The University of Zambia
School of Natural sciences
Department of Physics
PHY 1010
Lecture 1
Physical Quantities and Units in Physics

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Introduction

The **measurement and recording of quantities** is central to the whole of Physics. The skill of **making a reasonable estimate of a physical quantity** is very useful for any Physicist.

This lecture introduces the **SI System of Units**, which provides a universal framework of measurement that is common to all scientists internationally.

Learning Outcomes

The student should be able to:

1. understand the **nature of all physical quantities** i.e. any physical quantity consists of a numerical magnitude (or size) and a unit;
2. make **reasonable estimates of physical quantities** included within the syllabus;
3. recall **SI base quantities** and their units;
4. express **derived units** as products or quotients of the SI base units and use their named units as listed in this syllabus;
5. use the **power prefixes** and their symbols to indicate decimal sub-multiples or multiples of both SI base and derived units;

Table 1: The SI base quantities and their units

SI Base Quantity	SI Base Quantity Symbol	Unit Name	Unit Symbol
mass	m	kilogram	kg
length	l	metre	m
time	t	second	s
electric current	I	ampere	A
temperature	T	kelvin	K
amount of substance		mole	mol
light intensity	I	candela	cd

6. make and record measurements of physical quantities in the laboratory;
7. understand Avogadro's constant and use molar masses.
8. distinguish between scalar and vector quantities and give examples of each;
9. represent a vector as two perpendicular components;
10. add and subtract coplanar vectors.

SI Base Quantities and Units

By convention, modern scientists and engineers use the **International System of Units** (**SI** - for Systeme Internationale d'Unites). The **SI System of Units** uses 7 **base quantities** and various **derived quantities** obtained from these base quantities. The base quantities used in the SI System of Units are **mass**, **length**, **time**, **electric current**, **temperature**, **amount of substance** and **light intensity**. These SI base quantities and their units are listed in Table 1.

Definitions of SI Base Quantities and Units prior to May 2019

Mass

mass refers to an amount of matter with no definite shape. The SI base unit for mass is **kilogram** abbreviated **kg**. **1 kilogram** is equal to the mass of the international standard kilogram, officially known as the **International Prototype Kilogram (IPK)**. The **IPK** shown in Figure 1 is a mixture of platinum and iridium and is held at the International Bureau of Weights and Measures (website: https://www.bipm.org) in Paris, France. All other masses were defined by comparing with this metal cylinder.

Time

time refers to a measured period during which an action, process or condition exits or continues. The SI base unit for time is **second**. **1 second** is the duration

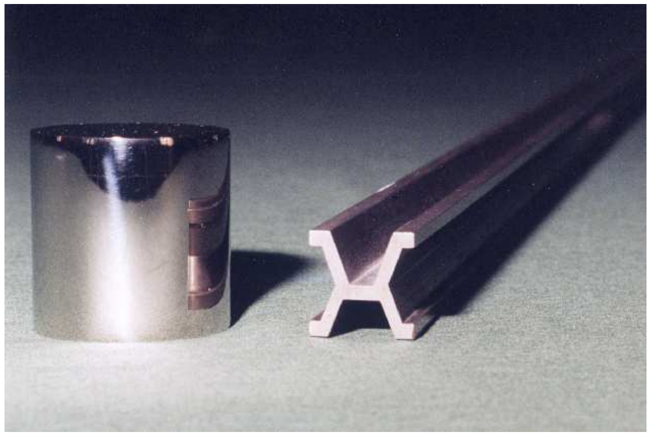

Figure 1: The International Prototype Kilogram (IPK) and Standard metre held at BIPM in Paris, France. The standard metre is made to be exactly the length that light could travel in 1/299 792 458 of a second.

of 9 192 631 770 periods of the radiation corresponding to the transition between two hypefine levels of ground state of the caesium-133 atom (see Figure 2).

Length

length refers to the extent of something from end to end. The SI base unit for length is the **metre** abbreviated **m**. **1 metre** is defined as the length light travels in vacuum during a time interval of $\frac{1}{299792458}$ of a second. Figure 1 shows an example of a standard metre bar held at the International Bureau of Weights and Measures in Paris.

Electric current

electric current (see Figure 3 top panel) refers to the rate of flow of electric charge past a point or region. The SI base unit for electric current is **Ampere** abbreviated **A**. **1 Ampere** is defined as a constant electric current which, if maintained in two straight parallel conductors of infinite length, of negligible cross-section, and placed 1 m apart in vacuum, would produce between the conductors a force equal to 2×10^{-7} newton per metre of length (see Figure 3 bottom panel).

Temperature

temperature refers to the degree of hotness or coldness measured on a definite scale. The SI base unit for temperature is **kelvin** abbreviated **K**. **1 kelvin** is the fraction of $\frac{1}{273.16}$ of the thermodynamic temperature of the **triple point**

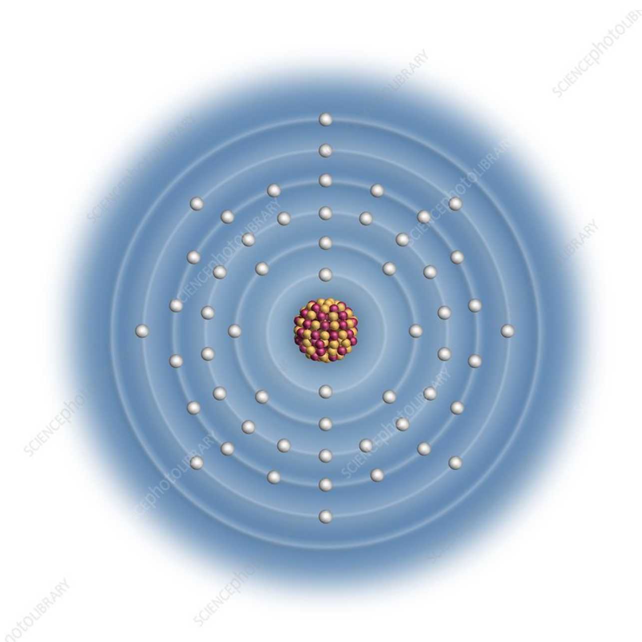

Figure 2: The caesium-133 atom is used to define a second. When the outermost electron in caesium transitions (or moves) between any two hyperfine electron energy levels, it emits electromagnetic radiation.

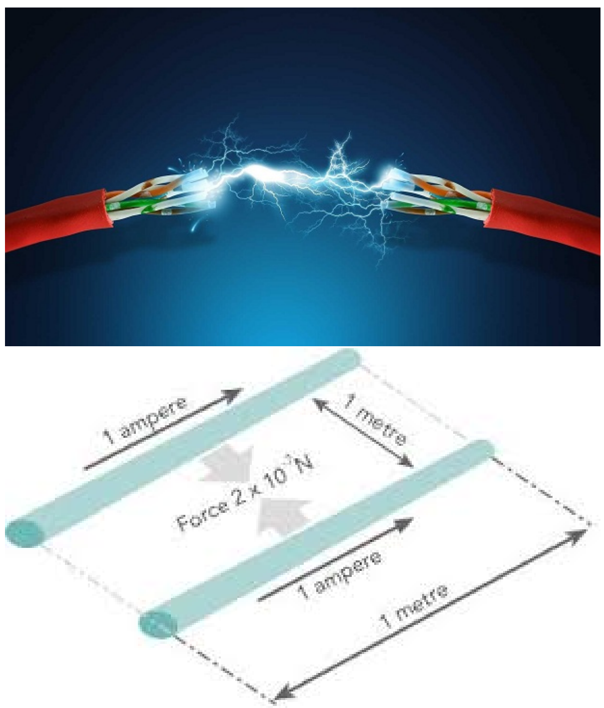

Figure 3: *top panel*: An electric current jumping from one conductor to another conductor through an air gap. *bottom panel*: Definition of an Ampere.

of water. The triple point is defined as the temperature and pressure at which a substance can exist in equilibrium in the liquid, solid and gaseous states. The triple point of water is at 0.01°C (or 273.16 K) and 4.58 mm (or 611.2 Pa) of mercury. The triple point is used to calibrate thermometers.

Figure 4: Water at the triple point can co-exist as a solid, liquid and gas. The triple point of water is used to define the kelvin, the SI base unit for temperature.

Amount of substance

The SI base unit for amount of substance is the **mole** abbreviated **mol**. **1 mole** is defined as the amount of substance which contains as many elementary particles (e.g., atoms, molecules, ions, or electrons) as they are atoms in 0.012 kg (or 12 grams) of carbon-12.

Light Intensity

light intensity refers to the amount of visible light that is emitted in 1 second (also known as Luminosity, denoted L) per solid angle,Ω. The SI units for Luminosity is **Watt** abbreviated W.

The solid angle Ω is defined as the ratio of the area A substended by a cone to the square of distance r of the source of light. The SI unit for solid angle is **steradians** abbreviated **sr** The SI base unit for light intensity is the standard candle or **candela** abbreviated **cd** (see Figure 6). **1 candela** is defined as 1 Watt per steradian.

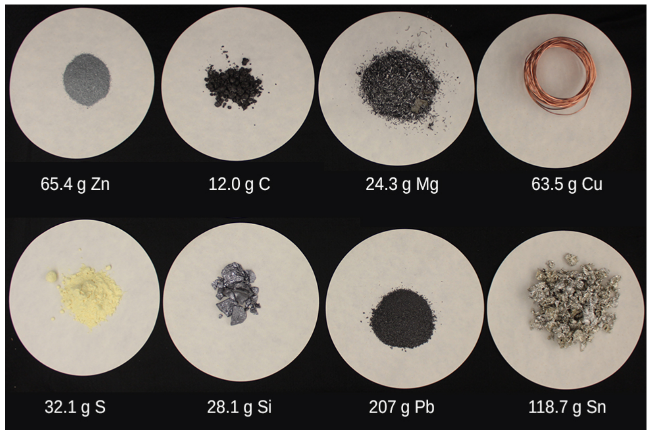

Figure 5: 1 mole of different substances has different masses but has the same number of elementary particles i.e. $6.02214086 \times 10^{23}$ particles per mol. This number is denoted N_A is called **Avogadro's number**.

