

$$x = \frac{3}{8} - \frac{6}{8}t, \quad y = t, \quad z = -\frac{15}{4} + \frac{y}{4}$$

THE UNIVERSITY OF ZAMBIA
SCHOOL OF NATURAL SCIENCES
DEPARTMENT OF MATHEMATICS AND STATISTICS

MAT 2100 TUTORIAL SHEET 8
October 2018

1. Show that the following functions are solutions of the accompanying differential equation.

   (a) $xy'' - y' = 0,\qquad y = x^2 + 3$ (b) $y' + \frac{1}{x}y = 1,\qquad y = \frac{C}{x} + \frac{x}{2}$

   (c) $x^3y''' + 4x^2y'' + xy' + y = x,\qquad y = (1/2)x$

2. Separate the variables and solve the following differential equations:

   (a) $x(2y-3)dx + (x^2+1)dy = 0$ (b) $x^2(y^2+1)dx + y\sqrt{x^3+1}dy = 0$

   (c) $\frac{dy}{dx} = e^{x-y}$ (d) $\sin x\frac{dx}{dy} + \cosh 2y = 0$

   (e) $\sqrt{2xy}\frac{dy}{dx} = 1$ (f) $(\ln x)\frac{dy}{dy} = \frac{x}{y}$

3. Show that the following equations are homogeneous and solve them

   (a) $(x^2+y^2)dx + xydy = 0$ (b) $x^2dy + (y^2 - xy)dx = 0$

4. Solve the equation

$$(x+y+1)dx + (y-x-3)dy = 0$$

by making a change of variable of the form

$$x = r + a, \qquad y = s + b$$

and choosing the constants a and b so that the resulting equation is

$$(r+s)dr + (r-s)ds = 0$$

Then solve this equation and express its solution in terms of x and y.

5. Solve the following linear differential equations

   (a) $\frac{dy}{dx} + 2y = e^{-x}$ (b) $2\frac{dy}{dx} - y = e^{x/2}$ (c) $x\frac{dy}{dx} + 3y = \frac{\sin x}{x^2}$

   (d) $xdy + ydx = ydy$ (e) $(x-1)^3\frac{dy}{dx} + 4(x-1^2)y = x+1$

6. Test the following ordinary differential equations for exactness and solve them

   (a) $2xy^3dx + (3x^2y^2+2)dy = 0$ (b) $\cos xdx - 2\sin ydy = 0$

   (c) $(xe^x + e^x + y)dx + xdy = 0$

7. If a, b and c are constants and

$$(ax^2 + by^2)dx + cxydy = 0$$

is an exact equation, what relations(s) among the constants must hold? Solve the equation subject to such conditions.

FIZIE MATH LAB

8. If a, b and c are constants and
$$(ax^2 + by^2)dx + cxydy = 0$$
is not an exact differential equation but has the integrating factor $1/x^2$, what relations among the constants must hold? Solve the equation subject to such conditions.

9. Solve the following differential equations

(a) $\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$ (b) $\frac{d^2y}{dx^2} + y\frac{dy}{dx} = 0$ (c) $\frac{d^2y}{dx^2} + \omega^2 y = 0$ (ω = constant $\neq 0$)

(d) $xy'' - 2y'' = 0$ (Hint: Substitute $y'' = q$)

10. Solve the following differential equations subject to the given initial conditions

(a) $y'' - y = 0$ $\quad y(0) = 1, \quad y'(0) = -2$
(b) $y'' + 2y' + y = 0$ $\quad y(0) = 0, \quad y'(0) = 1$
(c) $2y'' - y' - y = 0$ $\quad y(0) = -1, \quad y'(0) = 0$

11. Solve the following differential equations by variation of parameters:

(a) $\frac{d^2y}{dx^2} + \frac{dy}{dx} = x$ (b) $\frac{d^2y}{dx^2} + y = \tan x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$ (c) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = e^x$

12. Solve the following equations by the method of undetermined coefficients:

(a) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 10y = -3$ (b) $\frac{d^2y}{dx^2} - \frac{dy}{dx} = \sin x$ (c) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = x^2$

13. Solve the following Bernoulli's equation $\frac{dy}{dx} + y = (xy)^2$.

14. Solve the integral equation $y(x) + \int_0^t y(t)dt = x$.

15. Use power series to solve the following differential equations and find the interval of convergence of the power series.

