6. ORDINARY DIFFERENTIAL EQUATIONS

Certain mathematical relations are given in terms of unknown quantities and their derivatives. Such relations or equations involving one or more derivatives are referred to as differential equations.

6.1. *CLASSIFICATION OF ORDINARY DIFFERENTIAL EQUATIONS*

Definition 6.1.1

An equation involving derivatives of one or more dependent variables with respect to one or more independent variables is called a differential equation. ◊

Example 6.1.1

The following are differential equations:

1. $\frac{dy}{dx} = ax$
2. $\frac{d^2y}{dx^2} + xy\left(\frac{dy}{dx}\right)^7 + y = 2x$
3. $\left(\frac{d^3y}{dx^3}\right)^4 + \left(\frac{d^2y}{dx^2}\right)^5 + \frac{y}{x^2+1} = e^x$
4. $\frac{\partial v}{\partial s} + \frac{\partial v}{\partial t} = v$
5. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

△

Definition 6.1.2

A differential equation involving ordinary derivatives of one or more dependent variables with respect to a single independent variable is called an ordinary differential equation (ODE). ◊

Definition 6.1.3

An equation involving partial derivatives of one or more dependent variables with respect to more than one independent variables is called a partial differential equation (PDE). ◊

Example 6.1.2

From Example (6.1.1), equations 1, 2 and 3 are ordinary differential equations while equations 4 and 5 are partial differential equations. △

Definition 6.1.4

The order of the highest ordered derivative involved in a differential equation is called the order of the differential equation. ◊

Example 6.1.3

From Example (6.1.1), equation 1 is a first-order ODE, equation 2 is second-order ODE, equation 3 is a third-order ODE, equation 4 is a first-order PDE and equation 5 is a second-order PDE. △

Definition 6.1.5

The degree of the differential equation is the highest exponent appearing on the derivative of highest order. ◊

Example 6.1.4

From Example (6.1.1), equation 1 is of first degree, equation 2 is of first degree, equation 3 is of fourth degree, equation 4 is of first degree and equation 5 is of first degree. △

Definition 6.1.6

A linear ODE of order $n,$ in the dependent variable y and the independent variable $x,$ is an equation that is, or can be expressed in the form

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\ldots+a_1(x)\frac{dy}{dx}+a_0(x)y=b(x),$$

where a_n is not identically zero. If $a_0(x),\ a_1(x),\ldots,a_n(x)$ are constants, then the equation is said to have constant coefficients. Otherwise, it is said to have variable coefficients. ◊

Example 6.1.5

Which of the following are linear ODEs?

1. $\frac{d^2y}{dx^2}+5\frac{dy}{dx}+6y=0$
2. $\frac{d^4y}{dx^4}+x^2y\frac{d^3y}{dx^3}+y\frac{dy}{dx}=xe^x$
3. $\frac{d^2y}{dx^2}+5y\frac{dy}{dx}+6y=0$

4. $e^x\left(\frac{d^3y}{dx^3}\right)^2 - y^4x^2\cot^2 x + xy + 1$

Solutions:

Equation 1 is linear with constant coefficients. Equation 2 is non-linear because $a_3 = x^2y$ is a function of x and y. Equation 3 is not linear because $a_1 = 5y$ is not a function x. Equation 4 is non-linear because $\frac{d^3y}{dx^3}$ has power 2. △

Definition 6.1.7

If, in definition (6.1.6), $b(x) = 0$, for every number x in the domain of f, then such an ODE is said to be a linear homogeneous ODE. Otherwise, we say that the equation is non-homogeneous. ◊

Definition 6.1.8

Consider the n^{th}-order ODE

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}\right) = 0 \qquad \textbf{(6.1)}$$

where F is a real function of its $(n+2)$ arguments $x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}$.

1) Let f be a real function defined for all x in a real interval I and having an n^{th} derivative (and hence also all lower ordered derivative) of all $x \in I$. The function f is called an explicit solution of (6.1) on I if it satisfies the following conditions:
   (a) $F\left(x, f(x), f'(x), \ldots, f^{(n)}(x)\right)$ is defined for all $x \in I$, and
   (b) $F\left(x, f(x), f'(x), \ldots, f^{(n)}(x)\right) = 0$ for all $x \in I$.
2) A relation $g(x, y) = 0$ is called an implicit solution of (6.1) if this relation defines at least one real function f of the variable x on an interval I such that this function is an explicit solution of (6.1) on this interval.
3) Both explicit and implicit solutions are usually called solution of the ODE.

◊

Example 6.1.6

1. Show that $y = e^{4x}$ is an explicit solution of the ODE

$$\frac{dy}{dx} = 4y.$$

2. Show that the ODE

$$x + y\frac{dy}{dx} = 0$$

   has an implicit solution $x^2 + y^2 = r^2$.

3. Show that $y = 2\sin x + 3\cos x$ is an explicit solution of the ODE

$$\frac{d^2y}{dx^2} + y = 0.$$

Solutions:

1. Note that $y = e^{4x}$ is defined for all $x \in \mathbb{R}$ and $\frac{dy}{dx} = 4e^{4x}$. Thus,

$$\frac{dy}{dx} - 4y = 4e^{4x} - 4e^{4x} = 0.$$

Hence, $y = e^{4x}$ is an explicit solution of the ODE.

Note that in general $y = Ce^{4x}$, where C is any real constant, is also a solution.

2. Note that $x^2 + y^2 = r^2$ is defined for all $\{(x, y) : -r \le x, y \le r\}$.

$$x^2 + y^2 = r^2 \Rightarrow 2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow x + y\frac{dy}{dx} = 0,$$

showing that $x^2 + y^2 = r^2$ is an implicit solution of the ODE.

Note that from this implicit solution, we can get two explicit solutions

$y = f(x) = -\sqrt{r^2 - x^2}$ and $y = f(x) = \sqrt{r^2 - x^2}$.

3. Note that $y = 2\sin x + 3\cos x$ is defined for all real x. Then,

$$\frac{dy}{dx} = 2\cos x - 3\sin x \quad \text{and} \quad \frac{d^2y}{dx^2} = -2\sin x - 3\cos x.$$

Thus,

$$\frac{d^2y}{dx^2} + y = -2\sin x - 3\cos x + 2\sin x + 3\cos x = 0,$$

implying that $y = 2\sin x + 3\cos x$ is an explicit solution of the ODE.

Note that $y_1 = \sin x$, $y_2 = \cos x$ and $y_3 = y_1 + y_2 = \sin x + \cos x$ are also solutions. In general, $y = A\sin x + B\cos x$ is an explicit solution of the ODE.

△

6.2. ***FIRST-ORDER ODEs***

A first-order ODE of first degree

$$\frac{dy}{dx} = -f(x,y) \tag{6.2}$$

can be written as

$$M(x,y)dx + N(x,y)dy = 0, \tag{6.3}$$

where $f(x,y) = \dfrac{M(x,y)}{N(x,y)}$.

6.2.1. Variable Separable Equations:

Occasionally, we can find a function $D(x,y)$ such that when (6.3) is divided throughout by D we obtain

$$g(x)dx + h(y)dy = 0.$$

When this occurs, we say that the variables are separable and we can obtain the solution by integrating “directly”, i.e.

$$\int g(x)dx + \int h(y)dy = c$$

is the solution.

Example 6.2.1

Solve the following ODEs:

1. $(y^2 - 1)dx - (4y + 2xy)dy = 0$
2. $\dfrac{dy}{dx} = \left(1 + y^2\right)e^x.$

Solutions:

1. Choose $D(x,y)=(y^2-1)(2+x)$ so that dividing throughout by $D(x,y)$ gives

$$\frac{(y^2-1)}{(y^2-1)(2+x)}dx-\frac{(4y+2xy)}{(y^2-1)(2+x)}dy=0$$
$$\Rightarrow \frac{1}{2+x}dx-\frac{2y}{y^2-1}dy=0$$

$$\therefore \int\frac{1}{2+x}dx-\int\frac{2y}{y^2-1}dy=c_1$$
$$\Rightarrow \ln|2+x|-\ln|y^2-1|=c_1$$
$$\Rightarrow \ln\left|\frac{2+x}{y^2-1}\right|=\ln|c_1|$$
$$\Rightarrow \frac{2+x}{y^2-1}=c \quad \text{or} \quad c(y^2-1)=2+x$$

2. $\frac{dy}{dx}=\left(1+y^2\right)e^x \ \Rightarrow dy-e^x\left(1+y^2\right)dx=0$

Choosing $D(x,y)=(1+y^2),$ we get

$$\frac{1}{1+y^2}dy-e^xdx=0 \ \Rightarrow \int\frac{1}{1+y^2}dy-\int e^xdx=c$$
$$\Rightarrow \arctan\ y-e^x=c$$
$$\Rightarrow \arctan y=e^x+c$$
$$\Rightarrow y=\tan\left(e^x+c\right)$$

△

6.2.2. Homogeneous Equations:

Occasionally, an ODE whose variables are not separable can be transformed by a change of variable into an equation whose variables are separable. This can be done if M and N in equation (6.3), are both homogeneous functions of the same degree. Recall that a function $f(x,y)$ is said to be homogeneous of degree n in a region D if

$$f(tx,ty)=t^nf(x,y),$$

for all $t>0$ and (x,y) in $D.$

Theorem 6.2.1

If in equation (6.3), M and N are both homogeneous functions of the same degree, then the substitution $y=vx$ along with its corresponding derivative

$$\frac{dy}{dx}=v+x\frac{dv}{dx}$$

will transform the into one in v and x in which the variables are separable. Alternatively, rewriting the differential equation as

$$\frac{dx}{dy} = \frac{N(x,y)}{M(x,y)}$$

and substituting $x = yu$ along with its corresponding derivative

$$\frac{dx}{dy} = u + y\frac{du}{dy}$$

will transform the into one in u and y in which the variables are separable.

