THE UNIVERSITY OF ZAMBIA
SCHOOL OF NATURAL SCIENCES
Department of Physics
2019/2020 ACADEMIC YEAR
PHY 2231 - Properties of Matter and Thermodynamics Notes

....

CHAPTER 2: ELASTICITY

....

The action of forces applied on a rigid body produces a change in size and shape which is usually very small in magnitude. Any change from the normal configuration of a body in size or shape is a deformation and a corresponding fractional change is a strain.
If a force ΔF acts on an area ΔA at the same point on a body, the stress at that point within this area is stress $=\frac{\Delta F}{\Delta A}$. If the force is uniformly distributed over an area A, then stress is $\frac{F}{A}$.
Materials can be divided into two categories. These are isotropic and anisotropic materials. In isotropic materials, the strain is independent of the direction or the elastic properties of the material are the same in every direction. In non isotropic (anistropic) materials, the elastic properties at any point are different in different directions about that point. Quartz fibre and many crystalline substances are non isotropic. They also show different properties in different directions.
Most solids are polycrystalline i.e they consists of microscopic crystals called grains. Grains are joined together at boundaries. If the grains are oriented at random in the solid, the solid behaves as anisotropic substance.

HOOKES LAW

Hookes law states that provided the elastic limit is not exceeded, the stress is directly proportional to the strain i.e $\frac{stress}{strain}$ = constant. The constant is called modulus of elasticity. The relationship between stress and strain may be seen by plotting a graph between stress and strain. Such a graph is called a stress strain diagram. Let a wire be clamped or fixed at one end and gradually loaded at the other end till it breaks down.

