3. INTEGRAL CALCULUS OF FUNCTIONS OF ONE VARIABLE

As you may already know, differentiation and integration are closely connected. If $\frac{dy}{dx} = f(x)$, then integration is a process of determining a function $y = F(x)$ such that on differentiating $F(x)$ we obtain $f(x)$. If such a function $F(x)$ exists, it is called an indefinite integral of $f(x)$ or an antiderivative. In this chapter, we will discuss different methods of finding indefinite integrals of functions of one variable. Furthermore, we will look at definite integrals and their applications.

Definition 3.1.1

A function $F(x)$ is an antiderivative of a function $f(x)$ if

$$F'(x) = f(x)$$

for all x in the domain of f. The set of all antiderivatives of f is the indefinite integral of f with respect to x, denoted by

$$\int f(x)dx.$$

One antiderivative F differs from other antiderivatives by a constant, i.e.

$$\int f(x)dx = F(x) + c.$$

The function f is the integrand of the integral, x is the variable of integration and c is the constant of integration. ◇

3.1 METHODS OF INDEFINITE INTEGRATION

One common method of integration, is the so-called power rule, i.e. if u is any differentiable function, then

$$\int u^n du = \frac{u^{n+1}}{n+1} + c,$$

where $n \neq -1$, and n is rational. You have also seen integrals of most elementary functions such as

$$\int \tfrac{1}{x}\,dx = \ln|x| + c$$
$$\int e^x dx = e^x + c$$
$$\int a^x dx = \frac{a^x}{\ln a} + c,\ \ a > 0$$
$$\int \sin x\,dx = -\cos x + c$$
$$\int \cos x\,dx = \sin x + c$$
$$\int \tan x\,dx = -\ln|\cos x| + c$$
$$\int \cot x\,dx = \ln|\sin x| + c$$
$$\int \sec x\,dx = \ln|\sec x + \tan x| + c$$
$$\int \csc x\,dx = -\ln|\csc x + \cot x| + c$$
$$\int \sec^2 x\,dx = \tan x + c$$
$$\int \csc^2 x\,dx = -\cot x + c$$
$$\int \sec x \tan x\,dx = \sec x + c$$
$$\int \csc x \cot x\,dx = -\csc x + c$$
$$\int \sinh x\,dx = \cosh x + c$$
$$\int \cosh x\,dx = \sinh x + c.$$

We now examine some techniques that are frequently used to evaluate integrals which do not fit directly into any of the ones listed above.

3.1.1 ALGEBRAIC SUBSTITUTION

We can use change of variable to turn an unfamiliar integral into one which can easily be evaluated. Basic substitutions can be thought of as a way of running the chain rule backwards.

Example 3.1.1

Evaluate (a) $\int (ax+b)^n dx$ (b) $\int \sin(ax+b)dx$ (c) $\int 4ax\cos(ax^2+b)dx,$

assuming that a and b are constants, $a \neq 0$ and that n is a positive integer.

Solutions:

(a) Letting $u = ax + b \Rightarrow dx = \dfrac{du}{a}$. Then

$$\int (ax+b)^n\,dx = \int u^n \frac{du}{a} = \frac{1}{a}\int u^n du \;= \frac{1}{a}.\frac{u^{n+1}}{n+1} + c = \frac{1}{a}.\frac{(ax+b)^{n+1}}{n+1} + c.$$

(b) As done in Part (a) above, let $u = ax+b.$ Then

$$\int \sin(ax+b)dx = \frac{1}{a}\int \sin u\, du = -\frac{1}{a}\cos(ax+b) + c.$$

(c) Let $u = ax^2 + b \Rightarrow dx = \dfrac{du}{2ax}$

$$\int 4ax\cos(ax^2+b)\,dx = 2\int \cos u\, du = 2\sin(ax^2+b) + c.$$

3.1.2 INTEGRATION BY PARTS

Let u and v be differentiable functions. By product rule of differentiation

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}.$$

Integrating both sides with respect to $x,$ we get

$$uv = \int vdu + \int udv, \text{ i.e. } \int udv = uv - \int vdu.$$

Example 3.1.2

Evaluate the following

(1) $\int \ln x dx$ (2) $\int t^2 e^t dt$ (3) $\int 3^{\sqrt{2t+1}}\, dt.$

Solution

(1) Let $u = \ln x,\;\; dv = dx.$ Then

$$du = \frac{1}{x}dx,\;\; v = x.$$

$$\begin{aligned}\therefore \int udv = uv - \int vdu \Leftrightarrow \int \ln x dx &= x\ln x - \int x.\frac{1}{x}dx \\ &= x\ln x - \int dx \\ &= x\ln x - x + c.\end{aligned}$$

(2) Let $u = t^2,\;\; dv = e^t dx \Rightarrow du = 2tdt,\;\; v = e^t$

$$\begin{aligned}\int t^2e^t dt &= t^2e^t - 2\int te^t dt\\ &= t^2e^t - 2\left[te^t - \int e^t dt\right]\\ &= t^2e^t - 2\left[te^t - e^t\right] + c\\ &= t^2e^t - 2te^t + 2e^t + c.\end{aligned}$$

(3) Let $y=\sqrt{2t+1}$ so that $y^2=2t+1$ and $dt = ydy.$ Then

$$\int 3^{\sqrt{2t+1}}dt = \int y3^y dy.$$

Now, let $u=y,\ dv=3^y dy$

$$\Rightarrow du = dy,\ v=\frac{3^y}{\ln 3}$$

$$\begin{aligned}\therefore \int y3^y dy &= \frac{y.3^y}{\ln 3} - \frac{1}{\ln 3}\int 3^y dy\\ &= \frac{y.3^y}{\ln 3} - \frac{1}{\ln 3}.\frac{3^y}{\ln 3} + c.\end{aligned}$$

$$\therefore \int 3^{\sqrt{2t+1}}dt = \frac{\left(\sqrt{2t+1}\right)3^{\sqrt{2t+1}}}{\ln 3} - \frac{3^{\sqrt{2t+1}}}{(\ln 3)^2} + c.$$

△

3.1.3 REDUCTION FORMULAE

Sometimes integration by parts is used repeatedly to find a given integral. A reduction formula is useful in such situations because it gives a general formula for evaluating integrals of a given nature. A reduction formula is a formula which connects a given integral with another integral in which the integrand is of the same type but of lower degree.

Example 3.1.3

1. Find the reduction formula for

$$\int (x^2+1)^n dx, \text{ n is a constant}$$

and use it to evaluate

$$\int (x^2+1)^2 dx \text{ and } \int \frac{1}{(x^2+1)^2}dx.$$

2. Use the reduction formula for $\int \sin^n x dx$ to evaluate $\int_0^{\frac{\pi}{2}} \sin^3 x dx.$

Solutions:

1. Let $I_n = \int (x^2+1)^n dx.$ Letting $u = (x^2+1)^n$ and $dv = dx,$ we have that

   $du = 2nx(x^2+1)^{n-1}dx,\ v = x$ so that

   $$\begin{aligned} I_n &= x(x^2+1)^n - 2n\int x^2(x^2+1)^{n-1}dx \\ &= x(x^2+1)^n - 2n\int \left((x^2+1)-1\right)\left((x^2+1)^{n-1}\right)dx \end{aligned}$$

   i.e. $I_n = x(x^2+1)^n - 2nI_n + 2nI_{n-1}$

   $$\Rightarrow I_n = \frac{1}{2n+1}x(x^2+1)^n + \frac{2n}{2n+1}I_{n-1}.$$

   For $\int (x^2+1)^2 dx,\ \ n = 2$ and so

   $I_2 = \frac{1}{5}x(x^2+1)^2 + \frac{4}{5}I_1$ and

   $$I_1 = \int (x^2+1)dx = \frac{x^3}{3} + x + c$$

   $$\therefore \int (x^2+1)^2 dx = \frac{1}{5}x(x^2+1)^2 + \frac{4}{15}x^3 + \frac{4}{5}x + c.$$

   If we take $n = -2$ for the integral of $\int \frac{1}{(x^2+1)^2}dx,$ we notice that the powers in I_{n-1} will increase. So from the expression

   $I_n = \frac{1}{2n+1}x(x^2+1)^n + \frac{2n}{2n+1}I_{n-1},$ we get

   $I_{n-1} = -\frac{1}{2n}x(x^2+1)^n + \frac{2n+1}{2n}I_n.$ Letting $n = -1,$ we get

   $I_{-2} = \frac{x(x^2+1)^{-1}}{2} + \frac{1}{2}I_{-1}$ and $I_{-1} = \int \frac{1}{(x^2+1)}dx = \arctan x + c$

   $$\therefore \int \frac{1}{(x^2+1)^2}dx = \frac{x(x^2+1)^{-1}}{2} + \frac{1}{2}\arctan x + c.$$

2. Let $I_n = \int \sin^n x dx = \int \sin x \sin^{n-1} x dx.$ Then, let $u = \sin^{n-1} x,\ \ dv = \sin x dx$

   $\Rightarrow du = (n-1)\cos x \sin^{n-2} x dx,\ \ v = -\cos x$

$\therefore I_n = -\cos x \sin^{n-1} x + (n-1)\int \cos^2 x \sin^{n-2} x dx.$ Notice that

$$\int \cos^2 x \sin^{n-2} x dx = \int (1-\sin^2 x)\sin^{n-2} x dx = \int \sin^{n-2} x dx - \int \sin^n x dx$$

$$\therefore I_n = -\cos x \sin^{n-1} x + (n-1)I_{n-2} - (n-1)I_n$$

$$\Rightarrow I_n = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n} I_{n-2}.$$

Thus, with limits from 0 to $\frac{\pi}{2}$ and $n=3$,

$$I_3\Big|_0^{\frac{\pi}{2}} = \frac{2}{3} I_1\Big|_0^{\frac{\pi}{2}}, \quad I_1\Big|_0^{\frac{\pi}{2}} = \int_0^{\frac{\pi}{2}} \sin x dx = -\cos x\Big|_0^{\frac{\pi}{2}} = 1.$$

$$\therefore \int_0^{\frac{\pi}{2}} \sin^3 x dx = \frac{2}{3}.$$

△

3.1.3 PARTIAL FRACTIONS

Recall that any rational function $\frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials, with the degree of $P(x)$ less than that of $Q(x)$, $Q(x) \neq 0$, can be written as the sum of rational functions. If the degree of $P(x)$ is greater than or equal to that of $Q(x)$, then long division can be applied first after which the remainder can be resolved into partial fractions. The table below gives a summary of the rules involved in partial fractions:

Form of $Q(x)$	Expression	Form of partial fractions
Linear factors	$\frac{P(x)}{(ax+b)(cx+d)...}$	$\frac{A}{ax+b} + \frac{B}{cx+d} + ...$
Repeated linear factors	$\frac{P(x)}{(ax+b)^n}$	$\frac{A}{(ax+b)} + \frac{B}{(ax+b)^2} + ... + \frac{Z}{(ax+b)^n}$
Quadratic factors	$\frac{P(x)}{(ax^2+bx+c)(dx+e)...}$	$\frac{Ax+B}{ax^2+bx+c} + \frac{C}{dx+e}$
Repeated quadratic factors	$\frac{P(x)}{(ax^2+bx+c)^n}$	$\frac{Ax+B}{(ax^2+bx+c)} + \frac{Cx+D}{(ax^2+bx+c)^2} + ... + \frac{Yx+Z}{(ax^2+bx+c)^n}$

After resolving into partial fractions, the integral can then be integrated in terms of elementary functions.

Example 3.1.4

Evaluate the indefinite integral below using partial fractions:

$$\int \frac{4x^4+52x^3+151x^2+134x-37}{(x^2+2x+2)(4x-3)(2x+5)^2}\,dx$$

Solution:

Resolving the integrand into partial fractions, we get

$$\frac{4x^4+52x^3+151x^2+134x-37}{(x^2+2x+2)(4x-3)(2x+5)^2}=\frac{Ax+B}{x^2+2x+2}+\frac{C}{4x-3}+\frac{D}{2x+5}+\frac{E}{(2x+5)^2}$$

$$\Rightarrow (Ax+B)(4x-3)(2x+5)^2+C(x^2+2x+2)(2x+5)^2+D(x^2+2x+2)(4x-3)(2x+5)$$

$$+E(x^2+2x+2)(4x-3)=4x^4+52x^3+151x^2+134x-37$$

Letting $x=\frac{3}{4}$ gives $\frac{10985}{64}C=\frac{10985}{64}\Rightarrow C=1.$

Letting $x=-\frac{5}{2}$ gives $-\frac{169}{4}E=-\frac{169}{2}\Rightarrow E=2.$

Expanding and comparing coefficients, we get

$$16Ax^4+(68A+16B)x^3+(40A+68B)x^2+(40B-75A)x-75+4Cx^4+28Cx^3+73Cx^2+90Cx$$
$$+50C+8Dx^4+30Dx^3+29Dx^2-2Dx-30D+4Ex^3+5Ex^2+2Ex-6E=4x^4+52x^3+151x^2$$
$$+134x-37$$

$$\Rightarrow 16A+4C+8D=4\Rightarrow 2A=D\,.(*)$$

$$68A+16B+28C+30D+4E=52\ \Rightarrow 34A+8B+15D=8$$

$$\Rightarrow 8A+B=1\ .\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\ \ .\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.(**)$$

$$40A+68B+73C+29D+5E=151\Rightarrow 40A+68B+29D=68$$

$$\Rightarrow 49A+34B=34\ .\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.(***)$$

Solving (**) and (***) simultaneously, we get $A=0\Rightarrow D=0$ and $B=1.$

$$\therefore\ \frac{4x^4+52x^3+151x^2+134x-37}{(x^2+2x+2)(4x-3)(2x+5)^2}=\frac{1}{x^2+2x+2}+\frac{1}{4x-3}+\frac{2}{(2x+5)^2}$$

Thus,

$$\begin{aligned}\int \frac{4x^4+52x^3+151x^2+134x-37}{(x^2+2x+2)(4x-3)(2x+5)^2}\,dx &= \int\left(\frac{1}{x^2+2x+2}+\frac{1}{4x-3}+\frac{2}{(2x+5)^2}\right)dx \\ &= \int\frac{dx}{x^2+2x+2}+\int\frac{dx}{4x-3}+\int\frac{2}{(2x+5)^2}\,dx.\end{aligned}$$

Since $x^2+2x+2=(x+1)^2+1,$ letting $u=x+1$ gives

$$\int\frac{dx}{x^2+2x+2}=\int\frac{du}{u^2+1}=\arctan u+c=\arctan(x+1)+c,$$

letting $u=4x-3$ gives $\int\frac{dx}{4x-3}=\frac{1}{4}\int\frac{du}{u}=\frac{1}{4}\ln|4x-3|+c$

and letting $u=2x+5$ gives $\int\frac{2}{(2x+5)^2}\,dx=\int\frac{du}{u^2}=-\frac{1}{u}+c=-\frac{1}{2x+5}+c.$

Therefore,

$$\int\frac{4x^4+52x^3+151x^2+134x-37}{(x^2+2x+2)(4x-3)(2x+5)^2}\,dx=\arctan(x+1)+\frac{1}{4}\ln|4x-3|-\frac{1}{2x+5}+c$$

△

3.1.4 TRIGONOMETRIC INTEGRALS

In this section, we use trigonometric identities to evaluate trigonometric integrals.