(a) $y' + 3xy = 0$ (b) $y'' - xy' = 0$ (c) $y'' - xy' - y = 0$

16. Find the first three terms of each of the power series representing independent solutions of the following differential equations:

(a) $(x^2 + 4)y'' + y = 0$ (b) $y'' + x^2y = 0$

(i) (a) $y' + \frac{1}{x} y = 1$, $y = \frac{c}{x} + \frac{x}{2}$

Linear ODE

$$e^{\int \frac{1}{x} dx} = e^{\ln x} = x$$

$$\Rightarrow \quad xy' + y = x$$

$$\int \frac{d}{dx} xy = \int x \, dx$$

$$xy = \frac{x^2}{2} + c$$

$$y = \frac{c}{x} + \frac{x}{2}$$

Shown

(2) (a) $x(2y-3)\,dx + (x^2+1)\,dy = 0$

$$\frac{x}{x^2+1}\,dx + \frac{1}{2y-3}\,dy = 0 \qquad \text{seperated}$$

$$\int \frac{x}{x^2+1}\,dx + \int \frac{1}{2y-3}\,dy = C$$

$$\frac{1}{2}\ln|x^2+1| + \frac{1}{2}\ln|2y-3| = c$$

$$\ln|x^2+1| + \ln|2y-3| = 2c$$

$$e^{\ln|(x^2+1)(2y-3)|} = e^{2c} = K$$

$$|(x^2+1)(2y-3)| = K$$

$$(x^2+1)(2y-3) = \pm K = K'$$

$$2y-3 = \frac{K'}{x^2+1}$$

$$y = \frac{K'}{2(x^2+1)} + \frac{3}{2}$$

(b) $x^2(y^2+1)\,dx + y\sqrt{x^3+1}\,dy = 0$

$$\frac{x^2}{\sqrt{x^3+1}}\,dx + \frac{y}{y^2+1}\,dy = 0$$

$$\int \frac{x^2}{\sqrt{x^3+1}}\,dx + \int \frac{y}{y^2+1}\,dy = C$$

$$\frac{1}{3}\left(\frac{(x^3+1)^{1/2}}{1/2}\right) + \frac{1}{2}\ln|y^2+1| = C$$