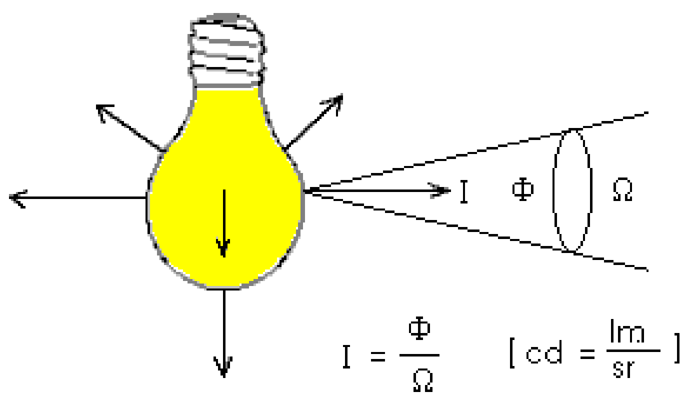

Figure 6: The definition of a standard candle or candela, the SI base unit for light intensity.

Table 2: Some well known SI derived quantities and their units

SI Derived Quantity	SI Derived Quantity Symbol	Unit Name	Unit Symbol	Base Units Equivalent
area	A			m^2
volume	V			m^3
density	ρ			$\mathrm{kg\ m^{-3}}$
speed	v			$\mathrm{m\ s^{-1}}$
momentum	p			$\mathrm{kg\ m\ s^{-1}}$
acceleration	a			$\mathrm{m\ s^{-2}}$
force	F	newton	N	$\mathrm{kg\ m\ s^{-2}}$
pressure	P	pascal	Pa	$\mathrm{kg\ m^{-1}\ s^{-2}}$
energy (or work)	E or W	joule	J	$\mathrm{kg\ m^2\ s^{-2}}$
power	P	watt	W	$\mathrm{kg\ m^2\ s^{-3}}$
frequency	ν	hertz	Hz	$\mathrm{s^{-1}}$
charge	Q	coulomb	C	A s
voltage	V	volt	V	$\mathrm{kg\ m^2\ s^{-3}\ A^{-1}}$
resistance	R	ohm	Ω	$\mathrm{kg\ m^2\ s^{-3}\ A^{-2}}$

SI Derived Quantities and their Units

The SI derived quantities enable us to measure more than the basic quantities of length, time, mass and etc. Table 2 gives some of the most common SI derived quantities, their units and equivalent SI base units.

Definitions of SI derived quantities and units

Area

$$\text{area} = \text{length} \cdot \text{length}$$
$$A = l^2$$

Volume

$$\text{volume} = \text{length} \cdot \text{length} \cdot \text{length}$$
$$V = l^3$$

Density

$$\text{density} = \frac{\text{mass}}{\text{volume}}$$
$$\rho = \frac{M}{V}$$

Speed

$$\text{speed} = \frac{\text{distance covered}}{\text{time taken}}$$
$$v = \frac{d}{t}$$

Linear Momentum

$$\text{momentum} = \text{mass} \cdot \text{speed}$$
$$p = mv$$

Acceleration

$$\text{acceleration} = \frac{\text{final speed} - \text{initial speed}}{\text{time}}$$
$$a = \frac{v - u}{t}$$

Force

$$\text{force} = \text{mass} \cdot \text{acceleration}$$
$$F = ma$$

Pressure

$$\text{pressure} = \frac{\text{force}}{\text{area}}$$
$$P = \frac{F}{A}$$

Work or Energy

$$\text{work} = \text{force} \cdot \text{distance}$$
$$W = Fd$$

Power

$$\text{power} = \frac{\text{work}}{\text{time}}$$
$$P = \frac{W}{t}$$

Frequency

$$\text{frequency} = \frac{1}{\text{Period}}$$
$$\nu = \frac{1}{T}$$

Period is the time to complete one oscilation or vibration.

Electric Charge

$$\text{charge} = \text{electric current} \cdot \text{time}$$
$$Q = It$$

Voltage

$$\text{voltage} = \frac{\text{power}}{\text{electric current}}$$
$$V = \frac{P}{I}$$

Resistance

$$\text{resistance} = \frac{\text{power}}{\text{electric current} \cdot \text{electric current}}$$
$$R = \frac{P}{I^2}$$

Standard Power Prefixes

Sometimes the values we have to work with for some quantities mean that the numbers involved are extremely large or small. To address this dilemma, scientists have made an easier system for writing very large and small numbers by adding a **prefix** to the unit which tells us that it has been multiplied by a very large or small amount. Table gives the prefixes used with SI units.

The Avogadro Constant, N_A

The **Avogadro Constant**, usually denoted N_A is the number of atoms in 0.012 kg of carbon-12. The Avogadro constant has a value $N_A = 6.02214086 \times 10^{23}$ atoms mol^{-1}. Multipying N_A by the number of moles of a substance gives you the number of atoms in sample.

For example, 1 mole of any substance has $6.02214086 \times 10^{23}$ particles. On a more informative note,

Table 3: Prefixes with SI units

Factor	Name	Symbol	Factor	Name	Symbol
10^1	deca	da	10^{-1}	deci	d
10^2	hecto	h	10^{-2}	cent	c
10^3	kilo	k	10^{-3}	mili	m
10^6	mega	M	10^{-6}	micro	μ
10^9	giga	G	10^{-9}	nano	n
10^{12}	tera	T	10^{-12}	pico	p
10^{15}	peta	P	10^{-15}	femto	f
10^{18}	exa	E	10^{-18}	atto	a
10^{21}	zetta	Z	10^{-21}	zepto	z
10^{24}	yotta	Y	10^{-21}	yocto	y

$0.5\ \text{mol} = 0.5\ \text{mol} \times \text{N}_\text{A} = 0.5\ \text{mol} \times \frac{6.02214086 \times 10^{23}\ \text{atoms}}{\text{mol}} = 3.01107043 \times 10^{23}\ \text{atoms}$

$2\ \text{mol} = 2\ \text{mol} \times \text{N}_\text{A} = 2\ \text{mol} \times \frac{6.02214086 \times 10^{23}\ \text{atoms}}{\text{mol}} = 1.204428172 \times 10^{24}\ \text{atoms}$

$5\ \text{mol} = 2\ \text{mol} \times \text{N}_\text{A} = 5\ \text{mol} \times \frac{6.02214086 \times 10^{23}\ \text{atoms}}{\text{mol}} = 3.01107043 \times 10^{24}\ \text{atoms}$

Scalars and Vectors

SI base and derived quantities may be divided into two groups:

scalars - these are physical quantities that have magnitude (size) only. Examples are *mass, length, time, energy, temperature* and *speed.*

vectors - these are physical quantities that with both magnitude and direction. Examples are *force, velocity, acceleration* and *momentum.*

Scalars may be added or subtracted together by simple arithmetic. When vectors are added or subtracted, the *direction* of the vector must be considered.

A vector may be represented by a line, the length of the line being the magnitude of the vector. The direction of the arrow indicates the direction of the vector.

Addition of vectors

The sum of adding two vectors is referred to as the **resultant** of the two vectors.

Vectors acting in the same line

The resultant of two vectors acting along the **same line** is the sum of the magnitudes of the two vectors. The direction of the resultant vector remains unchanged. If one of the two vectors acting along the same line is acting in the opposite direction, then its magnitude is given a negative sign.

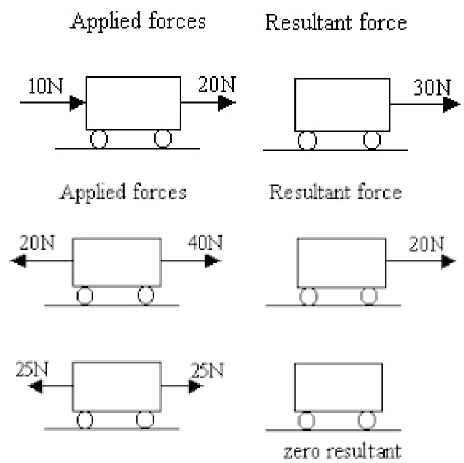

Figure 7: *top panel*: Adding two vectors acting along the same line in the same direction. *bottom panel*: Adding two vectors acting along the same line but in different direction.

Vectors acting in different directions

If two vectors are acting in different differentions, **a traingle of vectors** is used to find the resultant vector.

Components of vectors

It is often necessary to find the **components** of a vector, usually in two directions at right angles to each other. This process is called **resolution** of a vector. The directions are generally chosen to be the x and y-axes.

The component of a vector along any direction is the magnitude of the vector multiplied by the cosine of the angle between its direction and the direction of the vector. A component is the effective value of a vector along a particular direction.