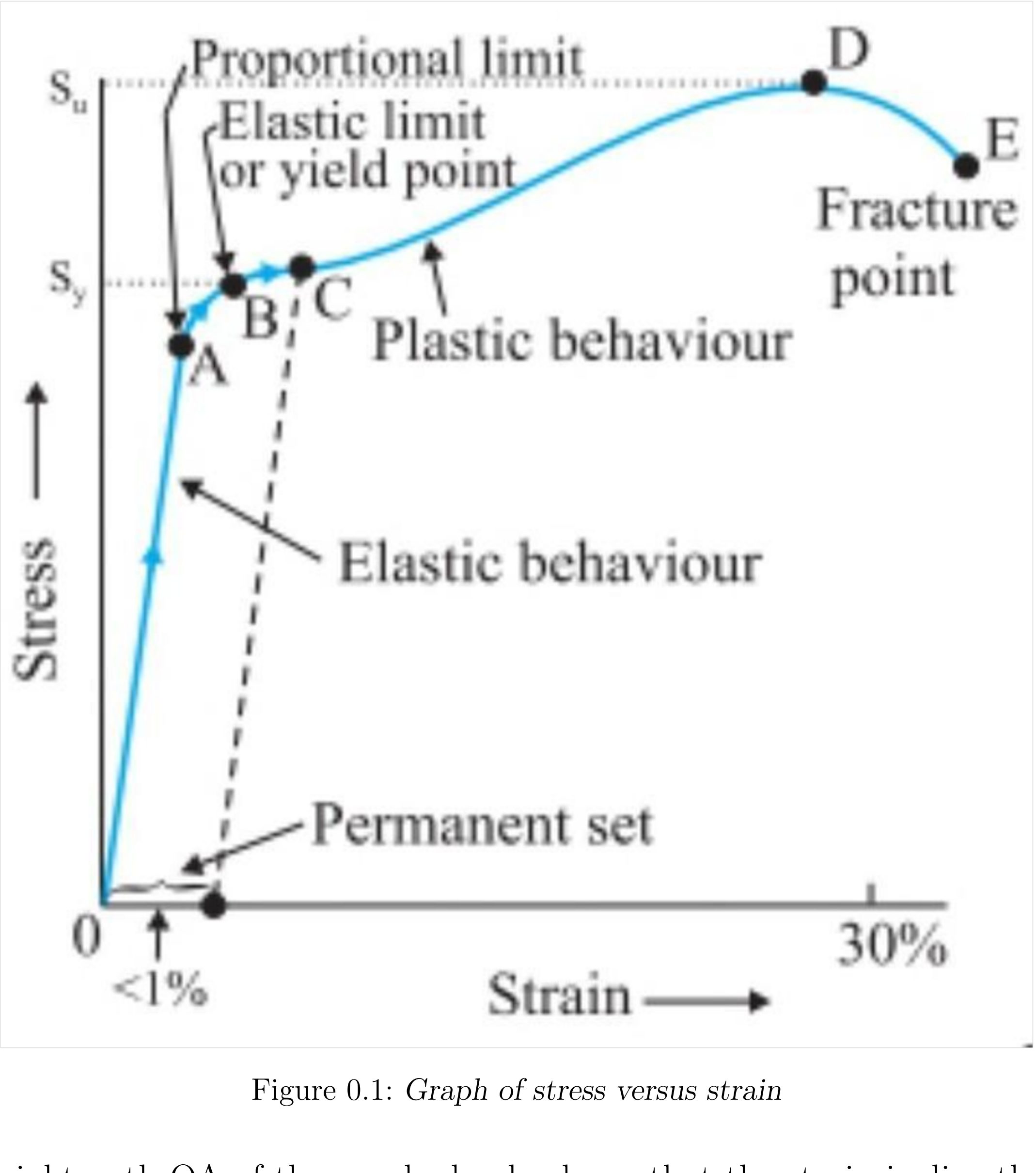


Figure 0.1: *Graph of stress versus strain*

Straight path OA of the graph clearly shows that the strain is directly proportional to the stress i.e obeys Hookes law. Up to A, on removal of the stress it will recover its original condition represented by O or the

wire is perfectly elastic up to A. The point A is called the elastic limit. As soon as the elastic limit is crossed, the strain increases more rapidly than the stress so the graph goes along AB, the extension of the wire being partially plastic and partially elastic hence when unloaded here it does not come come back to its original condition so that there remains a residual strain that is called permanent set.
Beyond point B, for practically little small or no increase in stress, there is a large increase in strain. BC is an irregular wave line. B is called the lower yield point and C is called the upper yield point. The correspoing stress is called yielding stress. Yielding ceases at C. At C, the wire begins to thin down (i.e become purely plastic). the cross section of the wire decreasing uniformly with extension up to D and hence the volume remains constant. The maximum load (force) to which the wire is subjected divided by the cross section area is called the tensile strength or ultimate strength of a wire.
The extension of a wire goes on increasing beyond D without any addition of load even if the load is reduced, that is called flowing down or plastic flow. The wire breaks down at E called the breaking point of a wire.

TYPES OF ELASTICITY

Corresponding to three types of strain, we have three types of elasticity:

- linear elasticity (Youngs modulus denoted by Y or E)
- elasticity of volume (bulk modulus denoted by K or B)
- elasticity of shape or shear modulus or rigidity modulus(η)

A. YOUNG MODULUS (Y)

When deforming force is applied to the body only in a particular direction, the change per unit length in that direction is called longitudinal strain. Force applied per unit cross section area is called longitudinal or linear stress. The ratio of longitudinal stress to longitudinal strain within the elastic limit is called Youngs modulus.

$$Y = \frac{F/A}{l/L} = \frac{FL}{Al} \tag{0.1}$$

The unit of Youngs modulus is the same as for stress, which is dynes/cm or N/m^2.

Poissons Ratio - When we stretch a string or wire, it becomes longer but thinner i.e increase in length is always accompanied by the decrease in its cross section i.e linear strain is accompanied by transverse or lateral strain.

Longitudinal strain $= \frac{l}{L}$

Transverse or lateral strain $= \frac{d}{D}$

Poissons ratio (σ) equals transverse strain (β) divided by longitudinal strain (α) i.e

$$\sigma = \frac{d/D}{l/L} = \frac{dL}{Dl} = \frac{\beta}{\alpha}. \tag{0.2}$$

WORK OF DEFORMATION

In order to deform a body, work must be done by the applied force. The energy so spent is stored up in the body and is called energy of strain. When the applied force is removed, the stress disappears and the strain appears as heat. Let F be the force applied to the wire fixed at the upper end. for a small increase in the length l of a wire , work done is $dW = Fdl$. During the all stretch of the wire from 0 to l, strain is directly proportional to stress.

$$W = \int_0^l Fdl. \tag{0.3}$$

but $Y = \frac{FL}{Al}$ from Youngs modulus. Therefore $F = \frac{YAl}{L}$

$$W = \int_0^l \frac{YAl}{L}dl = \frac{YA}{L}\int_0^l ldl = \frac{YA}{L}\frac{l^2}{2} = \frac{1}{2}\frac{YAl}{L}l = \frac{1}{2}Fl. \tag{0.4}$$

Work $= \frac{1}{2}\times$stretching force $\times$stretch.