$$\frac{2}{3}\sqrt{(x^3+1)} + \frac{1}{2}\ln|y^2+1| = c$$

$$e^{\ln\sqrt{|y^2+1|}} = e^{\left(c - \frac{2}{3}\sqrt{x^3+1}\right)}$$

$$|y^2+1| = e^{c} \cdot e^{-\frac{2}{3}\sqrt{x^3+1}}$$

$$y^2+1 = \frac{\pm e^{c}}{e^{\frac{2}{3}\sqrt{x^3+1}}}$$

$$y^2 = \frac{k}{e^{\frac{2}{3}\sqrt{x^3+1}}} - 1$$

$$y = \pm\sqrt{\frac{c}{e^{\frac{2}{3}\sqrt{x^3+1}}} - 1}$$

(e) $\frac{dy}{dx} = e^{x-y}$

$$\int e^{y}\,dy = \int e^{x}\,dx$$

$$\ln e^{y} = \ln(e^{x} + c)$$

$y = \ln(e^{x} + c)$, where $e^{x} + c > 0$

$e^{x} > 0 \quad \forall x \in \mathbb{R}$

(d) $\sin x \dfrac{dx}{dy} + \cosh 2y = 0$

$\int \sin x \, dx + \int \cosh 2y \, dy = 0$

$\int \sin x \, dx + \int \cosh 2y \, dy = c$

$-\cos x + \frac{1}{2} \sinh 2y = c$

$\frac{1}{2} \sinh 2y = c + \cos x$

$\sinh 2y = c' + 2\cos x \quad ; \quad c' = 2c$

$y = \frac{1}{2} \sinh^{-1} \left(c' + 2\cos x \right)$

(e) $\sqrt{2xy} \dfrac{dy}{dx} = 1$

$\sqrt{2} \int y^{1/2} \, dy = \int x^{-1/2} \, dx$

$$\frac{\sqrt{2} \, y^{3/2}}{3/2} = \frac{x^{1/2}}{1/2} + c$$

$$\frac{\sqrt{2} \, y^{3/2}}{3} = x^{1/2} + \frac{c}{2}$$

(f) $\ln x \dfrac{dy}{dx} = \dfrac{x}{y}$

$$\int y \frac{dy}{dx} = \int \frac{x}{\ln x} \, dx$$

$$\frac{y^2}{2} = \int \frac{x}{\ln x} \, dx$$

$$\frac{y^2}{2} = \int \frac{x}{\ln x}\, dx$$

let $u = (\ln x)^{-1} \Rightarrow du = -(\ln x)^{-2}\left(\frac{1}{x}\right)$

(3) (a) $(x^2+y^2)\,dx + xy\,dy = 0$

$$xy\,dy = -(x^2+y^2)\,dx$$

$$\frac{dy}{dx} = \frac{-(x^2+y^2)}{xy} = -\left(\frac{x}{y}+\frac{y}{x}\right) = -\left(\frac{1}{\frac{y}{x}}+\frac{y}{x}\right)$$

$$F(v) = -\left(\frac{1}{v}+v\right)$$

hence the ODE is homogeneous.

$$-\frac{1}{v} - v = v + x\frac{dv}{dx}$$

$$-x\frac{dv}{dx} = 2v + \frac{1}{v}$$

$$-x\frac{dv}{dx} = \frac{2v^2+1}{v}$$

$$\int \frac{-v}{2v^2+1}\,dv = \int -\frac{1}{x}\,dx$$

$$\frac{1}{4}\ln|2v^2+1| = -\ln|x| + c$$

$$\ln|2v^2+1| + 4\ln|x| = 4c$$

$$\ln|x^4(2v^2+1)| = 4c$$

$$\ln\left|x^4\left(\frac{2y^2}{x^2}+1\right)\right| = 4c$$

$$e^{\ln|2x^2y^2+x^4|} = e^{4c}$$

$$|2x^2y^2+x^4| = e^{4c}$$

$$2x^2y^2+x^4 = \pm e^{4c} = K$$

$$y^2 = \frac{K}{2x^2} + \frac{x^2}{2}, \quad \text{where } x \neq 0$$

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(b) $x^2 dy + (y^2 - xy)\,dx = 0$

$$\frac{dy}{dx} = \frac{xy - y^2}{x^2} = \frac{y}{x} - \frac{y^2}{x^2} = v - v^2 = F(v)$$

Hence the ODE is homogeneous.