$\square$

Example 6.2.2

Solve the following ODEs:

1. $\left(y^2 - 2xy + 4x^2\right)dx + 2x^2 dy = 0$
2. $2xydy + \left(x^2 + y^2\right)dx = 0$

Solutions:

1. Check that the variables are not separable and that

$$M\left(x,y\right) = y^2 - 2xy + 4x^2 \quad \text{and} \quad N\left(x,y\right) = 2x^2$$

are homogeneous of degree 2. Thus,

$$\frac{dy}{dx} = -\frac{\left(y^2 - 2xy + 4x^2\right)}{2x^2} = \frac{-y^2 + 2xy - 4x^2}{2x^2}$$

Letting $y = vx$ gives

$$\begin{aligned}
& v + x\frac{dv}{dx} = \frac{-(vx)^2 + 2x(vx) - 4x^2}{2x^2} \\
& \Rightarrow v + x\frac{dv}{dx} = \frac{-v^2 + 2v - 4}{2} \\
& \Rightarrow 2v + 2x\frac{dv}{dx} = -v^2 + 2v - 4 \\
& \Rightarrow 2xdv + (v^2 + 4)dx = 0 \\
& \Rightarrow \frac{1}{v^2 + 4}dv + \frac{1}{2x}dx = 0 \\
& \Rightarrow \int \frac{1}{v^2 + 4}dv + \frac{1}{2}\int \frac{1}{x}dx = c \\
& \Rightarrow \frac{1}{2}\arctan\left(\frac{v}{2}\right) + \frac{1}{2}\ln|x| = \frac{1}{2}\ln|c| \\
& \Rightarrow \arctan\left(\frac{v}{2}\right) = \ln\left|\frac{c}{x}\right| \\
& \Rightarrow \frac{v}{2} = \tan\left(\ln\left|cx^{-1}\right|\right)
\end{aligned}$$

Since $v=\frac{y}{x}$, it follows that

$$\frac{y}{2x}=\tan\left(\ln\left|cx^{-1}\right|\right) \quad \text{or} \quad y=(2x)\tan\left(\ln\left|cx^{-1}\right|\right)$$

is the solution.

2. Clearly, $M\left(x,y\right)=x^2+y^2$ and $N\left(x,y\right)=2xy$ are homogeneous of degree 2. Letting $x=yu$ gives

$$\begin{aligned}\frac{dx}{dy}=\frac{-2xy}{x^2+y^2} &\Leftrightarrow u+y\frac{du}{dy}=\frac{-2(yu)y}{(yu)^2+y^2}\\ &\Rightarrow u+y\frac{du}{dy}=\frac{-2u}{u^2+1}\\ &\Rightarrow\left(u+\frac{2u}{u^2+1}\right)dy+ydu=0\\ &\Rightarrow\left(\frac{u^3+3u}{u^2+1}\right)dy+ydu=0\\ &\Rightarrow\frac{1}{y}dy+\left(\frac{u^2+1}{u(u^2+3)}\right)du=0\end{aligned}$$

Note that

$$\frac{u^2+1}{u(u^2+1)}=\frac{1}{3}.\frac{1}{u}+\frac{1}{3}.\frac{2u}{u^2+3}.$$

$$\begin{aligned}\frac{1}{y}dy+\left(\frac{u^2+1}{u(u^2+1)}\right)du=0 &\Leftrightarrow\int\frac{1}{y}dy+\int\left(\frac{u^2+1}{u(u^2+1)}\right)du=c_1\\ &\Rightarrow\int\frac{1}{y}dy+\frac{1}{3}\int\frac{1}{u}du+\frac{1}{3}\int\frac{2u}{u^2+3}du=c_1\\ &\Rightarrow\ln|y|+\frac{1}{3}\ln|u|+\frac{1}{3}\ln\left(u^2+3\right)=\ln|c_1|\end{aligned}$$

Since $u=\frac{x}{y}$, it follows that

$$\ln|y| + \frac{1}{3}\ln|u| + \frac{1}{3}\ln\left(u^2+3\right) = \ln|c_1| \Leftrightarrow \ln|y| + \frac{1}{3}\ln\left|\frac{x}{y}\right| + \frac{1}{3}\ln\left(\frac{x^2}{y^2}+3\right) = \ln|c_1|$$

$$\Rightarrow \ln|y| + \frac{1}{3}\ln|x| - \frac{1}{3}\ln|y| + \frac{1}{3}\ln\left(x^2+3y^2\right) - \frac{2}{3}\ln|y| = \ln|c_1|$$

$$\Rightarrow \frac{1}{3}\ln|x| + \frac{1}{3}\ln\left(x^2+3y^2\right) = \frac{1}{3}\ln|c_1|$$

$$\Rightarrow \ln\left|x\left(x^2+3y^2\right)\right| = \ln|c_1|$$

$$\Rightarrow x\left(x^2+3y^2\right) = e^{c_1}$$

$\therefore x\left(x^2+3y^2\right) = c$

is the solution. △

6.2.3. Exact Equations and Integrating Factor:

Suppose that we take the total differential of the equation $g(x,y)=c.$ Then,

$$dg = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy = 0.$$

Reversing this "process", we can solve equation (6.3) by finding a function $g(x,y)=c$ such that

$$\frac{\partial g}{\partial x} = M(x,y) \text{ and } \frac{\partial g}{\partial y} = N(x,y),$$

i.e. $g(x,y)=c$ will be the general solution of equation (6.3). In this case, equation (6.3) is said to be an exact ODE.

TEST FOR EXACTNESS: If $M(x,y)$ and $N(x,y)$ are continuous functions and have continuous first order partial derivatives on some region $R,$ then equation (6.3) is exact if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

If (6.3) is not exact, it is occasionally possible to find a function $I(x,y)$ such that the equation

$$I(x,y)\left[M(x,y)dx + N(x,y)dy\right] = 0$$

is exact. This function $I(x,y)$ is called an integrating factor.

If

$$\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=f(x),$$

is a function of x only, then

$$I(x)=e^{\int f(x)\,dx}.$$

If

$$\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=h(y),$$

is a function of y only, then

$$I(y)=e^{\int h(y)\,dy}.$$

Example 6.2.3

Solve the following ODEs:

(a) $\left(3x^2+2xy^2+4y\right)dx+\left(2x^2y+4x+5y^4\right)dy=0$

(b) $\left(3x^2y+2xy+y^3\right)dx+\left(x^2+y^2\right)dy=0$

Solutions:

(a) Using the exactness test, with $M\left(x,y\right)=3x^2+2xy^2+4y$ and $N\left(x,y\right)=2x^2y+4x+5y^4$, we get

$$\frac{\partial M}{\partial y}=4xy+4=\frac{\partial N}{\partial x}$$

implying that the differential equation is exact. Using either

$$\frac{\partial g}{\partial x}=M\left(x,y\right) \text{ or } \frac{\partial g}{\partial y}=N\left(x,y\right)$$

we find the solution $g(x,y)=c$.

$$\begin{aligned}\frac{\partial g}{\partial x}=M\left(x,y\right)\Rightarrow g\left(x,y\right)&=\int M\left(x,y\right)\partial x\\&=\int\left(3x^2+2xy^2+4y\right)\partial x\\&=x^3+x^2y^2+4xy+h(y)\end{aligned}$$

Thus,

$$\frac{\partial g}{\partial y} = 0 + 2x^2 y + 4x + h'(y)$$

But

$$\begin{aligned}
\frac{\partial g}{\partial y} = N(x,y) &\Rightarrow 2x^2 y + 4x + h'(y) = 2x^2 y + 4x + 5y^4 \\
&\Rightarrow h'(y) = 5y^4 \\
&\Rightarrow h(y) = \int 5y^4\, dy = y^5 + c \\
\therefore g(x,y) = x^3 + x^2 y^2 + 4xy + y^5 + c = 0 \quad &\text{or} \quad x^3 + x^2 y^2 + 4xy + y^5 = c
\end{aligned}$$

is the general solution.