(a) Odd Powers of Sine or Cosine

We consider integrals of the form

$$\int\sin^m\theta\cos^n\theta d\theta, \qquad \textbf{(3.1)}$$

where either m or n is odd. Suppose n is odd. Then, write $n=2k+1,\ k\in\mathbb{Z}.$ It follows that (3.1) can be written as

$$\int\sin^m\theta\cos^n\theta d\theta=\int\sin^m\theta\cos^{2k+1}\theta d\theta=\int\sin^m\theta\cos^{2k}\theta\cos\theta d\theta.$$

Using the identity $\sin^2\theta+\cos^2\theta=1,$ we have that $\cos^{2k}\theta=(1-\sin^2\theta)^k.$ Letting $u=\sin\theta,$ we have that $d\theta=\frac{du}{\cos\theta}$ so that

$$\int \sin^m \theta (1-\sin^2\theta)\cos\theta d\theta = \int u^m (1-u^2)^k du,$$

which gives us a polynomial integrand. Similarly, if we have an odd power of sine, then letting $u = \cos\theta$ yields

$$\begin{aligned}\int \sin^{2k+1}\theta \cos^n \theta d\theta &= \int (1-\cos^2\theta)^k \cos^n\theta \sin\theta d\theta \\ &= \int (1-u^2)^k u^n (-du).\end{aligned}$$

<u>Example 3.1.5</u>

Evaluate (a) $\int \frac{\cos^7 x}{\sqrt{\sin x}} dx$ (b) $\int \sin^3 5x \cos^4 5x dx.$

<u>Solutions:</u>

(a) Here Cosine has odd power, so we write it as $\cos^7 x = \cos^6 x \cos x$ and $\cos^6 x = (\cos^2 x)^3 = (1-\sin^2 x)^3.$ Then,

$$\int \frac{\cos^7 x}{\sqrt{\sin x}} dx = \int \frac{(1-\sin^2 x)^3 \cos x}{\sqrt{\sin x}} dx.$$

Letting $u = \sin x,$ we get $dx = \frac{du}{\cos x}$ so that

$$\begin{aligned}\int \frac{\cos^7 x}{\sqrt{\sin x}} dx = \int \frac{(1-u^2)^3}{\sqrt{u}} du &= \int \left(u^{-\frac{1}{2}} - 3u^{\frac{3}{2}} + 3u^{\frac{7}{2}} - u^{\frac{11}{2}} \right) du \\ &= 2u^{\frac{1}{2}} - \frac{6}{5}u^{\frac{5}{2}} + \frac{6}{9}u^{\frac{9}{2}} - \frac{2}{13}u^{\frac{13}{2}} + c \\ &= 2\sqrt{\sin x} - \frac{6}{5}\left(\sqrt{\sin x}\right)^5 + \frac{2}{3}\left(\sqrt{\sin x}\right)^9 - \frac{2}{13}\left(\sqrt{\sin x}\right)^{13} + c.\end{aligned}$$

(b) $\int \sin^3(5x)\cos^4(5x)dx = \int \left(1-\cos^2(5x)\right)\cos^4(5x)\sin(5x)dx.$

Let $u = \cos 5x \Rightarrow du = -5\sin 5x dx$

$$\begin{aligned}\therefore \int \sin^3 5x \cos^4 5x dx = -\frac{1}{5}\int (1-u^2)u^4 du &= -\frac{1}{25}u^5 + \frac{1}{35}u^7 + c \\ &= -\frac{1}{25}\cos^5 5x + \frac{1}{35}\cos^7 5x + c.\end{aligned}$$

△

(b) Even Powers of Sine and Cosine

In this case, we use double-angle identities:

$$
\begin{aligned}
&\sin(A \pm B) = \sin A \cos B \pm \sin B \cos A \\
&\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B \\
&\sin 2A = 2\sin A \cos A \\
&\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A \\
&\cos^2 A = \frac{1}{2}\left(1 + \cos 2A\right) \\
&\sin^2 A = \frac{1}{2}\left(1 - \cos 2A\right).
\end{aligned}
$$

These identities can also be used to evaluate integrals with even powers of only cosine or only sine. For the power equal to $2,$ it can be shown that even integration by parts is applicable.

Example 3.1.6

Evaluate

(a) $\int \sin^2 3x \cos^2 3x dx$ (b) $\int \sin^6 x \cos^2 x dx.$,

Solution

(a) Since $\sin^2 A = \frac{1}{2}\left(1 - \cos 2A\right)$ and $\cos^2 A = \frac{1}{2}\left(1 + \cos 2A\right),$ letting $A = 3x,$ we get

$\sin^2 3x = \frac{1}{2}\left(1 - \cos 6x\right)$ and $\cos^2 3x = \frac{1}{2}\left(1 + \cos 6x\right)$ so that

$$
\begin{aligned}
\sin^2 3x \cos^2 3x &= \frac{1}{4}\left(1 - \cos 6x\right)\left(1 + \cos 6x\right) \\
&= \frac{1}{4}\left(1 - \cos^2 6x\right) \\
&= \frac{1}{4}\left(1 - \frac{1}{2}\left(1 + \cos 12x\right)\right) \\
&= \frac{1}{4} - \frac{1}{8} - \frac{1}{8}\cos 12x \\
&= \frac{1}{8} - \frac{1}{8}\cos 12x.
\end{aligned}
$$

$$\therefore \int \sin^2 3x\cos^2 3x dx = \int\left(\frac{1}{8}-\frac{1}{8}\cos 12x\right)dx$$
$$= \frac{x}{8}-\frac{1}{96}\sin 12x + c.$$

The same result can be obtained when the integrand is written as

$$\sin^2 3x\cos^2 3x = \left(\sin 3x\cos 3x\right)^2 = \frac{1}{4}\sin^2 6x = \frac{1}{4}\left(\frac{1}{2}\left(1-\cos 12x\right)\right) = \frac{1}{8}-\frac{1}{8}\cos 12x.$$

(b) $\sin^6 x\cos^2 x = \sin^6 x\left(1-\sin^2 x\right) = \sin^6 x - \sin^8 x$

$$= \left[\frac{1}{2}\left(1-\cos 2x\right)\right]^3 - \left[\frac{1}{2}\left(1-\cos 2x\right)\right]^4$$
$$= \frac{1}{8}\left[1-3\cos 2x+3\cos^2 2x-\cos^3 2x-\frac{1}{2}+2\cos 2x-3\cos^2 2x+2\cos^3 2x-\frac{1}{2}\cos^4 2x\right]$$
$$= \frac{1}{8}\left(\frac{1}{2}-\cos 2x+\cos^3 2x-\frac{1}{2}\cos^4 2x\right).$$

Since $\cos^2 2x = \frac{1}{2}\left(1+\cos 4x\right)$

$$\cos^3 2x = \cos 2x\left[\frac{1}{2}\left(1+\cos 4x\right)\right] = \frac{1}{2}\left(\cos 2x + \cos 2x\cos 4x\right)$$
$$= \frac{1}{2}\cos 2x + \frac{1}{2}\cos 2x\cos 4x.$$

Also, $\cos A\cos B = \frac{1}{2}\left(\cos(A+B)+\cos(A-B)\right)$. Thus, $\cos 2x\cos 4x = \frac{1}{2}\cos 6x + \frac{1}{2}\cos 2x$ and so $\cos^3 2x = \frac{1}{2}\cos 2x + \frac{1}{4}\cos 6x + \frac{1}{4}\cos 2x = \frac{3}{4}\cos 2x + \frac{1}{4}\cos 6x.$

Similarly, $\cos^4 2x = \frac{1}{4}\left(1+\cos 4x\right)^2 = \frac{1}{4}\left(1+2\cos 4x+\cos^2 4x\right)$ and since

$\cos^2 4x = \frac{1}{2}\left(1+\cos 8x\right)$, we have that

$$\cos^4 2x = \frac{1}{4}\left(1+2\cos 4x+\frac{1}{2}\left(1+\cos 8x\right)\right) = \frac{3}{8}+\frac{1}{2}\cos 4x+\frac{1}{8}\cos 8x.$$

$$\therefore \sin^6 x\cos^2 x = \frac{1}{8}\left[\frac{1}{2} - \cos 2x + \frac{3}{4}\cos 2x + \frac{1}{4}\cos 6x - \frac{1}{2}\left(\frac{3}{8} + \frac{1}{2}\cos 4x + \frac{1}{8}\cos 8x\right)\right]$$

$$= \frac{1}{16} - \frac{1}{8}\cos 2x + \frac{3}{32}\cos 2x + \frac{1}{32}\cos 6x - \frac{3}{128} - \frac{1}{32}\cos 4x - \frac{1}{128}\cos 8x$$

$$= \frac{5}{128} - \frac{1}{32}\cos 2x - \frac{1}{32}\cos 4x + \frac{1}{32}\cos 6x - \frac{1}{128}\cos 8x.$$

$$\therefore \int \sin^6 x\cos^2 x dx = \int\left(\frac{5}{128} - \frac{1}{32}\cos 2x - \frac{1}{32}\cos 4x + \frac{1}{32}\cos 6x - \frac{1}{128}\cos 8x\right)dx$$

$$= \frac{5}{128}x - \frac{1}{64}\sin 2x - \frac{1}{128}\sin 4x + \frac{1}{192}\sin 6x - \frac{1}{1024}\sin 8x + c.$$

△

(c) Even Powers of Secant or Odd Powers of Tangent

We consider the integral of the form

$$\int \sec^m \theta \tan^n \theta d\theta. \qquad \textbf{(3.2)}$$

Using the identity $\tan^2\theta + 1 = \sec^2\theta,$ two cases arise:

1. If (3.2) has an odd power of tangent, i.e. $n = 2k+1,\ k \in \mathbb{Z},$ let $u = \sec\theta$ so that $du = \sec\theta\tan\theta d\theta$ and the integral reduces to

$$\int \sec^m \theta \tan^n \theta d\theta = \int \sec^{m-1}\theta \tan^{2k}\theta \sec\theta\tan\theta d\theta = \int u^{m-1}(u^2-1)^k du,$$

which can easily be evaluated.

2. If (3.2) contains even powers of secant, then let $u = \tan\theta$ so that $du = \sec^2\theta d\theta,$ and the integral becomes

$$\int \sec^m \theta \tan^n \theta d\theta = \int \sec^{2(k-1)}\theta \tan^n\theta \sec^2\theta d\theta = \int (u^2+1)^{k-1}u^n du,$$

which can easily be evaluated.

(d) Even Powers of Cosecant or Odd Powers of Cotangent

For the integral of the type

$$\int \csc^m \theta \cot^n \theta d\theta, \qquad \textbf{(3.3)}$$

we use the identity $\cot^2\theta + 1 = \csc^2\theta.$ Then, two cases arise:

1. Since $\frac{d}{d\theta}(\csc\theta) = -\csc\theta\cot\theta,$ if (3.3) has an <u>odd power of Cotangent</u>, i.e. $n = 2k+1,\ k \in \mathbb{Z},$ then we let $u = \csc\theta$ so that

$$\begin{aligned}\int \csc^m \theta \cot^n \theta d\theta &= \int \csc^{m-1} \theta \cot^{2k} \theta \csc\theta \cot\theta d\theta \\ &= -\int u^{m-1}(u^2-1)^k du,\end{aligned}$$

which can easily be evaluated.

2. If (3.3) contains an <u>even power of Cosecant</u>, then let $u = \cot\theta$ so that $du = -\csc^2 \theta d\theta$ and so

$$\begin{aligned}\int \csc^m \theta \cot^n \theta d\theta &= -\int \csc^{2(k-1)} \theta \cot^n \theta(-\csc^2 \theta) d\theta \\ &= -\int u^{m-1}(u^2-1)^k du \\ &= -\int (\cot^2 \theta + 1)^{k-1} \cot^n \theta(-\csc^2 \theta) d\theta \\ &= -\int (u^2+1)^{k-1} u^n du,\end{aligned}$$

which can easily be evaluated.

<u>Example 3.1.7</u>

(a) $\int \sec^6 x \tan^8 x dx$ (b) $\int \tan^4 x dx$ (c) $\int \csc^3 \frac{x}{2} \cot^3 \frac{x}{2} dx$.

<u>Solutions:</u>

(a) Here, we have even powers of Secant. So we let $u = \tan x \Rightarrow du = \sec^2 x dx$

$$\begin{aligned}\therefore \int (\sec^2 x)^2 \tan^8 x \sec^2 x dx &= \int (u^2+1)^2 u^8 du \\ &= \int \left(u^{12} + 2u^{10} + u^8\right) du \\ &= \frac{u^{13}}{13} + \frac{2u^{11}}{11} + \frac{u^9}{9} + c \\ &= \frac{1}{13}\tan^{13} x + \frac{2}{11}\tan^{11} x + \frac{1}{9}\tan^9 x + c.\end{aligned}$$

(b) Since Tangent is not of odd power, we use the other case by introducing a special “1”.

$$\int \tan^4 x\ dx = \int \frac{\tan^4 x}{\sec^2 x}.\sec^2 x\ dx = \int \frac{\tan^4 x}{(\tan^2 x + 1)}.\sec^2 x\ dx.$$

Let $u=\tan x \Rightarrow du=\sec^2 x\,dx$

$$\therefore \int \tan^4 x\ dx=\int \frac{u^4}{u^2+1}du=\int\left(u^2-1+\frac{1}{u^2+1}\right)du$$

$$=\frac{u^3}{3}-u+\tan^{-1}u+c$$

$$=\frac{\tan^3 x}{3}-\tan x+\tan^{-1}(\tan x)+c$$

$$=\frac{\tan^3 x}{3}-\tan x+x+c.$$

(c) $\int \csc^3 \tfrac{x}{2}\cot^3 \tfrac{x}{2}\,dx=-\int \csc^2 \tfrac{x}{2}\cot^2 \tfrac{x}{2}(-\csc\tfrac{x}{2}\cot\tfrac{x}{2})dx.$

Let $u=\csc\tfrac{x}{2} \Rightarrow du=-\frac{1}{2}\csc\tfrac{x}{2}\cot\tfrac{x}{2}\,dx$

$$\Rightarrow \int \csc^3 \tfrac{x}{2}\cot^3 \tfrac{x}{2}\,dx=-2\int (u^2-1)u^2dx=-\frac{2}{5}u^5+\frac{2}{3}u^3+c$$

$$=-\frac{2}{5}\csc^5\tfrac{x}{2}+\frac{2}{3}\csc^3\tfrac{x}{2}+c.$$

△

3.1.5 INTEGRALS OF HYPERBOLIC FUNCTIONS

Recall that the hyperbolic sine and cosine are given by

$$\cosh x=\frac{1}{2}\left(e^x+e^{-x}\right)$$

$$\sinh=\frac{1}{2}\left(e^x-e^{-x}\right)$$

and from these we get other hyperbolic functions

$$\tanh x=\frac{\sinh x}{\cosh x}=\frac{e^{2x}-1}{e^{2x}+1}$$

$$\coth x=\frac{1}{\tanh x}=\frac{\cosh x}{\sinh x}=\frac{e^{2x}+1}{e^{2x}-1}$$

$$\operatorname{sech} x=\frac{1}{\cosh x}=2.\frac{e^x}{e^{2x}+1}$$

$$\operatorname{csch} x=\frac{1}{\sinh x}=2.\frac{e^x}{e^{2x}-1}.$$

Using Osborn's rule, which states that "cos" should be converted to "cosh" and "sin" into "sinh" and multiply a negative when there is a product of two sines, we get the following basic identities for hyperbolic functions:

$$\begin{aligned}
&\cosh^2 x - \sinh^2 x = 1 \\
&\tanh^2 x + \operatorname{sech} x = 1 \\
&\coth^2 x = 1 + \operatorname{csch}^2 x \\
&\sinh 2x = 2\sinh x \cosh x \\
&\cosh 2x = \cosh^2 x + \sinh^2 x \\
&\cosh^2 x = \frac{1}{2}\left(\cosh^2 2x + 1\right) \\
&\sinh^2 x = \frac{1}{2}\left(\cosh^2 2x - 1\right).
\end{aligned}$$

Other identities for the addition of angles can be obtained in a similar way.