Work done per unit volume $= \frac{\frac{1}{2}\times l\times F}{LA} = \frac{1}{2}\times\frac{F}{A}\times\frac{l}{L} = \frac{1}{2}\times$stress$\times$strain.

B. BULK MODULUS (INCOMPRESSIBILITY)

When a gas is subjected to an increased pressure, the substance contracts. The strain produced is change in volume divided by original volume.

$$B = \frac{F/A}{\Delta V/V} = \frac{P}{\Delta V/V} = \frac{PV}{\Delta V} \tag{0.5}$$

where F/A is bulk stress and $\Delta V/V$ is the bulk strain.
Bulk modulus (B or K) is also called incompressibility. Compressibility $= \frac{1}{K}$ (reciprocal of bulk modulus = compressibility).
There are two types of bulk modulus in terms of gas which are isothermal or adiabatic. When a fluid is compressed there is always some heat produced. If the heat is removed as fast as it is produced, the temperature of the fluid remains constant; this is called isothermal . But if the heat be allowed to remain in the fluid its temperature rises and is said to be adiabatic.

C. SHEAR MODULUS

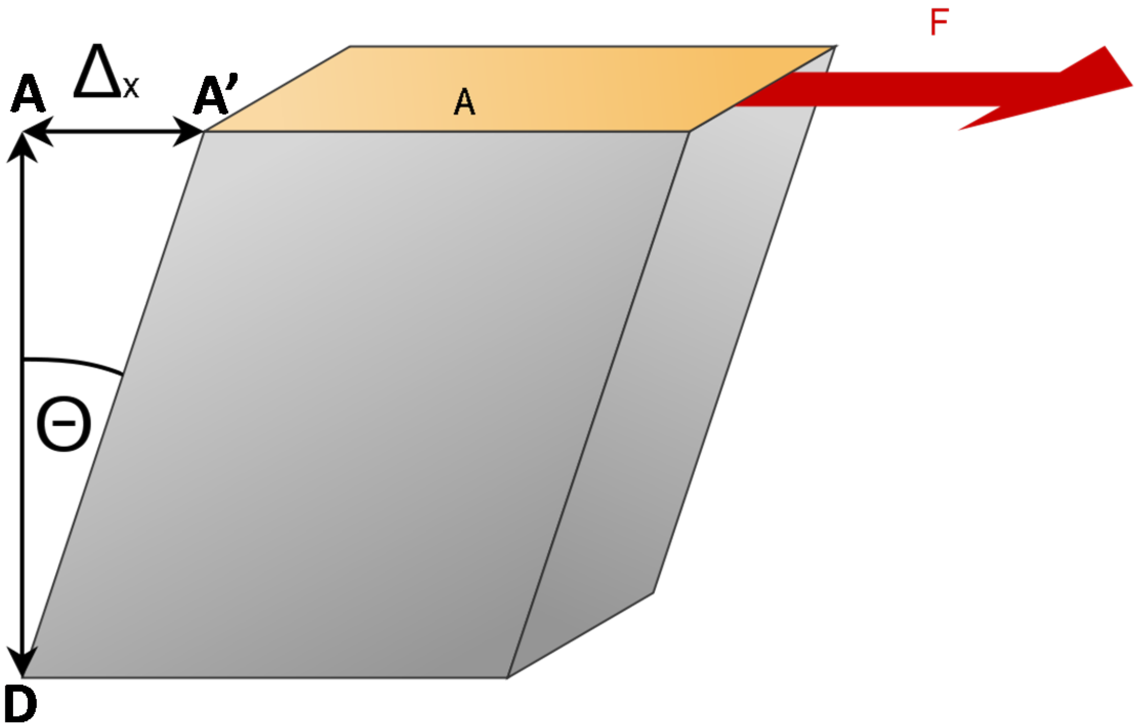


Figure 0.2

Strain $= \frac{AA'}{AD} = tan\theta \approx \theta$
Shear modulus $= \frac{stress}{strain} = \frac{F/A}{tan\theta}$

D. AXIAL MODULUS

Axial modulus is the longitudinal stress required to produce unit linear strain unaccompanied by any lateral strain (β).

