THE UNIVERSITY OF ZAMBIA
SCHOOL OF NATURAL SCIENCES
Department of Physics
2019/2020 ACADEMIC YEAR
PHY 2231 - Optics

CHAPTER 1: GEOMETRIC OPTICS (continued)

COLOUR DISPERSION

It is well known that refraction causes a separation of white light into its component colours. White light therefore when refracted give different values of ϕ'.

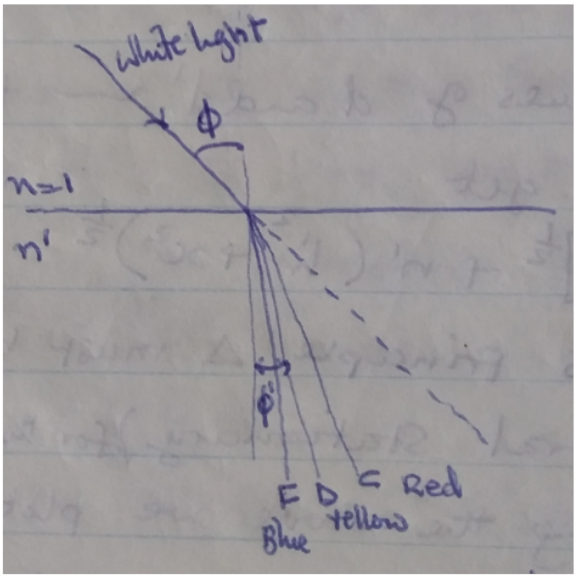
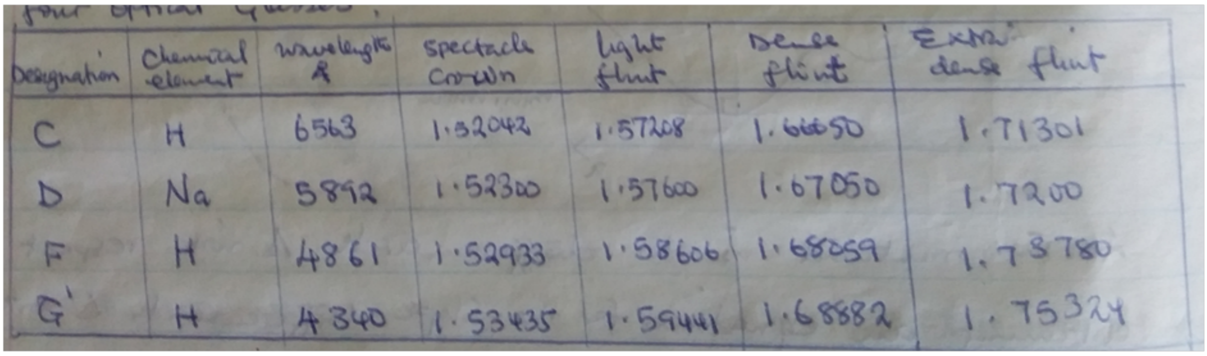


Figure 0.1

By Snells law $nsin\phi = n'sin\phi'$.
The table below shows Franhofers' designations, elements, source, wavelength and refractive indices for four optical glasses.



Designation	Chemical element	wavelength Å	spectacle Crown	light flint	Dense flint	Extra dense flint
C	H	6563	1.52042	1.57208	1.66650	1.71301
D	Na	5892	1.52300	1.57600	1.67050	1.7200
F	H	4861	1.52933	1.58606	1.68059	1.73780
G'	H	4340	1.53435	1.59441	1.68882	1.75324

Figure 0.2

For crown glass $n_F = 1.52933$, $n_D = 1.52300$, and $n_C = 1.52042$. For a given small angle ϕ, the dispersion of the F and the C rays $(\phi'_F - \phi'_C)$ is proportional to $n_F - n_C = 0.00891$, while the deviation of D ray $(\phi - \phi'_D)$ depends only on $n_D - 1 = 0.52300$.
To define the dispersive power of the glass we use

$$V = \frac{n_F - n_C}{n_D - 1} \tag{0.1}$$

The reciprocal of this value is called the refractive index ν

$$\nu = \frac{n_D - 1}{n_F - n_C}. \tag{0.2}$$

For most glasses ν ranges between 20 and 60.

SHERICAL REFRACTIVE SURFACES

Consider the figure below. SPM is the refracting surface. OPQ are on a straight line. Image may be formed any where between C and Q. Calculate the optical path length OSQ. Assume θ is small. From the triangle SOC we have

$$OS = [(x+r)^2 + r^2 - 2(x+r)rcos\theta]^{\frac{1}{2}} = [x^2 + 2xr + 2r^2 - 2(xr + r^2)cos\theta]^{\frac{1}{2}}. \tag{0.3}$$

But $cos^2\theta = 1 - sin^2\theta \rightarrow cos\theta = (1 - sin^2\theta)^{\frac{1}{2}} = (1 - \theta^2)^{\frac{1}{2}}$ since θ is small. We expand $(1 - \theta^2)^{\frac{1}{2}}$ using Binomial theorem and obtain

$(1-\theta^2)^{\frac{1}{2}} \approx (1-\frac{\theta}{2})$.

$$\therefore OS \approx \left[x^2 + 2xr + 2r^2 - 2(xr + r^2)\left(1 - \frac{\theta^2}{2}\right)\right]^{\frac{1}{2}}. \tag{0.4}$$

$$\therefore OS \approx \left[x^2 + 2xr + 2r^2 - (2xr + 2r^2)\left(1 - \frac{\theta^2}{2}\right)\right]^{\frac{1}{2}}. \tag{0.5}$$

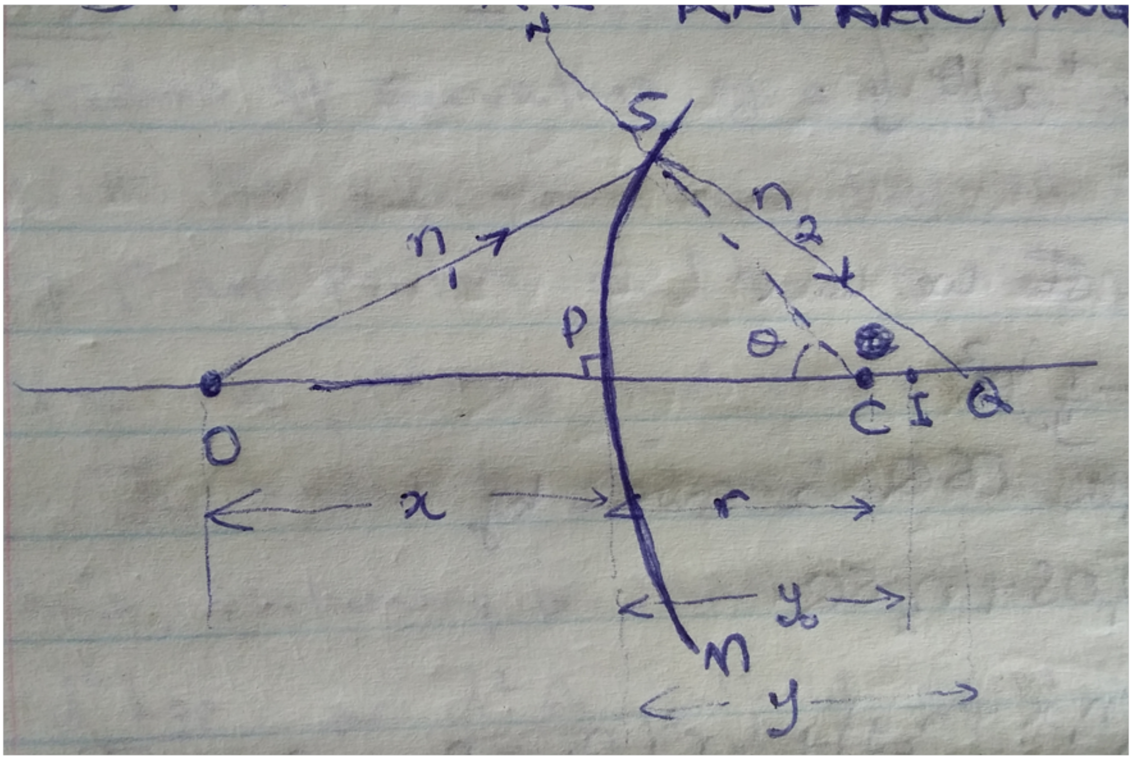


Figure 0.3

$$OS \approx \left[x^2 + 2xr + 2r^2 - (2xr - \frac{2xr\theta^2}{2} + 2r^2 - \frac{\theta^2 r^2}{2})\right]^{\frac{1}{2}}. \tag{0.6}$$

$$OS \approx \left[x^2 + xr\theta^2 + \theta^2 r^2\right]^{\frac{1}{2}} = x\left[1 + \frac{\theta^2 r}{x} + \frac{\theta^2 r^2}{x^2}\right]^{\frac{1}{2}}. \tag{0.7}$$

$$OS \approx x\left[1 + \frac{\theta^2 xr + \theta^2 r^2}{x^2}\right]^{\frac{1}{2}}. \tag{0.8}$$