Recall also that the logarithmic equivalents of inverse hyperbolic functions are:

$$\sinh^{-1} x = \ln\left\{x + \sqrt{x^2+1}\right\}, \text{ for all values of } x$$

$$\begin{aligned}
&\cosh^{-1} x = \ln\left\{x + \sqrt{x^2-1}\right\},\ x \geq 1 \\
&\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right),\ |x| < 1.
\end{aligned}$$

It can also be shown that the following are the derivatives of hyperbolic functions:

$$\begin{aligned}
&\frac{d}{dx}(\sinh x) = \cosh x \\
&\frac{d}{dx}(\cosh x) = \sinh x
\end{aligned}$$

$$\begin{aligned}
&\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x \\
&\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x \\
&\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \\
&\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x
\end{aligned}$$

and for inverse hyperbolic functions, we have

$$\frac{d}{dx}(\sinh^{-1}x)=\frac{1}{\sqrt{1+x^2}}$$
$$\frac{d}{dx}(\cosh^{-1}x)=\frac{1}{\sqrt{x^2-1}},\ x>1,\ \cosh^{-1}x>0$$
$$\frac{d}{dx}(\cosh^{-1}x)=-\frac{1}{\sqrt{x^2-1}},\ x>1,\ \cosh^{-1}x<0$$
$$\frac{d}{dx}(\tanh^{-1}x)=\frac{1}{1-x^2},\ |x|<1.$$

It then follows that

$$\int \sinh x dx=\cosh x+c$$
$$\int \cosh x dx=\sinh x+c$$
$$\int \operatorname{sech}^2 x dx=\tanh x+c$$
$$\int \operatorname{csch}^2 x dx=-\coth x+c$$
$$\int\left(\frac{1}{\sqrt{1+x^2}}\right)dx=\sinh^{-1}x+c$$

$$\int\left(\frac{1}{\sqrt{x^2-1}}\right)dx=\cosh^{-1}x+c,\quad x>1$$
$$\int\left(\frac{1}{1-x^2}\right)dx=\tanh^{-1}x+c,\ |x|<1$$
$$\int\left(\frac{1}{1-x^2}\right)dx=\coth^{-1}x+c,\ |x|>1.$$

We will use hyperbolic substitution later to evaluate integrals of this nature.

Example 3.1.8

Evaluate

(a) $\int x\sinh x dx$ (b) $\int \tanh^2 3x dx$ (c) $\int \frac{\operatorname{sech}\sqrt{x}.\tanh\sqrt{x}}{\sqrt{x}}dx.$

Solutions:

(a) Using integration by parts, let $u=x,\quad dv=\sinh x.$ Then, $du=dx$ and $v=\cosh x.$

$$\therefore\ \int x\sinh x dx=x\cosh x-\int \cosh x dx=x\cosh x-\sinh x+c.$$

(b) $\int \tanh^2 3x dx = \int \left(1-\operatorname{sech}^2 3x\right) dx = \int dx - \int \operatorname{sech}^2 3x dx = x - \frac{1}{3}\tanh 3x + c.$

(c) Let $u = \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}} dx$

$$\therefore \int \frac{\operatorname{sech}\sqrt{x}.\tanh\sqrt{x}}{\sqrt{x}} dx = 2\int \operatorname{sech} u \tanh u du = -2\operatorname{sech} u + c = -2\operatorname{sech}\sqrt{x} + c.$$

△

3.1.6 OTHER SUBSTITUTIONS

In this section, we will consider substitutions such as rationalising substitution, trigonometric and hyperbolic substitution and magic substitution.

1. RATIONALISING SUBSTITUTION

Some functions which are not rational can be changed into rational functions by means of appropriate substitution. In particular, when an integrand contains an expression of the form $\sqrt[n]{g(x)}$, then the substitution $u = \sqrt[n]{g(x)}$ may be effective.

Example 3.1.9

Evaluate (a) $\int \left(\frac{x}{\sqrt{1-x}}\right) dx$ (b) $\int \left(\frac{\sqrt{1+\sqrt{x}}}{x}\right) dx$ (c) $\int \left(\frac{1}{\sqrt{x}-\sqrt[3]{x}}\right) dx$

Solution

(a) Let $u = \sqrt{1-x} \Rightarrow x = 1-u^2 \Rightarrow dx = -2du$

$$\therefore \int \left(\frac{x}{\sqrt{1-x}}\right) dx = \int \left(\frac{1-u^2}{u}\right) - 2udu = 2\int (u^2-1) du = \frac{2}{3}u^3 - 2u + c$$

$$= \frac{2}{3}\left(\sqrt{1-x}\right)^3 - 2\sqrt{1-x} + c.$$

(b) Let $u = \sqrt{1+\sqrt{x}} \Rightarrow u^2 = 1+\sqrt{x}$

$$\Rightarrow x = (u^2-1)^2$$
$$\Rightarrow dx = 4u(u^2-1)du$$

$$\therefore \int\left(\frac{\sqrt{1+\sqrt{x}}}{x}\right)dx = \int \frac{u}{(u^2-1)^2}.4u(u^2-1)du = 4\int \frac{u^2}{u^2-1}dx$$

$$= 4\left[\frac{1}{2}\int \frac{1}{u-1}du - \frac{1}{2}\int \frac{1}{u+1}du + \int du\right]$$

$$= 2\ln|u-1| - 2\ln|u+1| + 4u + c$$

$$= 2\ln\left[\frac{\sqrt{1+\sqrt{x}}-1}{\sqrt{1+\sqrt{x}}+1}\right] + 4\sqrt{1+\sqrt{x}} + c.$$

(c) Let $u = \sqrt[6]{x} \Rightarrow x = u^6 \Rightarrow dx = 6u^5$

$$\therefore \int\left(\frac{1}{\sqrt{x}-\sqrt[3]{x}}\right)dx = \int \frac{1}{u^3-u^2}.6u^5du = 6\int \frac{u^3}{u-1}du$$

$$= 6\int\left[u^2+u+1+\frac{1}{u-1}\right]du$$

$$= 6\left[\frac{u^3}{3}+\frac{u^2}{2}+u+\ln|u-1|\right]+c$$

$$= 2\sqrt{x}+3\sqrt[3]{x}+6\sqrt[6]{x}+6\ln|\sqrt[6]{x}-1|+c.$$

△

2. TRIGONOMETRIC AND HYPERBOLIC SUBSTITUTION

So far, we have seen that sometimes it helps to replace a subexpression of a function by a single variable. Occasionally, it can help to replace the original variable by something more complicated. This seems like a "reverse" substitution, but it is really no different in principle than ordinary substitution. In this subsection, we will evaluate integrals by making substitutions involving trigonometric and hyperbolic functions. The following table summarises possible substitutions:

EXPRESSION	TRIGONOMETRIC SUBSTITUTION	HYPERBOLIC SUBSTITUTION
$\sqrt{a^2-x^2}$	$x = a\sin\theta,\ \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$	$x = a\tanh\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta,\ \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$	$x = a\sinh\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta,\ \left[0,\frac{\pi}{2}\right)$	$x = a\cosh\theta$

Example 3.1.10

Evaluate (a) $\displaystyle\int \frac{x^3}{\left(4x^2+9\right)^{\frac{3}{2}}}dx$ (b) $\displaystyle\int \frac{\sqrt{x^2+2x-3}}{x+1}dx$

using both trigonometric and hyperbolic substitution.

Solutions:

(a) $\left(4x^2+9\right)^{\frac{3}{2}} = \left[4\left(x^2+\frac{9}{4}\right)\right]^{\frac{3}{2}} = 8\left(x^2+\frac{9}{4}\right)^{\frac{3}{2}}.$

Using trigonometric substitution, let $x=\frac{3}{2}\tan\theta \Rightarrow dx=\frac{3}{2}\sec^2\theta d\theta$ and $\tan\theta=\frac{2x}{3}$

$$\therefore \int \frac{x^3}{\left(4x^2+9\right)^{\frac{3}{2}}}dx = \int\left(\frac{\frac{27}{8}\tan^3\theta}{3^3\sec^3\theta}.\frac{3}{2}\sec^2\theta\right)d\theta$$

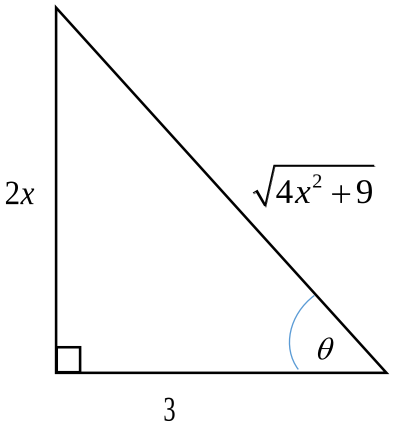
$$=\frac{3}{16}\int\frac{\tan^3\theta}{\sec\theta}d\theta=\frac{3}{16}\int\frac{\tan^2\theta\tan\theta\sec\theta}{\sec^2\theta}d\theta.$$

Let $u=\sec\theta \Rightarrow du=\sec\theta\tan\theta d\theta.$ Then,

$$\frac{3}{16}\int\frac{\tan^3\theta}{\sec\theta}d\theta=\frac{3}{16}\int\frac{(\sec^2-1)}{\sec^2\theta}\sec\theta\tan\theta d\theta=\frac{3}{16}\int\frac{u^2-1}{u^2}du=\frac{3}{16}\int(1-u^{-2})du$$

$$=\frac{3}{16}\left[u+u^{-1}\right]+c$$

$$=\frac{3}{16}\sec\theta+\frac{3}{16}.\frac{1}{\sec\theta}+c.$$



$$\therefore \cos\theta = \frac{3}{\sqrt{\left(4x^2+9\right)}},\ \sec\theta = \frac{\sqrt{\left(4x^2+9\right)}}{3}$$

$$\Rightarrow \int \frac{x^3}{\left(4x^2+9\right)^{\frac{3}{2}}}dx = \frac{3}{16}.\frac{\sqrt{\left(4x^2+9\right)}}{3} + \frac{3}{16}.\frac{3}{\sqrt{\left(4x^2+9\right)}} + c$$

$$= \frac{1}{16}\sqrt{\left(4x^2+9\right)} + \frac{9}{16}.\frac{1}{\sqrt{\left(4x^2+9\right)}} + c.$$

Using hyperbolic substitution, $x = \frac{3}{2}\sinh\theta \Rightarrow dx = \frac{3}{2}\cosh\theta d\theta$

$$\int \frac{x^3}{\left(4x^2+9\right)^{\frac{3}{2}}}dx = \frac{3}{16}\int \frac{\sinh^3\theta}{\cosh^2\theta}d\theta$$

Letting $u = \cosh\theta,$ we have that

$$\frac{3}{16}\int \frac{\sinh^2\theta\sinh\theta}{\cosh^2\theta}d\theta = \frac{3}{16}\int \frac{u^2-1}{u^2}du = \frac{3}{16}\int (1-u^{-2})du = \frac{3}{16}\cosh\theta + \frac{3}{16}.\frac{1}{\cosh\theta} + c$$

Since $\cosh\theta = \sqrt{\sinh^2+1},$ we have that $\cosh\theta = \frac{\sqrt{4x^2+9}}{3}$ so that

$$\int \frac{x^3}{\left(4x^2+9\right)^{\frac{3}{2}}}dx = \frac{1}{16}\sqrt{\left(4x^2+9\right)} + \frac{9}{16}.\frac{1}{\sqrt{\left(4x^2+9\right)}} + c.$$

(b) $x^2+2x-3=(x+1)^2-4.$ Then, using trigonometric substitution, let $x+1=2\sec\theta$

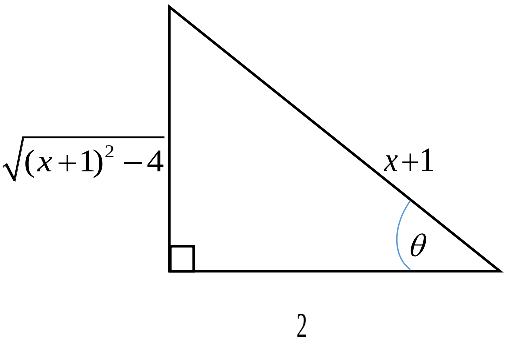
$\Rightarrow \sec\theta = \frac{x+1}{2}$ and $dx = 2\sec\theta\tan\theta d\theta.$

$$\int \frac{\sqrt{x^2+2x-3}}{x+1}dx = \int \frac{\sqrt{4\sec^2\theta-4}}{2\sec\theta}.2\sec\theta\tan\theta d\theta = 2\int \frac{\tan^2\theta\sec^2\theta}{(1+\tan^2\theta)}d\theta.$$

Let $u = \tan\theta \Rightarrow du = \sec^2\theta d\theta$

$$\therefore 2\int \frac{\tan^2\theta\sec^2\theta}{(1+\tan^2\theta)}d\theta = 2\int \frac{u^2}{1+u^2}du = 2u - 2\arctan u + c = 2\tan\theta - 2\theta + c.$$

Since $\sec\theta = \frac{x+1}{2}, \cos\theta = \frac{2}{x+1}$



$$\therefore \tan\theta = \frac{\sqrt{(x+1)^2-4}}{2}$$

$$\therefore \int \frac{\sqrt{x^2+2x-3}}{x+1}\,dx = \sqrt{x^2+2x-3} - 2\arccos\left(\frac{2}{x+1}\right) + c.$$

Exercise: Use hyperbolic substitution to evaluate the same integral.

△

3. MAGIC SUBSTITUTION

We can transform an integral of a rational function involving $\sin x$ and $\cos x$ into a rational function of just one variable, say z. The magic substitution we use in such situations is $z = \tan\frac{x}{2}$. We notice that

$dz = \frac{1}{2}\sec^2\frac{x}{2}\,dx = \frac{1}{2}\left(\tan^2\frac{x}{2}+1\right)dx = \frac{1}{2}\left(z^2+1\right)dx.$ Hence,

$$dx = \frac{2}{1+z^2}\,dz. \qquad \textbf{(3.4)}$$

Then, using the identity $\cos 2A = 2\cos^2 A - 1,$ we have

$$\cos x = 2\cos^2\tfrac{x}{2} - 1 = 2.\frac{1}{\frac{1}{\cos^2\frac{x}{2}}} - 1 = \frac{2}{\sec^2\frac{x}{2}} - 1 = \frac{2}{1+\tan^2\frac{x}{2}} - 1 = \frac{2}{1+z^2} - 1 = \frac{1-z^2}{1+z^2},$$

$$\therefore\ \cos x = \frac{1-z^2}{1+z^2} \tag{3.5}$$

$$\begin{aligned}
\sin x = 2\sin\tfrac{x}{2}\cos\tfrac{x}{2} = 2.\frac{\sin\tfrac{x}{2}}{\cos\tfrac{x}{2}}.\cos^2\tfrac{x}{2} \\
&= 2.\frac{\sin\tfrac{x}{2}}{\cos\tfrac{x}{2}}.\frac{1}{\frac{1}{\cos^2\frac{x}{2}}} = 2\tan\tfrac{x}{2}.\frac{1}{\sec^2\tfrac{x}{2}} \\
&= 2\tan\tfrac{x}{2}.\frac{1}{\tan^2\tfrac{x}{2}+1} \\
&= \frac{2z}{z^2+1},
\end{aligned}$$

$$\therefore\ \sin x = \frac{2z}{z^2+1} \tag{3.6}$$

<u>Example 3.1.11</u>

Evaluate $\displaystyle\int \frac{1}{4\cos x - 3\sin x}\,dx.$

<u>Solution</u>

We let $z = \tan\tfrac{x}{2}$ and use (3.4), (3.5) and (3.6).

$$4\cos x - 3\sin x = \frac{4(1-z^2)}{1+z^2} - \frac{3(2z)}{1+z^2} = \frac{4-4z^2-6z}{1+z^2}$$

$$\begin{aligned}
\int \frac{1}{4\cos x - 3\sin x}\,dx = \int \frac{1}{\frac{4-4z^2-6z}{1+z^2}}.\frac{2dz}{1+z^2} = \int \frac{1}{2-2z^2-3z}\,dz = -\left[\frac{2}{5}\int\frac{dz}{2z-1} - \frac{1}{5}\int\frac{dz}{z+2}\right] \\
&= -\frac{1}{5}\int\frac{dz}{2z-1} + \frac{1}{5}\int\frac{dz}{z+2} = -\frac{1}{5}\ln|2z-1| + \frac{1}{5}\ln|z+2| + c = \frac{1}{5}\ln\left|\frac{z+2}{2z-1}\right| + c \\
&= \frac{1}{5}\ln\left|\frac{\tan\frac{x}{2}+2}{2\tan\frac{x}{2}-1}\right| + c
\end{aligned}$$

△

3.2 DEFINITE INTEGRALS

Before we discuss definite integrals, recall the sum of n terms $a_1, a_2, ..., a_n$ can be written as

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + ... + a_n,$$

where i is the index of summation, a_i is the i^{th} term of the sum and the upper and lower bounds of the summation are n and 1 respectively. For example,

$$\sum_{i=1}^{4} i = 1 + 2 + 3 + 4 = 10$$

and

$$\sum_{i=1}^{n} f(x_i)\Delta x_i = f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + ... + f(x_n)\Delta x_n.$$

Two important properties of sums are:

1. $\sum_{i=1}^{n} ka_i = k\sum_{i=1}^{n} a_i$, where k is a constant.
2. $\sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$.