Usually, we resolve the components of a vector along the x and y-axis direction. The x-component and y-component of a vector $\mathbf{A}$ denoted A_x and A_y respectively are given by

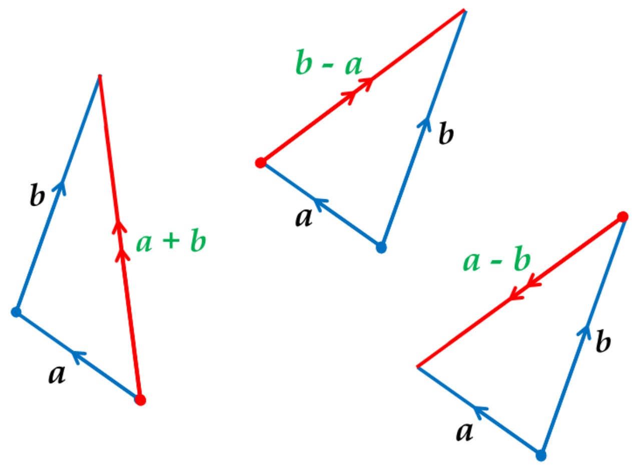

Figure 8: Adding two vectors acting along different directions with the aid of a triangle of vectors

Table 4: Resolving a vector into its components

vector	x-component	y-component
A	$A_x = A\cos\theta$	$A_y = A\cos(90-\theta) = A\sin\theta$

The magnitude of **A** denoted $|A|$ is given by

$$|A| = \sqrt{A_x^2 + A_y^2}$$

The direction of vector **A** denoted by θ is given by

$$\tan\theta = \frac{A_y}{A_x}$$

Consider vectors **A**, **B** and **C** which have magnitudes 20 m, 30 m and 10 m respectively. The vector **A** makes 45° angle anticlockwise with the positive $x-$ axis. The vector **B** makes a 120° angle anticlockwise with the positive $x-$ axis. The vector **C** lies along the negative $x-$axis.

These vectors **A**, **B** and **C** can be added or subtracted as shown in Table 5.

Table 5: Addition of vectors acting in different directions

vector	x-component	y-component
$\mathbf{A}$	A_x	A_y
$\mathbf{B}$	B_x	B_y
$\mathbf{C}$	C_x	C_y
$\mathbf{A}+\mathbf{B}$	$A_x + B_x$	$A_y + B_y$
$\mathbf{A}-\mathbf{B}$	$A_x - B_x$	$A_y - B_y$
$\mathbf{B}-\mathbf{A}$	$B_x - A_x$	$B_y - A_y$
$\mathbf{A}+\mathbf{B}+\mathbf{C}$	$A_x + B_x + C_x$	$A_y + B_y + C_y$
$\mathbf{A}+\mathbf{B}-\mathbf{C}$	$A_x + B_x - C_x$	$A_y + B_y - C_y$

Table 6: Addition of vectors acting in different directions

vector	x-component	y-component
$\mathbf{A}$	$20\cos 45^\circ = 14.14$ m	$(20)\sin 45^\circ = 14.14$ m
$\mathbf{B}$	$30\cos 120^\circ = -15$ m	$30\sin 120^\circ = 25.98$ m
$\mathbf{C}$	$10\cos 180^\circ = -10$ m	$10\sin 180^\circ = 0$
$\mathbf{A}+\mathbf{B}$	-0.86	40.12
$\mathbf{A}-\mathbf{B}$	29.14	-11.84
$\mathbf{B}-\mathbf{A}$	-29.14	-29.14
$\mathbf{A}+\mathbf{B}+\mathbf{C}$	-10.86	40.12
$\mathbf{A}+\mathbf{B}-\mathbf{C}$	9.14	40.12

The University of Zambia
School of Natural Sciences
Department of Physics
PHY 1010
Lecture 2
Scalars and Vectors

Mr. Gift L. Sichone

February 28, 2021

1 Introduction

This lecture will introduce two categroies in which all physical quantities can be categorised called **scalars** and **vectors**. **Scalars** are those physical quantities that only have magnitude while **vectors** have both magnitude and direction. The lecture will focus on decomposition of a vector into two perpendicular scalar quantities called **components** and how to **add** and **subtract** vectors.

2 Learning Outcomes

By the end of this lecture, the student should be able to:

1. define a scalar quantity and give examples of scalars;
2. define a vector quantity and give examples of vectors;
3. decompose or resolve a vector quantity into two scalar perpendicular components;
4. add and subtract coplanar vectors.

3 Scalars and Vectors

SI base and derived quantities may be divided into two groups: **scalars** and **vectors**. **Scalars** are physical quantities that have magnitude or size only. **Vectors** are those physical quantities that have magnitude and direction. Mathematically, a vector quantity A is denoted as $\vec{A}$ or **A**. A vector may also be represented pictorially as shown in Figure 1 by a line with an arrow at the end of the line. The length of the line represents the magnitude of the vector while the arrow indicates the direction of the vector. Table 1 shows some examples of scalars and vectors.

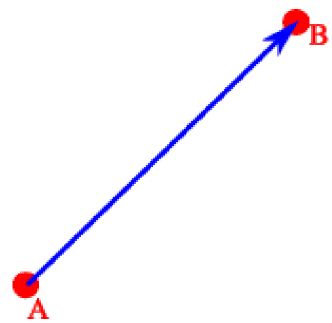

Figure 1: A pictorial representation of a vector

Scalars may be added or subtracted together by simple arithmetic as follows:

$$5 \text{ kg} + 3 \text{ kg} = 8 \text{ kg}$$

$$25 \text{ s} - 13 \text{ s} = 12 \text{ s}$$

$$25 \text{ m} + 50 \text{ m} = 75 \text{ m}$$

$$1.5 \text{ mol} + 0.7 \text{ mol} = 2.2 \text{ mol}$$

4 Vectors Addition

When vectors are added or subtracted, the **direction** of the vector must be considered. The sum of adding vectors is referred to as the **resultant**. We only consider **vector addition** because vector subtraction is a form of vector addition.

4.1 Adding vectors acting in the same line

The resultant of adding two vectors acting along the **same line** is the sum of the magnitudes of the two vectors. The direction of the resultant vector remains unchanged. If one of the two vectors acting along the same line is acting in the opposite direction, then its magnitude is given a negative sign.

Table 1: List of scalars and vectors

scalars	**vectors**
mass	displacement
length	velocity
time	acceleration
electric current	force
temperature	pressure
amount of substance	momentum
light intensity	torque
area	electric field
volume	magnetic field
density	
speed	
energy	
work	
power	
frequency	
charge	
voltage	
resistance	

4.2 Adding vectors acting along different lines of action

If two vectors are acting in different directions are added, the **resultant vector** is found using a **traingle of vectors** as shown in Figure 3.

5 Components of Vectors

It is often necessary to find the **components of a vector**, usually in two directions at right angles to each other. This process is called **resolving or decomposing a vector**. The directions along which the components are found is generally chosen to be the x and y-axes as shown in Figure 4.

The component of a vector along any direction is the magnitude of the vector multiplied by the cosine of the angle between its direction and the direction of the vector. A component is the effective value of a vector along a particular direction. Usually, we resolve the components of a vector along the x and y-axis direction. The x-component and y-component of a vector $\vec{A}$ are denoted A_x and A_y respectively. Table shows the decomposition of vector $\vec{A}$ into its components A_x and A_y.

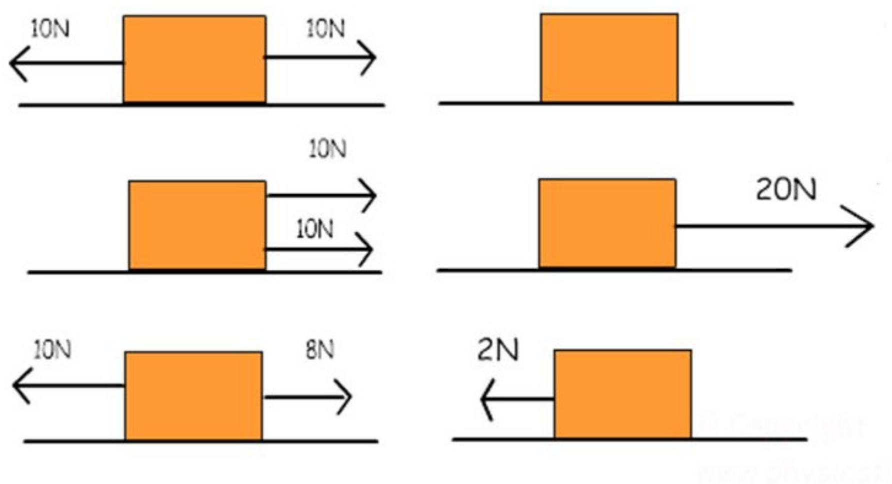

Figure 2: Adding two vectors acting along the same line of action

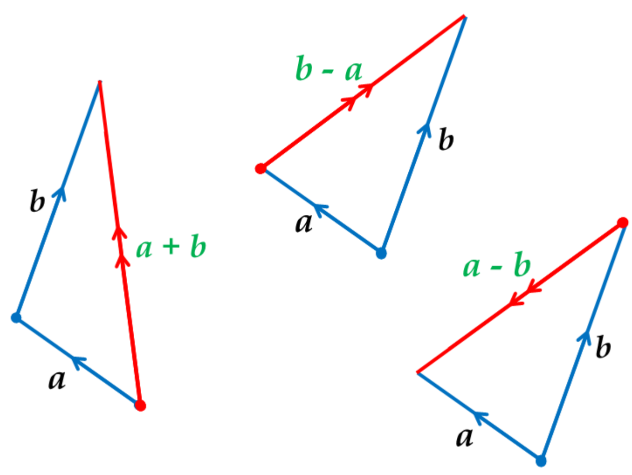

Figure 3: Adding two vectors acting along different directions with the aid of a triangle of vectors