But $\frac{\theta^2 xr+\theta^2 r^2}{x^2}$ is very small hence we can expand it using binomial theorem $(1+x)^n$ for n<1 is given by:
$(1+x)^n = 1+nx+\frac{n(n-1)x^2}{2!}+\frac{n(n-1)(n-2)x^3}{3!}+....$

$$OS \approx x\left[1+\frac{1r^2}{2x}\left(\frac{1}{r}+\frac{1}{x}\right)\theta^2\right]. \tag{0.9}$$

$$OS \approx \left[x+\frac{1}{2}r^2\left(\frac{1}{r}+\frac{1}{x}\right)\theta^2\right]. \tag{0.10}$$

For triangle SCQ we have

$$SQ \approx y-\frac{1}{2}r^2\left(\frac{1}{r}+\frac{1}{y}\right)\theta^2. \tag{0.11}$$

The optical path length OSQ is given by

$$\Delta = L_{op} = n_1 OS + n_2 SQ \approx (n_1 x + n_2 y) + \frac{1}{2}r^2\left[\frac{n_1}{x}+\frac{n_2}{y}-\frac{n-2-n_1}{r}\right]\theta^2 \tag{0.12}$$

For optical path to be an extremum we must have

$$\frac{d\Delta}{d\theta} = \frac{dL_{op}}{d\theta} = 0 \tag{0.13}$$

because θ is the one which is changing with Δ.

$$\therefore \frac{d\Delta}{d\theta} = \frac{dL_{op}}{d\theta} = r^2\left[\frac{n_1}{x}+\frac{n_2}{y}-\frac{n-2-n_1}{r}\right]\theta = 0 \tag{0.14}$$

$$\rightarrow r^2\left[\frac{n_1}{x}+\frac{n_2}{y}-\frac{n-2-n_1}{r}\right]\theta = 0 \tag{0.15}$$

If θ is zero, the light coming from a point O reaches Q without deviating (Snells Law).
If y was such that the quantity inside the square brackets was zero i.e if y was equal to y_0 such that $\frac{n_1}{x}+\frac{n_2}{y}=\frac{n-2-n_1}{r}$, then $\frac{d\Delta}{d\theta}=\frac{dL_{op}}{d\theta}$ would vanish for all θ. This is called paraxial approximation.
If the point I corresponds to $PI = y_0$ then ALL the paths like OSI are allowed ray paths implying that all paraxial rays leaving O will pass through I and I will represent the paraxial image point.

$\frac{n_2}{y_0}+\frac{n_1}{x}=\frac{n-2-n_1}{r}$ $(\frac{n_2}{v}-\frac{n_1}{u}=\frac{n-2-n_1}{R})$ is called the Lens equation, where $v=y_0$ is the image distance, $u=x$ is the object distance and $R=r$ is the radius of curvature with the sign convention that all distances measured to the right of P are positive and those to the left of P are negative. Therefore u=-x, v=+y and r = +R. This Lens equation is a particular form of the equation determining paraxial image point.
In order to determine whether OPQ corresponds to minimum time, stationary or maximum time, we must determine the sign of $\frac{d^2\Delta}{d\theta^2}$ or $\frac{d^2L_{op}}{d\theta^2}$ which gives

$$\frac{d^2\Delta}{d\theta^2}=\frac{d^2L_{op}}{d\theta^2}=r^2\left[\frac{n_1}{x}+\frac{n_2}{y}-\frac{n_2-n_1}{r}\right]=r^2n_2\left[\frac{1}{y}-\frac{1}{y_0}\right] \tag{0.16}$$

because for paraxial image $-x=r$ and for the minimum distance travelled $r=y_0$
If $y>y_0$ then $\frac{d^2\Delta}{d\theta^2}$ is negative. If $y=y_0$, $\frac{d^2\Delta}{d\theta^2}$ vanishes.
If I is the paraxial image point of P then $n_1OP+n_2PI=n_1OS+n_2SI$.
If Q lies on the right of the point I then we have $n_1OP+n_2PQ=$ $n_1OS+n_2(SI-PI+PQ)\rightarrow n_1OS+n_2(SI+IQ)>n_1OS+n_2SQ$
This means that ray OPQ corresponds to a maximum.

REFRACTION OF RAYS AT INTERFACE BETWEEN AN ISOTROPIC MEDIUM AND ANISOTROPIC MEDIUM

In isotropic medium, the properties remain the same in all directions, while in anisotropic medium, some of the properties such as speed of light may vary in different directions.
The optical path from A to B is

$$L_{OP}=n_1AP+n(\theta)PB=n_1[h_1^2+(L-x)^2]^{1/2}+n(\theta)[h_2^2+x^2]^{1/2} \tag{0.17}$$

where

$$n^2(\theta)=[n_o^2cos^2\theta+n_e^2sin^2\theta]. \tag{0.18}$$

According to our diagram

$$cos\theta=\frac{h_2}{(h_2^2+x^2)^{1/2}}, sin\theta=\frac{x}{(h_2^2+x^2)^{1/2}}. \tag{0.19}$$

$$n(\theta)=[n_o^2cos^2\theta+n_e^2sin^2\theta]^{1/2}. \tag{0.20}$$

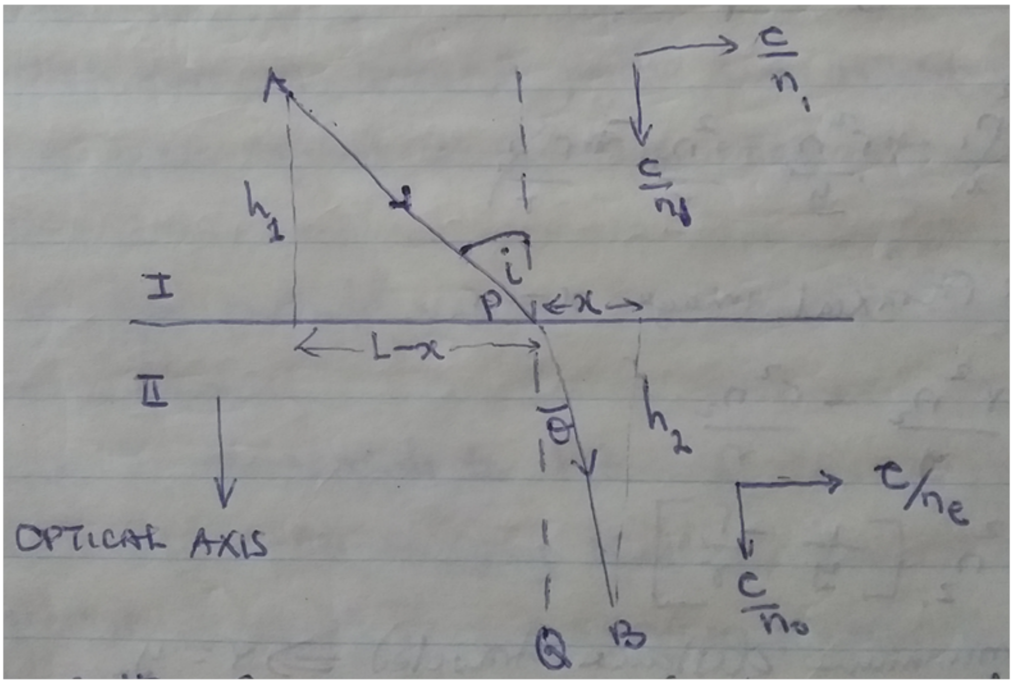


Figure 0.4

$$\therefore L_{OP} = n_1[h_1^2 + (L-x)^2]^{1/2} + [n_o^2 h_2^2 + n_e^2 x^2]^{1/2}. \tag{0.21}$$