The following summation formulas are very useful and can be proved by induction:

(a) $\sum_{i=1}^{n} k = k + k + k + ... + k = nk$

(b) $\sum_{i=1}^{n} i = 1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$

(c) $\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$

(d) $\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + 3^3 + ... + n^3 = \frac{n^2(n+1)^2}{4}$

Example 3.2.1

Evaluate

1. $\sum_{i=1}^{n}\left(\frac{i+1}{n^2}\right)$, for $n=7$

2. $\sum_{i=1}^{n}\left(\frac{4(i-1)}{n^2}-\frac{2}{n}\right)$, for $n=10.$

Solutions:

We use properties of sums and summation formulas.

1. $$\sum_{i=1}^{n}\left(\frac{i+1}{n^2}\right)=\frac{1}{n^2}\sum_{=1}^{n}(i+1)=\frac{1}{n^2}\sum_{=1}^{n}i+\frac{1}{n^2}\sum_{=1}^{n}1$$
$$=\frac{1}{n^2}\left[\frac{n(n+1)}{2}\right]+\frac{1}{n^2}(n)$$
$$=\frac{n+1}{2n}+\frac{1}{n}$$
$$=\frac{n+3}{2n}.$$

Thus, for $n=7$, $\sum_{i=1}^{n}\left(\frac{i+1}{n^2}\right)=\frac{7+3}{2(7)}=\frac{5}{7}.$

2. $$\sum_{i=1}^{n}\left(\frac{4(i-1)}{n^2}-\frac{2}{n}\right)=\frac{4}{n^2}\sum_{i=1}^{n}(i-1)-\frac{2}{n}\sum_{i=1}^{n}1$$
$$=\frac{4}{n^2}\left[\sum_{i=1}^{n}i-\sum_{i=1}^{n}1\right]-\frac{2}{n}\sum_{i=1}^{n}1$$
$$=\frac{4}{n^2}\left[\frac{n(n+1)}{2}-n\right]-\frac{2}{n}(n)$$
$$=-\frac{2}{n}.$$

When $n=10$,

$$\sum_{i=1}^{n}\left(\frac{4(i-1)}{n^2}-\frac{2}{n}\right)=-\frac{2}{10}=-\frac{1}{5}.$$

△

3.2.1 RIEMANN SUM

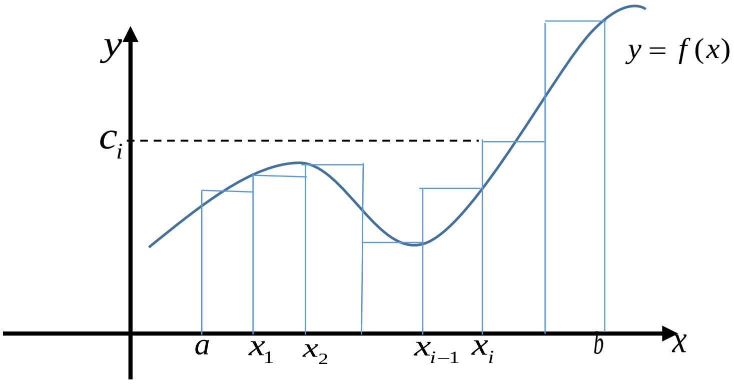
Let f be continuous on the closed interval $[a,b]$ and let Δ be a partition of $[a,b]$ given by

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b,$$

where Δx_i is the width of the i^{th} subinterval. If c_i is any point in the i^{th} subinterval $[x_{i-1}, x_i]$, then the sum

$$\sum_{i=1}^{n} f(c_i)\Delta x_i\,, \quad x_{i-1} \le c_i \le x_i$$

is called the Riemann sum of f for the partition Δ.



Definition 3.2.1

If f is defined and continuous on $[a,b]$, and the limit of the Riemann sums over Δ

$$\lim_{\Delta x_i \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

exists, then f is said to be integrable on $[a,b]$, and the limit

$$\lim_{\Delta x_i \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i = \int_a^b f(x)\, dx$$

is called the definite integral of f from a to b. The numbers a and b are the lower and upper limits of the integral.

◊

NOTE: 1. The limit is just the area under the graph. Since n is the number of subintervals obtained by partitioning by $[a,b]$, it follows that $\Delta x_i = x_i - x_{i-1} = \dfrac{b-a}{n}$ and as $\Delta x_i \to 0$, $n \to \infty$. Thus,

$$\text{Area} = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x_i = \int_a^b f(x)\,dx.$$

2. If we choose c_i to be the right endpoint of each subinterval, then $c_i = a + i(\Delta x_i)$.

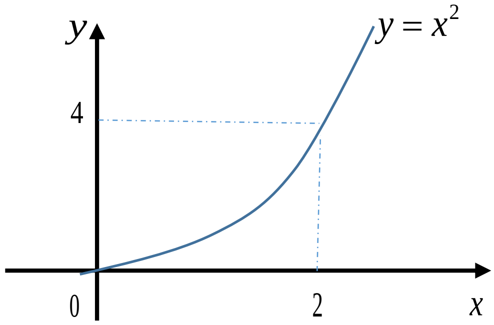
Example 3.2.2

1. Find the area under the graph $y = x^2$ on the interval $[0, 2]$.

2. Use Riemann sums to evaluate $\int_{-2}^{2} 2x\,dx.$

Solutions:

1.



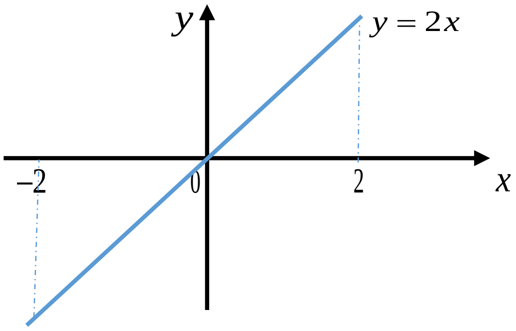
$$\Delta x_i = \frac{b-a}{n} = \frac{2}{n} \text{ and } c_i = a + i(\Delta x_i) = 0 + \frac{2i}{n} = \frac{2i}{n} \Rightarrow f(c_i) = f\left(\frac{2i}{n}\right) = \frac{4i^2}{n^2}$$

$$\therefore \int_0^2 x^2\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x_i = \lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(\frac{4i^2}{n^2}\right)\left(\frac{2}{n}\right)\right] = \lim_{n\to\infty}\frac{8}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)$$

$$= \frac{4}{3}\left(\lim_{n\to\infty}\frac{2n^2+3n+1}{n^2}\right) = \frac{4}{3}\left(\lim_{n\to\infty}(2+\tfrac{3}{n}+\tfrac{1}{n^2}\right)$$

$$= \frac{4}{3}(2) = \frac{8}{3}.$$

2.



$$\Delta x_i = \frac{b-a}{n} = \frac{4}{n} \text{ and } c_i = a + i(\Delta x_i) = -2 + \frac{4i}{n} \Rightarrow f(c_i) = f\left(-2+\frac{4i}{n}\right) = 2\left(-2+\frac{4i}{n}\right)$$

$$\begin{aligned}\therefore \text{ Area} &= \lim_{n\to\infty}\sum_{i=0}^{n} f(c_i)\Delta x_i = \lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(2\left(-2+\frac{4i}{n}\right)\right)\left(\frac{4}{n}\right)\right] = \lim_{n\to\infty}\frac{8}{n}\sum_{i=1}^{n}\left(-2+\frac{4i}{n}\right)\\ &= \lim_{n\to\infty}\frac{8}{n}\left(-2n+\frac{4}{n}\left(\frac{n(n+1)}{2}\right)\right)\\ &= \lim_{n\to\infty}\left(-16+16+\frac{16}{n}\right)\\ &= 0.\end{aligned}$$

It is clear from the diagram that the area cannot be zero. Direct computation has given us zero because the area above and below the $x-$ axis are equal but the area below the $x-$ axis is negative.

△

PROPERTIES OF DEFINITE INTEGRALS

If $f(x)$ and $g(x)$ are continuous on the interval of integration $[a,b]$, then

1. $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx,$ k is a constant
2. $\int_a^b [f(x)\pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,$ provided that $a<c<b$ and that f is integrable in $[a,c]$ and $[c,b]$.
4. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
5. $\int_a^a f(x)\,dx = 0$

<u>*Theorem 3.2.1*</u> (Fundamental Theorem of Calculus)

1. Let f be integrable on a closed interval $[a,b]$. Let c satisfy the condition $a \le c \le b$, and define a new function

   $$F(x) = \int_a^x f(t)dt, \ if \ a \le x \le b.$$

   Then the derivative $F'(x)$ exists at each point x in the open interval (a,b), where f is continuous and $F'(x) = f(x)$.

2. Assume that f is integrable on the closed interval $[a,b]$ and continuous in the open interval (a,b). Let F be any antiderivative so that $F'(x) = f(x)$,

   for each x in (a,b). If $a < c < b$, then for any x in (a,b)

   $$\int_c^x f(t)dt = F(x) - F(c).$$

   If the open interval on which f is continuous includes a and b, then we may write

   $$\int_a^b f(x)dx = F(b) - F(a).$$

□

<u>*Theorem 3.2.2*</u> (Mean Value theorem for integrals)

If f is continuous on the closed interval $[a,b]$, then there exists a number c in the closed interval $[a,b]$ such that

$$\int_a^b f(x)\,dx = f(c)(b-a).$$

<u>Proof:</u>

Define the function F by

$$F(x) = \int_a^x f(t)dt, \text{ for every } x \in [a,b].$$

Then by the fundamental theorem of calculus, F is continuous on $[a,b]$ and differentiable on (a,b) and that $F'(x) = f(x)$. By the mean value theorem for derivatives, there exists a number $c \in (a,b)$ such that

$$F'(c) = \frac{F(b) - F(a)}{b-a}.$$

implying that $F'(c) = f(c)$ and

$$F(a) = \int_a^a f(t)dt = 0$$

Thus,

$$f(c) = \frac{F(b)}{b-a} = \frac{1}{b-a}\int_a^b f(t)dt,$$

$$\text{i.e.} \qquad \int_a^b f(x)\,dx = f(c)(b-a).$$

□

Definition 3.2.1

If f is integrable over $[a,b]$, then the average value of f, denoted by $AV(f(x))$, is given by

$$AV(f(x)) = \frac{1}{b-a}\int_a^b f(x)dx.$$

◇

Example 3.2.3

1. Evaluate (a) $\int_{-2}^{2} 2x\,dx$ (b) $\int_0^2 |2x-1|\,dx.$
2. Find the average of $f(x) = \sqrt{x}$ over the interval $[1,9]$.

Solutions:

1. (a) We got zero when we evaluated this integral using Riemann sums. We now apply property (3):

   $$\int_{-2}^{2} 2x\,dx = -\int_{-2}^{0} 2x\,dx + \int_0^2 2x\,dx = -x^2\Big|_{-2}^{0} + x^2\Big|_0^2 = 4+4 = 8.$$

   (b) Since $|2x-1| = \begin{cases} 2x-1, & x \geq \frac{1}{2} \\ 1-2x, & x < \frac{1}{2} \end{cases}$ we have that

   $$\int_0^2 |2x-1|\,dx = \int_0^{\frac{1}{2}} (1-2x)\,dx + \int_{\frac{1}{2}}^{2} (2x-1)\,dx = (x-x^2)\Big|_0^{\frac{1}{2}} + (x^2-x)\Big|_{\frac{1}{2}}^{2} = \frac{5}{2}$$

2. $AV(f(x)) = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx = \dfrac{1}{9-1}\displaystyle\int_1^9 \sqrt{x}\;dx = \dfrac{1}{8}.\dfrac{2}{3}\left(x^{\frac{3}{2}}\right)\Big|_1^9 = \dfrac{13}{6}$

△

Theorem 3.2.3

If f is continuous on an open interval I containing $a,$ then for every x in the interval

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x).$$

□

Example 3.2.4

1. Evaluate $\frac{d}{dx}\left[\int_0^x \sqrt{t^2+1}\,dt\right]$
2. Find the derivative of $F(x) = \int_{\frac{\pi}{2}}^{x^3} \cos t\,dt.$

Solutions:

1. Note that $f(t) = \sqrt{t^2+1}$ is continuous on $\mathbb{R}.$ Thus,

$$\frac{d}{dx}\left[\int_0^x \sqrt{t^2+1}\,dt\right] = \sqrt{x^2+1}$$

2. Letting $u = x^3,$ we have that

$$F'(x) = \frac{dF}{du}.\frac{du}{dx} = \cos u.3x^2 = 3x^2\cos\left(x^3\right)$$

△

3.2.2 IMPROPER INTEGRALS

In the definition of a definite integral, $f(x)$ was assumed to be bounded in the interval $a \le x \le b.$ If $f(x)$ is not bounded in $a \le x \le b,$ then the integral $\int_a^b f(x)dx$ is said to be an improper integral. In other words, integrals with infinite limits of integration are improper integrals. In addition, integrals of functions that become infinite at a point within the interval of integration are improper integrals. When the limits involved exist, we evaluate such integrals with the following definitions:

1. If f is continuous on $[a,\infty),$ then

$$\int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx.$$

2. If f is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^{b} f(x)dx = \lim_{a\to-\infty}\int_{a}^{b} f(x)dx.$$

3. If f is continuous on $(a, b]$, then

$$\int_{a}^{b} f(x)dx = \lim_{c\to a^+}\int_{c}^{b} f(x)dx.$$

4. If f is continuous on $[a, b)$, then

$$\int_{a}^{b} f(x)dx = \lim_{c\to b^-}\int_{a}^{c} f(x)dx.$$

5. If f becomes infinite at an interior point d of $[a, b]$, then

$$\int_{a}^{b} f(x)dx = \int_{a}^{d} f(x)dx + \int_{d}^{b} f(x)dx.$$

The integral from a to b converges if the integrals from a to d and from d to b both converge. Otherwise, the integral from a to b diverges.

6. If f is continuous on $(-\infty, \infty)$, and if $\int_{-\infty}^{a} f(x)dx$ and $\int_{a}^{\infty} f(x)dx$ both converges, we say that $\int_{-\infty}^{\infty} f(x)dx$ converges and

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)dx + \int_{a}^{\infty} f(x)\,dx.$$

If either or both of the integrals on the right hand side diverges, the integral from $-\infty$ to ∞ is said to diverge as well.

Example 3.2.5

Evaluate

(a) $\int_2^{\infty} \frac{1}{x^3} dx$ (b) $\int_{-\infty}^{0} \tanh x dx$ (c) $\int_0^1 \frac{1}{\sqrt{x}} dx$ (d) $\int_0^1 \frac{1}{1-x} dx$ (e) $\int_0^6 \frac{2x}{\left(x^2-4\right)^{\frac{2}{3}}} dx$ (f) $\int_0^6 \frac{2x}{x^2-4} dx$

(g) $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx.$

Solutions:

(a) $\int_2^{\infty} \frac{1}{x^3} dx = \lim_{b\to\infty} \int_2^b x^{-3} dx = \lim_{b\to\infty} \left(-\frac{1}{2} x^{-2} \Big|_2^b \right) = -\frac{1}{2} \lim_{b\to\infty} \left(\frac{1}{b^2} - \frac{1}{4} \right) = \frac{1}{8}.$

(b) $\int_{-\infty}^{0} \tanh x dx = \lim_{a\to-\infty} \int_a^0 \tanh x dx = \lim_{a\to-\infty} \left(\left(\ln|\cosh x| \right)\Big|_a^0 \right) = \lim_{a\to-\infty} \left(\ln 1 - \ln\cosh a \right) = -\infty.$

$\therefore$ The integral diverges.