BENDING OF BEAMS

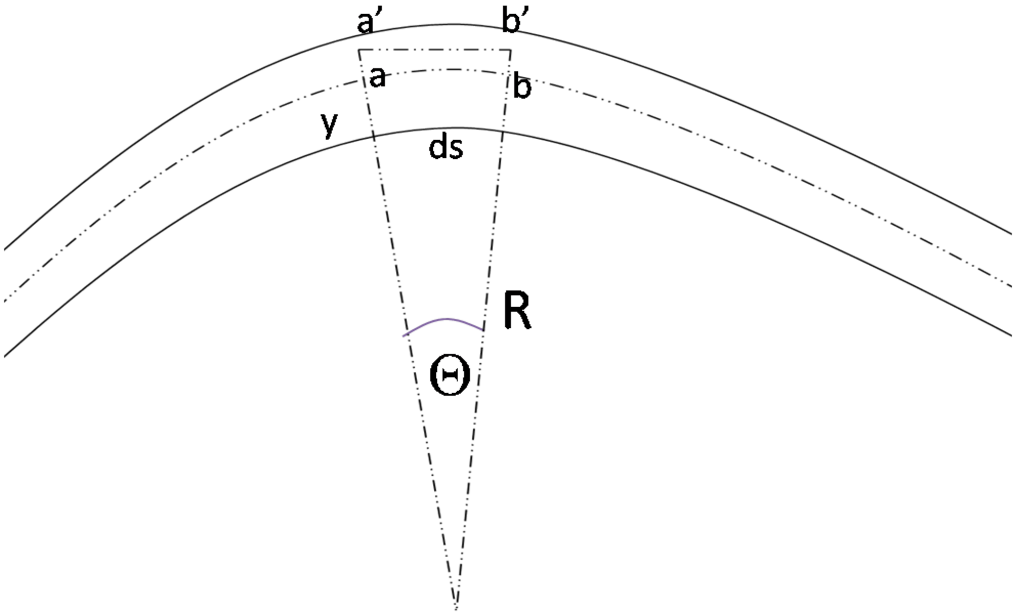


Figure 0.3

ab = $R\theta$
a'b' = $(R+y)\theta$
Change in length $(R+y)\theta$ - $R\theta$
Strain = change of length/original length = $\frac{y\theta}{R\theta} = \frac{y}{R}$.
E or $Y = \frac{Stress}{Strain}$
Stress = $Y\times$ Strain
Stress = $Y\frac{y}{R}$.
Stress = $\frac{F}{A}$, F = Stress $\times\delta A$
$F = \frac{Yy}{R}\delta A$.
Moment = force $\times$ distance = $Fy = \frac{Yy}{R}\delta Ay = \frac{Yy^2}{R}\delta A$
Total moment = $\sum\frac{Yy^2}{R}\delta A = \frac{Y}{R}\sum y^2\delta A = \frac{Y}{R}I_g$, where $\sum y^2\delta A = I_g$ is the moment of inertia = ak^2.

$$\tau = \frac{Y}{R}ak^2$$

For a rectangular rod, $a = b\times d$, $k^2 = \frac{d^2}{12}$.
For a circular rod, $a = \pi r^2$, $k^2 = \frac{r^2}{4}$.

i.e. for a rectangular rod

$$\tau = \frac{Y}{R} \times \frac{bd^3}{12}. \tag{0.6}$$

for a circular rod,

$$\tau = \frac{Y}{R} \pi r^2 \frac{r^2}{4}. \tag{0.7}$$

CANTILEVER

A cantilever is the uniform beam (bar) fixed horizontally at one end and loaded at the other end.

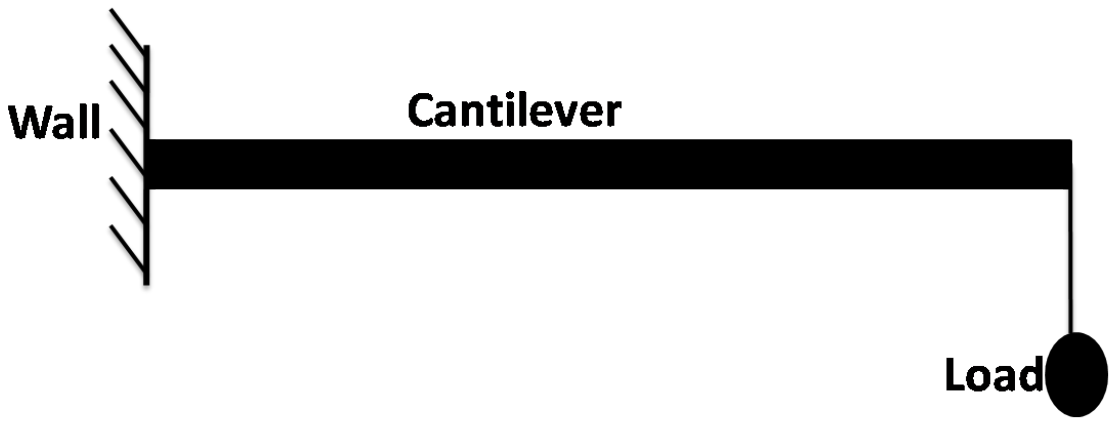


Figure 0.4

1*. When the weight of the beam is ineffective, then the bending is due to external load. Consider a point **P** a distance x from a fixed point. Torque at **P** due to weight $= W(L - x)$. At equilibrium, the external torque is balancéd by the internal torque i.e