(b) With $M(x,y) = 3x^2 y + 2xy + y^3$ and $N(x,y) = x^2 + y^2$, check that

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

implying that the differential equation is not exact.

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{x^2 + y^2}\left(3x^2 + 2x + 3y^2 - 2x\right) = 3$$

and

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{3x^2 y + 2xy + y^3}\left(2x - 3x^2 - 2x - 3y^2 - 2x\right) = \frac{-3\left(x^2 + y^2\right)}{3x^2 y + 2xy + y^3}$$

$$\therefore I(x) = e^{\int 3dx} = e^{3x}$$

Thus,

$$e^{3x}\left[\left(3x^2 y + 2xy + y^3\right)dx + \left(x^2 + y^2\right)dy\right] = 0$$

$$\Rightarrow M(x,y) = e^{3x}\left(3x^2 y + 2xy + y^3\right) = 3x^2 y e^{3x} + 2xye^{3x} + y^3 e^{3x},$$

$$N(x,y) = e^{3x}\left(x^2 + y^2\right) = x^2 e^{3x} + y^2 e^{3x}$$

and

$$\frac{\partial M}{\partial y} = 3x^2e^{3x} + 2xe^{3x} + 3y^2e^{3x} = \frac{\partial N}{\partial x}$$

$$\begin{aligned} g(x,y) &= \int N \partial y = \int \left(x^2e^{3x} + y^2e^{3x}\right)\partial y \\ &= x^2ye^{3x} + \frac{y^3}{3}e^{3x} + h(x) \\ \frac{\partial g}{\partial x} = M(x,y) &\Leftrightarrow 2xye^{3x} + 3x^2ye^{3x} + y^3e^{3x} + h'(x) = 3x^2ye^{3x} + 2xye^{3x} + y^3e^{3x} \\ &\Rightarrow h'(x) = 0 \\ \therefore g(x,y) &= x^2ye^{3x} + \frac{y^3}{3}e^{3x} = c \end{aligned}$$

is the general solution.

△

6.2.4. First-Order Linear ODE

Suppose that we have a first-order linear ODE

$$a_1(x)\frac{dy}{dx} + a_0(x)y = b(x). \qquad \textbf{(6.4)}$$

Equation (6.4) can also be written as

$$\frac{dy}{dx} + \frac{a_0(x)}{a_1(x)}y = \frac{b(x)}{a_1(x)},$$

and letting $p(x) = \frac{a_0(x)}{a_1(x)}$ and $q(x) = \frac{b(x)}{a_1(x)}$, we have that

$$\frac{dy}{dx} + p(x)y = q(x). \qquad \textbf{(6.5)}$$

We will consider (6.5) as a general equation of a first-order linear nonhomogeneous ODE.

If (6.5) is homogeneous, i.e. $q(x) = 0$, then using separation of variables, we get

$$\begin{aligned}\frac{dy}{dx}+p(x)y=0\Rightarrow dy+p(x)ydx=0\\ &\Rightarrow \frac{1}{y}dy+p(x)dx=0\\ &\Rightarrow \int\frac{1}{y}dy+\int p(x)dx=c_1\\ &\Rightarrow \ln|y|+\int p(x)dx=c_1\\ &\Rightarrow \ln|y|=c_1-\int p(x)dx\\ &\Rightarrow y=e^{c_1-\int p(x)dx},\end{aligned}$$

i.e. $y=ce^{-\int p(x)dx}$ is the general solution.

If (6.5) is nonhomogeneous, then by fundamental theorem of calculus,

$$\frac{d}{dx}\left(\int p(x)dx\right)=p(x)$$

so that

$$\frac{d}{dx}\left(e^{\int p(x)dx}\right)=p(x)e^{\int p(x)dx}.$$

Multiplying (6.5) by the integrating factor $I(x)=e^{\int p(x)dx}$, we get

$$e^{\int p(x)dx}\frac{dy}{dx}+p(x)ye^{\int p(x)dx}=q(x)e^{\int p(x)dx}$$

so that

$$\begin{aligned}\frac{d}{dx}\left(ye^{\int p(x)dx}\right)&=p(x)ye^{\int p(x)dx}+e^{\int p(x)dx}\frac{dy}{dx}\\ &=e^{\int p(x)dx}\left(\frac{dy}{dx}+p(x)y\right)\\ &=q(x)e^{\int p(x)dx}\end{aligned}$$

and since $q(x)e^{\int p(x)dx}$ is a function of x only, we have that

$$\begin{aligned}&ye^{\int p(x)dx}=\int q(x)e^{\int p(x)dx}dx\\ &\Rightarrow y=e^{-\int p(x)dx}\left[\int q(x)e^{\int p(x)dx}dx\right]\end{aligned}$$

is the general equation.

__Example 6.2.4__

Solve the following ODEs:

1. $\dfrac{dy}{dx}+3y=0$
2. $\dfrac{dy}{dx}+3y=x$
3. $x\dfrac{dy}{dx}-3y=x^2, \quad x>0$
4. $\dfrac{dy}{dx}+\dfrac{1}{x}y=\dfrac{4}{x^2}+10x, \;\; x>0.$

<u>Solutions</u>:

1. This is a homogeneous equation with $p(x)=3$ and so

$$y=ce^{-\int 3dx}=ce^{-3x}$$

is the general solution.

2. Exercise

3. Note that

$$x\frac{dy}{dx}-3y=x^2 \Leftrightarrow \frac{dy}{dx}-\frac{3}{x}y=x$$

implying that $p(x)=-\dfrac{3}{x},\; q(x)=x$ and so $I(x)=e^{\int -\frac{3}{x}dx}=e^{-3\ln x}=x^{-3}$

$$\therefore\; y=\left(x^{-3}\right)^{-1}\int x.x^{-3}dx=x^3\int x^{-2}dx=x^3\left(-x^{-1}+c\right)=-x^2+cx^3$$

4. Here $p(x)=\dfrac{1}{x},\; q(x)=\dfrac{4}{x^2}+10x$ implying that $I(x)=e^{\int \frac{1}{x}dx}=x$ and

$$y=x^{-1}\int x\left(\tfrac{4}{x^2}+10x\right)dx=x^{-1}\int\left(\tfrac{4}{x}+10x^2\right)dx=x^{-1}\left(4\ln x+\frac{10}{3}x^3+c\right)=\frac{4}{x}\ln x+\frac{10}{3}x^2+\frac{c}{x}$$

is the general solution. △

From Example (6.2.4), note that in part (3), the function cx^3 is a solution to the homogeneous equation

$$x\frac{dy}{dx}-3y=0$$

while $-x^2$ is one particular solution to the nonhomogeneous equation. Similarly, $\frac{c}{x}$ is a solution to the homogeneous equation

$$\frac{dy}{dx}+\frac{1}{x}y=0$$

while $\frac{4}{x}\ln x+\frac{10}{3}x^2$ is one particular solution to the nonhomogeneous equation. This is always true as the next theorem shows:

Theorem 6.2.2

1. Let y_1 and y_2 be two non-zero solutions to the homogeneous equation

   $$\frac{dy}{dx}+p(x)y=0.$$

   Then, for some constant α,

   $$y_1(x)=\alpha y_2(x), \text{ for every real number } x.$$

2. Let y_c be the solution to the homogeneous equation and y_p be one particular solution to the nonhomogeneous equation. Then, the general solution y, is given by

   $$y=y_c+y_p$$ □

6.2.5. Bernoulli's Equation

Certain non-linear first-order equations can be reduced to linear equations by a suitable change of variable. One such equation is

$$\frac{dy}{dx}+p(x)y=q(x)y^n, \tag{6.6}$$

which is known as Bernoulli's equation. To solve this equation, set $z=y^{1-n}$ so that

$$\frac{dz}{dx}=(1-n)y^{-n}\frac{dy}{dx}.$$

If we multiply both sides of equation (6.6) by $(1-n)y^{-n}$, we obtain

$$(1-n)y^{-n}\frac{dy}{dx}+(1-n)p(x)y^{1-n}=(1-n)q(x) \qquad \text{or} \qquad \frac{dz}{dx}+(1-n)p(x)z=(1-n)q(x),$$

which is now a linear equation.

Example 6.2.5

Solve the following ODEs:

1. $\frac{dy}{dx} + xy = xe^{-x^2} y^{-3}$
2. $\frac{dy}{dx} - \frac{y}{x} = -\frac{5}{2} x^2 y^3$

Solutions:

1. Here $n = -3,\ p(x) = x$ and $q(x) = xe^{-x^2}$. Setting $z = y^{1-n} = y^4$, we get

$$\frac{dz}{dx} + (1-n)p(x)z = (1-n)q(x) \Leftrightarrow \frac{dz}{dx} + (1-(-3))xz = (1-(-3))xe^{-x^2}$$

$$\Rightarrow \frac{dz}{dx} + 4xz = 4xe^{-x^2}.$$

Thus, $I(x) = e^{\int 4x dx} = e^{2x^2}$ so that

$$z = \left(e^{2x^2}\right)^{-1} \int e^{2x^2} 4xe^{-x^2}\, dx = e^{-2x^2} \int 4xe^{x^2}\, dx = e^{-2x^2}\left(2e^{x^2} + c\right) = 2e^{-x^2} + ce^{-2x^2}$$

Therefore, the general solution is

$$y^4 = 2e^{-x^2} + ce^{-2x^2}$$

2. Exercise.

△

6.3. SECOND-ORDER LINEAR ODEs

Suppose that we have a second-order linear ODE of first degree

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = b(x). \qquad \textbf{(6.7)}$$

We first consider equation (6.7) with constant coefficients.

