The University of Zambia

School of Natural Sciences

Department of Mathematics & Statistics

2021/2022 Academic Year Final Examinations

MAT 2100 - Analytic Geometry and Calculus

Time allowed : Three (3) hrs | Full marks : 100

Instructions:

- Indicate your **computer number** on all answer booklets.
- There are six (6) questions in this examination. Attempt **any five (5)** questions. All questions carry equal marks.
- **Full credit** will only be given when **necessary work** is shown.

This paper consists of 3 pages of questions.

✓1. (a) For the conic section given by

$$175x^2 - 210x + 400y^2 + 63 = 2800,$$

find the vertex/vertices, the focus/foci and the equation(s) of directrix/directrices. [7]

(b) For the conic section given by

$$\frac{9x^2}{29} - \frac{16xy}{29} - \frac{2x}{\sqrt{29}} + \frac{129y^2}{145} - \frac{4y}{5\sqrt{29}} - \frac{4}{5} = 0,$$

the equation of the directrix in the rotated $x'y'$-coordinate system is $x' = \frac{7}{2}$. Find the corresponding equation in the xy-coordinate system. [7]

(c) Find the normal line to the sphere $x^2+y^2+z^2 = 3$ at the point $(1,1,1)$. [6]

✓2. (a) The intrinsic equation of a curve is given by $s = \tan\psi$, where s is measured from the point $(0,1)$ on the xy-plane and ψ is the angle the tangent to the curve makes with the positive x-axis.

(i) Find curvature at $\psi = \frac{\pi}{4}$. [3]

(ii) Given that the centre of curvature is $\left(\ln\left(1+\sqrt{2}\right) - \sqrt{2},\ \ 2\sqrt{2}\right)$, find the equation of the circle of curvature. [2]

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(b) The function

$$f(x) = x\sqrt{2x}, \quad x \in [0,2]$$

satisfies the hypotheses of the mean value theorem. Find a real number $c \in (0,2)$ that satisfies the conclusion of the theorem. [7]

(c) Find the point of intersection of the lines

$$\frac{x}{1} = \frac{y+1}{-1} = \frac{z-1}{2}$$

and $x = -1+t,\ y = t,\ z = -1+2t,\ t \in \mathbb{R}$. [8] ✓

✓ 3. **(a)** Evaluate the following indefinite integrals:

(i) $\int \sec^5\left(\frac{\theta}{2}\right)\tan\left(\frac{\theta}{2}\right)\, d\theta$ [3] ×

(ii) $\int \frac{\cos t}{1+\cos t}\, dt$ [6] ✓

$I_y =$

(b) Find the moment about the y-axis of the region bounded by the curve $y = \cos x$ from $x = 0$ to $x = \frac{\pi}{2}$, and the x-axis. [6] ✓

(c) A student is tracking the paths of a planet and a comet around a star. The planet's orbit is described by the equation

$$r = \frac{3}{1+\frac{1}{3}\cos\theta}$$

and the comet's orbit is described by the equation

$$r = \frac{2}{1+\cos\theta}$$

with the sun being one of the focal point of their orbits. By sketching the two orbits, determine whether the comet intersects the orbit of the planet. [5]

4. **(a)** Find the equation of a plane containing the points $(2,5,1)$, $(1,5,3)$ and $(3,6,6)$. [6]

(b) For the space curve

$$R(t) = ti + (\ln\cos t)j, \quad -\pi/2 < t < \pi/2,$$

find, in terms of t, the tangential and normal components of the acceleration vector. [7]

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(c) The rate at which radioactive nuclei decay is represented by the differential equation

$$\frac{dy}{dt} = -ky$$

where y is the amount of radioactive nuclei present after t years. The number of radioactive nuclei initially present is 1200 and half of that amount disintegrates after 20 years. Find the amount present after 50 years. [7]

✓ 5. (a) If $z = f(x, y)$ satisfies the equation

$$x \sin(xy) + xz^2 + x = 10,$$

show that

$$x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} + \frac{5}{xz} = 0.$$ [7]

(b) Determine the nature of the critical points of the function

$$f(x, y) = 2x + 2y - 2xy - 2x^2 - y^2 + 4.$$ [6]

(c) Evaluate the following limits:

(i) $\lim_{x \to 1} \left(\frac{x \sin(\pi x)}{1 - \sqrt{x}} \right)$ [3]

(ii) $\lim_{x \to \infty} (\sqrt{x})^{\frac{1}{x}}$ [4]

✓ 6. (a) Solve the following ordinary differential equations:

(i) $x^2 \frac{dy}{dx} - 2xy = x^4 \cos x$ [4]

(ii) $14y'' + 5y' - y = 0$ [3]

$ar^2 + \cancel{b} + 0$

(b) Solve the differential equation

$$y'' + y = \tan^2 x$$ [8]

(c) Find the volume of the solid formed by resolving about the x-axis the region bounded by the curve $y = \frac{1}{x}$ and the lines $y = 0$, $x = -2$ and $x = -1$. [5]

END OF EXAMINATION!

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