For the actual ray path

$$\frac{dL_{OP}}{d\theta} = \frac{n_1(L-x)}{[h_1^2+(L-x)^2]^{1/2}} - \frac{n_e^2 x}{[n_o^2 h_2^2 + n_e^2 x^2]^{1/2}} = 0. \tag{0.22}$$

$$\frac{n_1(L-x)}{[h_1^2+(L-x)^2]^{1/2}} = \frac{n_e^2 x}{[n_o^2 h_2^2 + n_e^2 x^2]^{1/2}} \tag{0.23}$$

$$n_1 sini = \frac{n_e^2 tan\theta}{[n_o^2 + n_e^2 tan^2\theta]^{1/2}} \tag{0.24}$$

where $sini = \frac{(L-x)}{[h_1^2+(L-x)^2]^{1/2}}$ and $tan\theta = \frac{x}{h_2}$. We have $tanr = \frac{n_o n_1 sini}{n_e\sqrt{n_e^2 - n_1^2 sin^2 i}}$
or $r = tan^{-1}\left[\frac{n_o n_1 sini}{n_e\sqrt{n_e^2 - n_1^2 sin^2 i}}\right]$
As an example, consider $n_1 = 1.000$, then $r = tan^{-1}\left[\frac{n_o sini}{n_e\sqrt{n_e^2 - sin^2 i}}\right]$
For calcite $n_e = 1.48641$ and $n_o = 1.65838$; let i = 45°.
This gives us $r \approx 31.1^\circ$

If $n_o = n_e = n_2$, then $tanr = \frac{n_o n_1 sini}{n_e\sqrt{n_e^2 - n_1^2 sin^2 i}}$ becomes $tanr = \frac{n_1 sini}{\sqrt{n_2^2 - n_1^2 sin^2 i}}$.

$$sin^2 r(n_2^2 - n_1^2 sin^2 i) = n_1^2 sin^2 i cos^2 r \tag{0.25}$$

$$n_2^2 sin^2 r - n_1^2 sin^2 i sin^2 r = n_1^2 sin^2 i cos^2 r \tag{0.26}$$

$$n_2^2 sin^2 r = n_1^2 sin^2 i sin^2 r + n_1^2 sin^2 i cos^2 r \tag{0.27}$$

$$n_2^2 sin^2 r = n_1^2 sin^2 i(sin^2 r + cos^2 r) \tag{0.28}$$

$$n_2^2 sin^2 r = n_1^2 sin^2 i \tag{0.29}$$

Take the square root on both sides then

$$n_2 sinr = n_1 sini \tag{0.30}$$

which is the Snell's law

LENSES

REFRACTION AND REFLECTION AT SPHERICAL SURFACES

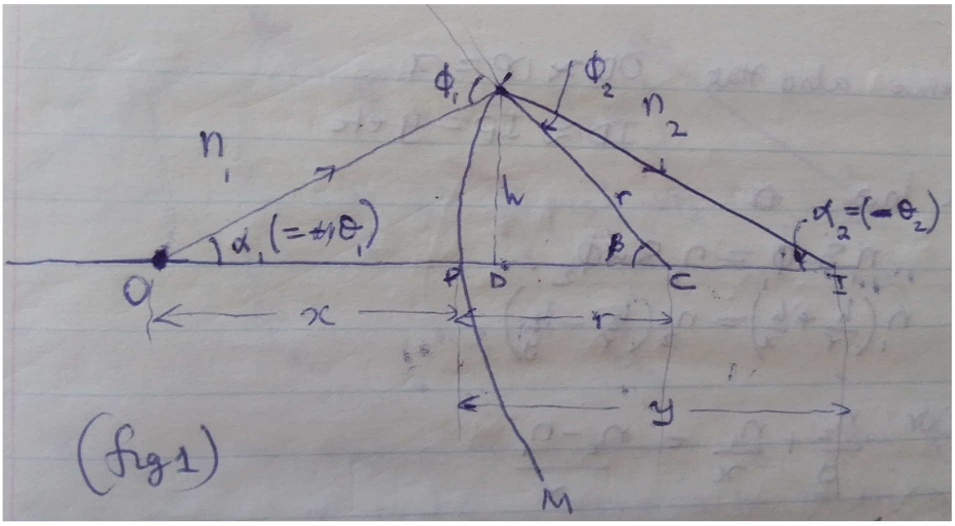


Figure 0.5

Consider the diagram above.

For this type of surface, we use the following convention. u<0, v>0, R>0.

$$n_1 sin\phi_1 = n_2 sin\phi_2 \tag{0.31}$$

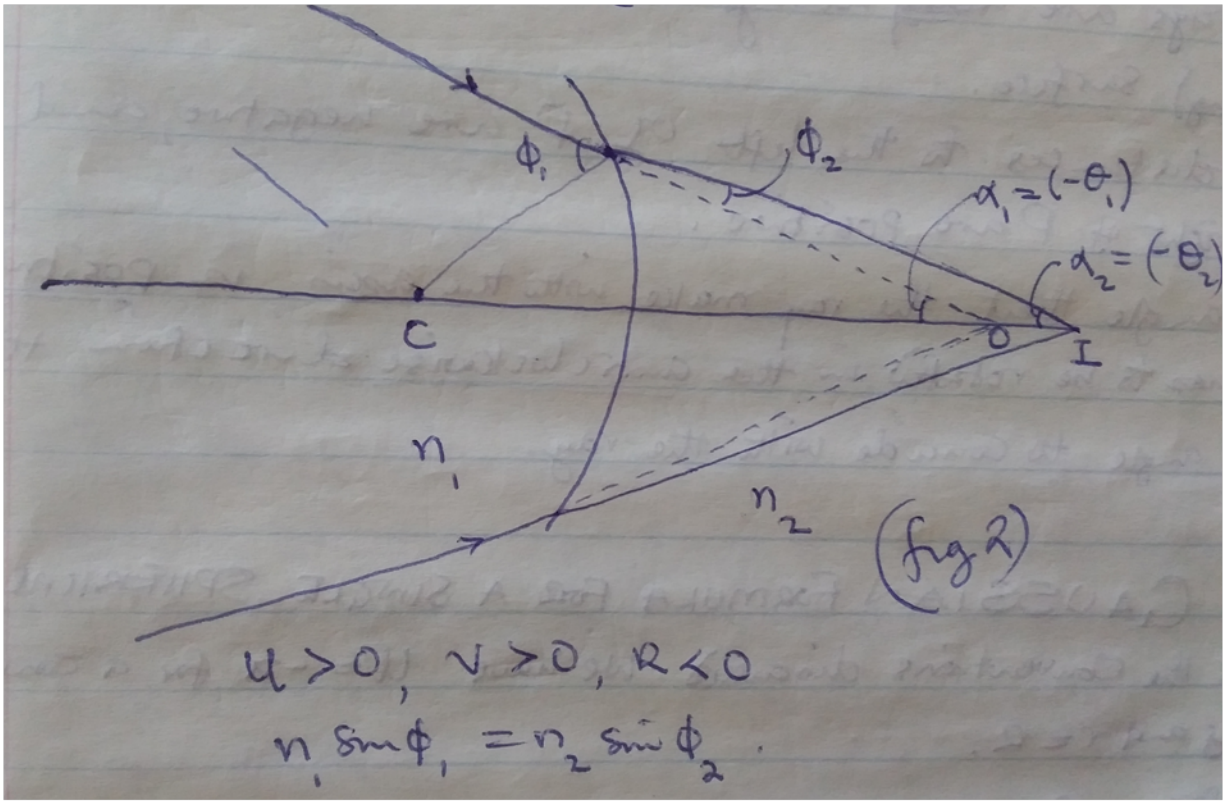


Figure 0.6

In terms of angles (figure 1), $\phi_1 = \beta + \alpha_1$ and $\phi_2 = \beta - \alpha_2$. We make use of the paraxial approximation that angles are small hence $sin\phi \approx tan\phi$ etc. where the angles are measured in radians.

$$sin\phi_1 \approx \phi_1 = \beta + \alpha_1 \approx tan\beta + tan\alpha_1 \approx \frac{h}{r} + \frac{h}{x} \tag{0.32}$$

$$sin\phi_2 \approx \phi_2 = \beta - \alpha_2 \approx tan\beta - tan\alpha_2 \approx \frac{h}{r} - \frac{h}{y} \tag{0.33}$$

We assume also that $OD \approx OP = x$, $ID \approx IP = y$ etc.
By Snells law $n_1 sin\phi_1 = n_2 sin\phi_2$. Therefore, $n_1\left(\frac{h}{r} + \frac{h}{x}\right) = n_2\left(\frac{h}{r} - \frac{h}{y}\right)$
or $\frac{n_2}{y} + \frac{n_1}{x} = \frac{n_2 - n_1}{r}$

Convention

- Point P is considered as the origin of the coordinate system.
- The rays are always incident from the left on the refracting (reflecting) surface.
- All distances to the left of P are negative, and all distances to the right of P are positive.
- The angle that the ray make with the axis is positive if the axis has to be rotated in the anticlockwise direction through the acute angle to coincide with the ray.