Definition 6.3.1

1. Let y_1 and y_2 be any two functions. By linear combination of y_1 and y_2 we mean a function y that can be written in the form

$$y = c_1 y_1 + c_2 y_2$$

for some constants c_1 and c_2 both not zero.

2. Two functions are linearly independent on a given interval whenever the relation

$$c_1 y_1 + c_2 y_2 = 0$$

for all x in the given interval implies that $c_1 = 0 = c_2$. Otherwise, they are linearly dependent.

In other words, two functions are linearly dependent on a given interval if and only if one of the functions is a constant multiple of the other. ◊

Example 6.3.1

1. The functions $y_1 = e^{-5x}$ and $y_2 = e^{2x}$ are linearly independent solutions of the ODE

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 10y = 0.$$

2. The functions $y_1 = \cos x$ and $y_2 = \sin x$ are independent solutions of

$$\frac{d^2y}{dx^2} = -y.$$

△

Theorem 6.3.1

Let y_1 and y_2 be two linearly independent solutions of the homogeneous equation

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = 0 \tag{6.8}$$

of equation (6.7). Then, any linear combination of y_1 and y_2 is also a solution of (6.8). If y_c represents a general solution to (6.8), then

$$y_c = c_1y_1 + c_2y_2$$

for some constants c_1 and c_2. □

Example 6.3.2

1. The general solution to the ODE $y'' + 3y' - 10y = 0$ is $y_c = c_1e^{-5x} + c_2e^{2x}$.
2. The general solution to the ODE $y'' + y = 0$ is $y_c = A\cos x + B\sin x.$

△

Definition 6.3.2

Let y_1 and y_2 be any two solutions to (6.8). The Wronskian of y_1 and y_2, denoted $W(y_1, y_2)$, is defined as

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'.$$

◊

Theorem 6.3.2

The solutions y_1 and y_2 of (6.8) are linearly independent on a given interval if $W(y_1, y_2) \neq 0.$ □

Exercise: Check that $W\left(y_1, y_2\right) \neq 0$ for Example (6.3.2).

Now, consider the nonhomogeneous equation (6.7) with constant coefficients. Let $y_p(x)$ be one particular solution to (6.7) and y_1 and y_2 be two linearly independent solutions to the corresponding homogeneous equation (6.8). Then, the general solution to (6.7) is given by

$$\begin{aligned} y &= y_c + y_p \\ &= c_1 y_1 + c_2 y_2 + y_p. \end{aligned}$$

6.3.1. Homogeneous With Constant Coefficients

We first assume that $y = e^{rx}$ is a solution to (6.8). Then, $\frac{dy}{dx} = re^{rx}$ and $\frac{d^2y}{dx^2} = r^2 e^{rx}$ so that

$$\begin{aligned} a_2 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = 0 &\Leftrightarrow a_2 r^2 e^{rx} + a_1 r e^{rx} + a_0 e^{rx} = 0 \\ &\Rightarrow e^{rx}\left(a_2 r^2 + a_1 r + a_0\right) = 0. \end{aligned}$$

Since $e^{rx} \neq 0$ for any real number $x,$ it follows that

$$a_2 r^2 + a_1 r + a_0 = 0 \qquad \textbf{(6.9)}$$

Equation (6.9) is called the characteristic equation and it can easily be solved since a_0, a_1 and a_2 are constants. This gives us three possible cases depending on the nature of the roots of (6.9).

Case 1: *Real and Distinct Roots*

If (6.9) has distinct real roots r_1 and r_2 then we get two linearly independent solutions $y_1 = e^{r_1 x}$ and $y_2 = e^{r_2 x}$ so that

$$y_c = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

is the general solution to (6.8).

Case 2: *Real Equal Roots*

If (6.9) has equal roots, then the general solution is

$$y_c = c_1e^{rx} + c_2xe^{rx}$$

is the general solution to (6.8).

<u>Case 3</u>: *Complex Conjugate Roots*

If the roots of (6.9) are complex conjugates $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$, then the solution can be written as

$$y_c = c_1e^{(\alpha+i\beta)x} + c_2e^{(\alpha-i\beta)x} = e^{\alpha x}\left(c_1e^{i\beta x} + c_2e^{-i\beta x}\right).$$

Using Euler's formula $e^{iz} = \cos z + i\sin z,$ we have that

$$\begin{aligned} y_c &= e^{\alpha x}\left[c_1\left(\cos\beta x + i\sin\beta x\right) + c_2\left(\cos\beta x - i\sin\beta x\right)\right] \\ &= e^{\alpha x}\left[(c_1+c_2)\cos\beta x + (ic_1 - ic_2)\sin\beta x\right]. \end{aligned}$$

Letting $A = c_1 + c_2$ and $B = i(c_1 - c_2)$ we get the general solution

$$y_c = e^{\alpha x}\left(A\cos\beta x + B\sin\beta x\right)$$

<u>Example 6.3.3</u>

Solve the following ODEs:

1. $y'' + 3y' - 10y = 0$
2. $4\dfrac{d^2y}{dx^2} + 20\dfrac{dy}{dx} + 25y = 0$
3. $y'' + 10y' + 29y = 0$
4. $y'' + y = 0$

<u>Solutions</u>:

1. The characteristic equation is

$$r^2 + 3r - 10 = 0 \Rightarrow r_1 = 2 \text{ and } r_2 = -5$$

$$\therefore\ y_c = c_1e^{2x} + c_2e^{-5x}$$

2. $4r^2 + 20r + 25 = 0$ is the characteristic equation implying that $r = -\dfrac{5}{2}.$

$$\therefore\ y_c = c_1e^{-\frac{5}{2}x} + c_2xe^{-\frac{5}{2}x}$$

3. The characteristic equation is

$$r^2+1=0 \Rightarrow r=\pm\sqrt{-1}$$
$$\Rightarrow r=\pm\, i$$
$$\therefore\ \alpha=0 \text{ and } \beta=1$$

$$\therefore\ y_c = A\cos x + B\sin x$$

4. Solving the characteristic equation, we get

$$r^2+10r+29=0 \Rightarrow r=\frac{-10\pm\sqrt{(10)^2-4(1)(29)}}{2(1)}$$
$$\Rightarrow r_1=-5+2i \quad\text{and}\quad r_2=-5-2i$$

$$\therefore\ y_c = e^{-5x}\left(A\cos 2x + B\sin 2x\right)$$

△

6.3.2. Nonhomogeneous With Constant Coefficients

We wish to find a solution to equation (6.7) with constant coefficients. We will use two methods to solve equation (6.7).

1. Method of Undetermined Coefficients

This method is applied when the function $b(x)$ is of the type we will call "Undetermined Coefficient" (UC) function.

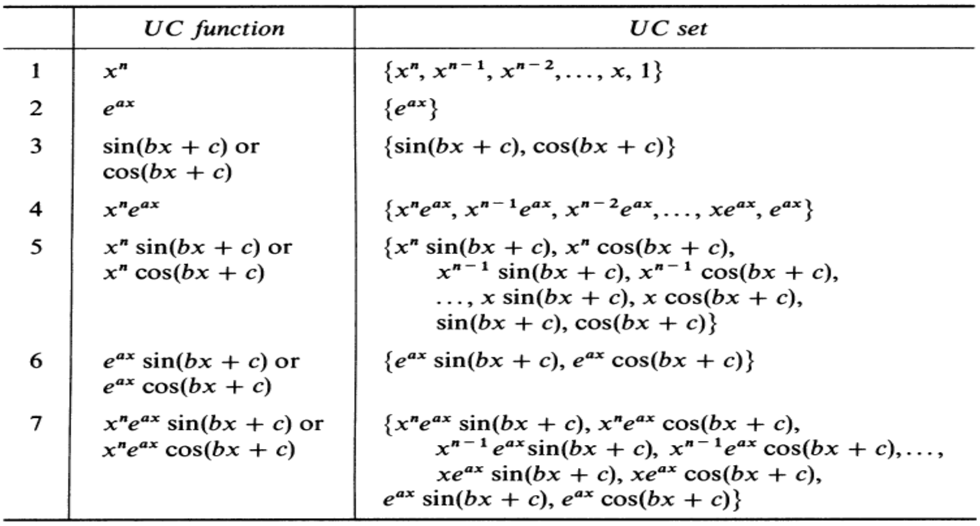
Definition 6.3.3

We call a function a UC function if it is either

1. a function defined by one of the following:
   - (i) a polynomial $p_n(x)$ of n^{th} order
   - (ii) e^{kx}, where k is a non-zero constant
   - (iii) $\sin(bx+c)$ or $\cos(bx+c)$, where b and v are non-zero constants
2. a function defined as a finite product of two or functions of these three types in (1).