$$W(L - x) = \frac{Y}{R} I_g. \tag{0.8}$$

$$dx = Rd\theta; R = \frac{dx}{d\theta} \tag{0.9}$$

Substituting Equation 0.8 into Equation 0.9 we have

$$W(L - x) = YI_g \frac{d\theta}{dx} \tag{0.10}$$

Making $d\theta$ the subject of formula we have

$$d\theta = \frac{W(L - x)dx}{YI_g} \tag{0.11}$$

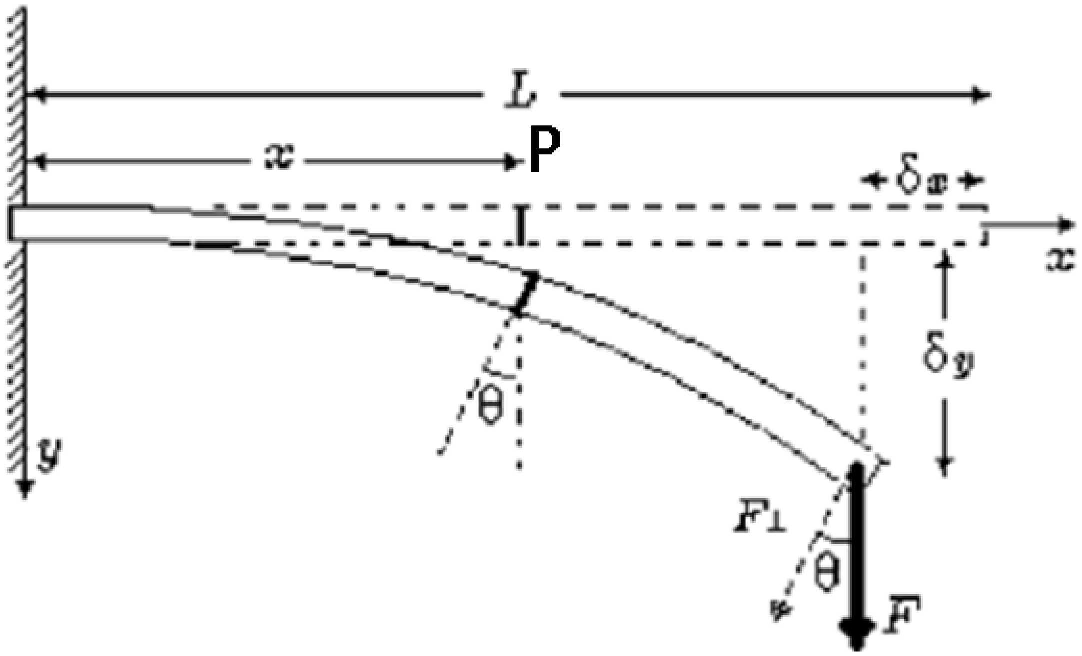


Figure 0.5

But

$$dy = (L - x)d\theta = \frac{(L - x)W(L - x)dx}{YI_g} = \frac{W}{YI_g}(L - x)^2 dx \tag{0.12}$$

Total depression (y) is given by

$$y = \int_0^L \frac{W}{YI_g}(L - x)^2 dx = \frac{W}{YI_g}\int_0^L (L^2 - 2xL + x^2)dx \tag{0.13}$$

$$y = \frac{W}{YI_g}\left[L^2x - 2L\frac{x^2}{2} + \frac{x^3}{3}\right]_0^L = \frac{W}{YI_g}\left[L^3 - \frac{L^3}{2} + \frac{L^3}{3}\right] = \frac{WL^3}{3YI_g} \tag{0.14}$$