THE GAUSSIAN FORMULA FOR A SINGLE SPHERICAL SURFACE

With the conventions discussed we have u=x for converging lens v=y and r=R.
Therefore

$$\frac{n_2}{y}+\frac{n_1}{x}=\frac{n_2-n_1}{r} \tag{0.34}$$

becomes

$$\frac{n_2}{v}-\frac{n_1}{u}=\frac{n_2-n_1}{R}. \tag{0.35}$$

This gives the image point due to refraction at a spherical surface. The equation $\frac{n_2}{v}-\frac{n_1}{u}=\frac{n_2-n_1}{R}$ is known as the Gaussian formular for a single spherical surface and is the exact form of that derived using Fermat's principle.

REFLECTION BY SINGLE SHERICAL SURFACE

Consider the figure below.

$$\phi_1=\beta-\alpha_1\approx\frac{h}{r}-\frac{h}{x}. \tag{0.36}$$

and

$$\phi_2=\beta-\alpha_2\approx\frac{h}{y}-\frac{h}{r}. \tag{0.37}$$

But $\phi_1=\phi_2$

$$\therefore\frac{h}{r}-\frac{h}{x}=\frac{h}{y}-\frac{h}{r}. \tag{0.38}$$

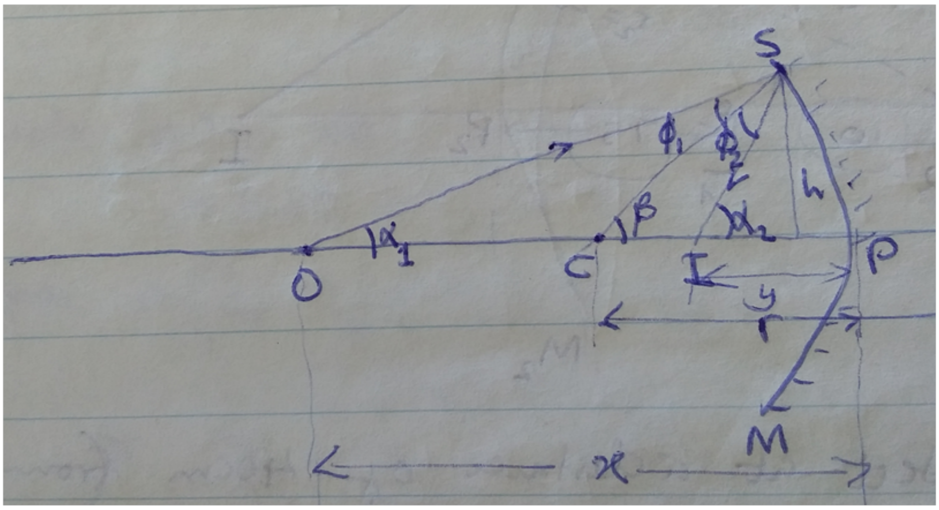


Figure 0.7

or

$$\frac{1}{x} + \frac{1}{y} = \frac{2}{r}. \tag{0.39}$$

Let $x = -u, y = -v$ and $r = R$

$$\therefore \frac{1}{u} + \frac{1}{v} = \frac{2}{R}. \tag{0.40}$$

If we set $n_2 = -n_1$ in $\frac{n_2}{v} - \frac{n_1}{u} = \frac{n_2 - n_1}{R}$, we get $\frac{1}{u} + \frac{1}{v} = \frac{2}{R}$. This follows from Snells law of refraction $\frac{sin\phi_1}{sin\phi_2} = \frac{n_2}{n_1}$. Therefore, for reflection we set $n_1 = n_2$.

THIN LENSES

A medium bounded by two spherical reflecting surfaces is refered to as a spherical lens. If the thickness of such a lens (t) is very small compared to object and image distances and the radii of curvature of the reflecting surfaces, then a lens is refered to as a thin lens.

$u = -x, v' = y', v = y$

R_1 and R_2 are the radii of curvature of the left and right surfaces of the lens.

If the second refracting surface was not present, then the image of the point object O would be formed at Q (the image distance v') and determined by $\frac{n_2}{v'} - \frac{n_1}{u} = \frac{n_2 - n_1}{R_1}$. For the second surface $\frac{n_1}{v} - \frac{n_2}{v'} = \frac{n_1 - n_2}{R_2}$, where v' is the object distance for the second surface. Adding the two equations, we obtain

$\frac{n_2}{v'} - \frac{n_1}{u} + \frac{n_1}{v} - \frac{n_2}{v'} = \frac{n_2 - n_1}{R_1} + \frac{n_1 - n_2}{R_2} \rightarrow n_1(\frac{1}{v} - \frac{1}{u}) = \frac{n_2 - n_1}{R_1} + \frac{n_1 - n_2}{R_2}$. Dividing

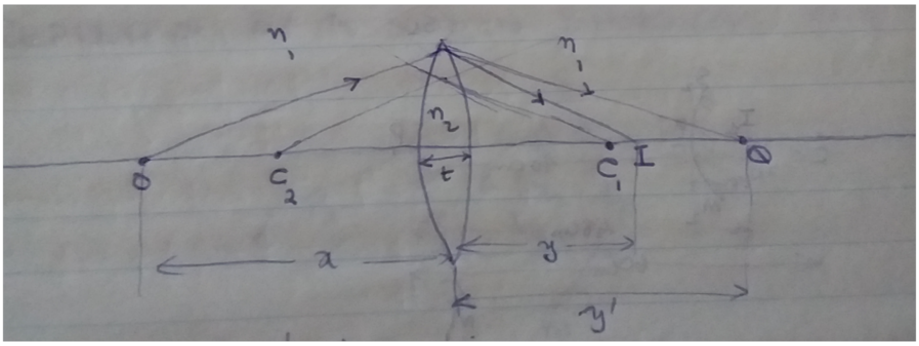


Figure 0.8

through out by n_1 we get $\frac{1}{v} - \frac{1}{u} = \frac{n_2}{n_1 R_1} - \frac{1}{R_1} + \frac{1}{R_2} - \frac{n_2}{n_1 R_2}$. If we let $\frac{n_2}{n_1} = n$, then we get $\frac{1}{v} - \frac{1}{u} = (n-1)(\frac{1}{R_1} - \frac{1}{R_2})$, which is the lens formular and is usually written as $\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$ where $\frac{1}{f} = (n-1)(\frac{1}{R_1} - \frac{1}{R_2})$. f is the focal length of the lens.

THE CRITICAL ANGLE AND TOTAL INTERNAL REFLECTION

The critical angle for a boundary separating two optical media is defined as the smallest angle of incidence in the medium of greater refractive index for which the light is totally reflected.

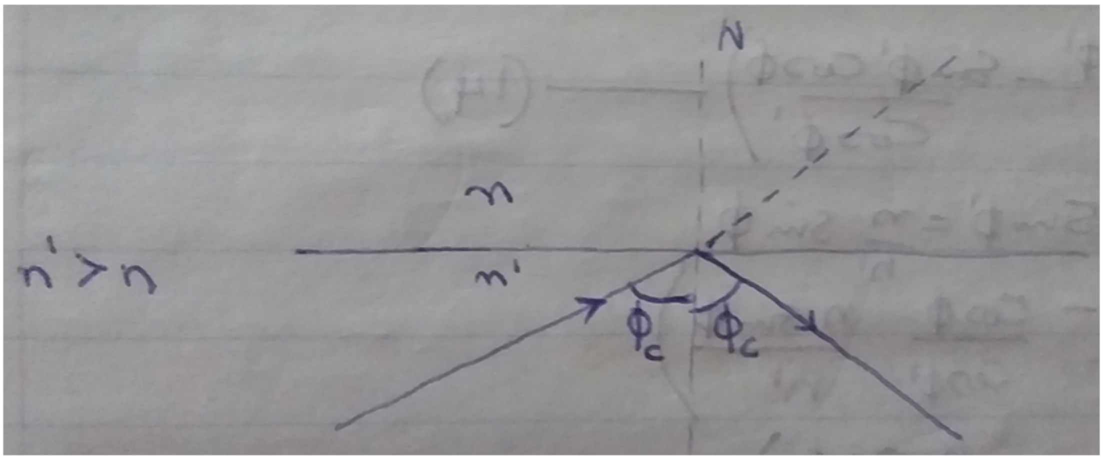


Figure 0.9

To calculate the critical angle, we use Snells law

$$nsin\phi = n'sin\phi' \tag{0.41}$$

For $\phi = 90$, $sin\phi = 1$.

$$\therefore nsin\phi = n'sin\phi' \tag{0.42}$$

or

$$sin\phi_c = \frac{n}{n'} \tag{0.43}$$

where $sin\phi' = sin\phi_c$.
Application of total internal reflection is also found in total reflection prisms.