◊

If $b(x)$ is a UC function, then we make a "scientific guess" of a particular solution with undetermined coefficients. We then use this particular solution in the differential equation and compare both sides to "determine" the unknown coefficients. The following table can help to make the correct guess:



	UC function	*UC set*
1	x^n	$\{x^n, x^{n-1}, x^{n-2}, \dots, x, 1\}$
2	e^{ax}	$\{e^{ax}\}$
3	$\sin(bx+c)$ or $\cos(bx+c)$	$\{\sin(bx+c), \cos(bx+c)\}$
4	$x^n e^{ax}$	$\{x^n e^{ax}, x^{n-1}e^{ax}, x^{n-2}e^{ax}, \dots, xe^{ax}, e^{ax}\}$
5	$x^n \sin(bx+c)$ or $x^n \cos(bx+c)$	$\{x^n \sin(bx+c), x^n \cos(bx+c),$ $x^{n-1}\sin(bx+c), x^{n-1}\cos(bx+c),$ $\dots, x\sin(bx+c), x\cos(bx+c),$ $\sin(bx+c), \cos(bx+c)\}$
6	$e^{ax}\sin(bx+c)$ or $e^{ax}\cos(bx+c)$	$\{e^{ax}\sin(bx+c), e^{ax}\cos(bx+c)\}$
7	$x^n e^{ax}\sin(bx+c)$ or $x^n e^{ax}\cos(bx+c)$	$\{x^n e^{ax}\sin(bx+c), x^n e^{ax}\cos(bx+c),$ $x^{n-1}e^{ax}\sin(bx+c), x^{n-1}e^{ax}\cos(bx+c), \dots,$ $xe^{ax}\sin(bx+c), xe^{ax}\cos(bx+c),$ $e^{ax}\sin(bx+c), e^{ax}\cos(bx+c)\}$

Example 6.3.4

Solve the following ODEs:

1. $y'' - 2y'' - 3y = 442\sin 5x$
2. $y'' + 6y' + 8y = 40\cos 2x + 30xe^x$
3. $y'' - 3y' + 2y = e^x \sin x$

Solutions:

1. The characteristic equation of the homogeneous equation is

$$r^2 - 2r - 3 = 0 \Rightarrow r_1 = -1 \text{ and } r_2 = 3$$
$$\therefore y_c = c_1 e^{-x} + c_2 e^{3x}$$

Since $b(x) = 442\sin 5x,$ we use $y_p = A\cos 5x + B\sin 5x$ implying that

$$y' = -5A\sin 5x + 5B\cos 5x \quad \text{and} \quad y'' = -25A\cos 5x - 25B\sin 5x.$$

Thus,

$$\begin{aligned} y'' - 2y'' - 3y = 442\sin 5x \Leftrightarrow -25A\cos 5x - 25B\sin 5x - 2\left(-5A\sin 5x + 5B\cos 5x\right) \\ -3\left(A\cos 5x + B\sin 5x\right) = 442\sin 5x \end{aligned}$$

$$\begin{aligned} &\Rightarrow -25A\cos 5x - 25B\sin 5x + 10A\sin 5x - 10B\cos 5x - 3A\cos 5x - 3B\sin 5x = 442\sin 5x \\ &\Rightarrow \left(-28A - 10B\right)\cos 5x + \left(10A - 28B\right)\sin 5x = 442\sin 5x \\ &\Rightarrow -28A - 10B = 0 \qquad \text{(i)} \end{aligned}$$

and

$$10A - 28B = 442 \qquad \text{(ii)}$$

Solving equation (i) and (ii) simultaneously, we get $A=5$ and $B=-14$ implying that

$$y_p = 5\cos 5x - 14\sin 5x.$$

Therefore,

$$y = y_c + y_p = c_1 e^{-x} + c_2 e^{3x} + 5\cos 5x - 14\sin 5x$$

is the general solution.

2. The homogeneous equation

$$y'' + 6y' + 8y = 0$$

has characteristic equation

$$\begin{aligned} &r^2 + 6r + 8 = 0 \Rightarrow r_1 = -4 \quad \text{and} \quad r_2 = -2 \\ &\therefore y_c = c_1 e^{-4x} + c_2 e^{-2x} \end{aligned}$$

$$\begin{aligned} b(x) = 4\cos 2x + 3xe^x \ \Rightarrow y_p &= A\cos 2x + B\sin 2x + (Cx+D)e^x \\ y_p' &= -2A\sin 2x + 2B\cos 2x + (Cx+D)e^x + Ce^x \\ y_p'' &= -4A\cos 2x - 4B\sin 2x + (Cx+D)e^x + 2Ce^x. \end{aligned}$$

Thus,

$$\begin{aligned} y'' + 6y' + 8y = 40\cos 2x + 30xe^x \Leftrightarrow &-4A\cos 2x - 4B\sin 2x + (Cx+D)e^x + 2Ce^x \\ &+6\left(-2A\sin 2x + 2B\cos 2x + (Cx+D)e^x + Ce^x\right) \\ &+8\left(A\cos 2x + B\sin 2x + (Cx+D)e^x\right) = 40\cos 2x + 30xe^x \end{aligned}$$

$$\begin{aligned} &\Rightarrow -4A + 12B + 8A = 40 \Rightarrow 3B + A = 10 && \text{(i)} \\ &\quad -4B - 12A + 8B = 0 \Rightarrow B - 3A = 0 && \text{(ii)} \\ &\Rightarrow A = 1 \quad \text{and} \quad B = 3 \\ &C + 6C + 8C = 30 \Rightarrow C = 2 \\ &D + 2C + 6D + 6C + 8D = 0 \Rightarrow D = -\frac{16}{15} \\ &\therefore y_p = \cos 2x + 3\sin 2x + \left(2x - \frac{16}{15}\right)e^x \end{aligned}$$

Therefore,

$$y = y_c + y_p = c_1 e^{-x} + c_2 e^{3x} + \cos 2x + 3\sin 2x + \left(2x - \frac{16}{15}\right)e^x$$

is the general solution.

3. Check that $y_c = c_1e^x + c_2e^{2x}$. Since $b(x) = e^x \sin x$, we have that

$$y_p = (A\cos x + B\sin x)e^x = Ae^x \cos x + Be^x \sin x$$

$$\Rightarrow y_p{}' = Ae^x \cos x - Ae^x \sin x + Be^x \sin x + Ae^x \cos x = (A+B)e^x \cos x + (-A+B)e^x \sin x$$

$$\begin{aligned}\Rightarrow y_p{}'' &= Ae^x \cos x - Ae^x \sin x - Ae^x \sin x - Ae^x \cos x + Be^x \sin x + Be^x \cos x + Be^x \cos x - Be^x \sin x \\ &= -2Ae^x \sin x + 2Be^x \cos x.\end{aligned}$$

Thus,

$$\begin{aligned}y'' - 3y' + 2y = e^x \sin x &\Leftrightarrow -2Ae^x \sin x + 2Be^x \cos x - 3\left[(A+B)e^x \cos x + (-A+B)e^x \sin x\right] \\ &\qquad + 2\left[Ae^x \cos x + Be^x \sin x\right] = e^x \sin x\end{aligned}$$

$$\Rightarrow -2A - 3B + 3A + 2B = 1 \Rightarrow A - B = 1 \qquad \text{(i)}$$

$$2B - 3A - 3B + 2A = 0 \Rightarrow -A - B = 0 \qquad \text{(ii)}$$

$$\Rightarrow B = -\frac{1}{2} \quad and \quad A = \frac{1}{2}$$

$$\therefore y_p = \frac{1}{2}e^x \cos x - \frac{1}{2}e^x \sin x.$$

Therefore,

$$y = c_1e^x + c_2e^{2x} + \frac{1}{2}e^x \cos x - \frac{1}{2}e^x \sin x$$

is the general solution.

△

Modification of the Method of Undetermined Coefficients

If any term of a particular solution y_p is a solution of the homogeneous equation, then we multiply y_p by x repeatedly until no term of the product $x^k y_p$ is a solution the homogeneous equation, where k is the smallest integer.