To find the depression at a particular point x from a fixed point

$$W(L - x) = \frac{Y}{R}I_g; \frac{1}{R} = \frac{d^2y}{dx^2} \tag{0.15}$$

$$W(L - x) = YI_g\frac{d^2y}{dx^2} \tag{0.16}$$

$$\frac{d^2y}{dx^2} = \frac{W(L - x)}{YI_g} \tag{0.17}$$

Integrating

$$\frac{dy}{dx} = \frac{W}{YI_g}\left(Lx - \frac{x^2}{2}\right) + C \tag{0.18}$$

At $x = 0$, $y = 0$ and $dy = 0$, $\therefore C = 0$

$$\therefore \frac{dy}{dx} = \frac{W}{YI_g}\left(Lx - \frac{x^2}{2}\right) \tag{0.19}$$

Integrating again we have

$$y = \frac{W}{YI_g}\left(L\frac{x^2}{2} - \frac{x^3}{6}\right) + C_2 \tag{0.20}$$

At $x = 0$, $y = 0$ and $dy = 0$, $\therefore C_2 = 0$

$$\therefore y = \frac{W}{YI_g}\left(L\frac{x^2}{2} - \frac{x^3}{6}\right) \tag{0.21}$$

which is depression at a point x. Depression at the free end $x = L$

$$\therefore, y = \frac{W}{YI_g}\left(\frac{L^3}{2} - \frac{L^3}{6}\right) = \frac{W}{YI_g}\left(\frac{2L^3}{6}\right) = \frac{W}{YI_g}\left(\frac{L^3}{3}\right) = \frac{WL^3}{3YI_g} \tag{0.22}$$

2*. When the weight of the beam is effective
Let W' be the weight of the beam per unit length. The weight of the beam from the centre to the free end $= W'(L - x)$. There are two forces acting which are W'(L-x) and W (weight at the free end). At equilibrium these must balance.

$$\tau = W(L - x) + W'(L - x)\left(\frac{L - x}{2}\right) = \frac{Y}{R}I_g \tag{0.23}$$

Taking a point Q near to P, $dx = rd\theta$; $R = \frac{dx}{d\theta}$, we have

$$W(L - x) + \frac{W'}{2}(L - x)^2 = YI_g\frac{d\theta}{dx} \tag{0.24}$$

$$d\theta = \frac{\left[W(L - x) + \frac{W'}{2}(L - x)^2\right]}{YI_g}dx. \tag{0.25}$$

But $dy = (L-x)d\theta$.

$$dy = \frac{(L-x)\left[W(L-x) + \frac{W'}{2}(L-x)^2\right]}{YI_g}dx. \tag{0.26}$$

$$dy = \frac{\left[W(L-x)^2dx + \frac{W'}{2}(L-x)^3dx\right]}{YI_g}. \tag{0.27}$$

Integrating, we have

$$y = \int \frac{W}{YI_g}(L-x)^2dx + \int \frac{W'}{2YI_g}(L-x)^3dx \tag{0.28}$$

$$y = \frac{WL^3}{3YI_g} + \frac{W'L^4}{8YI_g} \tag{0.29}$$

but $W'L = W_1$

$$\therefore y = \frac{WL^3}{3YI_g} + \frac{W_1L^3}{8YI_g} \tag{0.30}$$

where W_1 is the weight of the cantilever.

$$\therefore y = \frac{L^3}{3YI_g}\left[W + \frac{3W_1}{8}\right] \tag{0.31}$$

3*. For a cantilever loaded uniformly and bending by its own weight, we have

$$y = \frac{WL^3}{8YI_g} \tag{0.32}$$

TRANSVERSE VIBRATION OF A CANTILEVER

If you pull down a cantilever and release it, it will move up and down. To show that such motion is simple harmonic , you have to show that acceleration is directly proportional to the negative of its displacement i.e $a \propto -y$

$$y = \frac{WL^3}{3YI_g} \rightarrow W = \frac{y3YI_g}{L^3}. \tag{0.33}$$

According to Newtons second law, $F = ma$

$$ma = \frac{3YI_g}{L^3}y \rightarrow a = -\frac{3YI_g}{mL^3}y \tag{0.34}$$

$$a = -\mu y \tag{0.35}$$

since Y, I_g, m and L are constants, hence $\frac{3YI_g}{mL^3} = \mu$
Therefore $a \propto -y$, hence motion is simple harmonic motion.
Period (T) of such motion is given by