(c) The function $\frac{1}{\sqrt{x}}$ is discontinuous at $x = 0.$

$$\therefore \int_0^1 \frac{1}{\sqrt{x}} dx = \lim_{c\to 0^+} \int_c^1 x^{-\frac{1}{2}} dx = \lim_{c\to 0^+} \left(2x^{\frac{1}{2}} \Big|_c^1 \right) = \lim_{c\to 0^+} \left(2 - 2c^{\frac{1}{2}} \right) = 2.$$

(d) The function $\frac{1}{1-x}$ is discontinuous at $x = 1.$

$$\therefore \int_0^1 \frac{1}{1-x} dx = \lim_{c\to 1^-} \int_0^c \frac{1}{1-x} dx = \lim_{c\to 1^-} \left(\left(-\ln|1-x| \right)\Big|_0^c \right) = -\lim_{c\to 1^-} \left(\ln|1-c| \right) = \infty$$

$\therefore$ The integral diverges.

(e) In the interval $[0,6],$ the function $\frac{2x}{(x^2-4)^{\frac{2}{3}}}$ is discontinuous at $x = 2.$

$$\begin{aligned}\therefore \int_0^6 \frac{2x}{\left(x^2-4\right)^{\frac{2}{3}}} dx &= \int_0^2 \frac{2x}{\left(x^2-4\right)^{\frac{2}{3}}}\, dx + \int_2^6 \frac{2x}{\left(x^2-4\right)^{\frac{2}{3}}}\, dx \\ &= \lim_{c\to 2^-} \int_0^c \frac{2x}{\left(x^2-4\right)^{\frac{2}{3}}}\, dx + \lim_{c\to 2^+} \int_c^6 \frac{2x}{\left(x^2-4\right)^{\frac{2}{3}}}\, dx\end{aligned}$$

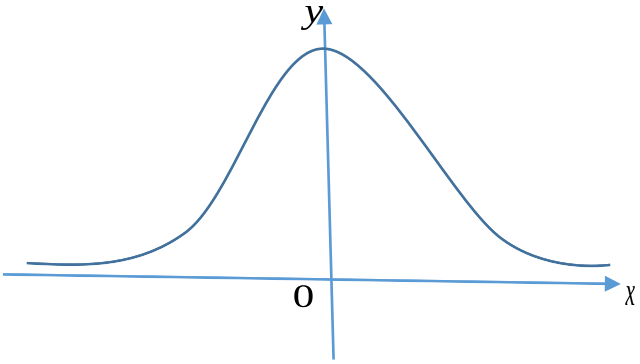
$$
\begin{aligned}
&= \lim_{c\to 2^-}\left(3\left(x^2-4\right)^{\frac{1}{3}}\Big|_0^c\right)+\lim_{c\to 2^+}\left(3\left(x^2-4\right)^{\frac{1}{3}}\Big|_c^6\right)\\
&= \lim_{c\to 2^-}\left[3\left(c^2-4\right)^{\frac{1}{3}}-3(-4)^{\frac{1}{3}}\right]+\lim_{c\to 2^+}\left[3\left(32\right)^{\frac{1}{3}}-3(c^2-4)^{\frac{1}{3}}\right]\\
&= 3.4^{\frac{1}{3}}+3.(32)^{\frac{1}{3}}\\
&= 9.4^{\frac{1}{3}}.
\end{aligned}
$$

(f) The integrand is discontinuous at $x=2$. Thus,

$$
\begin{aligned}
\int_0^6 \frac{2x}{x^2-4}dx &= \lim_{c\to 2^-}\int_0^c \frac{2x}{\left(x^2-4\right)}dx+\lim_{c\to 2^+}\int_c^6 \frac{2x}{\left(x^2-4\right)}dx\\
&= \lim_{c\to 2^-}\left(\left(\ln|x^2-4|\right)\Big|_0^c\right)+\lim_{c\to 2^+}\left(\left(\ln|x^2-4|\right)\Big|_c^6\right)\\
&= \lim_{c\to 2^-}\left(\ln|c^2-4|-\ln 4\right)+\lim_{c\to 2^+}\left(\ln 32-\ln|c^2-4|\right).
\end{aligned}
$$

Since $\lim_{c\to 2^-}\left(\ln|c^2-4|-\ln 4\right)=-\infty$ and $\lim_{c\to 2^+}\left(\ln 32-\ln|c^2-4|\right)=\infty$, the integral diverges.

(g) We try to sketch the graph of $\frac{1}{1+x^2}$.



The graph is symmetric at $x=0$. So

$$
\begin{aligned}
\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx &= \int_{-\infty}^{0}\frac{1}{1+x^2}dx+\int_0^{\infty}\frac{1}{1+x^2}dx=\lim_{a\to-\infty}\int_a^0\frac{1}{1+x^2}dx+\lim_{b\to\infty}\int_0^b\frac{1}{1+x^2}dx\\
&= \lim_{a\to-\infty}\left(\arctan x\big|_a^0\right)+\lim_{b\to\infty}\left(\arctan x\big|_0^b\right)=-\left(-\frac{\pi}{2}\right)+\frac{\pi}{2}=\pi.
\end{aligned}
$$

△

3.3 APPLICATION OF INTEGRATION

The use of the integral as a limit of a sum enables us to solve many physical and geometrical problems such as determination of areas, volumes, arc length, surface area, centre of mass and moments of inertia.

3.3.1 THE AREA BETWEEN TWO CURVES

Suppose we want to find the area A of the region R bounded by two curves $y_1 = f_1(x)$ and $y_2 = f_2(x),$ and two vertical lines $x = a$ and $x = b.$ We assume that the curve $y_2 = f_2(x)$ lies above the curve $y_1 = f_1(x)$ such that $f_2(x) \geq f_1(x),$ for $a \leq x \leq b.$ Then

$$A = \int_a^b [\, f_2(x) - f_1(x)]dx.$$

Similarly, if R is bounded by two curves $x_1 = f_1(y)$ and $x_2 = f_2(y)$ and two horizontal lines $y = c$ and $y = d$ in such a way that $f_2(y) \geq f_1(y),$ for $c \leq x \leq d,$ then

$$A = \int_c^d [\, f_2(y) - f_1(y)]dy.$$

Example 3.3.1

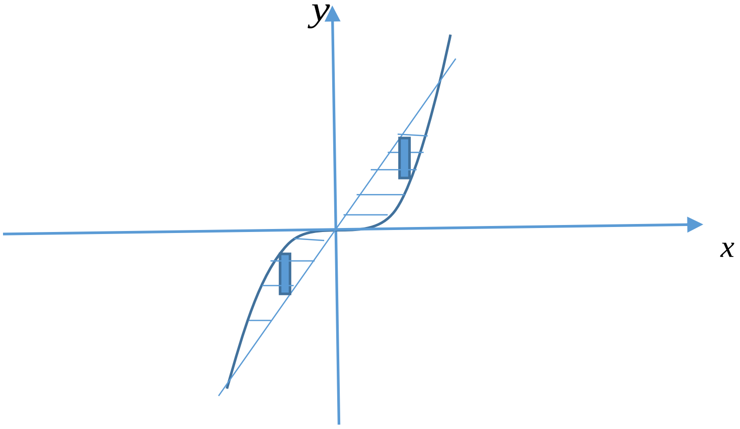
Find the area of the finite region bounded by the curves $x = y$ and $y = \dfrac{x^5}{16}$ using

(a) vertical strips

(b) horizontal strips.

Solutions:

(a)

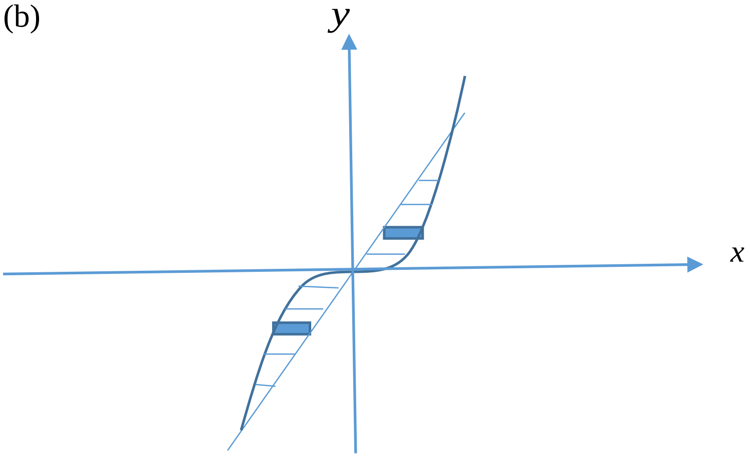


Solving the two equations simultaneously, we find

$$16x = x^5 \text{ or } x(16 - x^4) = 0 \Rightarrow x = 0 \text{ or } x = \pm 2.$$

$$\therefore A = \int_{-2}^{0}\left(\frac{x^5}{16} - x\right)dx + \int_{0}^{2}\left(x - \frac{x^5}{16}\right)dx = \left(\frac{x^6}{96} - \frac{x^2}{2}\right)\Bigg|_{-2}^{0} + \left(\frac{x^2}{2} - \frac{x^6}{96}\right)\Bigg|_{0}^{2} = \frac{8}{3}.$$

(b)



$$x = y, \quad x = (16y)^{\frac{1}{5}}$$

$$\therefore A = \int_{-2}^{0}\left(y - (16y)^{\frac{1}{5}}\right)dy + \int_{0}^{2}\left((16y)^{\frac{1}{5}} - y\right)dy = \left(\frac{y^2}{2} - \frac{5}{6}.(16)^{\frac{1}{5}}y^{\frac{6}{5}}\right)\Bigg|_{-2}^{0} + \left(\frac{5}{6}.(16)^{\frac{1}{5}}y^{\frac{6}{5}} - \frac{y^2}{2}\right)\Bigg|_{0}^{2} = \frac{8}{3}.$$

△

3.3.2 VOLUMES OF SOLIDS OF REVOLUTION

If a region in the plane is revolved about a given line, the resulting solid is a solid of revolution and the line is called axis of revolution.

1. DISC METHOD

The simplest solid of revolution is a right circular cylinder which is found by revolving a rectangle about an axis adjacent to one side of the rectangle. In this case, the cross sections of the solid (slice or disc) are perpendicular to the $x-$ axis of revolution. Two cases arise:

(i) The volume of the solid generated by a region under $y = f(x)$ bounded by the $x-$ axis and vertical lines $x = a$ and $x = b$ which is revolved about the $x-$ axis is

$$V=\int_a^b \pi r^2 dx=\int_a^b \pi y^2 dx=\pi\int_a^b \left[f(x)\right]^2 dx.$$

Here the disc is with respect to x and $r=y=f(x)$.

(ii) The volume of the solid generated by a region under $x=f(y)$ bounded by the $y-$axis and horizontal lines $y=c$ and $y=d$ which is revolved about the $y-$axis is

$$V=\pi\int_c^d \left[f(y)\right]^2 dy.$$

2. <u>WASHER METHOD</u>

Suppose that we are dealing with a hollowed object and two functions have been given instead of one. Then we extend the disc method and take into account the difference of the two given functions. This is the washer method and we have two cases:

(i) If the washer is with respect to x then the volume of the solid generated by a region between $y_1=f(x)$ and $y_2=g(x)$ bounded by the vertical lines $x=a$ and $x=b,$ which is revolved about the $x-$ axis is

$$V=\int_a^b \pi\left(R^2-r^2\right)dx=\pi\int_a^b \left|\left[f(x)\right]^2-\left[g(x)\right]^2\right|dx,$$

where R is the outer radius and r is the inner radius.

(ii) If the washer being considered is with respect to $y,$ then the volume of the solid generated by a region between $x_1=f(y)$ and $x_2=g(y)$ bounded by the horizontal lines $y=c$ and $y=d,$ which is revolved about the $y-$axis is

$$V=\pi\int_a^b \left|\left[f(y)\right]^2-\left[g(y)\right]^2\right|dy.$$

<u>NOTE</u>: The axis of revolution does not necessarily need be the $x-$ or $y-$axis.

<u>Example 3.3.2</u>

1. Find the volume of the solid of revolution obtained by rotating the region under the curve $y=x^{\frac{2}{3}}$ between $x=0$ and $x=8$

(a) about the $x-$ axis (b) about the $y-$axis.

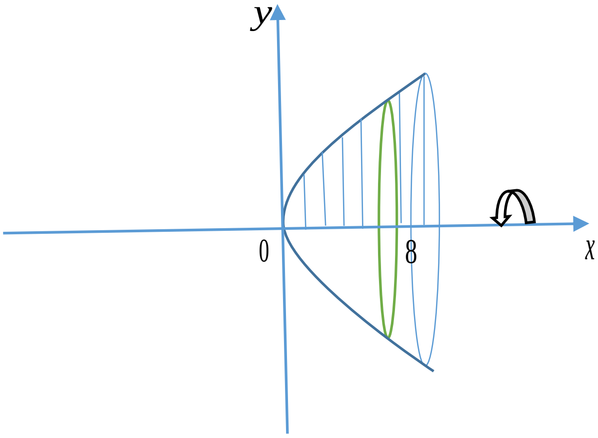
2. Determine the volume of the solid generated by rotating the finite region bounded by $y=\sqrt[3]{x}$ and $y=\frac{x}{4}$ that lies in the first quadrant.

(a) about the $x-$axis (b) about the $y-$axis.

3. Determine the volume of the solid obtained by rotating the finite region bounded by $y=x^2-2x$ and $y=x$ about the line $y=4$.