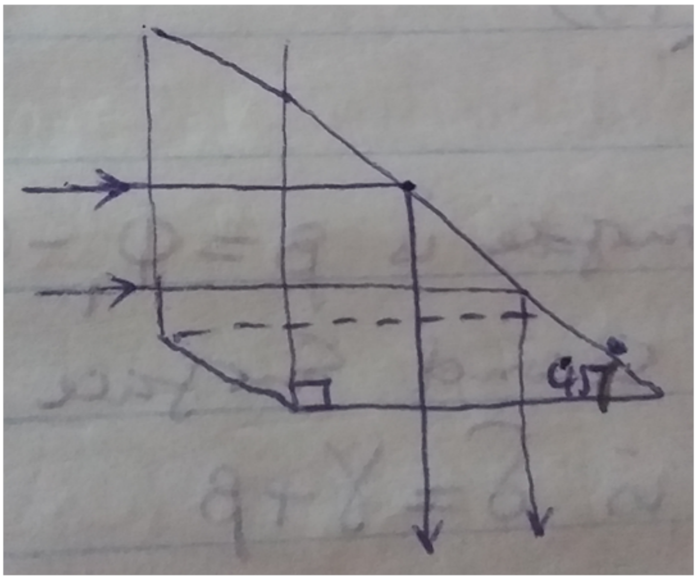


Figure 0.10

PLANE PARALLEL PLATE

From triangle ABE, $d = lsin(\phi - \phi')$ where $l=$ distance AB.
$d = l(sin\phi cos\phi' - sin\phi' cos\phi)$.
From triagle ABC, we have $lcos\phi' = t$ or $l = \frac{t}{cos\phi'}$

$$\therefore d = l(sin\phi cos\phi' - sin\phi' cos\phi) = \frac{t}{cos\phi'}(sin\phi cos\phi' - sin\phi' cos\phi) \tag{0.44}$$

From Snells law $sin\phi' = \frac{n}{n'}sin\phi$.

$$\therefore d = t\left(sin\phi - \frac{cos\phi}{cos\phi'}\frac{n}{n'}sin\phi\right). \tag{0.45}$$

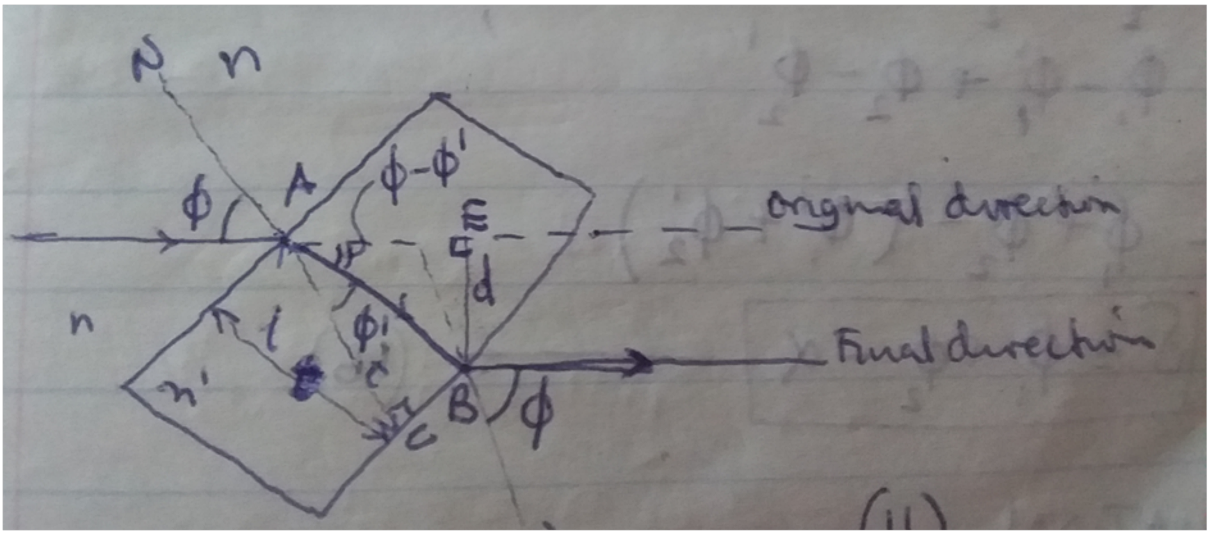


Figure 0.11

$$\therefore d = t sin\phi\left(1-\frac{n}{n'}\frac{cos\phi}{cos\phi'}\right). \tag{0.46}$$

REFRACTION BY PRISM

In a prism the two faces are inclined at a certain angle α.

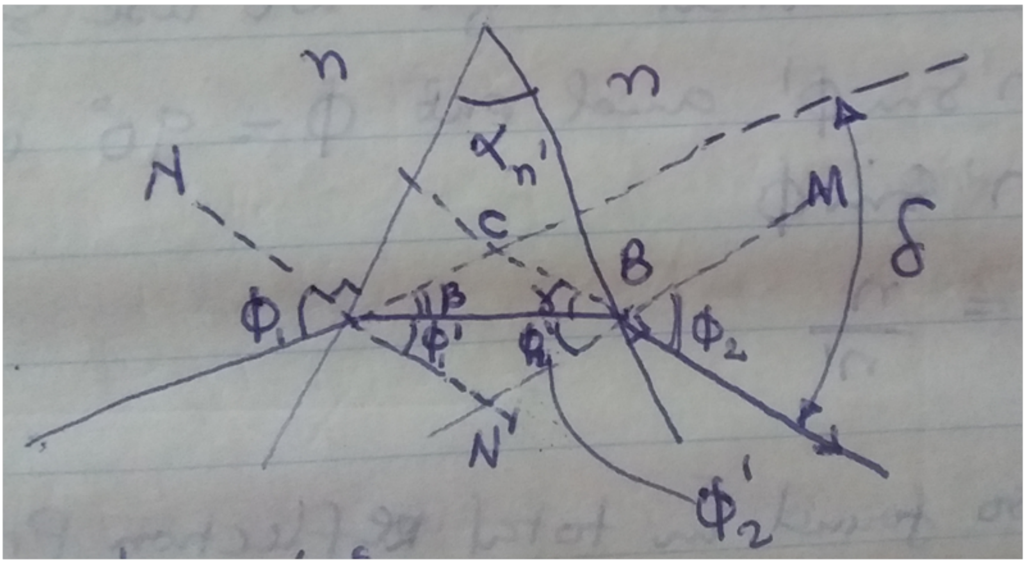


Figure 0.12

$$\frac{sin\phi_1}{sin\phi_1'} = \frac{n'}{n} = \frac{sin\phi_2}{sin\phi_2'} \tag{0.47}$$

The deiviation produced by first surface is $\beta = \phi_1 - \phi_1'$; while deviation produced by the second surface is $\gamma = \phi_2 - \phi_2'$. Total deviation by the two surfaces is $\delta = \gamma + \beta$.
According to geometry,

$$\alpha = \phi_1' + \phi_2' \tag{0.48}$$

With this, then $\delta = \beta + \gamma = \phi_1 - \phi_1' + \phi_2 - \phi_2' = \phi_1 + \phi_2 - (\phi_1' + \phi_2')$

$$\therefore \delta = \phi_1 + \phi_2 - \alpha \tag{0.49}$$

MINIMUM DEVIATION
When the total angle of deviation δ for a given prism is calculated by using the formular $\delta = \phi_1 + \phi_2 - \alpha$, δ varies considerably with the angle of incidence.