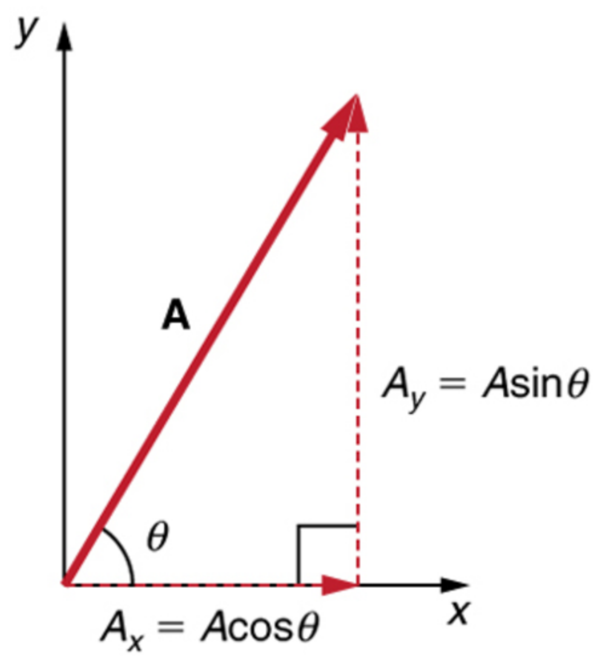

Figure 4: Resolving a vector **A** into its x and y-components

Table 2: Vector resolution or decomposition

vector	x-component	y-component
$\vec{A}$	$A_x = A\cos\theta$	$A_y = A\cos(90-\theta) = A\sin\theta$

The magnitude of $\vec{A}$ denoted $|A|$ is given by

$$|A| = \sqrt{A_x^2 + A_y^2}$$

The direction of vector $\vec{A}$ denoted by θ is given by

$$\tan\theta = \frac{A_y}{A_x}$$

The angle θ by convention is measured from the positive x-axis in an anti-clockwise direction.

6 Addition of vectors acting along different directions using table of components

Consider vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ which have magnitudes 20 m, 30 m and 10 m respectively. The vector $\vec{A}$ makes 45° angle anticlockwise with the positive $x-$ axis. The vector $\vec{B}$ makes a 120° angle anticlockwise with the positive $x-$ axis. The vector $\vec{C}$ lies along the negative $x-$axis.

These vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ can be added or subtracted as shown in Table 3.

The magnitude of $\vec{A}+\vec{B}$ denoted $|\vec{A}+\vec{B}|$ is given by

$$|\vec{A}+\vec{B}| = \sqrt{(A_x + B_x)^2 + (A_y + B_y)^2}$$

$$|\vec{A}+\vec{B}| = \sqrt{(-1\text{ m})^2 + (40\text{ m})^2}$$

$$|\vec{A}+\vec{B}| = \sqrt{1601\text{ m}^2}$$

$$|\vec{A}+\vec{B}| \simeq 40\text{ m}$$

The direction of the vector $\vec{A}+\vec{B}$ is given by

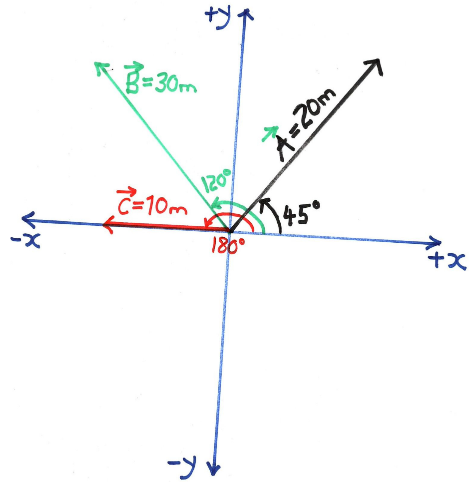

Figure 5: The coplanar vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ on the xy-plane.

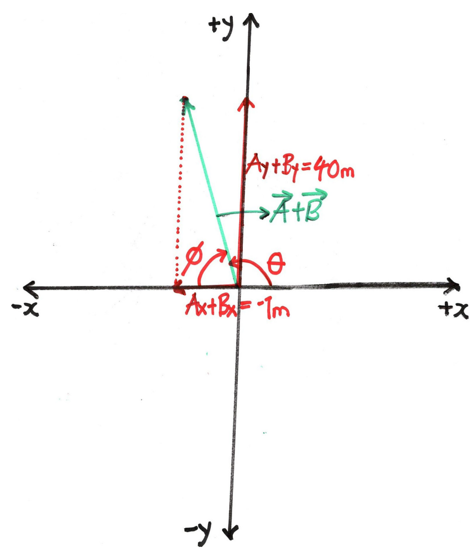

Figure 6: The resultant vector $\vec{A}+\vec{B}$

Table 3: Addition of vectors acting in different directions

vector	x-component	y-component
$\vec{A}$	A_x	A_y
$\vec{B}$	B_x	B_y
$\vec{C}$	C_x	C_y
$\vec{A}+\vec{B}$	$A_x + B_x$	$A_y + B_y$
$\vec{A}-\vec{B}$	$A_x - B_x$	$A_y - B_y$
$\vec{B}-\vec{A}$	$B_x - A_x$	$B_y - A_y$
$\vec{A}+\vec{B}+\vec{C}$	$A_x + B_x + C_x$	$A_y + B_y + C_y$
$\vec{A}+\vec{B}-\vec{C}$	$A_x + B_x - C_x$	$A_y + B_y - C_y$

$$\tan\phi = \frac{A_y + B_y}{A_x + B_x}$$

$$\tan\phi = \frac{40 \text{ m}}{-1 \text{ m}} = -40$$

$$\phi = \tan^{-1}(-40)$$

$$\phi \simeq -89^\circ$$

$$\theta = 180^\circ + \phi = 180^\circ + (-89^\circ)$$

$$\theta = 91^\circ$$

The magnitude of $\vec{A}-\vec{B}$ denoted $|\vec{A}-\vec{B}|$ is given by

$$|\vec{A}-\vec{B}| = \sqrt{(A_x - B_x)^2 + (A_y - B_y)^2}$$

$$|\vec{A}-\vec{B}| = \sqrt{(29 \text{ m})^2 + (-12 \text{ m})^2}$$

$$|\vec{A}-\vec{B}| = \sqrt{985 \text{ m}^2}$$

$$|\vec{A}-\vec{B}| \simeq 31.4 \text{ m}$$

The direction of the vector $\vec{A}-\vec{B}$ is given by

Table 4: Addition of vectors acting in different directions

vector	x-component	y-component
$\vec{A}$	$A_x = (20\text{ m})\cos 45° \simeq 14\text{ m}$	$A_y = (20\text{ m})\sin 45° \simeq 14\text{ m}$
$\vec{B}$	$B_x = (30\text{ m})\cos 120° = -15\text{ m}$	$B_y = (30\text{ m})\sin 120° \simeq 26\text{ m}$
$\vec{C}$	$C_x = (10\text{ m})\cos 180° = -10\text{ m}$	$C_y = (10\text{ m})\sin 180° = 0$
$\vec{A}+\vec{B}$	$A_x + B_x = -1\text{ m}$	$A_y + B_y = 40\text{ m}$
$\vec{A}-\vec{B}$	$A_x - B_x = 29\text{ m}$	$A_y - B_y = -12\text{ m}$
$\vec{B}-\vec{A}$	$B_x - A_x = -29\text{ m}$	$B_y - A_y = 12\text{ m}$
$\vec{A}+\vec{B}+\vec{C}$	$A_x + B_x + C_x = -11\text{ m}$	$A_y + B_y + C_y = 40\text{ m}$
$\vec{A}+\vec{B}-\vec{C}$	$A_x + B_x - C_x = 9\text{ m}$	$A_y + B_y - C_y = 40\text{ m}$