Example 6.3.5

Solve the following ODEs:

1. $y'' - 2y' + y = e^x + xe^x$
2. $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 2x^2 + e^x + 2xe^x + 4e^{3x}$
3. $y'' + y = x\sin x$

Solutions:

1. Clearly, $y_c = c_1 e^x + c_2 x e^x$ and since $b(x) = e^x + xe^x = (x+1)e^x$, we may guess that $y_p = (Ax+B)e^x$. But note that both terms in y_p are contained in y_c. We first multiply by x to get $y_p = \left(Ax^2 + Bx\right)e^x$. Again note that the new y_p contains the term xe^x which is part of y_c. Multiplying by x again, we get $y_p = \left(Ax^3 + Bx^2\right)e^x = Ax^3e^x + Bx^2e^x$ implying that

$$y_p{}' = Ax^3e^x + (3A+B)x^2e^x + 2Bxe^x \quad \text{and} \quad y_p{}'' = Ax^3e^x + (6A+B)x^2e^x + (6A+4B)xe^x + 2Be^x$$

$$\begin{aligned}
\therefore y'' - 2y' + y = e^x + xe^x \Leftrightarrow{}& Ax^3e^x + (6A+B)x^2e^x + (6A+4B)xe^x + 2Be^x \\
&- 2\left[Ax^3e^x + (3A+B)x^2e^x + 2Bxe^x\right] + Ax^3e^x + Bx^2e^x = e^x + xe^x
\end{aligned}$$

$$\begin{aligned}
\Rightarrow {}& A - 2A + A = 0, \quad \text{which is true for all values of } A \\
& 6A + B - (6A + 2B) + B = 0, \quad \text{which is true for all values of } A \text{ and } B \\
& 6A + 4B - 4B = 1 \ \Rightarrow A = \frac{1}{6} \\
& 2B = 1 \ \Rightarrow B = \frac{1}{2}
\end{aligned}$$

$$\therefore y_p = \left(\frac{1}{6}x^3 + \frac{1}{2}x^2\right)e^x$$

Thus,

$$y = y_c + y_p = (c_1 + c_2 x)e^x + \left(\frac{1}{6}x^3 + \frac{1}{2}x^2\right)e^x$$

is the general solution.

2. Check that $y_c = c_1 e^x + c_2 e^{2x}$. Since $b(x) = 2x^2 + e^x + 2xe^x + 4e^{3x} = 2x^2 + (2x+1)e^x + 4e^{3x}$, we could take $y_p = Ax^2 + Bx + C + (Dx + E)e^x + Fe^{3x}$. But e^x is contained in y_c. Thus, we take

$$\begin{aligned}
& y_p = Ax^2 + Bx + C + \left(Dx^2 + Ex\right)e^x + Fe^{3x} \\
\Rightarrow {}& y_p{}' = 2Ax + B + Dx^2e^x + 2Dxe^x + Exe^x + Ee^x + 3Fe^{3x} \\
& y_p{}'' = 2A + Dx^2e^x + 2Dxe^x + 2Dxe^x + 2De^x + Exe^x + Ee^x + Ee^x + 9Fe^{3x}
\end{aligned}$$

$$\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=2x^2+e^x+2xe^x+4e^{3x} \;\Leftrightarrow\; 2A+Dx^2e^x+2Dxe^x+2Dxe^x+2De^x$$

$$+Exe^x+Ee^x+Ee^x+9Fe^{3x}-3\left[2Ax+B+Dx^2e^x+2Dxe^x+Exe^x+Ee^x+3Fe^{3x}\right]$$

$$+2\left[Ax^2+Bx+C+\left(Dx^2+Ex\right)e^x+Fe^{3x}\right]=2x^2+e^x+2xe^x+4e^{3x}$$

$$\Rightarrow 2A-3B+2C=0$$

$$-6A+2B=0$$

$$2A=2 \;\Rightarrow A=1 \Rightarrow B=3 \text{ and } C=\frac{7}{2}$$

$$D-3D+2D=0 \quad \left(\text{True for all } D\right)$$

$$4D+E-6D-3E+2E=2 \Rightarrow D=-1$$

$$2D+2E-3E=1 \;\Rightarrow E=-3$$

$$2F=4 \;\Rightarrow F=2$$

$$\therefore y_p=x^2+3x+\frac{7}{2}+(-x^2-3x)e^x+2e^{3x}$$

Therefore,

$$y=y_c+y_p=c_1e^x+c_2e^{2x}+x^2+3x+\frac{7}{2}+(-x^2-3x)e^x+2e^{3x}$$

is the general solution.

3. Since $y_c=A\cos x+B\sin x$ and $b\left(x\right)=x\sin x,$ taking

$$y_p=Cx\cos x+Dx\sin x+E\cos x+F\sin x$$

shows that y_p contains $\cos x$ and $\sin x$ which are solutions of the homogeneous equation. Thus, we take

$$y_p=Cx^2\cos x+Dx^2\sin x+Ex\cos x+Fx\sin x$$

$$\Rightarrow y_p{}'=\left(Dx^2+\left(2C+F\right)x+E\right)\cos x+\left(-Cx^2+\left(2D-E\right)x+F\right)\sin x$$

$$y_p{}''=\left[-Cx^2+\left(4D-E\right)x+2C+2F\right]\cos x+\left[-Dx^2+\left(-4C-F\right)x+2D-2E\right]\sin x$$

$$y''+y=x\sin x \Leftrightarrow \left[-Cx^2+\left(4D-E\right)x+2C+2F\right]\cos x+\left[-Dx^2+\left(-4C-F\right)x+2D-2E\right]\sin x$$
$$+Cx^2\cos x+Dx^2\sin x+Ex\cos x+Fx\sin x=x\sin x$$

$$\Rightarrow -4C-F+F=1 \Rightarrow C=-\frac{1}{4}$$
$$4D-E+E=0 \Rightarrow D=0$$
$$2C+2F=0 \Rightarrow F=\frac{1}{4}$$
$$-2E+2D=0 \Rightarrow E=0$$
$$\therefore y_p=-\frac{1}{4}x^2\cos x+\frac{1}{4}x\sin x$$

Therefore,

$$y=y_c+y_p=A\cos x+B\sin x-\frac{1}{4}x^2\cos x+\frac{1}{4}x\sin x$$

is the general solution.

△

2. <u>Method of Variation of Parameters</u>

The method of variation of parameters is sometimes called variation of constant. We will consider a procedure developed by Lagrange who noticed that any particular solution y_p, to (6.7) must have the property that $\frac{y_p}{y_1}$ and $\frac{y_p}{y_2}$ are not constants, suggesting that we look for a particular of the form

$$y_p=v_1(x)y_1+v_2(x)y_2, \tag{6.10}$$

where y_1 and y_2 are linearly independent solutions of the homogeneous equation. Differentiating (6.10) gives

$$y_p{}'=v_1y_1{}'+v_2y_2{}'+v_1{}'y_1+v_2{}'y_2.$$

We now impose the condition that

$$v_1{}'y_1+v_2{}'y_2=0, \tag{6.11}$$

which does not violate the fact that (6.10) is a particular solution to (6.7). Thus,

$$y_p{}'=v_1y_1{}'+v_2y_2{}'. \tag{6.12}$$

Differentiating (6.12) again, we obtain

$$y_p'' = v_1 y_1'' + v_2 y_2'' + v_1' y_1' + v_2' y_2'. \qquad \textbf{(6.13)}$$

Using (6.10), (6.12) and (6.13) we get

$$a_2\left(v_1 y_1'' + v_2 y_2'' + v_1' y_1' + v_2' y_2'\right) + a_1\left(v_1 y_1' + v_2 y_2'\right) + a_0\left(v_1 y_1 + v_2 y_2\right) = b(x)$$
$$\Rightarrow v_1\left(a_2 y_1'' + a_1 y_1' + a_0 y_1\right) + v_2\left(a_2 y_2'' + a_1 y_2' + a_0 y_2\right) + a_2\left(v_1' y_1' + v_2' y_2'\right) = b(x).$$

Since y_1 and y_2 are linearly independent solutions of the corresponding homogeneous equation (6.8), we have that

$$a_2\left(v_1' y_1' + v_2' y_2'\right) = b(x)$$

or

$$v_1' y_1' + v_2' y_2' = \frac{b(x)}{a_2} \qquad \textbf{(6.14)}$$

Solving equation (6.11) and (6.14) simultaneously, we obtain

$$v_1' = \frac{\begin{vmatrix} 0 & y_2 \\ \frac{b(x)}{a_2} & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} \quad \text{and} \quad v_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & \frac{b(x)}{a_2} \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}.$$

Note that

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = W(y_1, y_2) \neq 0$$

and integration gives

$$v_1 = \int \frac{-y_2(x)b(x)}{a_2 W(y_1, y_2)}\, dx \quad \text{and} \quad v_2 = \int \frac{y_1(x)b(x)}{a_2 W(y_1, y_2)}\, dx$$

<u>Example 6.3.6</u>

Determine the solution of each of the following ODEs:

1. $y'' + y = \tan x$
2. $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 5y = e^{-2x}\sec x$

Solutions:

1. Clearly, $y_c = A\cos x + B\sin x$ implying that $y_1 = \cos x$, $y_2 = \sin x$ and $a_2 = 1$. Then,

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$\therefore v_1 = \int \frac{-y_2(x)b(x)}{a_2 W(y_1, y_2)}\, dx = \int \frac{-\sin x \tan x}{1.1} dx = -\int \sin x \tan x \, dx = -\int \frac{\sin^2 x}{\cos x} dx$$
$$= \int \frac{\cos^2 x - 1}{\cos x} dx = \int \cos x \; dx - \int \sec x \, dx$$
$$= \sin x - \ln\left|\sec x + \tan x\right|$$

and

$$v_2 = \int \frac{y_1(x)b(x)}{a_2 W(y_1, y_2)}\, dx = \int \cos x \tan x \, dx = \int \sin x \, dx = -\cos x$$

$$\therefore y_p = v_1 y_1 + v_2 y_2 = \left(\sin x - \ln\left|\sec x + \tan x\right|\right)\cos x - \sin x \cos x = -\left(\cos x\right)\ln\left|\sec x + \tan x\right|$$
$$\therefore y = y_c + y_p = A\cos x + B\sin x - \left(\cos x\right)\ln\left|\sec x + \tan x\right|.$$

2. The characteristic equation of the corresponding homogeneous equation is $r^2 + 4r + 5 = 0$

$$\Rightarrow r_1 = \frac{-4 + \sqrt{16 - 20}}{2} = -2 + i \quad \text{and} \quad r_2 = -2 - i$$

$$\Rightarrow \alpha = -2 \quad \text{and} \quad \beta = 1$$
$$\Rightarrow y_c = e^{-2x}\left(A\cos x + B\sin x\right)$$
$$\Rightarrow y_1 = e^{-2x}\cos x \quad \text{and} \quad e^{-2x}\sin x$$
$$\therefore \; W(y_1, y_2) = \begin{vmatrix} e^{-2x}\cos x & e^{-2x}\sin x \\ -2e^{-2x}\cos x - e^{-2x}\sin x & -2e^{-2x}\sin x + e^{-2x}\cos x \end{vmatrix} = e^{-4x}$$

$$\therefore v_1 = \int \frac{-y_2(x)b(x)}{a_2 W(y_1, y_2)}\, dx = -\int \frac{e^{-2x}\sin x . e^{-2x}\sec x}{e^{-4x}} dx = -\int \tan x \, dx = \ln\left|\cos x\right|$$

and

$$v_2 = \int \frac{y_1(x)b(x)}{a_2 W(y_1, y_2)}\, dx = \int \frac{e^{-2x}\cos x . e^{-2x}\sec x}{e^{-4x}} dx = \int dx = x$$

$$\therefore y_p = v_1 y_1 + v_2 y_2 = \left(e^{-2x}\cos x\right)\ln\left|\cos x\right| + xe^{-2x}\sin x$$
$$\Rightarrow y = y_c + y_p = e^{-2x}\left(A\cos x + B\sin x\right) + \left(e^{-2x}\cos x\right)\ln\left|\cos x\right| + xe^{-2x}\sin x$$
$$= e^{-2x}\left[A\cos x + B\sin x + \left(\cos x\right)\ln\left|\cos x\right| + x\sin x\right]$$

△

6.4. ***SOLUTIONS BY SERIES***

We can also use power series to find the solution to ODEs. We assume that there is a solution in the form of a Maclaurin series

$$y=\sum_{n=0}^{\infty}a_nx^n=a_0+a_1x+a_2x^2+...+a_nx^n+...,$$

where the unknown coefficient are to be determined. The next example illustrates how this method is used.

Example 6.4.1

Use series to solve the following ODEs:

1. $\frac{dy}{dx}=y$ (b) $\frac{d^2y}{dx^2}+y=0$ (c) $y''+xy'+y=0$

Solutions:

(a) Assume that

$$y=\sum_{n=0}^{\infty}a_nx^n=a_0+a_1x+a_2x^2+...+a_nx^n+...,$$

is the solution. Then

$$y'=\sum_{n=1}^{\infty}na_nx^{n-1}=a_1+2a_2x+3a_3x^2+...+na_nx^{n-1}+...,$$

$$\frac{dy}{dx}-y=0\Leftrightarrow\sum_{n=1}^{\infty}na_nx^{n-1}-\sum_{n=0}^{\infty}a_nx^n=0.$$

Let $n-1=N\Rightarrow n=N+1.$ Thus,

$$\sum_{n=1}^{\infty}na_nx^{n-1}-\sum_{n=0}^{\infty}a_nx^n=0\Leftrightarrow\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n-\sum_{n=0}^{\infty}a_nx^n=0$$

$$\Rightarrow\sum_{n=0}^{\infty}\left[(n+1)a_{n+1}-a_n\right]x^n=0$$

The series on the LHS will be zero if each coefficient is zero, i.e.

$$n=0:\ a_1-a_0=0\ \Rightarrow a_1=a_0$$

$$n=1:\ 2a_2-a_1=0\ \Rightarrow a_2=\frac{a_1}{2}=\frac{a_0}{2}=\frac{a_0}{2!}$$

$$n=2:\ 3a_3-a_2=0\ \Rightarrow a_3=\frac{a_2}{3}=\frac{a_0}{6}=\frac{a_0}{3!}$$

$$\vdots$$

and so on. Letting $a_0=c,$ we get

$a_0 = c,\ a_1 = c,\ a_2 = \frac{c}{2!},\ a_3 = \frac{c}{3!}, \ldots, a_n = \frac{c}{n!}$. Therefore,

$$
\begin{aligned}
y = \sum_{n=0}^{\infty} a_n x^n &= a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots \\
&= c + cx + \frac{c}{2!}x^2 + \frac{c}{3!}x^3 + \ldots + \frac{c}{n!}x^n + \ldots \\
&= c\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots\right) \\
&= c\sum_{n=0}^{\infty} \frac{x^n}{n!}.
\end{aligned}
$$

Recall that

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x,$$

implying that

$$y = ce^x$$

is the general solution.

(b) Assuming that

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots,$$

is the solution, we have that

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\therefore\ y'' + y = 0 \Leftrightarrow \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0.$$

Leting $n - 2 = N \Rightarrow n = N + 2$ so that

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n = 0 \Rightarrow \sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + a_n\right] x^n = 0.$$

Thus,

$$
\begin{aligned}
n = 0:&\ 2a_2 + a_0 = 0 \ \Rightarrow a_2 = -\frac{a_0}{2} \\
n = 1:&\ 6a_3 + a_1 = 0 \ \Rightarrow a_3 = -\frac{a_1}{6} = -\frac{a_1}{3!} \\
n = 2:&\ 12a_4 + a_2 = 0 \ \Rightarrow a_4 = -\frac{a_2}{12} = \frac{a_0}{24} = \frac{a_0}{4!}
\end{aligned}
$$

$$n=3:\ 20a_5+a_3=0\ \Rightarrow a_4=-\frac{a_3}{20}=\frac{a_1}{120}=\frac{a_1}{5!}$$
$$\vdots$$

and so on. Letting $a_0=A$ and $a_1=B,$ we get

$$\begin{aligned}
y=\sum_{n=0}^{\infty}a_nx^n &= a_0+a_1x+a_2x^2+\ldots+a_nx^n+\ldots \\
&= A+Bx-\frac{A}{2!}x^2-\frac{B}{3!}x^3+\frac{A}{4!}x^4+\frac{B}{5!}x^5+\ldots \\
&= A-\frac{A}{2!}x^2+\frac{A}{4!}x^4+\ldots+Bx-\frac{B}{3!}x^3+\frac{B}{5!}x^5+\ldots \\
&= A\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots\right)+B\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}+\ldots\right) \\
&= A\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}+B\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}.
\end{aligned}$$

Recall that

$$\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}=\cos x \quad\text{and}\quad \sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}=\sin x.$$

Therefore,

$$y=A\cos x+B\sin x$$

is the general solution.