$$\cancel{v} - v^2 = \cancel{v} + x\frac{dv}{dx}$$

$$x\frac{dv}{dx} = -v^2$$

$$-\int \frac{1}{v^2}\,dv = \int \frac{1}{x}\,dx$$

$$\frac{1}{v} = \ln|x| + c$$

$$\frac{x}{y} = \ln|x| + c$$

$$y = \frac{x}{\ln|x| + c} \;;\; |x| > 0 \Rightarrow |x| \neq 0$$

(4) $(x + y + 1)\,dx + (y - x - 3)\,dy = 0$

$x = r + a \qquad y = s + b$

$dx = dr \qquad dy = ds$

$$(r + a + s + b + 1)\,dr + (s + b - r - a - 3)\,ds = 0$$

Let $a + b + 1 = p$ and $b - a - 3 = q$

where $p, q \in \mathbb{R}$

$$(r + s + p)\,dr + (s - r + q)\,ds = 0$$

$$(r + s)\,dr + (s - r)\,ds + p\,dr + q\,ds = 0$$

$$(r + s)\,dr + (s - r)\,ds + 0 = 0$$

Observe that $p\,dr + q\,ds$ is exact so

$\Rightarrow$ $p\,dr + q\,ds = 0$

because $\frac{\partial p}{\partial s} = \frac{\partial q}{\partial r} = 0$

By the the fact that $s\,ds = r\,ds$

we have the $(s-r)\,ds = (r-s)\,ds$

$$\Rightarrow \quad (r+s)\,dr + (r-s)\,ds = 0$$

$$(r+s)\,dr = (s-r)\,ds$$

$$\frac{dr}{ds} = \frac{s-r}{r+s} = \frac{\frac{s}{r} - 1}{1 + \frac{s}{r}}$$

Let $v = \frac{s}{r}$

we get $\frac{dr}{ds} = \frac{v-1}{1+v} = F(v)$

homogeneous ODE

$$\frac{v-1}{1+v} = v + s\frac{dv}{ds}$$

$$s\frac{dv}{ds} = \frac{v-1}{1+v} - v$$

$$s\frac{dv}{ds} = \frac{\cancel{v} - 1 - \cancel{v} - v^2}{1+v}$$

$$-\frac{1+v}{(1+v^2)}\,dv = \frac{1}{s}\,ds$$

$$- \int \frac{(1+v)}{(1+v^2)}\, dv = \int \frac{1}{s}\, ds = \ln|s| + c$$

Let $v = \operatorname{Tan}\theta \Rightarrow \theta = \operatorname{Tan}^{-1} v$

$dv = \sec^2\theta \, d\theta$

$$- \int \frac{(1 + \operatorname{Tan}\theta)\cancel{(\sec^2\theta)}}{\cancel{\sec^2\theta}}\, d\theta = \ln|s| + c$$

$$- \int (1 + \operatorname{Tan}\theta)\, d\theta = \ln|s| + c$$

$$- \left(\theta \pm \ln|\cos\theta| \right) = \ln|s| + c$$

$$- \operatorname{Tan}^{-1} v + \ln|\cos(\operatorname{Tan}^{-1} v)| = \ln|s| + c$$

$$- \operatorname{Tan}^{-1} \frac{s}{r} + \ln\left|\cos\left(\operatorname{Tan}^{-1} \frac{s}{r}\right)\right| = \ln|s| + c$$

$$- \operatorname{Tan}^{-1} \left(\frac{\cancel{y+b}}{x-a}\right) + \ln\left|\cos \operatorname{Tan}^{-1}\left(\frac{y-b}{x-a}\right)\right| = \ln|y-b| + c$$

(5) (a) $\dfrac{dy}{dx} + 2y = e^{-x}$

$$e^{\int 2\,dx} = e^{2x}$$

$$e^{2x}\frac{dy}{dx} + 2ye^{2x} = e^{x}$$

$$\int \frac{d}{dx} ye^{x} = \int e^{x}$$

$$ye^{x} = \int e^{x}\,dx$$

$$ye^{x} = e^{x} + c$$

$$y = 1 + ce^{-x}$$

(b) $2\dfrac{dy}{dx} - y = e^{x/2}$

$$\frac{dy}{dx} - \frac{1}{2}y = \frac{1}{2}e^{x/2}$$

$$e^{-\int 1/2\,dx} = e^{-\frac{1}{2}x}$$

$$e^{-\frac{1}{2}x}\frac{dy}{dx} - \frac{1}{2}e^{-\frac{1}{2}x}y = \frac{1}{2}e^{\frac{x}{2}}e^{-\frac{1}{2}x}$$