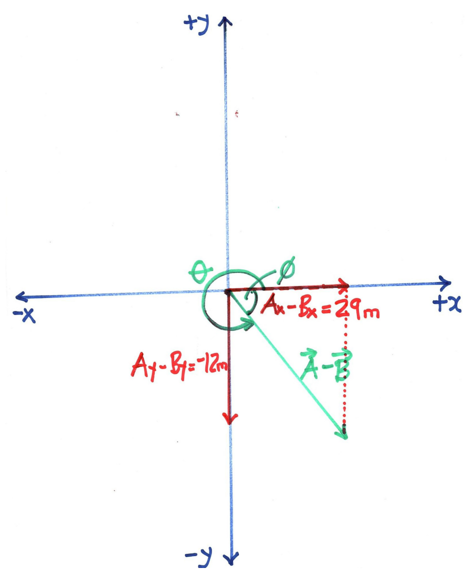

Figure 7: The resultant vector $\vec{A}-\vec{B}$

$$\tan\phi = \frac{A_y - B_y}{A_x - B_x}$$

$$\tan\phi = \frac{-12\text{ m}}{29\text{ m}} = -0.41$$

$$\phi = \tan^{-1}(-0.41)$$

$$\phi \simeq -22.3°$$

$$\theta = 360° + \phi = 360° + (-22.3°)$$

$$\theta = 337.7°$$

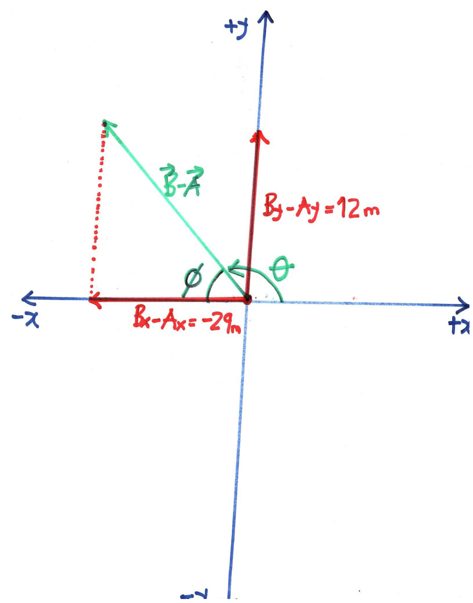

Figure 8: The resultant vector $\vec{B} - \vec{A}$

The magnitude of $\vec{B} - \vec{A}$ denoted $|\vec{B} - \vec{A}|$ is given by

$$|\vec{B} - \vec{A}| = \sqrt{(B_x - A_x)^2 + (B_y - A_y)^2}$$

$$|\vec{B} - \vec{A}| = \sqrt{(-29 \text{ m})^2 + (12 \text{ m})^2}$$

$$|\vec{B} - \vec{A}| = \sqrt{985 \text{ m}^2}$$

$$|\vec{B} - \vec{A}| \simeq 31.4 \text{ m}$$

The direction of the vector $\vec{B} - \vec{A}$ is given by

$$\tan\phi = \frac{B_y - A_y}{B_x - A_x}$$

$$\tan\phi = \frac{12\text{ m}}{-29\text{ m}} = -0.41$$

$$\phi = \tan^{-1}(-0.41)$$

$$\phi \simeq -22.3°$$

$$\theta = 180° + \phi = 180° + (-22.3°)$$

$$\theta = 157.7°$$

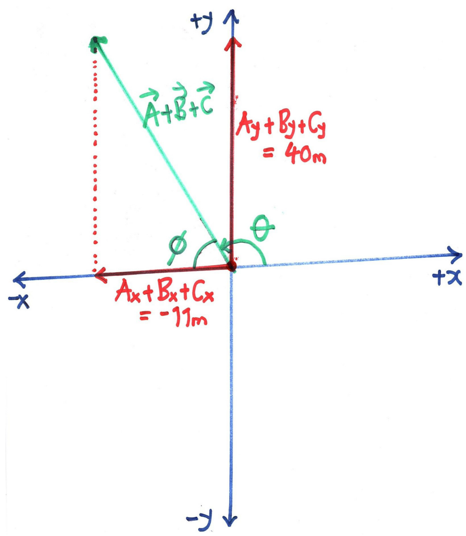

Figure 9: The resultant vector $\vec{A}+\vec{B}+\vec{C}$

The magnitude of $\vec{A}+\vec{B}+\vec{C}$ denoted $|\vec{A}+\vec{B}+\vec{C}|$ is given by

$$|\vec{A}+\vec{B}+\vec{C}| = \sqrt{(A_x+B_x+C_x)^2+(A_y+B_y+C_y)^2}$$

$$|\vec{A}+\vec{B}+\vec{C}| = \sqrt{(-11\text{ m})^2+(40\text{ m})^2}$$

$$|\vec{A}+\vec{B}+\vec{C}| = \sqrt{1\ 721\text{ m}^2}$$

$$|\vec{A}+\vec{B}+\vec{C}| \simeq 41.5\text{ m}$$

The direction of the vector $\vec{A}+\vec{B}+\vec{C}$ is given by

$$\tan\phi = \frac{A_y + B_y + C_y}{A_x + B_x + C_x}$$

$$\tan\phi = \frac{40 \text{ m}}{-11 \text{ m}} = -3.64$$

$$\phi = \tan^{-1}(-3.64)$$

$$\phi \simeq -74.6^\circ$$

$$\theta = 180^\circ + \phi = 180^\circ + (-74.6^\circ)$$

$$\theta = 105.4^\circ$$

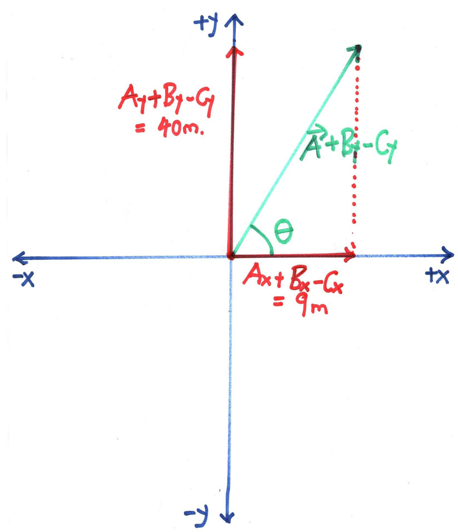

Figure 10: The resultant vector $\vec{A} + \vec{B} - \vec{C}$

The magnitude of $\vec{A} + \vec{B} - \vec{C}$ denoted $|\vec{A} + \vec{B} - \vec{C}|$ is given by

$$|\vec{A} + \vec{B} - \vec{C}| = \sqrt{(A_x + B_x - C_x)^2 + (A_y + B_y - C_y)^2}$$

$$|\vec{A} + \vec{B} - \vec{C}| = \sqrt{(9 \text{ m})^2 + (40 \text{ m})^2}$$

$$|\vec{A} + \vec{B} - \vec{C}| = \sqrt{1\ 681 \text{ m}^2}$$

$$|\vec{A} + \vec{B} - \vec{C}| \simeq 41 \text{ m}$$

The direction of the vector $\vec{A} + \vec{B} - \vec{C}$ is given by

$$\tan\theta = \frac{A_y + B_y - C_y}{A_x + B_x - C_x}$$

$$\tan\theta = \frac{40 \text{ m}}{9 \text{ m}} = 4.44$$

$$\theta = \tan^{-1}(4.44)$$

$$\theta \simeq 77.3^\circ$$

The University of Zambia
School of Natural Sciences
Department of Physics
PHY 1010
Lecture 3A
Description of Uniformly Accelerated Motion

Mr. Gift L. Sichone

March 6, 2021

1 Introduction

The movement of objects (living and non-living) is a part of our everyday experience. So it is important that we are able to study, analyse and predict the motion of objects in our surroundings. In this course, we are particularly interested in a type of motion referred to as **uniform accelerated motion**. This uniform acelerated motion of objects can easily be studied both graphically and through mathematical like expressions called **equations of motion**. The study of the motion of objects without referring to the cause of the motion is called **kinematics**.

2 Learning Outcomes

By the end of this lecture, the student should be able to:

1. define and use distance, displacement, speed, velocity and acceleration;
2. use graphical methods to represent distance, displacement, speed, velocity and acceleration;
3. determine displacement from the area under a velocity-time graph;
4. determine velocity using the gradient of a displacement-time graph;
5. determine acceleration using the gradient of a velocity-time graph;

3 Description of Uniformly Accelerated Motion

3.1 Distance and Displacement

Distance and displacement are closely related in that both physical quantities are a form of length. **Distance** denoted d is a scalar while displacement denoted $\mathbf{s}$ is a vector. **Displacement** refers to the distance moved by an object in a specific direction, for example, 120 m west. The SI units for distance and displacement are metres.

3.2 Speed and Velocity

Speed and **velocity** are also closely related but do not refer to the same physical quantity. Speed is a scalar while velocity is a vector. Speed only refers to how fast an object is moving (i.e. rate of movement) and does not say anything about the direction in which the object is moving. Velocity, on the other hand, refers to the speed of an object in a specific direction, for example, a car moving at 150 m/s east.

There are two type of speed or velocity. The first type is called **instantaneous speed or velocity** while the second type is the **average speed or velocity**.**Instantaneous speed or velocity** refers to the speed or velocity of an object in motion at a specific point in time. The speed limit on the side of a road shows the maximum speed or velocity a motorist is lawfully allowed to drive a vehicle in that section of road. Instantaneous speed or velocity is what traffic police use to determine if a motorist has been overspeeding on a section of road or not.

Average speed or velocity, on the other hand, is obtained by dividing the total distance travelled by an object by the total time taken to travel that distance. **Average speed or velocity** is the preferred physical quantity we tend to use to describe how fast an object was moving because it convieniently gives us only one value of speed to work with. Instantaneous speed or velocity on the other hand generates different values of speed or velocity at each point in time which can complicate our ability to analyse and predict the motion of an object.

The average speed denoted $\bar{v}$ is defined as follows

$$\text{average speed} = \frac{\text{total distance covered}}{\text{total time taken}}$$

$$\bar{v} = \frac{d}{t} \tag{1}$$

where

$\bar{v}$ is the average speed in metres per second (m/s)

d is the total distance covered in metres (m)

t is the total time taken in seconds (s)

The average velocity denoted $\bar{\mathbf{v}}$ is defined as follows

$$\text{average velocity} = \frac{\text{total displacement covered}}{\text{total time taken}}$$

$$\bar{\mathbf{v}} = \frac{\mathbf{s}}{t} \tag{2}$$

where

$\bar{\mathbf{v}}$ is the average velocity in metres per second (m/s)

$\mathbf{s}$ is the total displacement covered in metres (m)

t is the time taken in seconds (s)

Alternatively, average velocity can also be defined in terms of the initial velocity denoted u and the final velocity denoted v of a moving object as follows:

$$\text{average velocity} = \frac{1}{2}\,(\text{initial velocity} + \text{final velocity})$$

$$\bar{v} = \frac{1}{2}\,(u + v) \tag{3}$$

where

$\bar{v}$ is the average velocity in metres per second (m/s)

u is the initial velocity in metres per second (m/s)

v is the initial velocity in metres per second (m/s)

Example 1

An athlete takes 10 s to run a distance of 100 m. Calculate the average speed of the athlete during this 10 s.

$$\text{average speed} = \frac{\text{total distance covered}}{\text{total time taken}}$$

$$\bar{v} = \frac{d}{t}$$

$$\bar{v} = \frac{100\ \text{m}}{10\ \text{s}}$$

$$\bar{v} = 10\ \text{m/s}$$

On average, the athlete covers a distance of 10 m per second. In reality, the athlete does not actually cover a distance of 10 m every second of the 10 s he or she covers the 100 m distance.