(c) If $y=\sum_{n=0}^{\infty}a_nx^n=a_0+a_1x+a_2x^2+\ldots+a_nx^n+\ldots$ is the solution, then

$$y'=\sum_{n=1}^{\infty}na_nx^{n-1} \quad\text{and}\quad y''=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}$$

$$\begin{aligned}
\therefore\ y''+xy'+y=0 &\Leftrightarrow \sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}+x\sum_{n=1}^{\infty}na_nx^{n-1}+\sum_{n=0}^{\infty}a_nx^n=0 \\
&\Rightarrow \sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}+\sum_{n=1}^{\infty}na_nx^n+\sum_{n=0}^{\infty}a_nx^n=0.
\end{aligned}$$

Letting $n-2=N\Rightarrow n=N+2,$ we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=1}^{\infty}na_nx^n+\sum_{n=0}^{\infty}a_nx^n=0 \\
&\Rightarrow 2a_2+\sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=1}^{\infty}na_nx^n+a_0+\sum_{n=1}^{\infty}a_nx^n=0. \\
&\Rightarrow 2a_2+a_0+\sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2}+(n+1)a_n\right]x^n=0
\end{aligned}$$

Thus,

$$2a_2 + a_0 = 0 \Rightarrow a_2 = -\frac{a_0}{2}$$

$$n = 1:\ 3(2)a_3 + 2a_1 = 0 \Rightarrow a_3 = -\frac{a_1}{3}$$

$$n = 2:\ 4(3)a_4 + 3a_2 = 0 \Rightarrow a_4 = -\frac{a_2}{4} = \frac{a_0}{2.4}$$

$$n = 3:\ 5(4)a_5 + 4a_3 = 0 \Rightarrow a_5 = -\frac{a_3}{5} = \frac{a_1}{3.5}$$

$$n = 4:\ 6(5)a_6 + 5a_4 = 0 \Rightarrow a_6 = -\frac{a_4}{6} = -\frac{a_0}{2.4.6}$$

$$n = 5:\ 7(6)a_7 + 6a_5 = 0 \Rightarrow a_7 = -\frac{a_5}{7} = -\frac{a_1}{3.5.7}$$

$$\vdots$$

and so on. Letting $a_0 = c_1$ and $a_1 = c_2$, we get

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n \\
&= c_1 + c_2 x - \frac{c_1}{2}x^2 - \frac{c_2}{3}x^3 + \frac{c_1}{2.4}x^4 + \frac{c_2}{3.5}x^5 + \ldots + (-1)^n \frac{c_1}{2^n (n!)} x^{2n} + (-1)^n \frac{c_2}{3.5.7\ldots(2n+1)} x^{2n+1} + \ldots \\
&= c_1\left(1 - \frac{x^2}{2} + \frac{x^4}{2.4} + \ldots + (-1)^n \frac{x^{2n}}{2^n (n!)} + \ldots\right) + c_2\left(x - \frac{x^3}{3} + \frac{x^5}{3.5} + \ldots + (-1)^n \frac{x^{2n+1}}{3.5.7\ldots(2n+1)} + \ldots\right) \\
&= c_1 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^n (n!)} + c_2 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{3.5.7\ldots(2n+1)}.
\end{aligned}$$

△

6.5. APPLICATION OF ODEs

Ordinary differential equations can be used to solve many physical problems and such problems lead to what are known as Initial-Value Problems (IVPs) and Boundary-Value Problems (BVPs).

Definition 6.5.1

A differential equation solved by applying initial conditions to the general solution is known as an IVP. If the conditions are given at the end points of a finite interval, then the problem is a BVP. ◊

Example 6.5.1

1. Solve the following IVPs and BVPs:

   (a) $\frac{dy}{dx}+y=f(x), \quad y(0)=1,$

   where $f(x)=\begin{cases} e^{-x}, & 0\le x<2 \\ e^{-2}, & x\ge 2 \end{cases}$

   (b) $3\frac{d^2y}{dx^2}+4\frac{dy}{dx}-4y=0, \quad y(0)=2, \; y'(0)=-4$

   (c) $y''+y=0, \quad y(0)=1, \; y\left(\frac{\pi}{2}\right)=5$

   (d) $\frac{d^2y}{dx^2}+y=0, \quad y(0)=1, \; y(\pi)=5$

   (e) $y''-y=-x^2+1, \quad y(0)=2, \; y(1)=2e-\frac{1}{e}+2$

2. The rate at which radioactive nuclei decay is represented by the ODE

$$\frac{dy}{dt}=-ky,$$

where y is the amount of radioactive nuclei present after t years. The number of radioactive nuclei initially present is 1200 and half of that amount disintegrates after 20 years. Find the amount present after 50 years.

Solutions:

1 (a) Since this is a linear nonhomogeneous first-order ODE, we have that $I(x)=e^{\int dx}=e^x.$

We now choose $f(x)$ in the interval where the initial point $x=0.$ Thus,

$$y=e^{-x}\int e^x e^{-x}dx=e^{-x}\int dx=e^{-x}\left(x+c\right)=xe^{-x}+ce^{-x}$$
$$\therefore \; y(0)=1 \Leftrightarrow 0e^0+ce^0=1 \Rightarrow c=1$$
$$\therefore y=xe^{-x}+e^{-x}$$

(b) The characteristic equation is

$$3r^2+4r-4=0 \Rightarrow r_1=\frac{2}{3} \text{ and } r_2=-2$$
$$\Rightarrow y=y_c=c_1e^{\frac{2}{3}x}+c_2e^{-2x}$$

Applying initial conditions, we get

$$y(0)=2 \Leftrightarrow c_1e^{\frac{2}{3}(0)}+c_2e^{-2(0)}=2 \Rightarrow c_1+c_2=2 \qquad \text{(i)}$$
$$y=c_1e^{\frac{2}{3}x}+c_2e^{-2x} \Rightarrow y'(x)=\frac{2}{3}c_1e^{\frac{2}{3}x}-2c_2e^{-2x}$$

$$y'(0) = -4 \Leftrightarrow \frac{2}{3}c_1e^{\frac{2}{3}(0)} - 2c_2e^{-2(0)} = -4$$
$$\Rightarrow c_1 - 3c_2 = -6 \qquad \text{(ii)}$$

Solving equation (i) and (ii) simultaneously, we get $c_1 = 0$ and $c_2 = 2.$ Therefore,

$$y = 2e^{-2x}$$

(c) Clearly, $y = A\cos x + B\sin x$ is the general solution. Thus,

$$y(0) = 1 \Leftrightarrow A\cos 0 + B\sin 0 = 1 \Rightarrow A = 1$$

and

$$y(\tfrac{\pi}{2}) = 5 \Rightarrow A\cos\tfrac{\pi}{2} + B\sin\tfrac{\pi}{2} = 5 \Rightarrow B = 5.$$

Therefore,

$$y = \cos x + 5\sin x.$$

(d) The general equation is the same as part (c)'s solution. Applying boundary conditions, we get

$$y(0) = 1 \Leftrightarrow A\cos 0 + B\sin 0 = 1 \Rightarrow A = 1$$

and

$$y(\pi) = 5 \Rightarrow A\cos\pi + B\sin\pi = 5 \Rightarrow A = -5.$$

Therefore, we conclude that the BVP has no solution.

(e) $r^2 - 1 = 0$ is the characteristic equation of the corresponding homogeneous equation implying that $y_c = c_1e^x + c_2e^{-x}.$ Using the method of undetermined coefficients, we let

$y_p = Ax^2 + Bx + C$ so that $y_p{}' = 2Ax + B$ and $y_p{}'' = 2A.$

Substituting these in the differential equation, we get $y_p = x^2 + 1.$ Thus, $y = y_c + y_p = c_1e^x + c_2e^{-x} + x^2 + 1.$ Applying boundary conditions, we get

$$y(0) = 2 \Leftrightarrow c_1e^0 + c_2e^0 + 0^2 + 1 = 2$$
$$\Rightarrow c_1 + c_2 = 1$$
$$\Rightarrow c_2 = 1 - c_1 \qquad \text{(i)}$$

$$
\begin{aligned}
y(1) = 2e - \frac{1}{e} + 2 \Leftrightarrow c_1 e + c_2 e^{-1} + 1^2 + 1 &= 2e - \frac{1}{e} + 2 \\
\Rightarrow c_1 e + e^{-1}\left(1 - c_1\right) + 2 &= 2e - \frac{1}{e} + 2 \\
\Rightarrow \left(e - e^{-1}\right) c_1 + e^{-1} + 2 &= 2e - \frac{1}{e} + 2 \\
\Rightarrow \left(e - e^{-1}\right) c_1 &= 2e - \frac{2}{e} \\
\Rightarrow \left(e - e^{-1}\right) c_1 &= 2\left(e - e^{-1}\right) \\
\Rightarrow c_1 = 2 \quad \Rightarrow c_2 &= 1 - 2 = -1
\end{aligned}
$$

$$\therefore\ y = 2e^{x} - e^{-x} + x^2 + 1$$

2. The given ODE has the general solution

$$y\left(t\right) = ce^{-kt}.$$

Applying the first condition $y\left(0\right) = 1200,$ we have that

$$1200 = ce^{-k(0)} \quad \Rightarrow c = 1200.$$

Since half of that amount disintegrates after 20 years, the second condition is $y\left(20\right) = 600$ implying that

$$
\begin{aligned}
1200e^{-k(20)} = 600 \ \Leftrightarrow e^{-k(20)} &= \frac{1}{2} \\
\Rightarrow -20k &= \ln\left(\tfrac{1}{2}\right) \\
\Rightarrow k &= -\frac{1}{20}\ln\left(\tfrac{1}{2}\right)
\end{aligned}
$$

$$\therefore\ \ y(t) = 1200e^{-\left(-\frac{t}{20}\right)\ln\left(\frac{1}{2}\right)} = 1200e^{\ln\left(\frac{1}{2}\right)^{\frac{t}{20}}} = 1200\left(\frac{1}{2}\right)^{\frac{t}{20}} = 1200\left(2\right)^{-\frac{t}{20}}.$$

Thus, after 50 years

$$y\left(50\right) = 1200\left(2\right)^{-\frac{50}{20}} = 212.1320344 \approx 212.$$

△

THE END!