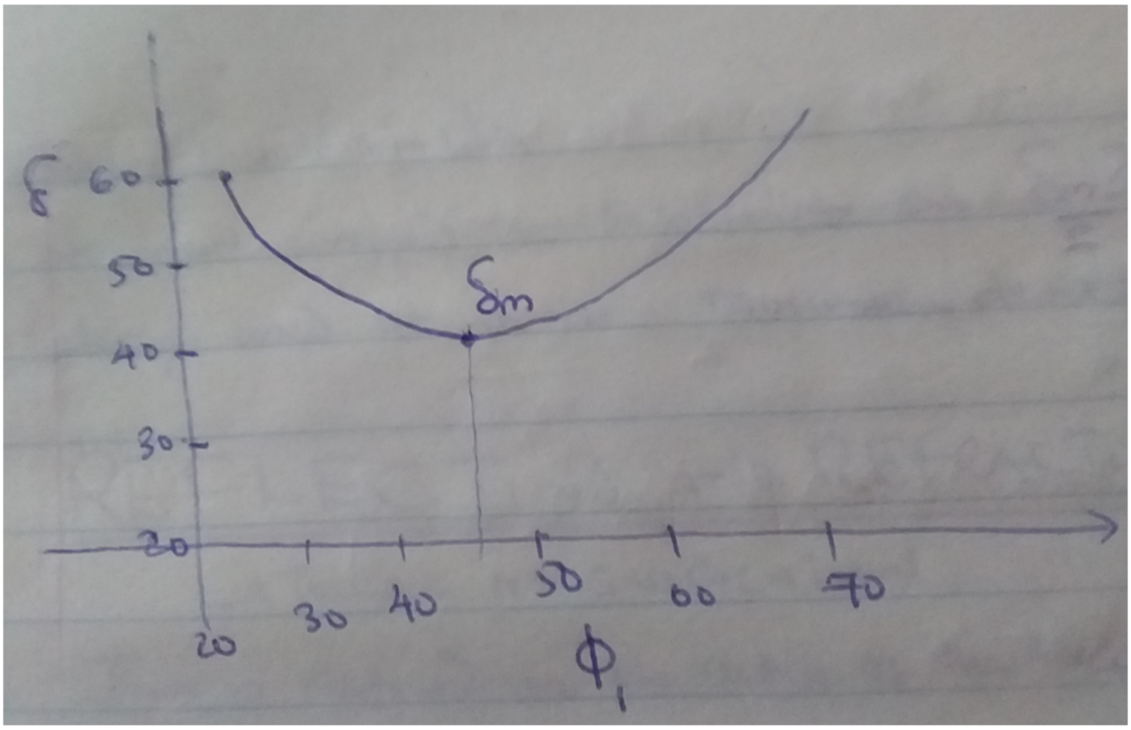


Figure 0.13: *A graph of deviation produced by 60° glass prism of refractive index 1.50.*

At minimum deviation $\delta_m = 37.2, \phi_1 = 48.6$ and $\phi_1' = 30$.
δ_m occurs at a particular value of the angle of incidence where the ray inside the prism makes equal angles with the two faces of the prism.
In this special case

$$\phi_1 = \phi_2, \phi_1' = \phi_2', \beta = \gamma \tag{0.50}$$

To prove that this is the case, assume $\phi_1 \neq \phi_2$.

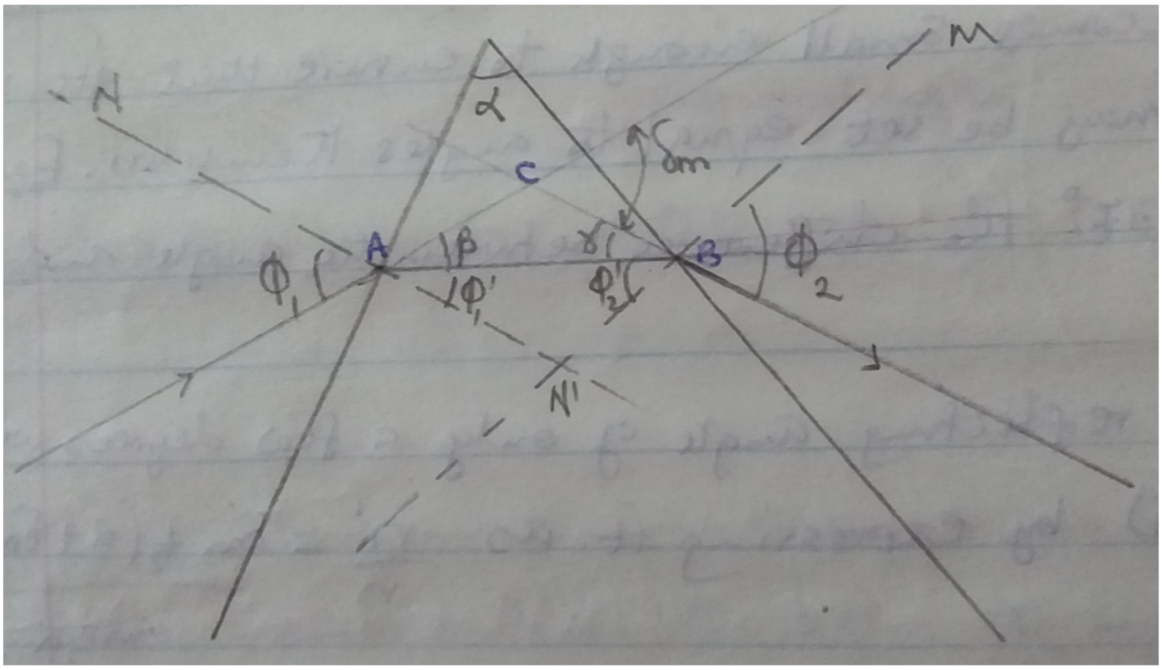


Figure 0.14

In the triangle ABC, the exterior angle δ_m equals the sum of opposite angles $\beta + \gamma$. Similarly for $\Delta ABN'$ $\alpha = \phi'_1 + \phi'_2$.
Therefore, $\alpha = 2\phi'_1$, $\delta_m = 2\beta$ and $\phi_1 = \phi'_1 + \beta$
Solve for ϕ'_1 and ϕ_1 to get $\phi'_1 = \frac{1}{2}\alpha$

$$\phi_1 = \frac{1}{2}\alpha + \beta, \beta = \frac{\delta_m}{2} \tag{0.51}$$

$$\therefore \phi_1 = \frac{1}{2}\alpha + \frac{\delta_m}{2} = \frac{1}{2}(\alpha + \delta_m) \tag{0.52}$$

According to Snells law

$$\frac{n'}{n} = \frac{sin\phi_1}{sin\phi'_1} \tag{0.53}$$

$$\frac{n'}{n} = \frac{sin\frac{1}{2}(\alpha + \delta_m)}{sin\frac{1}{2}\alpha} \tag{0.54}$$

Therefore if we know n, α and we can measure δ_m, we can determine the refractive index of a second medium.

THIN PRISMS

The equation for the prism become much simpler when the refracting angle α becomes small enough to ensure that its sine of the deviation δ may be set equal to angles themselves.
For prisms having a reflecting angle of only a few degrees we can simplify $\frac{n'}{n} = \frac{sin\frac{1}{2}(\alpha+\delta_m)}{sin\frac{1}{2}\alpha} = \frac{\alpha+\delta_m}{\alpha}$.

$$\therefore \alpha n' = (\alpha + \delta_m)n \tag{0.55}$$

$$\delta_m n + \alpha n = \alpha n' \tag{0.56}$$

or

$$\delta_m n = \alpha n' - \alpha n \tag{0.57}$$

$$\delta_m = \alpha\left(\frac{n'}{n} - 1\right) \tag{0.58}$$

or

$$\delta_m = \left(\frac{n'}{n} - 1\right)\alpha \tag{0.59}$$

If the surrounding medium is air, then $n = 1.000$, then $\delta = (n-1)\alpha$. We have dropped the subscript on δ because such prisms are always used at or near minimum deviation.

REFLECTING AND REFRACTING CURVED SURFACES

LATERAL MAGNIFICATION

Lateral magnification m is defined as the ratio of the height of the image to that of the object.

$$m = \frac{y'}{y} = \frac{v}{u} = \frac{f_2 + x_2}{f_1 + x_1} = -\frac{f_1}{x_1} = -\frac{x_2}{f_2} \tag{0.60}$$

which is the Newtons formular, where f_1 is the first focal length and f_2 is the second focal length.
Consider triangle AOC and ICB. We get $\frac{-y'}{y} = \frac{v-R}{-u+R} \to \frac{\frac{v}{R}-1}{\frac{-u}{R}+1}$.
But $\frac{n_2}{v} - \frac{n_1}{u} = \frac{n_2-n_1}{R}$. Dividing this equation by n_1 and then multiplying by v we get $\frac{n_2}{n_1} - \frac{v}{u} = \frac{n_2-n_1}{n_1}\frac{v}{R}$. Dividing $\frac{n_2}{v} - \frac{n_1}{u} = \frac{n_2-n_1}{R}$ by n_2 and then multiplying by u we get $\frac{u}{v} - \frac{n_1}{n_2} = \frac{n_2-n_1}{n_2}\frac{u}{R}$.