3.3 Acceleration

Acceleration refers to the rate of change of speed or velocity. In this course, we are interested in a type of acceleration called **average acceleration** which is defined as follows:

$$\text{average acceleration} = \frac{\text{change in velocity or speed}}{\text{time taken}}$$

$$a = \frac{\Delta \mathbf{v}}{t} \tag{4}$$

where

a is the average acceleration in metres per second squared (m/s^2)

$\Delta \mathbf{v}$ is the change velocity in metres per second (m/s)

t is the time taken in seconds (s)

The average acceleration can also be expressed in terms of the initial and final velocity as follows:

$$\text{average acceleration} = \frac{\text{final velocity or speed} - \text{initial velocity or speed}}{\text{time taken}}$$

$$a = \frac{v - u}{t} \tag{5}$$

where

a is the average acceleration in metres per second squared (m/s^2)

v is the initial velocity in metres per second (m/s)

u is the initial velocity in metres per second (m/s)

t is the time taken in seconds (s)

Example 2

A car accelerates from rest to 28 m/s in 10 s. Calculate the average acceleration the car experiences.

initial velocity of car $u = 0$ m/s

final velocity of car $v = 28$ m/s

time $t = 10$ s

Table 1: The velocity of an accelerating car in 10 s.

t (s)	0	1	2	3	4	5	6	7	8	9	10
v (m/s)	0	2.8	5.6	8.4	11.2	14	16.8	19.6	22.4	25.2	28

$$a = \frac{v-u}{t}$$

$$a = \frac{28\text{ m/s} - 0\text{ m/s}}{10\text{ s}}$$

$$a = \frac{28\text{ m/s}}{10\text{ s}}$$

$$a = 2.8\text{ m/s}^2$$

The car experiences an average acceleration of $a = 2.8$ m/s^2. This means that the speed or velocity of the car increases by 2.8 m/s every second during this duration of 10 s. Table 1 shows how the velocity of this accelerating car changes during these 10 s.

4 Graphical Representation of Uniformly Accelerated Motion

The motion of a moving object can usually be displayed graphically. The most useful motion graph is the velocity-time graph denoted $v{-}t$ graph. Other motion graphs of note are distance-time or displacement-time graphs and acceleration-time graph.

The acceleration of a moving object can be obtained from the $v{-}t$ graph and is equal to the slope or gradient of the $v{-}t$ graph. The distance or displacement covered by a moving body is equal to the area under the $v - t$ graph.

For example, consider a car that starts from rest and accelerates to 28 m/s in 10 s. The acceleration of the car is 2.8 m/s^2. Physically, this acceleration means that velocity of the car increases by 2.8 m/s every second. Table 1 shows how the velocity of the car changes with time in these 10 s.

From the $v - t$ graph shown in Figure 1, if we pick the points $(t_3, v_3) = (3\,\text{s}, 8.4\,\text{m/s})$ and $(t_8, v_8) = (8\,\text{s}, 22.4\,\text{m/s})$, we can calculate the slope or gradient as follows:

$$\text{slope or gradient} = \frac{\text{change in velocity}}{\text{change in time}}$$

$$\text{slope} = \frac{v_8 - v_3}{t_8 - t_3}$$

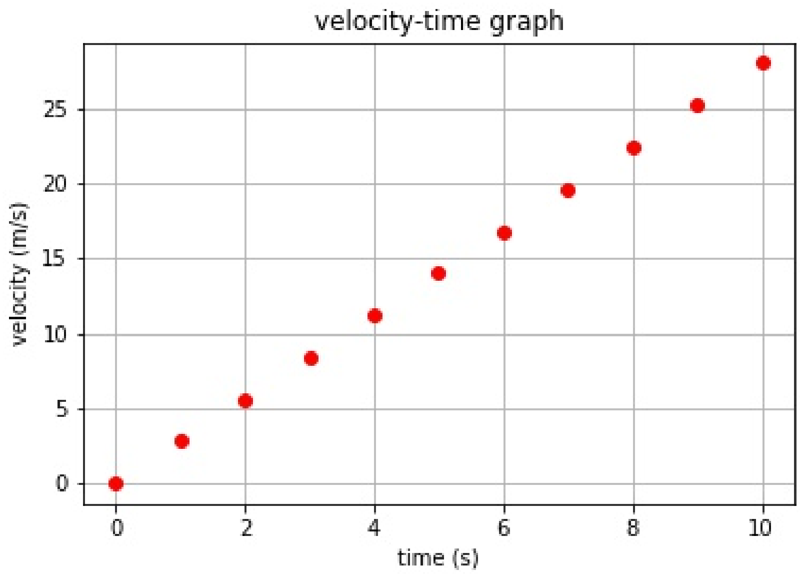

Figure 1: The velocity-time graph of a car accelerating at $a = 2.8$ m/s^2 for 10 s.

$$\text{slope} = \frac{22.4 \text{ m/s} - 8.4 \text{ m/s}}{8 \text{ s} - 3 \text{ s}}$$

$$\text{slope} = \frac{14 \text{ m/s}}{5 \text{ s}}$$

$$\text{slope} = 2.8 \text{ m/s}^2$$

The slope of this $v - t$ graph is equal to the acceleration of the car, that is, 2.8 m/s^2. Every second, the velocity of this car increases by 2.8 m/s.

The total area under the $v - t$ graph in Figure 1 gives the total distance or displacement covered by the car in 10 s. In this case, the area under the $v - t$ graph, corresponds to the area of the triangle. The area of a triangle is given by:

$$\text{area of triangle} = \frac{1}{2} \times \text{base} \times \text{height}$$

In this case, the base = 10 s and the height = 28 m/s. Therefore, we obtain an area of triangle given by

$$\text{area of triangle} = \frac{1}{2} \times 10 \text{ s} \times 28 \text{ m/s} = 140 \text{ m}.$$

Table 2: The displacement of an accelerating car in 10 s.

t (s)	0	1	2	3	4	5	6	7	8	9	10
s (m)	0	1.4	5.6	12.6	22.4	35	50.4	68.6	89.6	113.4	140

The area under an acceleration-time $a - t$ graph gives the average velocity. Note that we are only concerned with uniform accelerated motion i.e. the acceleration is constant and does not change with time.

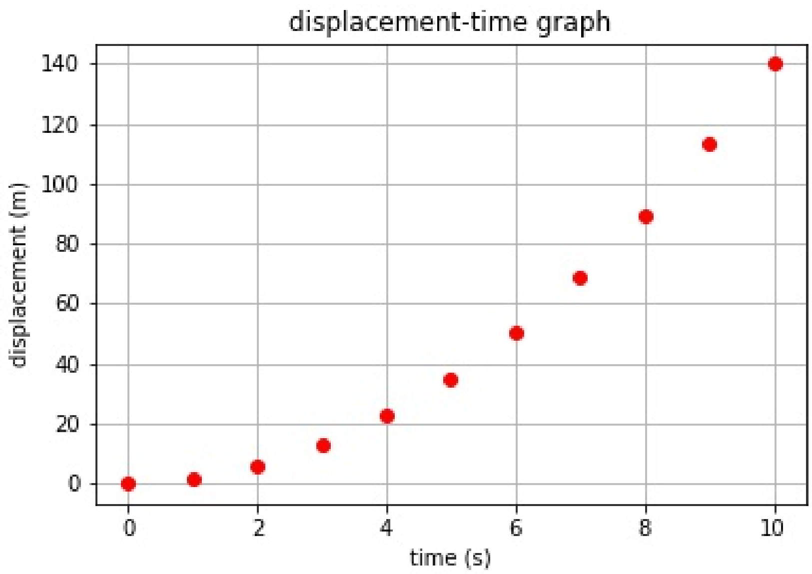

Figure 2: The $s - t$ graph of a car accelerating at $a = 2.8$ m/s^2 for 10 s.

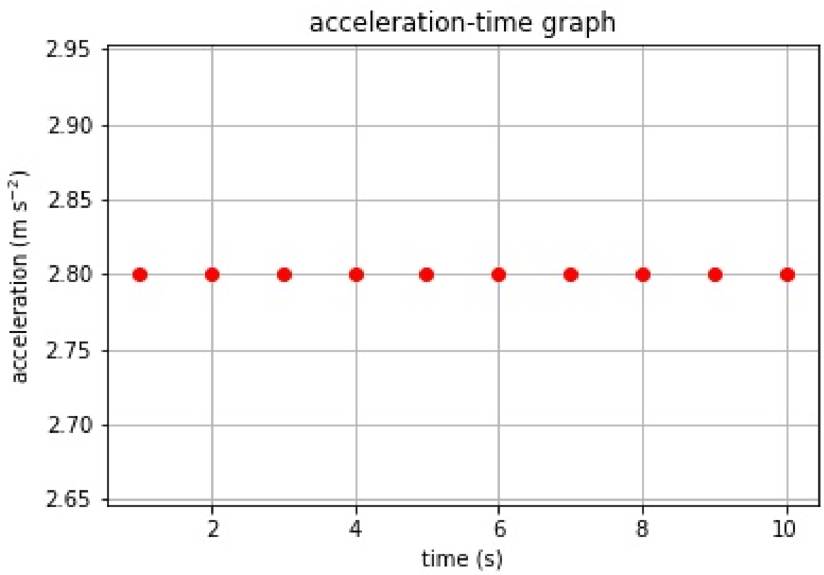

Figure 3: The $a - t$ graph of a car accelerating at $a = 2.8$ m/s^2 for 10 s.