$$T = 2\pi\sqrt{\frac{1}{\mu}} = 2\pi\sqrt{\frac{mL^3}{3YI_g}} \tag{0.36}$$

DEPRESSION OF A BEAM SUPPORTED AT THE ENDS

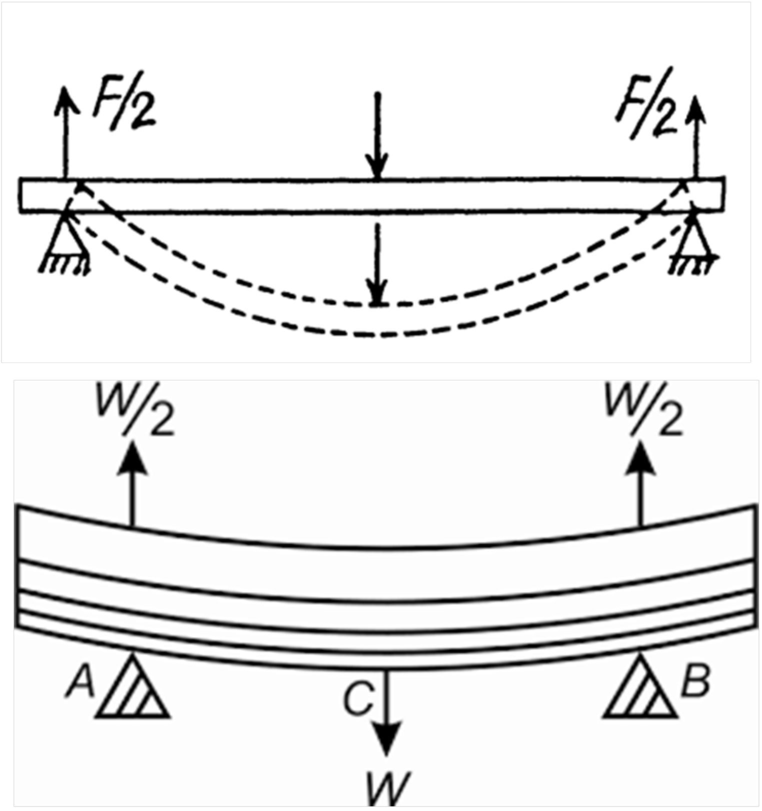


Figure 0.6

This acts like two cantilevers. In such a case $y = \frac{WL^3}{3YI_g}$, but $W = \frac{W}{2}$ and $L = \frac{L}{2}$

$$y = \frac{\frac{W}{2}\left(\frac{L}{2}\right)^3}{3YI_g} = \frac{WL^3}{48YI_g} \tag{0.37}$$

If a beam is loaded uniformly, then

$$y = \frac{5WL^3}{384YI_g} \tag{0.38}$$

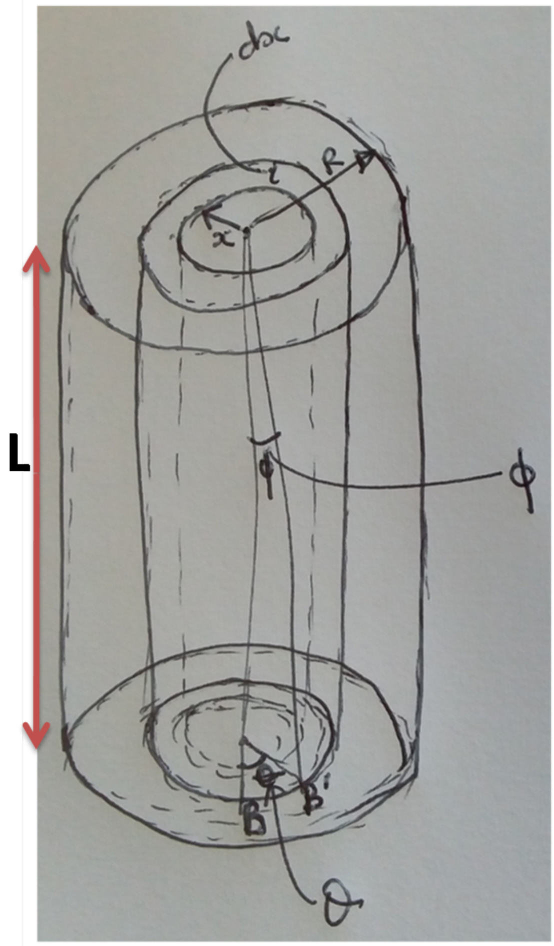
The stiffness of the beam is the ratio between the maximum deflection and its span (length).

TORSION

If we consider a cylindrical rod or wire with one end fixed and you rotate the lower free end, we say the wire is undergoing torsion motion. When ever you disturb an object from its mean position, an internal force in the opposite direction is set up in the object and the object goes back to its original position. At equilibrium, the internal force balances external force. In torsion, the modulus involved is rigidity modulus or shear modulus because the length remains constant only the shape changes.