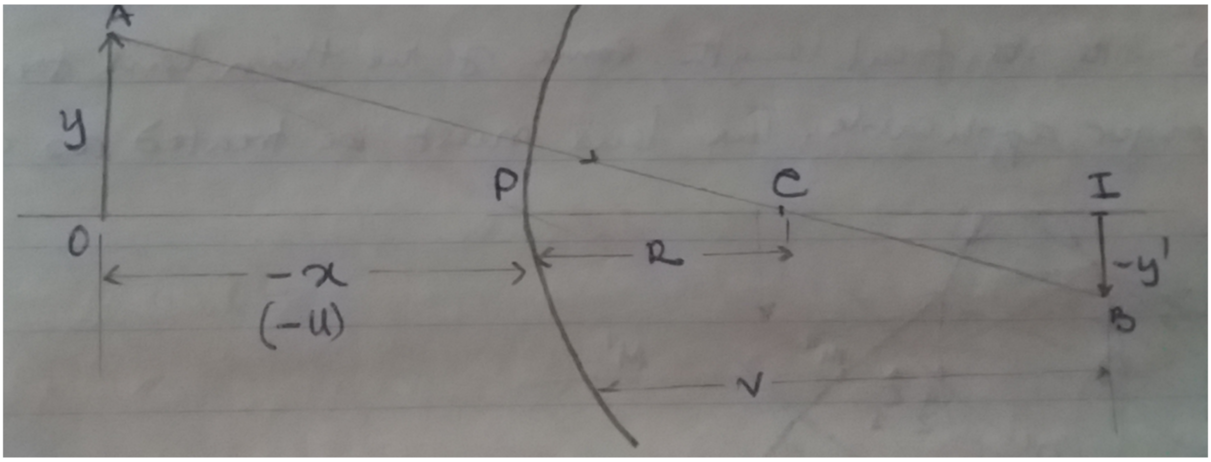


Figure 0.15

Substituting for $\frac{v}{R}$ and $\frac{u}{R}$ in $\frac{-y'}{y} = \frac{v-R}{-u+R} \to \frac{\frac{v}{R}-1}{\frac{-u}{R}+1}$ we get $m = \frac{y'}{y} = \frac{n_1 v}{n_2 u}$ for a single surface.

If m_1 and m_2 represents the magnifications produced by two refracting surfaces in the converging lens diagram, then $m_1 = \frac{n_1 v'}{n_2 u}$ and $m_2 = \frac{n_2 v}{n_1 v'}$. The total magnification is $m = m_1 m_2 = \frac{v}{u}$

THICK LENSES

Each surface acts as an image forming component contributes to the final image formed by the system as a whole. Let n, n', and n'' be the refractive indices as shown by radii of curvatures r_1 and r_2.

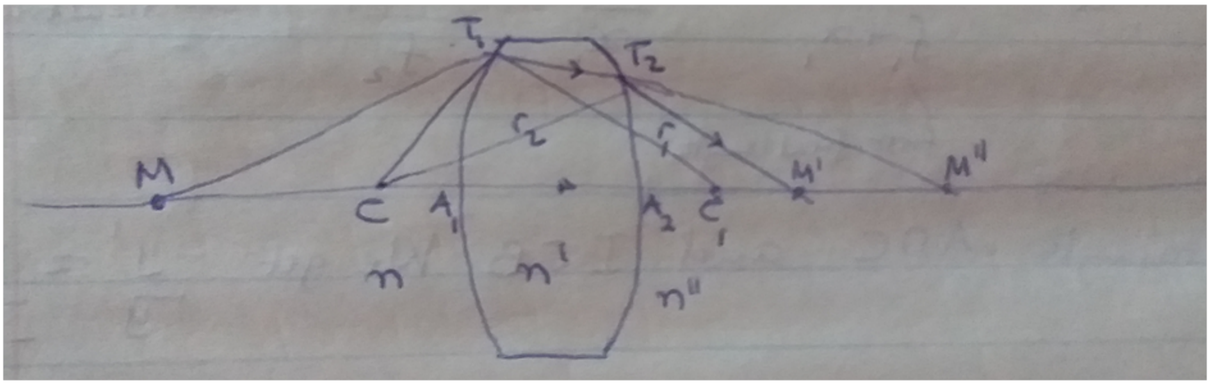


Figure 0.16

For the first surface

$$\frac{n}{s_1} + \frac{n'}{s_1'} = \frac{n'-n}{r_1} \to \frac{n}{u} + \frac{n'}{v} = \frac{n'-n}{r_1} \tag{0.61}$$

For the second surface

$$\frac{n'}{s'_2} + \frac{n''}{s''_2} = \frac{n'' - n'}{r_2} \tag{0.62}$$

ABERRATIONS

Aberrations can be described as problems associated with the sharpness/position of actual images formed by lenses/prisms/curved refracting and reflecting surfaces.
The departure of the actual images from the ideal images lead to what is known as aberrations. There are five coefficients for specifying primary aberrations for any symmetric system. The coefficients represent the:

- Spherical aberrations
- Comma
- Astigmation
- Curvature of field and
- Distortion

These represent what are known as Seidel aberrations.
Because aberrations are present even for light of a single wavelength, they are also called monochromatic aberrations.
If a polychromatic source (white light) is used for image formation then in general, the image will be coloured. This effect is called chromatic aberration. Physically, chromatic aberration is due to the dependence of the refractive index of a material on different colours of light.
Since the image formation is accompanied by refraction at refractive index discontinuities, the wavelength dependence of refractive index results in coloured image.

CHROMATIC ABERRATION

Consider a parallel beam of white light incident on a thin convex lens. Since the blue light gets refracted more than red light, the point at which the blue light would focus is nearer the lens than the point at which the red light would focus.
For the case of the thin lens, we found out that $\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_1}\right)$.
If a change of n occurs by δn (the change of n is due to the change in wavelength of the light) then this results in change of f by δf.

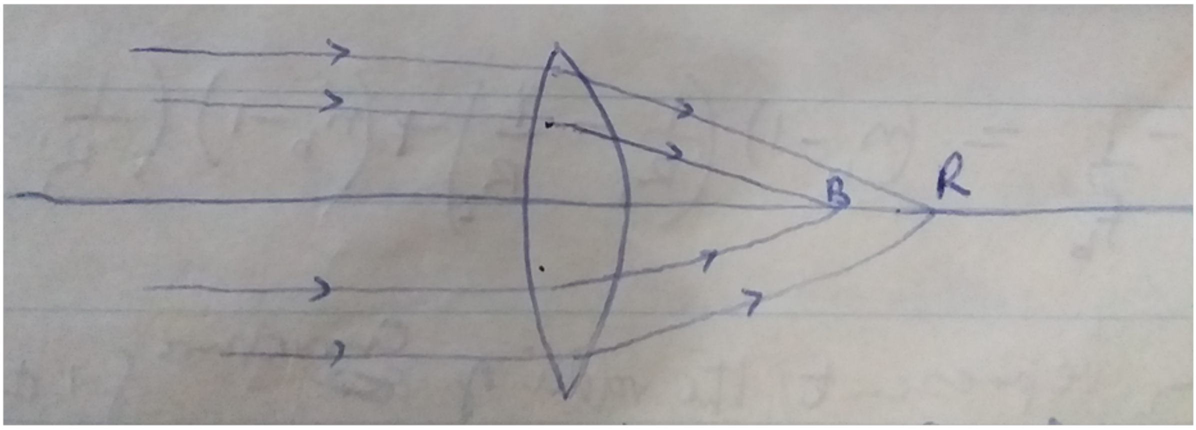


Figure 0.17

We must have

$$\frac{\delta f}{f^2} = \delta n \left(\frac{1}{R_1} - \frac{1}{R_2} \right) \tag{0.63}$$

But $\frac{1}{R_1} - \frac{1}{R_2} = \frac{1}{f(n-1)}$.

$$\therefore \frac{-\delta f}{f^2} = \frac{\delta n}{f(n-1)}. \tag{0.64}$$

So $\delta f = \frac{-f\delta n}{\frac{1}{(n-1)}}$ which represents the chromatic aberration of a thin lens. If n_b and n_r represent the refractive indices for blue and red colours respectively, then $f_r - f_b = f\left(\frac{n_b - n_r}{n-1}\right)$.

To rectify chromatic aberration one uses what is known as the **chromatic doublet**.

Consider an optical system of two thin lenses made of different materials placed in contact with each other. We need to know the condition for the combination to have the same focal length for the blue and red light. We go about this in the following manner . Let n_b, n_y and n_r represent the refractive indices for the material of the first lens corresponding to blue, yellow and red colours, respectively.

Let n'_b, n'_y and n'_r represent the refractive indices for the material of the second lens corresponding to blue, yellow and red colours, respectively.

