THE UNIVERSITY OF ZAMBIA
SCHOOL OF NATURAL SCIENCES
DEPARTMENT OF MATHEMATICS & STATISTICS
MAT 2100–Analytic Geometry and Calculus
Tutorial Sheet 1

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1. For each given parabola below, find the vertex, focus, and directrix. Then, sketch the parabola.
   (a) $y^2 = -6x$
   (b) $x^2 = 4y$
   (c) $x = \dfrac{y^2}{28}$
   (d) $6y = -2x^2$

2. Find an equation of the parabola with vertex at the origin if
   (a) focus is at $(-12, 0)$
   (b) focus is at $(0, 7)$
   (c) directrix is $y = -1$
   (d) directrix is $x = \frac{2}{3}$

3. A parabola has focus at $(3, -4)$ and the $x-$axis is the equation of the directrix. Find the equation of the parabola and sketch it, clearly showing the vertex and the $y-$intercepts.

4. Show that the line $4x + 12y = -21$ passes through the focus of the parabola $x^2 = -7y$. Hence, find the coordinates at which this line intersects the parabola.

5. Prove that the length of latus rectum of any parabola with vertex at the origin is $4p$. Hence, find the length of latus rectum for each of the following parabolas:
   (i) $2y^2 = 3x$ (ii) $4x^2 = 6y$ (iii) $x^2 = -10y$ (iv) $5x^2 = -2y$ (v) $y^2 = -\frac{3}{10}$

6. A beam is supported at its ends by two supports which are 16 metres apart. Since the load is concentrated at the center, there is a deflection of 1 metre at the center. How far from the center is the deflection of 0.5 metres?

7. Find the foci for of each of the following ellipses and sketch the graph:
   (a) $\frac{2x^2}{9} + \frac{4y^2}{25} = 1$ (b) $\frac{x^2}{25} + 4y^2 = 1$ (c) $100x^2 + 5100y^2 = 1$

8. An ellipse is centred at the origin and passes through the points $(3, \sqrt{7})$ and $(-\sqrt{3}, 3)$. Find the equation and sketch it.

9. Write down the standard form for the equation of the hyperbola with
   (a) $\pm(|PF_1| - |PF_2|) = 6$ and foci at $(0, \pm 5)$
   (b) $\pm(|PF_1| - |PF_2|) = \frac{1}{2}$ and foci at $(\pm 2, 0)$

10. Find the equation of the hyperbola that goes through the points $(\sqrt{6}, 4)$ and $(4, 6)$ if its foci are on the y-axis and the hyperbola has the x-axis as the axis of symmetry.

11. Find the eccentricity of each of the following conics:

    (a) $169x^2 + 25y^2 = 4225$ (b) $y^2 - 4y - 8x - 12 = 0$ (c) $64x^2 - 36y^2 = 2304$

12. Find the standard equation of the following conic sections:

(a) Foci : $(0, \pm 3)$; Eccentricity : 0.5
(b) Vertices : $(0, \pm 70)$; Eccentricity : 0.1
(c) Foci : $(\pm 8, 0)$; Eccentricity : 0.2
(d) Vertices : $(\pm 10, 0)$; Eccentricity : 0.24
(e) Eccentricity : 3; Vertices : $(0, \pm 1)$
(f) Eccentricity : 2; Vertices : $(\pm 2, 0)$
(g) Eccentricity : 3; Foci : $(\pm 3, 0)$
(h) Eccentricity : 1.25; Foci : $(0, \pm 5)$

13. Find the standard equation of the conic section described below:

    (a) major axis is 10 and $e = \frac{1}{2}$
    (b) Transverse axis is 12 and $e = 2$
    (c) distance between the foci is 32, major axis is along x-axis and $e = \frac{1}{3}$
    (d) vertices are at $(\pm 2, 0)$ and foci at $(\pm 6, 0)$
    (e) minor axis is 10 and foci at $(\pm 4, 0)$
    (f) distance between the foci is 16 and distance between directrices is 30.
    (g) focus is at $(\sqrt{5}, 0)$ and asymptotes are $2y = \pm x$.

14. Show that the vertical distance between the asymptotes and the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ approaches 0.

15. Discuss the graph of each of the following conic sections:
    (a) $x^2 + 4x + y^2 = 12$
    (b) $y^2 + 20 = x^2 + 10y + 4x$
    (c) $x^2 + 2y^2 - 2x - 4y = -1$
    (d) $x^2 - y^2 - 2x + 4y = 4$
    (e) $y^2 - 4x^2 + 16x = 24$
    (f) $2x^2 - y^2 + 6y = 3$
    (g) $x + 20y^2 + 40y + 27 = 0$
    (h) $2x^2 + 2y^2 - 28x + 12y = -114$
    (i) $2y + 180x = x^2 + 7950$

16. Find a Cartesian equation for the hyperbola centred at $(3, 0)$ with $x = 1$ as one equation of the directrix and distance from vertex to the directrix equal to $\frac{3}{2}$.