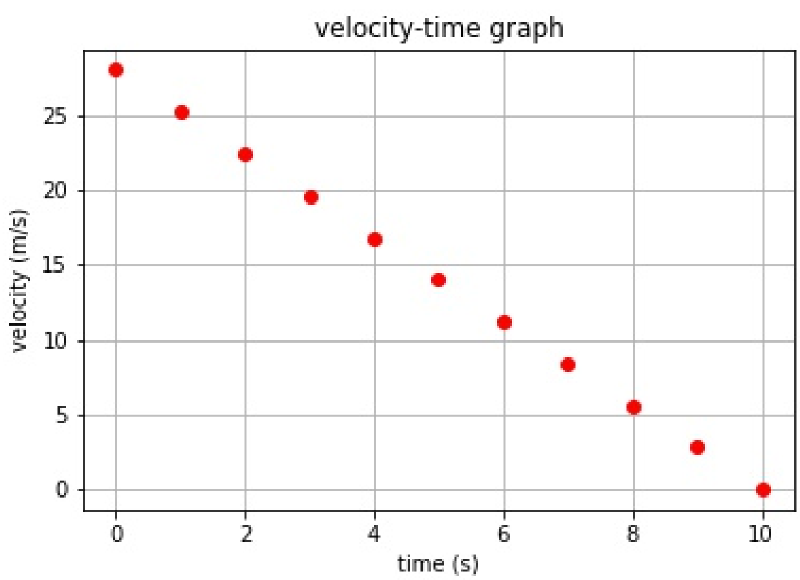

Figure 4: The $v - t$ graph of a car accelerating at $a = -2.8$ m/s^2 for 10 s.

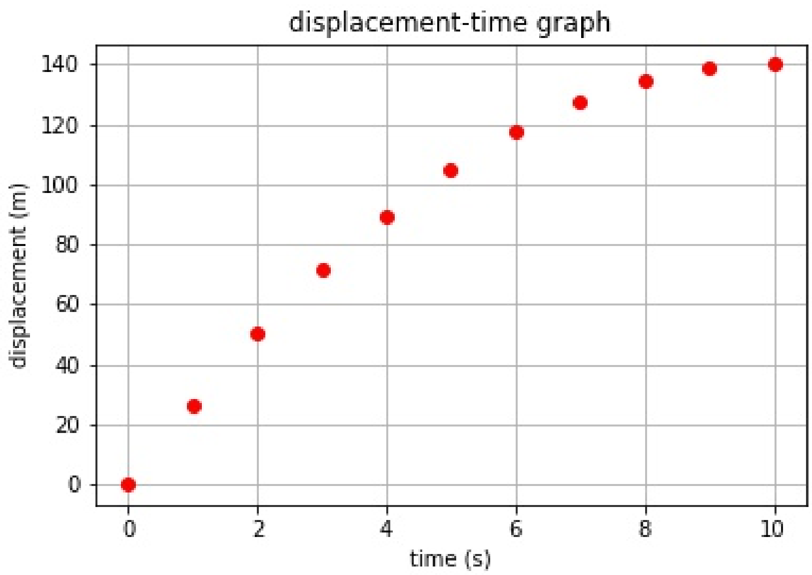

Figure 5: The $s - t$ graph of a car accelerating at $a = -2.8$ m/s^2 for 10 s.

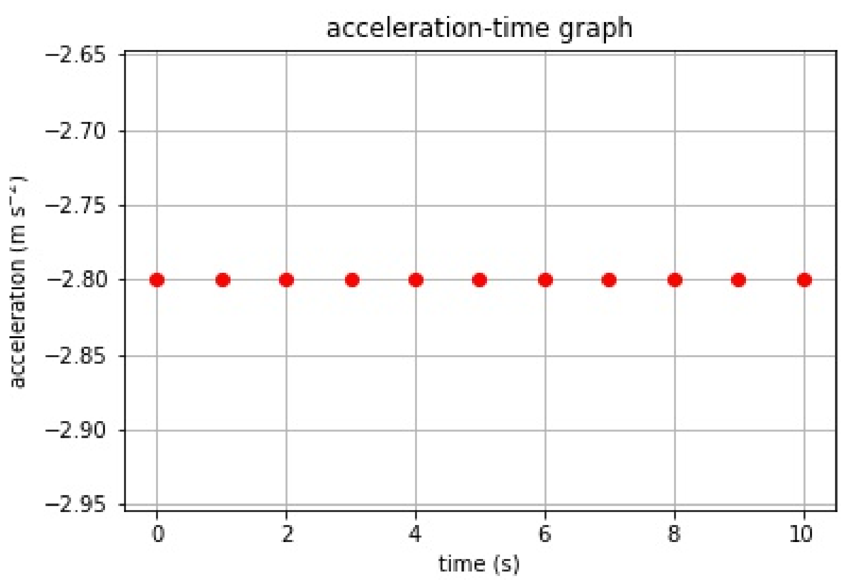

Figure 6: The $a - t$ graph of a car accelerating at $a = -2.8$ m/s^2 for 10 s.

The University of Zambia
School of Natural Sciences
Department of Physics
PHY 1010
Lecture 3B
Equations of Uniform Accelerated Motion

Mr. Gift L. Sichone

March 6, 2021

1 Introduction

In this lecture, we will derive the equations of motion that are used to describe uniformly accelerated motion in a straight line.

2 Learning outcomes

By the end of this lecture, the student should be able to:

1. state the five equations of motion that are used to represent uniformly accelerated motion in a straight line;
2. use the equations of motion used to describe uniformly accelerated motion to solve problems.

3 Equations of uniformly accelerated motion in a straight line

The velocity-time $(v - t)$ graph is particularly important for deriving the equations of uniform accelerated motion because its gradient is equal to the acceleration and the area under the graph is equal to the displacement.

$$\text{acceleration} = \text{slope of velocity time graph}$$

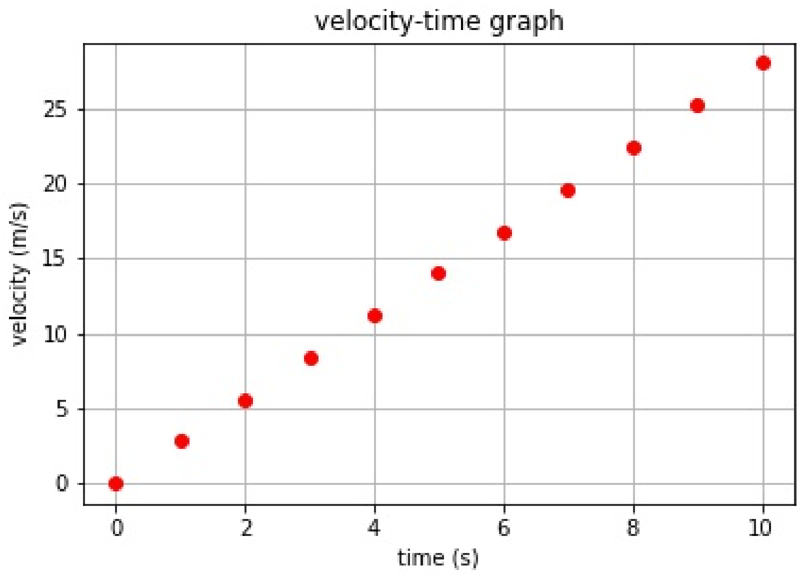

Figure 1: The velocity-time graph of a car accelerating at $a = 2.8$ m/s^2 for 10 s.

To obtain the gradient of a $v - t$ graph, we need to define the following physical quantities:

s - displacement (m)

t - time (s)

u - initial velocity (m/s) at $t = 0$

v - final velocity (m/s) after t

a - acceleration (m/s^2)

$$\text{acceleration} = \frac{\text{change in velocity}}{\text{time taken}}$$

$$a = \frac{\Delta v}{t}$$

where

a is the average acceleration in m/s^2;

Δv is the change in velocity in m/s

t is the time taken in s

$$a = \frac{v - u}{t} \tag{1}$$

where

a is the average acceleration in m/s^2;

v is the final velocity in m/s;

u is the final velocity in m/s;

t is the time taken in s.

This is the first equation of motion used to describe uniform accelerated motion in a straight line.

Multiplying both side of the equation (1) by t, we obtain:

$$at = v - u$$

Moving u to the other side of the equation, we obtain the second equation of motion used to describe uniform accelerated motion in a straight line.

$$v = u + at \tag{2}$$

The area under the $v - t$ graph is equal to the total displacement s distance d moved. Recall, that we defined average velocity denoted $\bar{v}$ as follows

$$\bar{v} = \frac{1}{2}(u + v) \tag{3}$$

Next, we obtain the total displacement s in terms of average velocity $\bar{v}$ and time t as follows:

$$s = \bar{v}t \tag{4}$$

Substituting for the average velocity $\bar{v}$, we obtain

$$s = \frac{1}{2}\left(v + u\right)t \tag{5}$$

This is the third equation of motion used to describe uniformly accelerated motion in a straight line.

$$s = \frac{vt + ut}{2}$$

Substituting for v from the first equation of motion, we obtain:

$$s = \frac{(u + at)t + ut}{2}$$

$$s = \frac{ut + at^2 + ut}{2}$$

$$s = \frac{2ut + at^2}{2}$$

$$s = ut + \frac{1}{2}at^2 \tag{6}$$

This is the fourth equation of motion used to describe uniformly accelerated motion in a straight line.