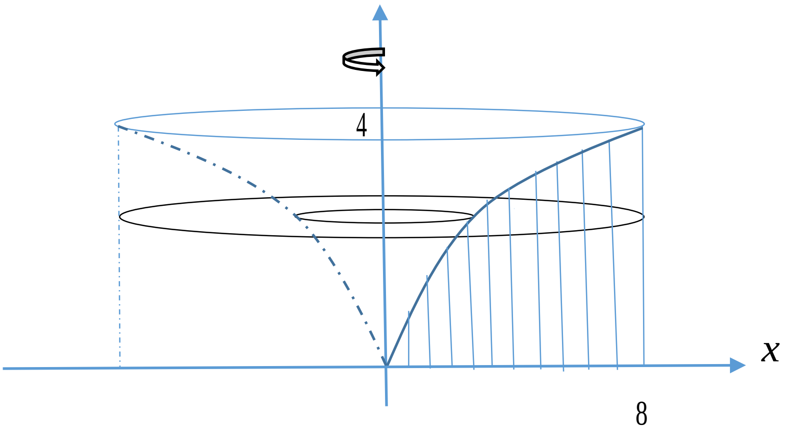
Solution

1. (a)



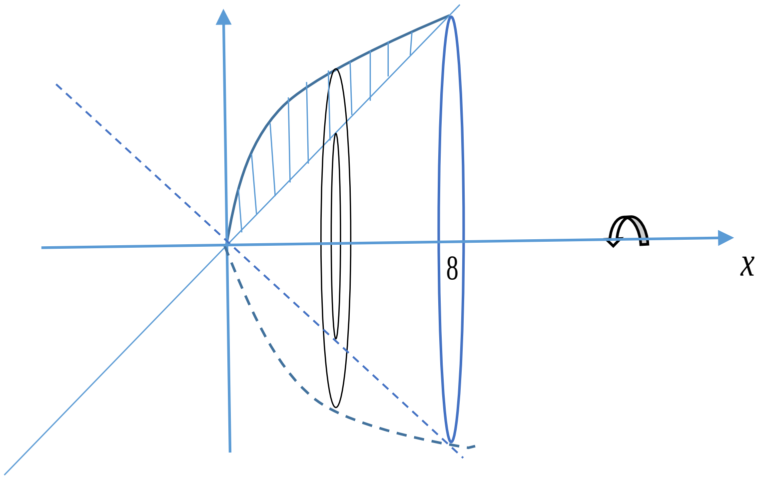
$$V=\pi\int_0^8\left(x^{\frac{2}{3}}\right)^2dx=\pi.\frac{3}{7}x^{\frac{7}{3}}\bigg|_0^8=\frac{384\pi}{7}.$$

(b) When $x=8,\ y=4$ and $x=y^{\frac{3}{2}}$



$$V=\pi\int_0^4\left(8^2-\left(y^{\frac{3}{2}}\right)\right)dy=\pi\left(64y-\frac{y^4}{4}\right)\bigg|_0^4=192\pi.$$

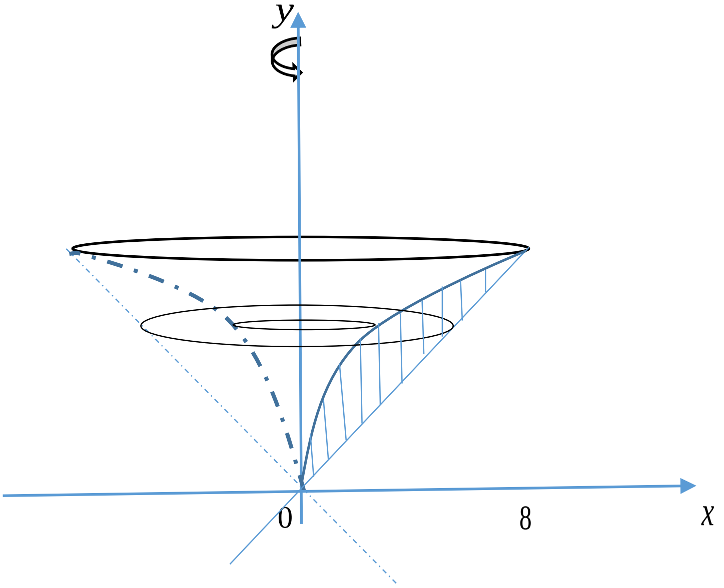
2. (a)



$y=\frac{x}{4},\quad y=x^{\frac{1}{3}} \Rightarrow \frac{x}{4}=x^{\frac{1}{3}} \Rightarrow x=0$ or $x=\pm 8.$

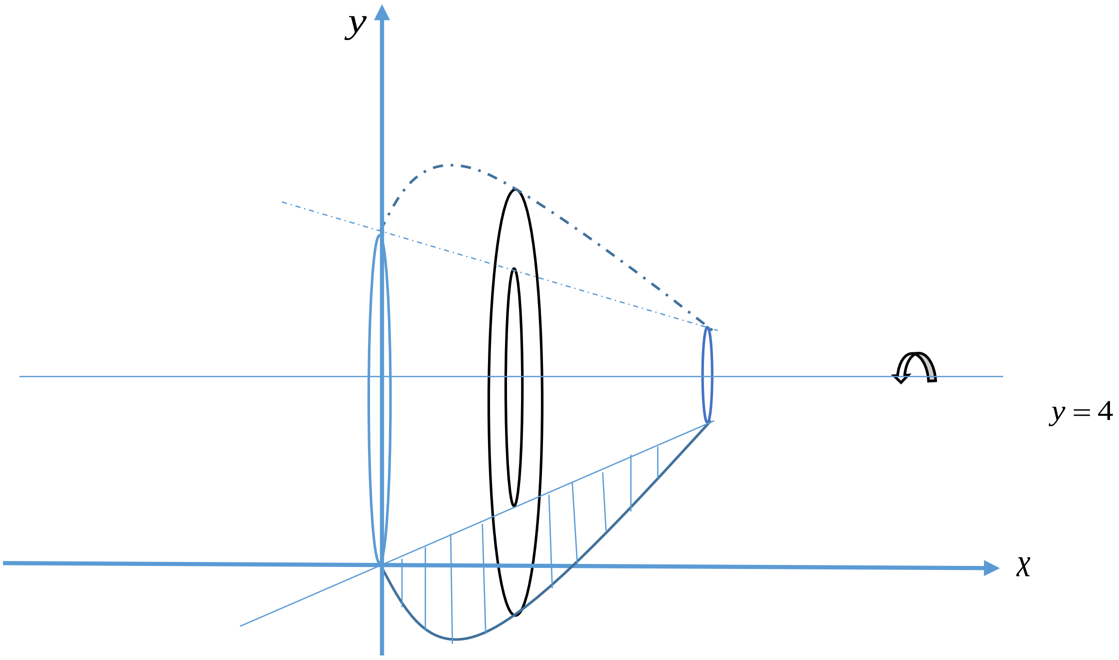
$$V=\pi\int_0^8\left(R^2-r^2\right)dx=\pi\int_0^8\left(\left(x^{\frac{1}{3}}\right)^2-\left(\frac{x}{4}\right)^2\right)dx=\pi\left(\frac{3}{5}x^{\frac{5}{2}}-\frac{x^3}{48}\right)\Bigg|_0^8=\frac{128\pi}{15}.$$

(b) $y=\frac{x}{4},\quad y=x^{\frac{1}{3}} \Rightarrow x=4y$ and $x=y^3$. When $x=8,\ y=2.$



$$V=\pi\int_0^2\left[\left((4y)^2-(y^{3)^2}\right)\right]dy=\pi\left(\frac{16y^3}{3}-\frac{y^7}{7}\right)\Bigg|_0^2=\frac{512\pi}{21}.$$

3.



$$y=x,\ y=x^2-2x \Rightarrow x=x^2-2x=x \Rightarrow x^2-3x=0 \Rightarrow x=0 \text{ or } x=3.$$

$$V=\pi\int_0^3\left(R^2-r^2\right)dx=\pi\int_0^3\left[\left(4-(x^2-2x)\right)^2-\left(4-x\right)^2\right]dx=\pi\left(\frac{1}{5}x^5-x^4-\frac{5}{3}x^3+12x^2\right)\Bigg|_0^3=\frac{153\pi}{5}.$$

△

3. SHELL METHOD

The shell method is a method of calculating the volume of a solid of revolution when integrating along an axis parallel to the axis of revolution. The idea is that a “representative rectangle” can be rotated about the axis of revolution, thus generating a hollow cylinder-a shell. Then

$$\text{Volume}(V)=\int 2\pi.\text{radius}.\text{height}.\text{thickness}$$

(i) The volume of the solid generated by a region bounded by the $x-$axis and vertical lines $x=a$ and $x=b,$ which is revolved about the $y-$axis is

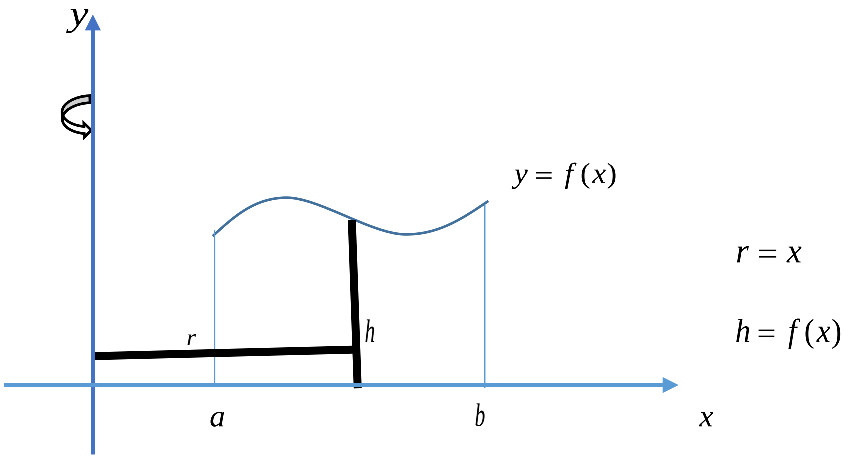
$$V=\int_a^b 2\pi xydx=2\pi\int_a^b x\left[f(x)\right]dx,$$

where radius $(r)=x,$ height $(h)=y=f(x)$ and thickness $=dx.$

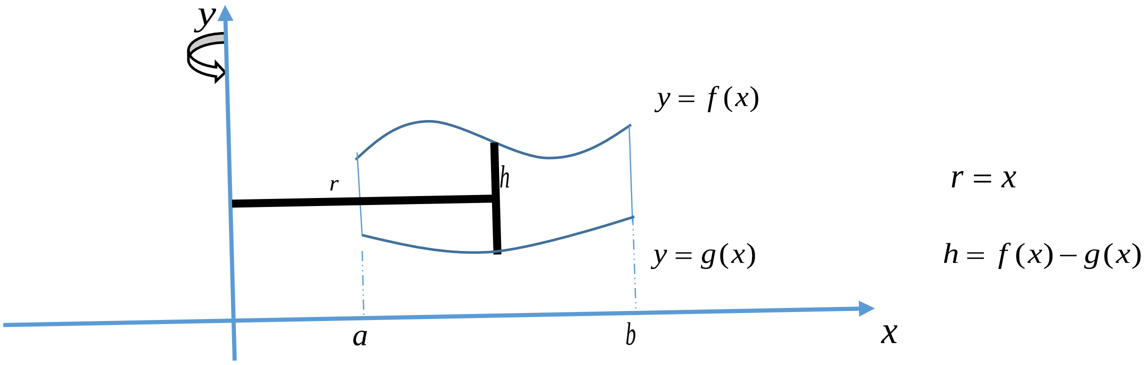
(ii) The volume of the solid generated by a region bounded by the $y-$axis and horizontal lines $y=c$ and $y=d,$ which is revolved about the $x-$axis is

$$V = 2\pi\int_{c}^{d} y\left[f(y)\right]dy.$$

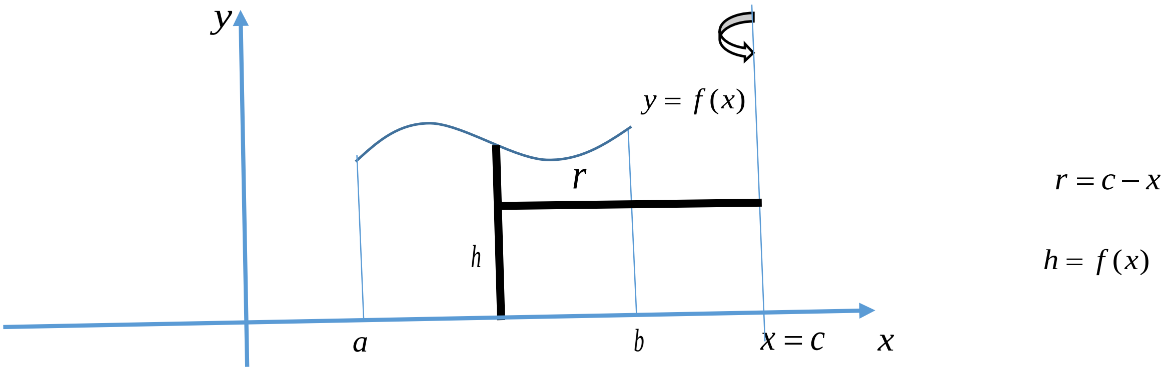
NOTE: Sometimes two functions are involved and the axis of revolutions is not always the $x-$ or $y-$axis. These and other cases discussed above are summarised in the following graphs below:



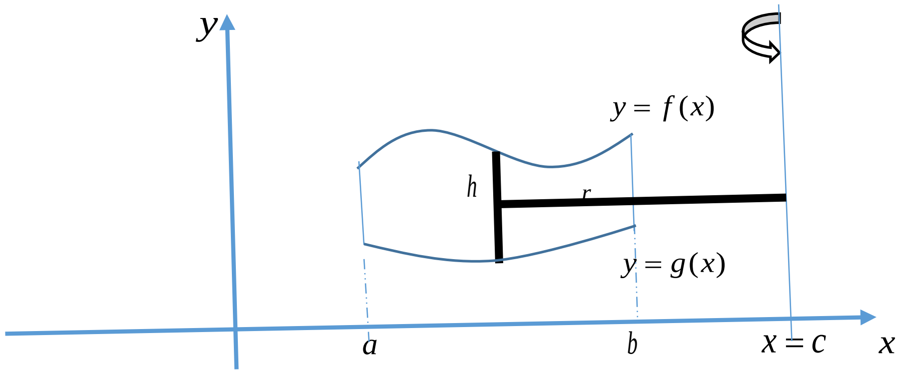
$$V = 2\pi\int_{a}^{b} x\left[f(x)\right]dx$$



$$V = 2\pi\int_{a}^{b} x\left[f(x)-g(x)\right]dx$$



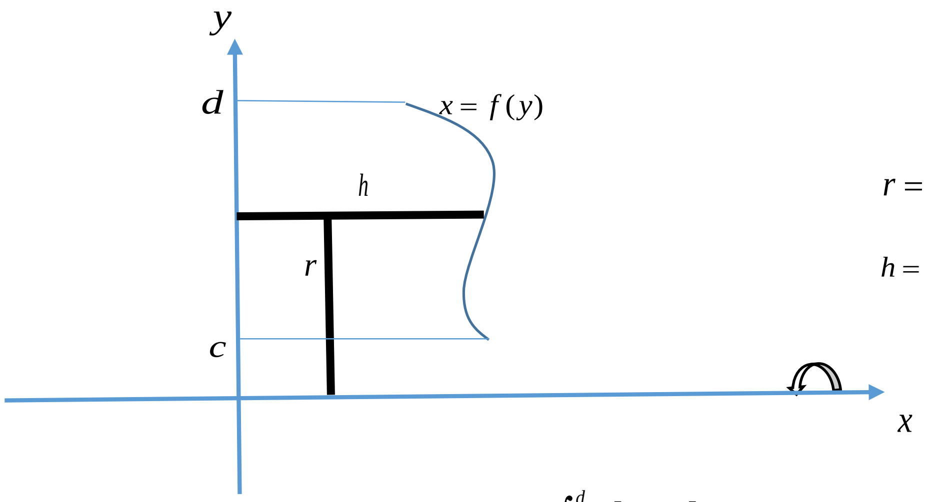
$$V = 2\pi\int_{a}^{b} (c-x)\left[f(x)\right]dx$$



$r = c - x$

$h = f(x) - g(x)$

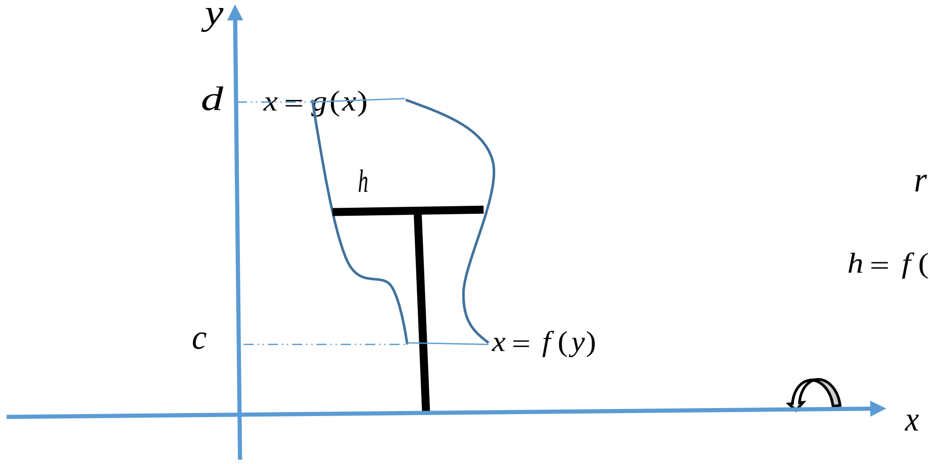
$$V = 2\pi\int_a^b (c-x)\left[f(x)-g(x)\right]dx$$



