THE UNIVERSITY OF ZAMBIA
SCHOOL OF NATURAL SCIENCES
DEPARTMENT OF MATHEMATICS & STATISTICS
MAT 2100 – Analytic Geometry and Calculus

Tutorial Sheet 2

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1. Find the equation of the tangent line and normal line to the curve with the indicated conditions:

   (a) $\frac{x^2}{9} - y^2 = 1$ at $x = 6$

   (b) $f(x) = \sqrt{3 + x + \sqrt{x}}$ at $x = 4$

   (c) $y^2(y^2 - 4) = x^2(x^2 - 5)$ at the point $\left(\sqrt{5}, 0\right)$

2. Determine whether the functions below satisfy the hypotheses of the Rolle's theorem on the given interval. If they do, find all numbers c that satisfy the conclusion of the theorem.

   (a) $f(x) = \frac{1}{2}x - \sqrt{x}$ on $[0, 4]$

   (b) $f(x) = |x|$ on $[1, 2]$

   (c) $h(x) = 2^{2x+1} - 3.2^x + 1$ on $[-1, 0]$

   (d) $f(x) = 4x - \tan \pi x$ on $[-\frac{1}{4}, \frac{1}{4}]$.

3. Decide whether the Mean Value Theorem applies to the given function on the given interval. If it does, find all possible values of c.

   (a) $f(x) = \sqrt{25 - x^2}, \quad [-5, 3]$

   (b) $f(x) = x + \frac{1}{x}, \quad [\frac{1}{2}, 2]$

   (c) $f(t) = \dfrac{1}{t-1}, \quad [0, 2]$.

4. For each of the given functions f and g, apply the Cauchy Generalized Mean Value theorem to find the value(s) c inside the given interval.

   (a) $f(x) = x^2, \quad g(x) = x^3, \quad [1, 2]$

   (b) $f(x) = \cos x, \quad g(x) = \sin x, \quad [0, \frac{\pi}{2}]$

5. Suppose that f is a differentiable function such that $f'(x) \leq -2$ for all $x \in [0, 4]$ and that $f(1) = 6$.

   (a) Prove that $f(4) \leq 0$.

   (b) Prove that $f(4) \geq 8$.

6. Use differentials to approximate the following:

   (a) $\sqrt[6]{65}$ **(b)** $\sqrt{0.037}$ **(c)** $\sqrt{3.9} + (3.9)^2 + 1$

7. The measurement of the side of a square floor tile is 60 centimetres, with a possible error of $\frac{1}{100}$ centimetres. Use differentials to approximate the possible propagated error in computing the area of the square.

8. Use the definition of the limit to show that:

   (a) $\lim_{x\to 2}(x^2+4x-5)=7$ **(b)** $\lim_{x\to -1}\left[\dfrac{2x^2+5x-3}{x^2+3x-2}\right]=\dfrac{3}{2}$

9. Evaluate the following limits:

   (a) $\lim_{x\to 0^+}\left[\dfrac{\ln\sin^2 x}{\ln\tan x}\right]$ **(b)** $\lim_{x\to 0^+}\left[\dfrac{\sin^{-1}x}{\sin^2(3x)}\right]$ **(c)** $\lim_{x\to 0^+}\left[(e^x-1)^{\sin x}\right]$ **(d)** $\lim_{x\to 0}\left[\left(\dfrac{\sin x}{x}\right)^{\frac{1}{x^2}}\right]$

10. Determine convergence or divergence of the following sequences with the given $n^{th}-$term:

    (a) $a_n=\left(\frac{-2}{3}\right)^n$ **(b)** $a_n=(-1)^n\left(\frac{n}{n+1}\right)$ **(c)** $a_n=n\sin\left(\frac{1}{n}\right)$ **(d)** $a_n=\frac{\ln(n^3)}{2n}$

11. For each of the following series, find the sum if it converges:

    (a) $\sum_{n=1}^{\infty}\frac{5(2^n)}{3^n}$ **(b)** $\sum_{n=1}^{\infty}\frac{n+10}{10n+1}$ **(c)** $\sum_{n=1}^{\infty}(\sin 1)^n$ **(d)** $\sum_{n=1}^{\infty}\frac{2}{(n+2)(n+4)}$ **(e)** $\sum_{n=1}^{\infty}\arctan n$