Let f_b and f'_b represent the focal length for the first and second lens corresponding to the blue colour. Let the focal length for blue light due to both lenses be F_b (combination focal length). Then

$$\frac{1}{F_b} = \frac{1}{f_b} + \frac{1}{f'_b} \tag{0.65}$$

$$\frac{1}{F_b} = \frac{1}{f_b} + \frac{1}{f_b'} = (n_b - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) + (n_b' - 1)\left(\frac{1}{R_1'} - \frac{1}{R_2'}\right) \tag{0.66}$$

where R_1 and R_2 represent the radii of curvature for the first lens; and R_1' and R_2' represent the radii of curvature for the second lens.
We may rewrite the equation as $\frac{1}{F_b} = \left(\frac{(n_b-1)}{n-1}\right)\frac{1}{f} + \left(\frac{(n_b'-1)}{n'-1}\right)\frac{1}{f'}$, where $\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$ and $\frac{1}{f'} = (n'-1)\left(\frac{1}{R_1'} - \frac{1}{R_2'}\right)$ and $n = \frac{n_b+n_r}{2} \approx n_y$; $n' = \frac{n_b'+n_r'}{2} \approx n_y'$.
f and f' represent focal length of the first and second lens corresponding to a mean colour which is around the yellow region.
Similarly; the focal length of the combination corresponding to the red colours we must have

$$\frac{1}{F_r} = \left(\frac{(n_r-1)}{n-1}\right)\frac{1}{f} + \left(\frac{(n_r'-1)}{n'-1}\right)\frac{1}{f'} \tag{0.67}$$

For the focal length of the combination to be equal for the blue and red colours, we must have

$$\left(\frac{(n_b-1)}{n-1}\right)\frac{1}{f} + \left(\frac{(n_b'-1)}{n'-1}\right)\frac{1}{f'} = \left(\frac{(n_r-1)}{n-1}\right)\frac{1}{f} + \left(\frac{(n_r'-1)}{n'-1}\right)\frac{1}{f'}. \tag{0.68}$$

or

$$\frac{\omega}{f} + \frac{\omega'}{f'} = 0 \tag{0.69}$$

where $\omega = \frac{n_b-n_r}{n-1}$ and $\omega' = \frac{n_b'-n_r'}{n'-1}$ are known as dispersive power (V) for different surfaces.
Recall that $V = \frac{n_f-n_c}{n_D-1}$.
Since ω and ω' are both positive and f and f' must have opposite signs for validity of $\left(\frac{(n_b-1)}{n-1}\right)\frac{1}{f} + \left(\frac{(n_b'-1)}{n'-1}\right)\frac{1}{f'} = \left(\frac{(n_r-1)}{n-1}\right)\frac{1}{f} + \left(\frac{(n_r'-1)}{n'-1}\right)\frac{1}{f'}$. A lens which satisfies the equation, $\left(\frac{(n_b-1)}{n-1}\right)\frac{1}{f} + \left(\frac{(n_b'-1)}{n'-1}\right)\frac{1}{f'} = \left(\frac{(n_r-1)}{n-1}\right)\frac{1}{f} + \left(\frac{(n_r'-1)}{n'-1}\right)\frac{1}{f'}$, is called an **Achromatic doublet**.

If the two lenses are made of the same material, $\omega = \omega'$ and the equation $\left(\frac{(n_b-1)}{n-1}\right)\frac{1}{f} + \left(\frac{(n'_b-1)}{n'-1}\right)\frac{1}{f'} = \left(\frac{(n_r-1)}{n-1}\right)\frac{1}{f} + \left(\frac{(n'_r-1)}{n'-1}\right)\frac{1}{f'}$ would imply that $f = -f'$. Such a lens would have an infinite focal length.

REMOVAL OF CHROMATIC ABERRATION OF A SEPARATED DOUBLET

Consider two thin lenses of focal length f and f' separated by a distance t.

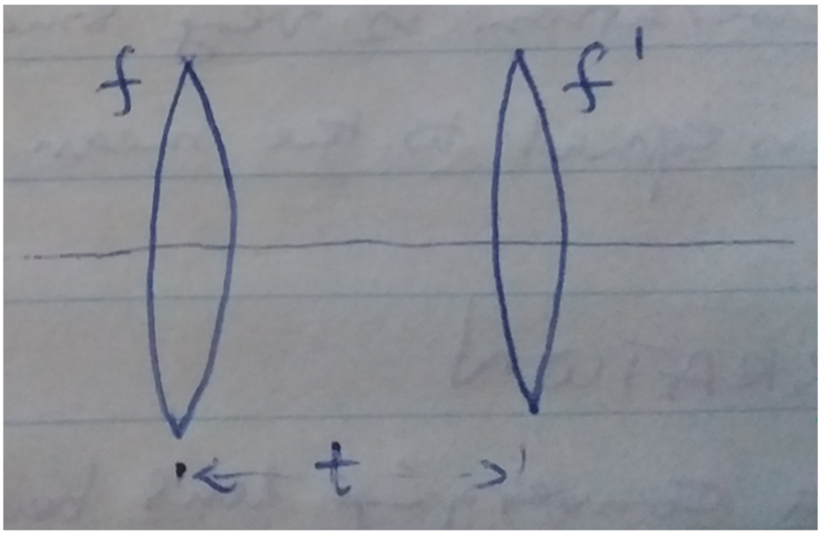


Figure 0.18

The focal length of a combination F would be

$$\frac{1}{F} = \frac{1}{f} + \frac{1}{f'} - \frac{t}{ff'} \tag{0.70}$$

The focal length of the first lens is

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \tag{0.71}$$

For the second lens we have a similar expression for $\frac{1}{f'}$. If Δf and Δn represents the changes in the focal length and the refractive index due to the change in $\Delta\lambda$ in the wavelength, then differentiating $\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$, we get

$$-\frac{\Delta f}{f^2} = (\Delta n)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = \frac{\Delta n}{(n-1)}\frac{1}{f} \tag{0.72}$$

Differentiating $\frac{1}{F} = \frac{1}{f} + \frac{1}{f'} - \frac{t}{ff'}$ we obtain

$$-\frac{\Delta F}{F^2} = -\frac{\Delta f}{f^2} - \frac{\Delta f'}{f'^2} + \frac{t}{f}\frac{\Delta f'}{f'^2} + \frac{t}{f'}\frac{\Delta f}{f^2} \tag{0.73}$$

$$-\frac{\Delta F}{F^2} = \frac{\Delta n}{(n-1)f} + \frac{\Delta n'}{(n'-1)f'} - \frac{t}{f}\frac{\Delta n'}{(n'-1)f'} - \frac{t}{f'}\frac{\Delta n}{(n-1)f} \tag{0.74}$$

which looks like $\frac{\omega}{f} + \frac{\omega'}{f'} - \frac{t}{ff'}(\omega + \omega')$.
For the combination to have the same focal length for blue and red colours, we must have $\frac{t(\omega+\omega')}{ff'} = \frac{\omega}{f} + \frac{\omega'}{f'}$ or $t = \frac{\omega f' + \omega' f}{\omega + \omega'}$ which must be satisfied to get rid of chromatic aberration.
If both lenses are made of the same material then $\omega = \omega'$ and the above expression satisfies $t = \frac{f+f'}{2}$ implying that the chromatic aberation is very small if the distance between the two lenses is equal to the mean of the focal lengths.

SPHERICAL ABERRATION

Light rays passing through a converging lens bend towards the axis and cross the axis at some point. If we restrict ourselves to the paraxial region (point close to the axis), then all rays cross the z-axis at the same point which is at a distance f_p from the lens. f_p represents the paraxial focal length of the lens.

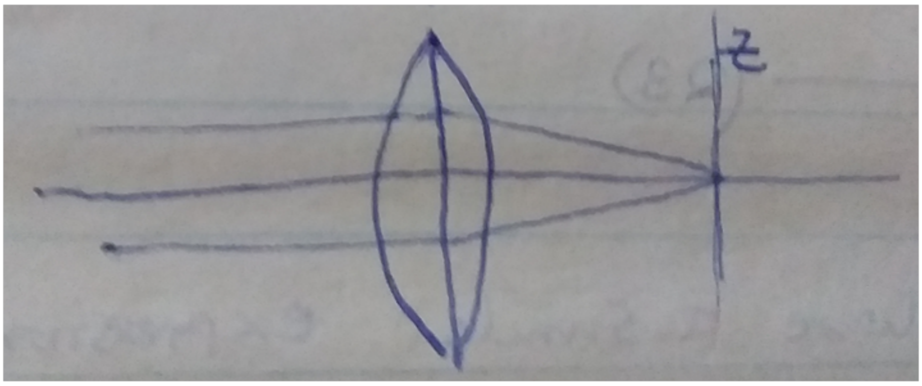


Figure 0.19

Spherical aberration comes about when rays are restricted to the paraxial region.

COMMA: The effect of comma is that the rays which proceed near the axis of the lens focus at points different from that of the marginal rays and tend to cause different magnifications for different parts of the object.

ASTIGMATISM AND CURVATURE OF FIELD

When an optical system is free from spherical and comma, the image will be focussed sharply by these points lying close to the axis. For points far away from the axis, the image of a point will not be a point and the system is said to be afflicted with Astimatism.

DISTORTION: This is caused by non uniform magnification of the optical system.

Further reading

- Spherical mirrors - Jenkins and White page 98-112 (chapter 6)
- Ray tracing - Jenkins and White page 130-143 (chapter 8)