17. Discuss the graph of each of the following conics:
    (a) $3x^2 + 4\sqrt{3}xy - y^2 = 7$
    (b) $x^2 + xy + y^2 = 1$
    (c) $3x^2 + 2\sqrt{3}xy + y^2 - 8x + 8\sqrt{3}y = 0$
    (d) $x^2 - \sqrt{3}xy + 2y^2 = 1$
    (e) $3x^2 + 2\sqrt{3}xy + y^2 - 8x + 8\sqrt{3}y = 0$
    (f) $3x^2 - 2\sqrt{3}xy + y^2 = 1$
    (g) $\sqrt{2}x^2 + 2\sqrt{2}xy + \sqrt{2}y^2 - 8x + 8y = 0$
    (h) $xy - y - x + 1 = 0$
    (i) $x^2 - 4xy + 4y^2 - 5 = 0$
    (j) $3x^2 + 4\sqrt{3}xy - y^2 = 7$
    (k) $x^2 + y^2 + 2xy + 2\sqrt{2}x - 2\sqrt{2}y + 2 = 0$
    (l) $9x^2 + 6y^2 + 4xy - 20 = 0$
    (m) $4x^2 + 12xy + 9y^2 = 52$
    (n) $5x^2 - 2xy + 5y^2 - 12 = 0$
    (o) $9x^2 + 24xy + 16y^2 + 80x - 60y = 0$
    (p) $xy - 2y - 4x = 0$

18. Under what conditions is the graph

$$y = Ax^2 + Bx + C$$

    a parabola? If it is indeed a parabola and passes through the points $(1, 3)$, $(2, 4)$ and $(3, 7)$, find the unknown constants.

19. Use the discriminant test to identify the following conic sections:

(a) $16x^2 - 24xy + 9y^2 - 30x - 40y = 0$
(b) $x^2 - 4xy - 2y^2 - 6 = 0$
(c) $13x^2 - 8xy + 7y^2 - 45 = 0$
(d) $2x^2 + 4xy + 5y^2 + 3x - 4y - 20 = 0$
(e) $x^2 - 6xy - 5y^2 + 4x - 22 = 0$
(f) $36x^2 - 60xy + 25y^2 + 9y = 0$
(g) $x^2 + 4xy + 4y^2 - 5x - y - 3 = 0$
(h) $x^2 + xy + 4y^2 + x + y - 4 = 0$

20. Find all the polar coordinates of each of the following points:
(a) $(3, 0))$
(b) $(-2, \frac{\pi}{3}$
(c) $(5, \pi)$
(d) $(7, 135°)$
(e) $(4, 310°)$
(f) $(-3, -\frac{3\pi}{2})$

21. Find the Cartesian coordinates of each of the following points:
(a) $(\sqrt{2}, 120°)$
(b) $\left(4, \frac{\pi}{4}\right)$
(c) $(2\sqrt{3}, \frac{2\pi}{3})$
(d) $(-1, 480°)$
(e) $\left(5, \arctan(\frac{4}{3})\right)$
(f) $(7, \sin^{-1}(\frac{3}{5}))$

22. For each of the following given polar equations, find the equivalent Cartesian equation. Then identify and describe the graph.

(a) $r = 4\csc\theta$
(b) $r = 4\tan\theta\sec\theta$
(c) $r\sin\theta = \ln r + \ln\cos\theta$
(d) $r = \dfrac{7}{1-\cos\theta}$

23. Replace the following Cartesian equations by their equivalent polar equations:

(a) $x^2 - y^2 = 1$
(b) $xy = 2$
(c) $(x-5)^2 + (y+1)^2 = 4$
(d) $x^2 + xy + y^2 = 1$

24. Identify and describe the following conics:
(a) $r = \dfrac{-1}{1-\sin\theta}$
(b) $r = \dfrac{6}{1+\cos\theta}$
(c) $r = \dfrac{6}{2+\cos\theta}$
(d) $r = \dfrac{1}{1+\cos\theta}$
(e) $r = \dfrac{5}{-1+2\cos\theta}$
(f) $r(2+\sin\theta) = 4$
(g) $r(3-2\cos\theta) = 6$
(h) $r = \dfrac{400}{16+8\sin\theta}$
(i) $r = \dfrac{6\sec\theta}{2\sec\theta - 2}$
(j) $r = \dfrac{4}{1+2\cos\theta}$

25. Show that the polar equation for
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$
is
$$r^2 = \frac{-b^2}{1-e^2\cos^2\theta}.$$

26. Show that the polar equation
$$r = \frac{k}{1+e\cos\theta}$$
is equivalent to the Cartesian equation
$$(1-e^2)x^2 + y^2 + 2kex - k^2 = 0.$$

27. Halley's Comet has an elliptical orbit with the sun at one focus. Its orbit is approximately given by

$$r = \frac{1.069}{1 + 0.967 \sin \theta}.$$

Find the distance from Halley's Comet to the sun at its shortest and greatest distance from the sun.