12. Determine convergence or divergence of the following series:

    (a) $\sum_{n=1}^{\infty}\frac{2+\sin n}{n}$ **(b)** $\sum_{n=1}^{\infty}\frac{\arctan n}{n^2+1}$ **(c)** $\sum_{n=1}^{\infty}\frac{n^{k-1}}{n^k+1},\quad k>2$ **(d)** $\sum_{n=1}^{\infty}\frac{3^n}{2^n-1}$

    (e) $\sum_{n=1}^{\infty}\frac{(-1)^n n}{\ln(n+1)}$ **(f)** $\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}$ **(g)** $\sum_{n=2}^{\infty}\frac{n}{(\ln n)^n}$ **(h)** $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n!}{1.3.5...(2n+1)}$

13. Find the n^{th} Maclaurin polynomial of the following functions:

    (a) $f(x)=\sin x$ **(b)** $f(x)=\frac{1}{(1+2x)^2}$ **(c)** $f(x)=\ln[(1-x)(1-2x)]$

14. Find the n^{th} Taylor polynomial of the following functions at $x_0=c$:

    (a) $f(x)=\frac{1}{(1-x)^2},\quad c=2$

    (b) $f(x)=\cos x,\quad c=\frac{\pi}{2}$

15. Find the radius and the interval of convergence (if possible) for the following power series:

    (a) $\sum_{n=1}^{\infty}\frac{(4x)^n}{n^2}$ **(b)** $\sum_{n=1}^{\infty}\frac{(-1)^n(2x-3)^n}{n(2^n)}$ **(c)** $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(x-c)^n}{n(c^n)},\quad c\in\mathbb{R}$ **(d)** $\sum_{n=1}^{\infty}n!(x-a)^n,\quad a\in\mathbb{R}$

16. Find the curvature and the radius of curvature of the curves below at the indicated point:

    (a) $y=x^2+2x,\quad x=0$

    (b) $x=\ln|\cos y|,\quad y=2\pi$

    (c) $y=\frac{3}{4}\sqrt{16-x^2},\quad x=0$

17. Sketch the circle of curvature for each of the following curves at the given point:

    (a) $y=2x^2+3,\quad x=-1$

    (b) $y=\cos 2x,\quad x=2\pi$

18. A curve is given by

$$y = \ln|\sec x|, \ \ 0 \leq x < \frac{\pi}{2}.$$

Find an intrinsic equation of the curve in the form $s = g(\psi)$, where s is measured from the point $(0,0)$ and ψ is the angle the tangent to the curve makes with the positive $x-$axis.

19. A curve is given by

$$y = \frac{1}{3}(2x+1)^{\frac{3}{2}}, \ \ x \geq 0.$$

Find an intrinsic equation of the curve in the form $s = g(\psi)$, where s is measured from the point $(0, \frac{1}{3})$ and ψ is the angle the tangent to the curve makes with the positive $x-$axis.

20. The radius of curvature at a point on a curve C is given by

$$\rho = \frac{e^{2\psi}+1}{e^{\psi}},$$

where ψ is the angle the tangent to C makes with the positive $x-$axis. If s is the length of the arc measured from a fixed point and that $s = \pi$ when $\psi = 0$, find an intrinsic equation for C in the form $s = f(\psi)$.

21. A curve is given by

$$y = \ln|\sin x|, \ \ \frac{\pi}{2} \leq x < \pi.$$

Find an intrinsic equation of the curve in the form $s = g(\psi)$, where s is measured from the point $(0,0)$ and ψ is the angle the tangent to the curve makes with the positive $x-$axis.

22. The intrinsic equation of a curve is given by $s = \tan\psi$, where s is measured from the point $(0,1)$ on the xy-plane and ψ is the angle the tangent to the curve makes with the positive $x-$axis.

   (a) Find curvature at $\psi = \frac{\pi}{4}$.

   (b) Given that the centre of curvature is $\left(\ln\left(1+\sqrt{2}\right) - \sqrt{2}, \ \ 2\sqrt{2}\right)$, find the equation of the circle of curvature.