$r = y$

$h = f(y)$

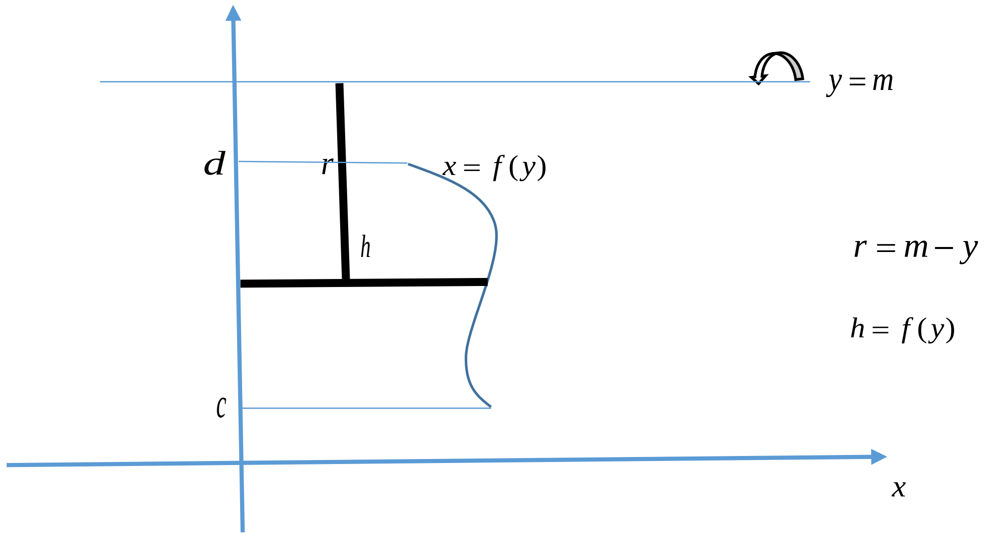
$$V = 2\pi\int_c^d y\left[f(y)\right]dy$$



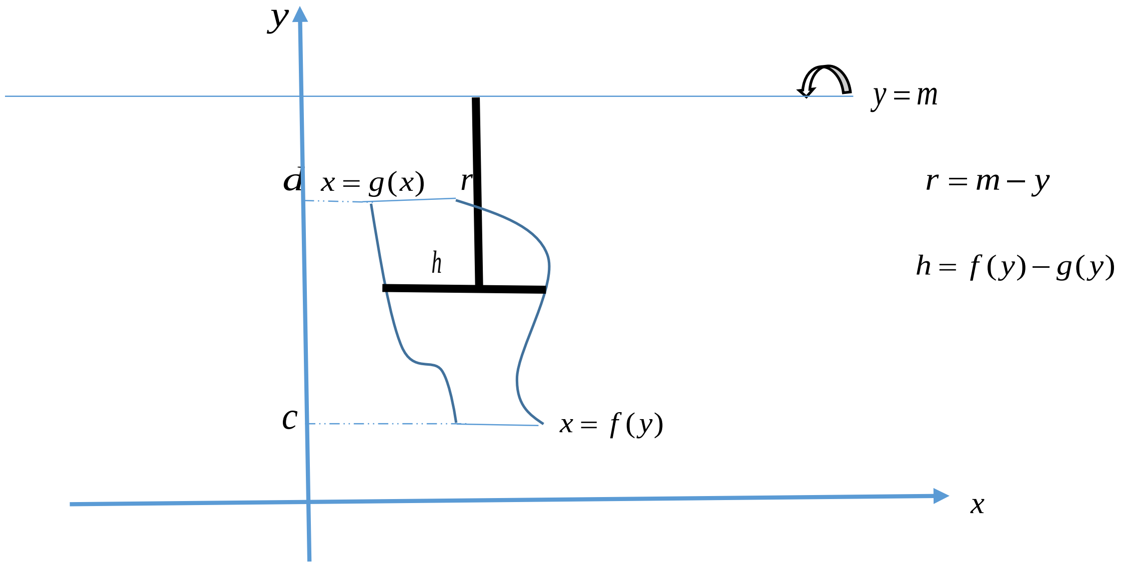
$r = y$

$h = f(y) - g(y)$

$$V = 2\pi\int_c^d y\left[f(y)-g(y)\right]dy$$



$$V=2\pi\int_c^d (m-y)\left[f(y)\right]dy$$



$$V=2\pi\int_c^d (m-y)\left[f(y)-g(y)\right]dy$$

Example 3.3.3

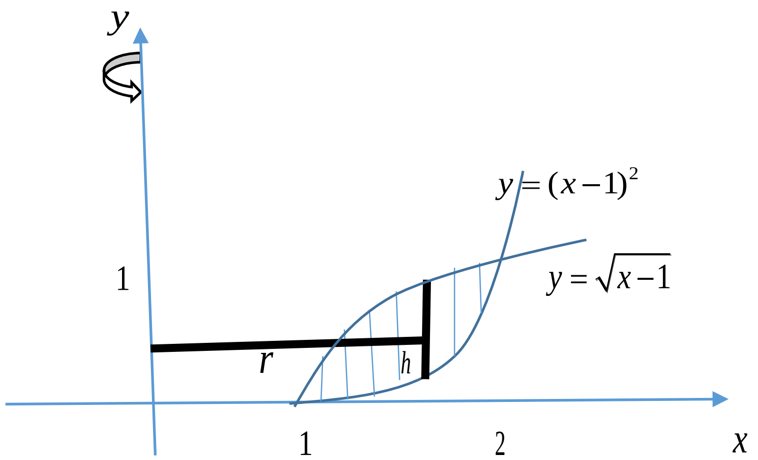
1. Find the volume of the solid of revolution formed by rotating the finite region bounded by the graphs of $y=\sqrt{x-1}$ and $y=(x-1)^2$

(a) about the $y-$axis (b) about the $x-$ axis.

2. A region is bounded by $y=\cos x,\ y=\sin x,\ x=0,\ x=\frac{\pi}{6}$. Find the volume when this region is rotated about the line $x=2$.

Solutions:

1 (a)



$$r = x,\ \ h = \sqrt{x-1} - (x-1)^2$$

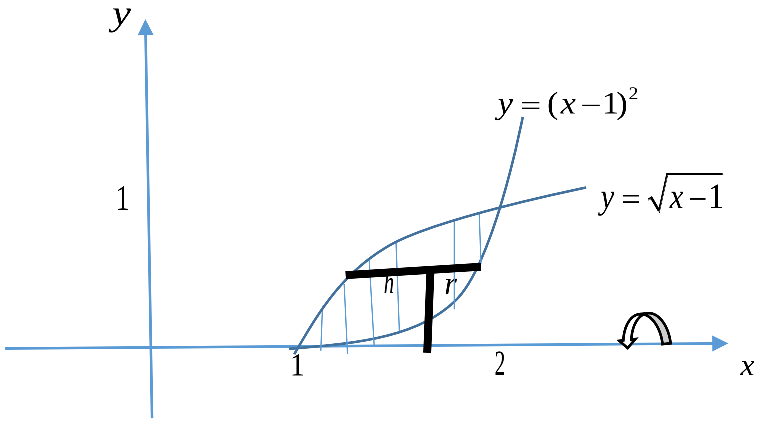
$$\therefore V = 2\pi\int_1^2 x\left(\sqrt{x-1}-(x-1)^2\right)dx = 2\pi\left[\int_1^1 x\sqrt{x-1}dx - \int_1^2 (x^3-2x+x)dx\right]$$

$$= 2\pi\left[\left(\frac{2}{5}(x-1)^{\frac{5}{2}} + \frac{2}{3}(x-1)^{\frac{3}{2}}\right)\Bigg|_1^2 - \left(\frac{x^4}{4} - \frac{2}{3}x^3 + \frac{1}{2}x^2\right)\Bigg|_1^2\right]$$

$$= 2\pi\left(\frac{29}{60}\right)$$

$$= \frac{29\pi}{30}.$$

(b)

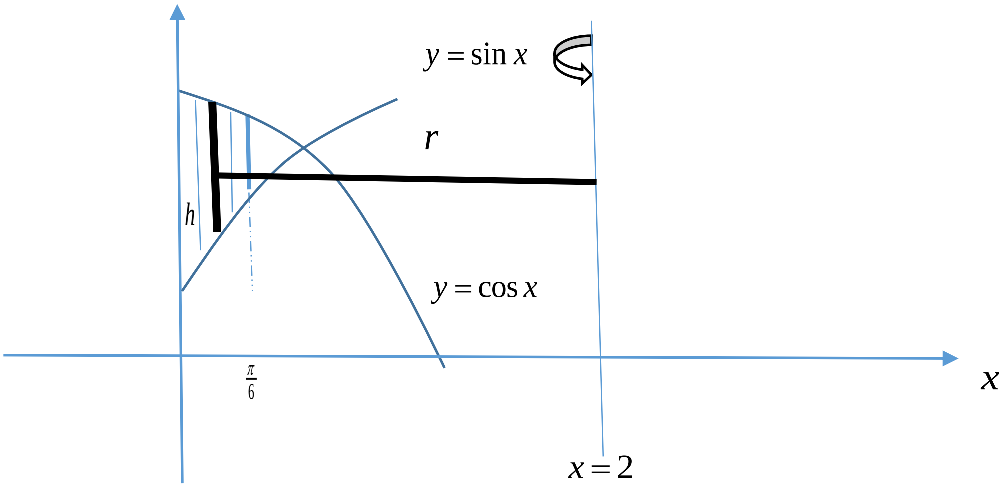


$$y=\sqrt{x-1} \Rightarrow x = y^2+1.\ \ y=(x-1)^2 \Rightarrow x = y^{\frac{1}{2}}+1.$$

$$r = y,\ \ h = y^{\frac{1}{2}}+1-(y^2+1) = y^{\frac{1}{2}} - y^2$$

$$\therefore V = 2\pi\int_0^1 y\left(y^{\frac{1}{2}}-y^2\right)dy = 2\pi\int_0^1\left(y^{\frac{3}{2}}-y^3\right)dy = \frac{3\pi}{10}.$$

2.



$$h=\cos x-\sin x,\ r=2-x$$

$$\therefore V=2\pi\int_0^{\frac{\pi}{6}}(2-x)(\cos x-\sin x)dx=2\pi\left[(2-x)(\sin x+\cos x)\Big|_0^{\frac{\pi}{6}}+(\sin x-\cos x)\Big|_0^{\frac{\pi}{6}}\right]$$

$$=2\pi\left[\left(2-\frac{\pi}{6}\right)\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\right)+\left(\frac{1}{2}-\frac{\sqrt{3}}{2}\right)-(2-1)\right]=\frac{\pi\left[6+12\sqrt{3}-\pi-\pi\sqrt{3}\right]}{6}.$$

△

3.3.3 LENGTH OF THE ARC

Recall from chapter two that if $f'(x)$ is continuous in $a\le x\le b,$ then the length $L,$ of the arc of the curve $y=f(x)$ between $x=a$ and $x=b$ is given by

$$L=\int_a^b\sqrt{1+\left[f'(x)\right]^2}dx.$$

If the curve is given by $x=g(y)$ between $y=c$ and $y=d$, then

$$L=\int_c^d\sqrt{1+\left[g'(x)\right]^2}dy.$$

If a plane curve is given by $\left\{\left(x(t),y(t)\right):t\in\left[a,b\right]\right\},$ where x and y are continuous functions, then

$$L=\int_a^b\sqrt{(x'(t))^2+(y'(t))^2}dt.$$

If a curve is given in polar form $r=f(\theta),\ f(\theta)>0,\ \alpha\le\theta\le\beta,$ then $x(\theta)=r(\theta)\cos\theta,$

$y(\theta) = r(\theta)\sin\theta$ and

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

<u>Example 3.3.4</u>

1. Find the length of the arc of the curve $y = 2\sqrt{x^3}$ between $x = \dfrac{1}{3}$ and $x = \dfrac{5}{3}$.

2. Find the arc length of the cardioid $r = 2 + 2\cos\theta,\ 0 \le \theta \le 2\pi.$

<u>Solution</u>

1. $y = \sqrt{x^3},\ x \ge 0$ and $\dfrac{dy}{dx} = 3\sqrt{x}$

$$\begin{aligned}\therefore L = \int_{\frac{1}{3}}^{\frac{5}{3}} \sqrt{1 + [y'(x)]^2}\, dx &= \int_{\frac{1}{3}}^{\frac{5}{3}} \sqrt{1 + \left[3\sqrt{x}\right]^2}\, dx \\ &= \int_{\frac{1}{3}}^{\frac{5}{3}} \sqrt{1 + 9x^2}\, dx \\ &= \frac{1}{9}.\frac{2}{3}\left(1 + 9x\right)\Big|_{\frac{1}{3}}^{\frac{5}{3}} \\ &= \frac{112}{27}.\end{aligned}$$

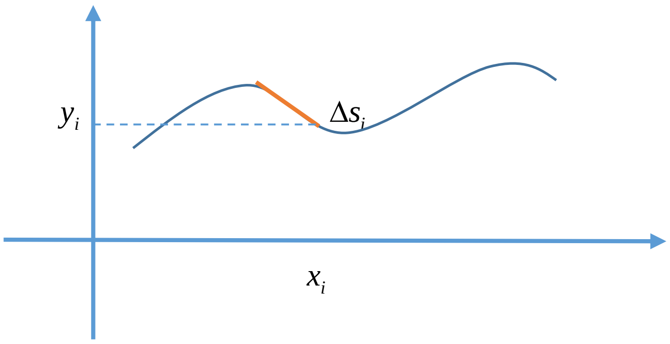
2. $\quad r = 2 + 2\cos\theta \Rightarrow \dfrac{dr}{d\theta} = -2\sin\theta$

$$\begin{aligned}\therefore L = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta &= \int_0^{2\pi} \sqrt{(2 + 2\cos\theta)^2 + \left(-2\sin\theta\right)^2}\, d\theta \\ &= \int_0^{2\pi} \sqrt{4 + 8\cos\theta + 4\cos^2\theta + 4\sin^2\theta}\, d\theta \\ &= \int_0^{2\pi} \sqrt{8 + 8\cos\theta}\, d\theta \\ &= 2\int_0^{2\pi} \sqrt{2 + 2\cos\theta}\, d\theta \\ &= 2\int_0^{2\pi} \sqrt{4\cos^2\tfrac{\theta}{2}}\, d\theta \\ &= 4\int_0^{2\pi} \cos\tfrac{\theta}{2}\, d\theta \\ &= 8\int_0^{\pi} \cos\tfrac{\theta}{2}\, d\theta \\ &= 16\sin\tfrac{\theta}{2}\Big|_0^{\pi} \\ &= 16.\end{aligned}$$

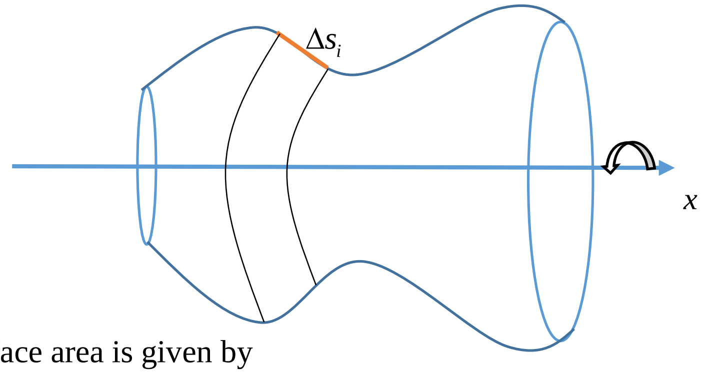
△

3.3.4 AREA OF A SURFACE OF REVOLUTION

Let $y = f(x),\ a \le x \le b$ determine a curve in the upper half of the xy plane.



Partition the interval $[a,b]$ into n pieces by means of the points $a = x_0 < x_1 < x_2 < ... < x_n = b,$ thereby also dividing the curve into n pieces. Let Δs_i denote the length of the i^{th} piece and ley y_i be the $y-$coordinate of a point on this piece. When this curve is rotated about the $x-$ axis, it generates a surface called a surface of revolution.