EXPRESSION FOR COUPLE/UNIT TWIST

Consider a cylinder of length L and radius R. Consider a small cylinder within this cylinder with radius x and thickness dx.



$$BB' = L\phi \tag{0.39}$$

$$BB' = x\theta \tag{0.40}$$

$$L\phi = x\theta \tag{0.41}$$

$$\phi = \frac{x\theta}{L} = strain \tag{0.42}$$

Rigidity modulus $=\frac{Stress}{Strain}$
Stress = $\eta \times Strain$, where η is the rigidity modulus.
Stress $=\eta\frac{x\theta}{L}$
Force = stress $\times$ Area.

$$F = \eta\frac{x\theta}{L} \times 2\pi x dx \tag{0.43}$$

$$F = \eta\frac{2\pi x^2\theta}{L}dx \tag{0.44}$$

$$\tau = F \times x = \frac{2\pi\eta x^3\theta}{L}dx \tag{0.45}$$

Total torque $=\int_0^R \frac{2\pi\eta x^3\theta}{L}dx = \frac{2\pi\eta\theta}{L}\int_0^R x^3 dx = \frac{2\pi\eta\theta}{L}\frac{R^4}{4} = \frac{\pi\eta R^4}{2L}\theta.$
$\tau = C\theta$ where $C = \frac{\pi\eta R^4}{2L}$ is the couple twist.
Suppose it is a hallow cylinder with inner radius r_1 and outer radius r_2 then

$$\tau = \frac{\pi\eta(r_2^4 - r_1^4)}{2L}\theta, \tag{0.46}$$

where τ is the torque and η is the rigidity modulus and θ should be in radians.

Work done in twisting

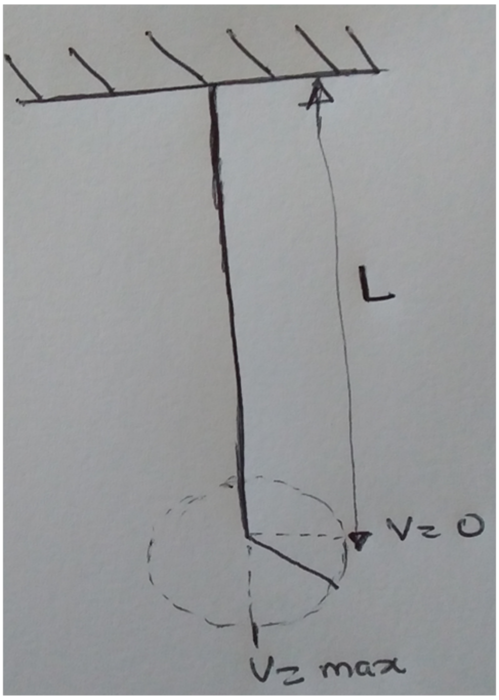
$$dW = \tau d\theta \tag{0.47}$$

but $\tau = C\theta$

$$\therefore W = \int C\theta d\theta = \frac{1}{2}\theta^2 \tag{0.48}$$

TORSIONAL PENDULUM

Consider a long cylindrical rod. In order to twist the rod, you do some work which is stored as internal energy. To show that torsional pendulum motion is simple harmonis we have to show that $a \propto -y$.
Rotational kinetic energy = $\frac{1}{2}I\omega^2$
$PE = \frac{1}{2}C\theta^2$
Total energy $T = PE + KE = \frac{1}{2}I\omega^2 + \frac{1}{2}C\theta^2 = \frac{1}{2}I\left(\frac{d\theta}{dt}\right)^2 + \frac{1}{2}C\theta^2$



Differentiating we have $\frac{1}{2}I2\frac{d\theta}{dt}\frac{d^2\theta}{dt^2}+\frac{1}{2}C2\frac{d\theta}{dt}=0$

$$I\frac{d^2\theta}{dt^2}+C\theta=0 \tag{0.49}$$

$$I\frac{d^2\theta}{dt^2}=-C\theta \tag{0.50}$$

$$\frac{d^2\theta}{dt^2}=-\left(\frac{C}{I}\right)\theta \tag{0.51}$$

$$\therefore \frac{d^2\theta}{dt^2}\propto -\theta \tag{0.52}$$

hence simple harmonic.
Period of torsional pendulum $T=2\pi\sqrt{\frac{I}{C}}=2\pi\sqrt{\frac{2LI}{\pi\eta R^4}}$