$$\int \frac{dy}{dx}\left(ye^{-\frac{1}{2}x}\right) = \int \frac{1}{2}$$

$$ye^{-\frac{1}{2}x} = \frac{1}{2}x + c$$

$$y = \frac{1}{2}xe^{\frac{1}{2}x} + ce^{\frac{1}{2}x}$$

(c) $x\dfrac{dy}{dx} + 3y = \dfrac{\sin x}{x^2}$

$$\frac{dy}{dx} + \frac{3}{x}y = \frac{\sin x}{x^3}$$

$$e^{\int \frac{3}{x}dx} = e^{3\ln x} = x^3$$

$$x^3\frac{dy}{dx} + 3x^2y = \sin x$$

$$\int \frac{d}{dx}(x^3y) = \int \sin x \, dx$$

$$x^3y = -\cos x + c$$

$$y = -\frac{\cos x}{x^3} + \frac{c}{x^3}$$

(d) $x\,dy + y\,dx = y\,dy$

$$(x-y)\,dy + y\,dx = 0$$

$$(x-y)\,dy = -y\,dx$$

$$\frac{dy}{dx} = \frac{y}{y-x} = \frac{\frac{y}{x}}{\frac{y}{x}-1} = \frac{v}{v-1} = F(v)$$

$$\frac{v}{v-1} = v + x\frac{dv}{dx}$$

$$\frac{v - v^2 + v}{v-1} = x\frac{dv}{dx}$$

$$\frac{2v - v^2}{v-1} = x\frac{dv}{dx}$$

$$\int \frac{1}{x}\,dx = \int \frac{v-1}{2v-v^2}\,dv$$

$$= \int \left(-\frac{1}{2v} + \frac{1}{2(2-v)}\right) dv$$

$$\ln|x| + C = -\frac{1}{2}\ln|v| - \frac{1}{2}\ln|2-v|$$

$$2\ln|x| + 2C = -\left(\ln|2v - v^2|\right)$$

$$-\left(\ln\left|2\frac{y}{x} - \frac{y^2}{x^2}\right|\right) = 2\ln|x| + 2c$$

$$-\left(\ln\left|\frac{2y}{x} - \frac{y^2}{x^2}\right|\right) = 2\ln|x| + 2c$$

$$\ln\left|\frac{x^2}{2xy - y^2}\right| = 2\ln|x| + 2c$$

$$e^{\ln\left|\frac{1}{2xy-y^2}\right|} = e^{2c}$$

$$\left|\frac{1}{2xy - y^2}\right| = e^{2c}$$

$$\frac{1}{2xy - y^2} = k \;; \quad k = \pm e^{2c}$$

(e) $(x-1)^3 \frac{dy}{dx} + 4(x-1)^2 y = x+1$

$$\frac{dy}{dx} + \frac{4}{x-1} y = \frac{x+1}{(x-1)^3}$$

$$e^{\int \frac{4}{x-1} dx} = e^{4 \ln|x-1|} = (x-1)^4$$

$$(x-1)^4 \frac{dy}{dx} + 4(x-1)^3 y = (x+1)(x-1)$$

$$(x-1)^4 \frac{dy}{dx} + 4(x-1)^3 y = x^2 - 1$$

$$\int \frac{d}{dx}\left(y(x-1)^4\right) = \int (x^2 - 1) dx$$

$$y(x-1)^4 = \frac{x^3}{3} - x + c$$

$$y = \frac{x^3}{3(x-1)^4} - \frac{x}{(x-1)^4} + \frac{c}{(x-1)^4}$$

(6) (a) $2xy^3 dx + (3x^2y^2 + 2) dy = 0$

$M(x,y) = 2xy^3 \quad N(x,y) = 3x^2y^2 + 2$

$\frac{\partial M}{\partial y} = 6xy^2 \qquad \frac{\partial N}{\partial x} = 6xy^2$

since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ thus ODE is Exact

$f(x,y) = \int 2xy^3 \, dx$

$f(x,y) = x^2y^3 + g(y)$

$\frac{\partial f(x,y)}{\partial y} = 3x^2y^2 + g'(y) \equiv 3x^2y^2 + 2$

$\Rightarrow \quad g'(y) = 2 \quad \Rightarrow \int g'(y) = \int 2 \, dy$

$= \quad g(y) = 2y$

$\therefore \quad f(x,y) = x^2y^3 + 2y$

GOD BLESS YOU IN YOUR EXAMS