The surface area is given by

$$\begin{aligned}\delta &= \lim_{\Delta s_i \to 0} \sum_{i=1}^{n} 2\pi y_i \Delta s_i \\ &= 2\pi \int_a^b y\ ds \\ &= 2\pi \int_a^b f(x)\sqrt{1+\left[f'(x)\right]^2}\ dx\end{aligned}$$

Thus, we have the following theorem:

Theorem 3.3.1

Suppose that $f(x) \ge 0$ and $f'(x)$ is continuous in the interval $\left[a,b\right]$. Then, the area of the surface of revolution generated by rotating the curve $y = f(x)$ between $x = a$ and $x = b$ about the $x-$ axis is given by

$$\delta = \int_a^b 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx.$$

If $x=g(y)$ is the curve between $y=c$ and $y=d$ rotated about the $y-$axis, then

$$\delta=\int_c^d 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy.$$

NOTE: The axis of rotation could be any line. If $y=n$ and $x=m$ are axes of rotation, then

$$\delta=\int_a^b 2\pi(n-y)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

and

$$\delta=\int_c^d 2\pi(m-x)\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

respectively.

□

Example 3.3.5

The arc of the curve $y=x^3$ lying between $x=0$ and $x=2$ is rotated about the $x-$ axis. Find the area of the surface generated.

Solution:

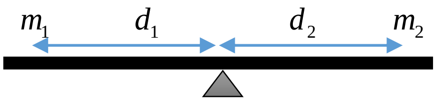
$$y=x^3\Rightarrow\frac{dy}{dx}=3x^2$$

$$\therefore\delta=2\pi\int_0^2 x^3\sqrt{1+(3x^2)^2}\,dx=2\pi\int_0^2 x^3\sqrt{1+9x^4}\,dx=\frac{2\pi}{36}\cdot\frac{2}{3}\left(1+9x^4\right)^{\frac{3}{2}}\Big|_0^2=\frac{\pi}{27}\left[(145)^{\frac{3}{2}}-1\right].$$

△

3.3.5 CENTROID AND MOMENT OF INERTIA

To discuss the concept of the moment a body, consider the example of two bodies of masses m_1 and m_2 placed on the seesaw at their respective distances d_1 and d_2 from the fulcrum.



Then the seesaw will balance if and only if

$$d_1 m_1 = d_2 m_2.$$

If the directed distance d_2 is negative, then the condition for balance is that $d_1 m_1 + d_2 m_2 = 0$, that is, the turning effect is zero. The moment of a body is a measure of its tendency to rotate about a point.

Definition 3.3.1

Let $\mathcal{P}$ be a plane system of particles of masses $m_1, m_2, ..., m_n$ located at the points $p_1, p_2, ..., p_n$, respectively. Let the line $\mathcal{L}$ in the plane be taken as an axis and let $l_1, l_2, ..., l_n$ be the directed distances from the line $\mathcal{L}$ of the points $p_1, p_2, ..., p_n$, respectively. Then, the moment of this system of particles about the axis $\mathcal{L}$, denoted by $M_{\mathcal{L}}$, is given by

$$M_{\mathcal{L}} = \sum_{k=1}^{n} m_k l_k = m_1 l_1 + m_2 l_2 + ... + m_n l_n.$$

Thus, moments about $x-$ and $y-$axes are given by

$$M_x = \sum_{k=1}^{n} m_k y_k$$

and

$$M_y = \sum_{k=1}^{n} m_k x_k,$$

respectively. ◇

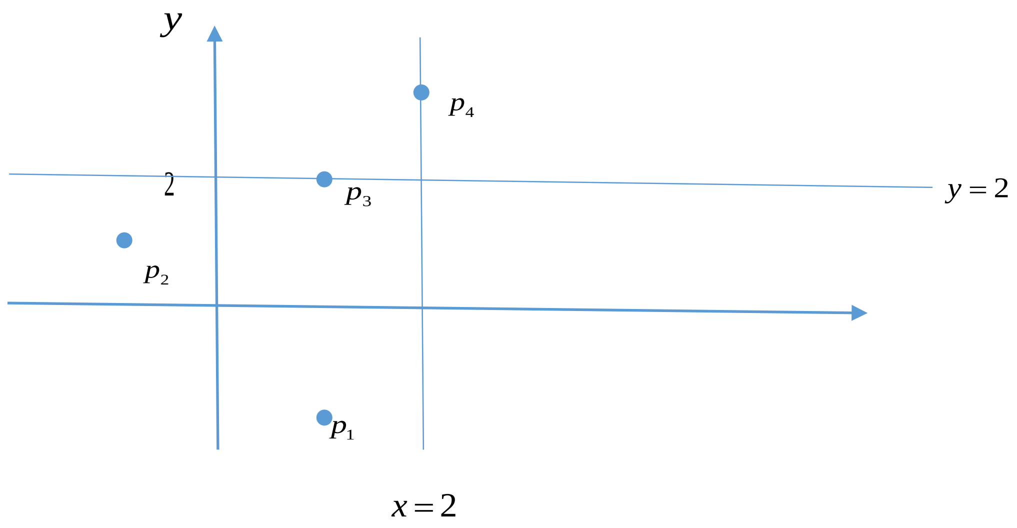
For example, for the points $p_1(1,-1)$, $p_2(-1,1)$, $p_3(1,2)$, and $p_4(2,3)$ with their respective masses $m_1 = 2$, $m_2 = 1$, $m_3 = 4$ and $m_4 = 7$, we have that

$$M_x = 2(-1) + 1(1) + 4(2) + 7(3) = 28$$
$$M_y = 2(1) + 1(-1) + 4(1) + 7(2) = 19.$$

If $\mathcal{L}_1$ is the line $x = 2$ and $\mathcal{L}_2$ is the line $y = 2$, then

$$M_{\mathcal{L}_1} = 2(-1) + 1(-3) + 4(-1) + 7(0) = -9$$

$$M_{\mathcal{L}_2} = 2(-3) + 1(-1) + 4(0) + 7(1) = 0$$



Definition 3.3.2

Let $\mathcal{P}$ be a plane system of particles and let $\bar{\mathcal{P}}$ be the system obtained by concentrating the total mass of the system at a single point $\bar{\mathcal{P}}(\bar{x},\bar{y})$. Let $M_{\mathcal{L}}$ and $\bar{M}_{\mathcal{L}}$ denote the moments about $\mathcal{L}$ of the systems $\mathcal{P}$ and $\bar{\mathcal{P}}$ respectively. If $M_{\mathcal{L}}=\bar{M}_{\mathcal{L}}$ for each axis $\mathcal{L}$, then $\bar{\mathcal{P}}(\bar{x},\bar{y})$ is called the centre of mass of the system $\mathcal{P}$. ◇

If the moment of the system $\bar{\mathcal{P}}$ is equal to the moment of the original system, then we must have $m\bar{x}=M_y$ and $m\bar{y}=M_x$, where m is the total mass of the system. Thus, we have the following theorem:

Theorem 3.3.3

If $\bar{\mathcal{P}}(\bar{x},\bar{y})$ is called the centre of mass of a system of particles of mass at $p_k(x_k,y_k)$, $k=1,2,\ldots,n$, then

$$\bar{x}=\frac{M_y}{m}$$

and

$$\bar{y}=\frac{M_x}{m},$$

where $m=m_1+m_2+\ldots+m_n$.

□

From our previous example,

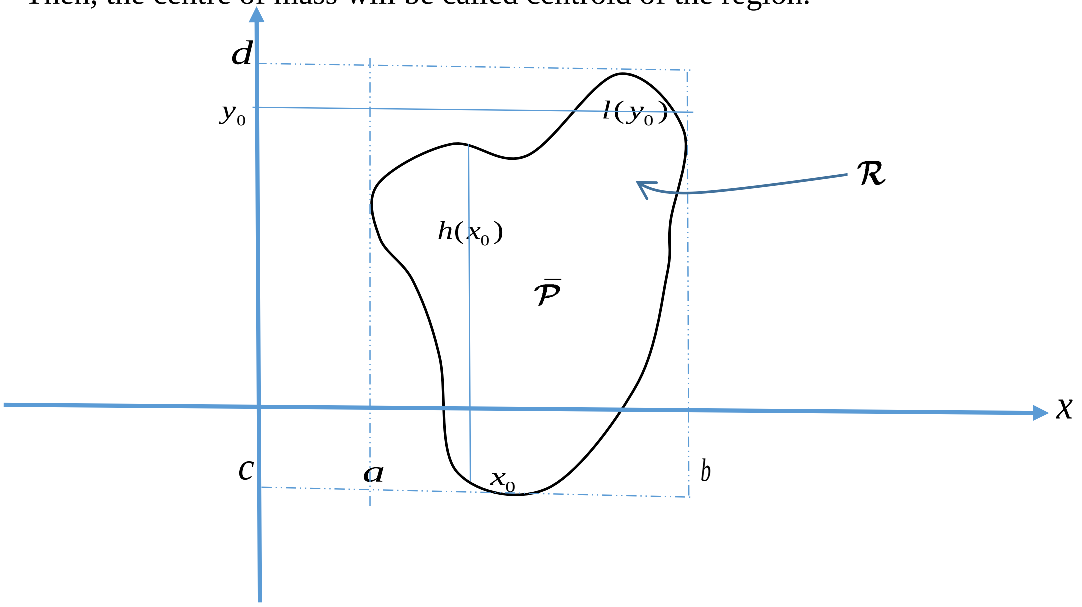
$$m = 2 + 1 + 4 + 7 = 14.$$

Hence,

$$\bar{x} = \frac{M_y}{m} = \frac{19}{14}$$

$$\bar{y} = \frac{M_x}{m} = \frac{28}{14} = 2.$$

Instead of a plane system of particles $\mathcal{P}$, suppose that we have a plane region of particles $\mathcal{R}$. Then, the centre of mass will be called centroid of the region.



Definition 3.3.3

Let $\mathcal{R}$ be a plane region lying in the rectangle $a \le x \le b,\ \ c \le y \le d$. Let the line $x = x_0$ intersect this region in a line segment of length $h(y_0)$ for each x_0 in $a \le x \le b$, and let the line $y = y_0$ intersect this region in a line segment of length $l(y_0)$ for each y_0 in $c \le y \le d$. Then, M_x and M_y, the moments of $\mathcal{R}$ about the $x-$ and the $y-$axes respectively, are given by

$$M_x = \int_c^d y\, l(y) dy,$$

$$M_y = \int_a^b x\, h(x) dx$$

and the centroid $\bar{\mathcal{P}}(\bar{x},\bar{y})$ of the region is such that

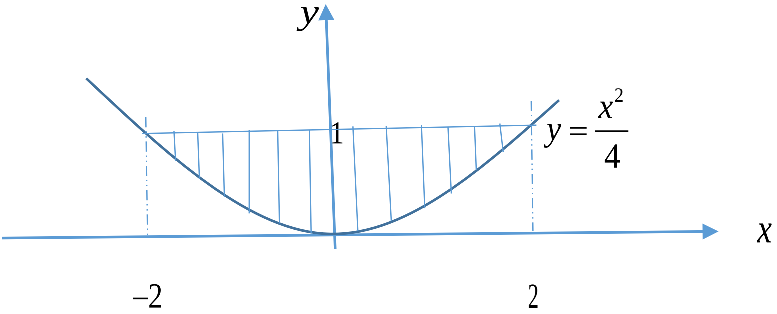
$$\bar{x}=\frac{M_y}{A} \text{ and } \bar{y}=\frac{M_x}{A}, \text{ where } A \text{ is the area of } \mathcal{R}.$$

◇

Example 3.3.6

Find the centroid of the finite region bounded above by $y=1$ and below by $y=\frac{x^2}{4}$.

Solutions:



$$A=\int_{-2}^{2}\left(1-\frac{x^2}{4}\right)dx=2\int_{0}^{2}\left(1-\frac{x^2}{4}\right)dx=2\left(x-\frac{x^3}{12}\right)\Bigg|_0^2=\frac{8}{3}$$

$$\therefore M_x=\int_0^1 yl(y)dy=\int_0^1 y\left(2\sqrt{y}+2\sqrt{y}\right)dy=4\int_0^1 y^{\frac{3}{2}}dy=4.\frac{2}{5}.y^{\frac{5}{2}}\Bigg|_0^1=\frac{8}{5}.$$

$$M_y=\int_{-2}^{2}x\left(1-\frac{x^2}{4}\right)dx=\left(\frac{x^2}{2}-\frac{x^4}{16}\right)\Bigg|_{-2}^{2}=0.$$

$$\therefore \bar{x}=\frac{M_y}{A}=0 \text{ and } \bar{y}=\frac{M_x}{A}=\frac{8}{5}.\frac{3}{8}=\frac{3}{5}.$$

$$\therefore \bar{P}(\bar{x},\bar{y})=\bar{P}\left(0,\frac{3}{5}\right).$$

△

We have discussed the moment of a particle about an axis as ml, where m is the mass and l is the directed distance from the axis. This is also called the first moment of a particle. We can also find the second moment ml^2, the third moment ml^3 and so on. The second moment is important in dynamics and in the mechanics of materials. It is called moment of inertia. The moment of inertia is a measure of the resistance of a rotating body to change in motion.

Definition 3.3.4

With the notation and the condition of definition 3.3.3, the moment of inertia (I) of the region $\mathcal{R}$ about the $x-$axis, denoted I_x, and about the $y-$axis (I_y), is given by

$$I_x = \int_c^d y^2 l(y)dy$$

and

$$I_y = \int_a^b x^2 h(x)dx,$$

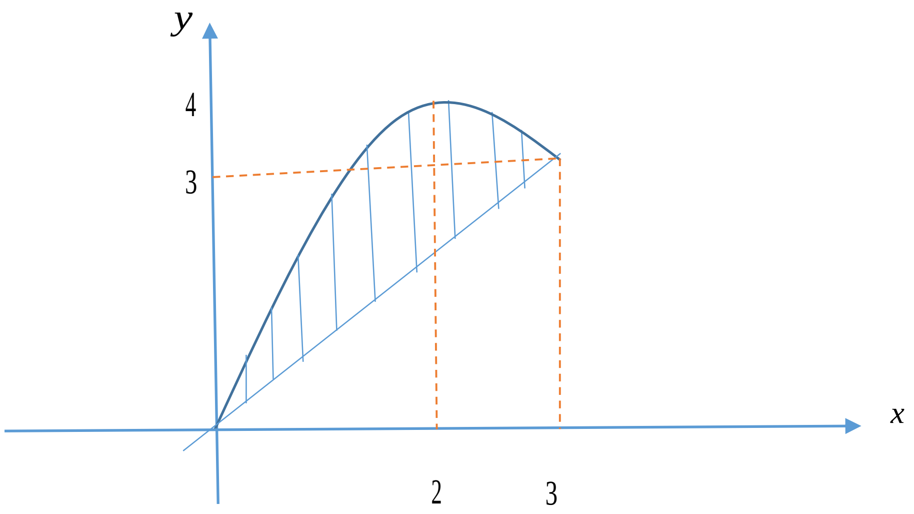
respectively.

◇

Example 3.2.7

Find the moment of inertia for the finite region bounded by the parabola $y = 4x - x^2$ and the line $y = x$.

Solutions:



$$\begin{aligned} I_x &= \int_0^3 y^2\left(y-2+\sqrt{4-y}\right)dy + \int_3^4 y^2\left(2+\sqrt{4-y}-2+\sqrt{4-y}\right)dy \\ &= \int_0^3 \left(y^3 - 2y^2 + y^2\sqrt{4-y}\right)dy + 2\int_3^4 y^2\sqrt{4-y}\,dy \end{aligned}$$

$$=\left(\frac{y^4}{4}-\frac{2}{3}y^3-\frac{32}{3}(4-y)^{\frac{3}{2}}+\frac{32}{5}(4-y)^{\frac{5}{2}}-\frac{2}{7}(4-y)^{\frac{7}{2}}\right)\Bigg|_0^3+2\left(-\frac{32}{3}(4-y)^{\frac{3}{2}}+\frac{32}{5}(4-y)^{\frac{5}{2}}-\frac{2}{7}(4-y)^{\frac{7}{2}}\right)\Bigg|_3^4$$

$$=-\frac{30167}{420}.$$

$$I_y=\int_0^3 x^2h(x)dx=\int_0^3 x^2(4x-x^2-x)dx=\int_0^3(3x^3-x^4)dx=\left(\frac{3}{4}x^4-\frac{1}{5}x^5\right)\Bigg|_0^3=\frac{243}{20}.$$

△

THE END!