We can re-write equation (1) as:

$$t = \frac{v - u}{a}$$

and substitute it into equation (6), we get.

$$s = u(\frac{v - u}{a}) + \frac{1}{2}a(\frac{v - u}{a})^2$$

This gives us

$$s = \frac{uv - u^2}{a} + \frac{1}{2}a\frac{(v - u)^2}{a^2}$$

$$s = \frac{uv - u^2}{a} + \frac{(v - u)(v - u)}{2a}$$

$$s = \frac{uv - u^2}{a} + \frac{(v^2 - uv - uv + u^2)}{2a}$$

$$s = \frac{2uv - 2u^2 + v^2 - 2uv + u^2}{2a}$$

$$s = \frac{v^2 - u^2}{2a}$$

$$2as = v^2 - u^2$$

$$v^2 = u^2 + 2as \tag{7}$$

This is the fifth equation of motion used to describe uniform linearly accelerated motion in a straight line.

4 Uniform accelerated motion due to gravity

Uniform accelerated motion due to gravity is divided into upward motion from the surface of the Earth and downward motion towards the surface of the Earth. When an object is launched or thrown upwards, its velocity reduces at a steady 9.8 m/s every second until it reaches its maximum height, where its velocity is 0 m/s for a very short time. Every body thrown vertically upwards therefore accelerates at -9.8/m s^2.

On the other hand, if an object is dropped from a height, it accelerates towards the surface of the Earth at a steady 9.8 m/s every second. At the time, of being droped the object has an initial velocity of 0 m/s and once released will increase its velocity at a rate of 9.8/m s every second. This means that that the body will accelerate at 9.8 m/s^2.

In both cases (i.e, upward and downward motion), the five equations of motion for uniformly accelerated motion still apply because the acceleration is constant i.e. 9.8 m/s^2. Since our acceleration $a = g = 9.8\text{m/s}^2$, our equations of motion become:

$$g = \frac{v - u}{t} \tag{8}$$

$$v = u + gt \tag{9}$$

$$s = \frac{1}{2}(u + v)t \tag{10}$$

$$s = ut + \frac{1}{2}gt^2 \tag{11}$$

$$v^2 = u^2 + 2gs \tag{12}$$

During the upward motion of a body, the acceleration due to gravity has a value of $g = -9.8$ m/s^2. On the other hand, the acceleration due of gravity has a value of $g = 9.8$ m/s^2 in the downward motion of a body.

Example 3

A bullet from a rifle shot upwards has a muzzle velocity of 550 m/s. Calculate:

(a) the time it takes the bullet to reach its maximum height.

(b) the maximum height reached by the bullet.

(c) the total time of flight of the bullet

(d) the velocity with which the bullet strikes the ground.

Solutions

(a) The bullet exits the rifle with a muzzle velocity (i.e. initial velocity) $u =$ 550 m /s. When the bullet reaches its maximum height, its final velocity $v = 0$ m/s. Since the bullet is undergoing upward motion, acceleration due to gravity g becomes -9.8 m/s^2.

We can obtain, the time of flight t from the equation $v = u + gt$.

$$v = u + gt$$

$$0 \text{ m/s} = 550 \text{ m/s} + (-9.8 \text{ m/s}^2)t_{\text{rise}}$$

Dropping off units, we obtain

$$0 = 550 + (-9.8)t_{\text{rise}}$$

$$0 = 550 - 9.8t_{\text{rise}}$$

$$9.8t_{\text{rise}} = 550$$

Dividing both sides by 9.8, we get

$$t_{\text{rise}} = \frac{550}{9.8}$$

$$t_{\text{rise}} = 56.1 \text{ s}$$

(b) the maximum height reached by the bullet, s_{max}

Now that we know the rise time of the bullet, $t_{\text{rise}} = 56.1$ s and the initial velocity, $u = 550$ m/s, we can use the equation of motion

$$s = ut + \frac{1}{2}gt^2$$

to get the maximum height denoted s_{max}

$$s_{\text{max}} = ut + \frac{1}{2}gt^2$$

$$s_{\text{max}} = (550 \text{ m/s})(56.1 \text{ s}) + \frac{1}{2}(-9.8 \text{ m/s}^2)(56.1 \text{ s})^2$$

Dropping off units, we get

$$s_{\text{max}} = (550)(56.1) - \frac{1}{2}(9.8)(56.1)^2$$

$$s_{\text{max}} \approx 15\ 434 \text{ m} \approx 15.434 \text{ km}$$

The bullet rises to a maximun height of $s_{\text{max}} \approx 15.434$ km

(c) the total time of flight of the bullet.

At maximum height, the bullet has an initial velocity $u = 0$ m/s. As the bullet returns to Earth it gathers velocity as its accelerates at $g = 9.8\ \text{m/s}^2$. The bullet on its downward motion covers a distance of $s = 15\ 434$ m. Therefore, we can work out the time of fall on its downward journey using

$$s = ut + \frac{1}{2}gt^2$$

. Substituting the values, we have

$$s_{\text{max}} = ut + \frac{1}{2}gt^2$$

$$15\ 434 \text{ m} = (0 \text{ m/s})t_{\text{fall}} + \frac{1}{2}(9.8 \text{ m/s}^2)t_{\text{fall}}^2$$

Dropping off units, we get

$$15\ 434 = (0)t_{\text{fall}} + \frac{1}{2}(9.8)t_{\text{fall}}^2$$

$$15\ 434 = 0 + 0.5(9.8)t_{\text{fall}}^2$$

$$15\ 434 = (4.9)t_{\text{fall}}^2$$

$$4.9t_{\text{fall}}^2 = 15\ 434$$

Dividing both sides by 4.9, we get

$$t_{\text{fall}}^2 = \frac{15\ 434}{4.9} = 3\ 149.8$$

Getting the square root on both sides, we obtain

$$t_{\text{fall}} = \sqrt{3\ 149.8}$$

$$t_{\text{fall}} = 56.1 \text{ s}$$

the fall time on the downward motion is again $t_{\text{fall}} = 56.1$ s.

Therefore, the total time of flight for both the upward and downward motion of the bullet is

$$t_{\text{flight}} = t_{\text{rise}} + t_{\text{fall}}$$

$$t_{\text{f}} = 56.1 \text{ s} + 56.1 \text{ s}$$

$$t_{\text{f}} = 112.2 \text{ s}$$

(d) the velocity with which the bullet strikes the ground.

Now that we know the fall time of the bullet on its downward motion, $t_{\text{fall}} = 56.1$ s, and its initial velocity at maximum height $u = 0$ m/s, we can calculate the final velocity with which the bullet strikes the ground, v.

$$v = u + gt$$
$$v = 0 \text{ m/s} + 9.8 \text{ m/s}^2 \times 56.1 \text{ s}$$

Dropping off units, we get

$$v = 0 + 9.8(56.1)$$

$$v = 549.78$$

$$v \sim 550 \text{ m/s}$$

The bullet strikes the ground with a velocity of 550 m/s. This is the same as the muzzle velocity with which the bullet is fired upwards from the gun.

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Lecture 3C
Projectile Motion

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1 Introduction

This lecture introduces the motion of projectiles under the influence of gravity. A projectile is a body that has been projected by an external force and continues in motion by its own inertia. In this lecture, we will learn how to describe the motion of a projectiles which is characterized to movement in two perpendicular directions.

2 Learning outcomes

By the end of this lecture, the student should be able to:

1. describe in words the motion of a projectile due to a uniform velocity in horizontal direction and a uniform acceleration in the vertical direction;
2. calculate the rise time of a projectile to maximum height;
3. calculate the maximum height the projectile rises;
4. calculate the fall time of a projectile from maximum height;
5. calculate the flight time of a projectile i.e. sum of rise time and fall time;
6. calculate the horizontal distance travelled by a projectile i.e. the range
7. calculate the velocity with which a projectile strikes the ground.

Table 1: Decomposition of projectile muzzle velocity $u = 550$ m/s at $\theta = 45^\circ$ into u_x and u_y components

vector	x-component	y-component
$\vec{u}$	$u_x = (550 \text{ m/s}) \cos 45^\circ = 389$ m/s	$u_y = (550 \text{ m/s}) \sin 45^\circ = 389$ m/s

3 Projectile Motion of a shell

Consider a shell fired from an anti-aircraft gun with a muzzle velocity of 550 m/s at an angle of 45° with the ground. The muzzle velocity of the shell is a vector as it has magnitude of 550 m/s and direction of 45° from the horizontal. In the case of projectiles, we are generally interested in finding out how high the shell will go up into the sky i.e. maximum height and how far the shell will travel horizontally i.e. the range.

Projectiles always carries out two types of motion at the same time. The first type of motion is vertical (i.e. up and down) motion along the y-direction and the second type of motion is horizontal motion along the x-direction. These two types of motions are connected to one another by the flight time of the projectile. The **flight time** is defined as the time it takes the projectile to rise from the ground, reach maximum height and then return to the ground. This is the same time it takes the projectile of travel the horizontal distance referred to as the **range**.

Another important point to note is that for a projectile to reach maximum height, it upward or vertical velocity at maximum height must be equal to 0 m/s. Otherwise, the projectile will continue to rise. Also at maximum height, the projectile is not stationary but continues to move along the horizontal with a non-zero velocity.

To solve any problem of projectile motion, we first need to decompose the initial muzzle velocity u into its perpendicular components. We first need to find out the magnitude of the initial muzzle velocity along the the x-direction (i.e. x-component) denoted u_x and y-direction (i.e. y-component) denoted u_y. Table 1 shows the decompostion of the initial muzzle velocity u of the shell fired from an anti-aircraft gun into u_x and u_y.

The second step is to consider only the vertical motion of the projectile along the $y-$direction and ignore the horizontal motion of the projectile. In the y-direction, the velocity of the projectile will continue to decrease at a steady 9.8 m/s every second i.e. 9.8 m/s^2 until it becomes 0 m/s at maximum height, s_{max}. Concerning this upward motion of the projectile, we can find out the maximum height as well as the time it will take the projectile to reach maximum height using our equations of motion in a straight line.

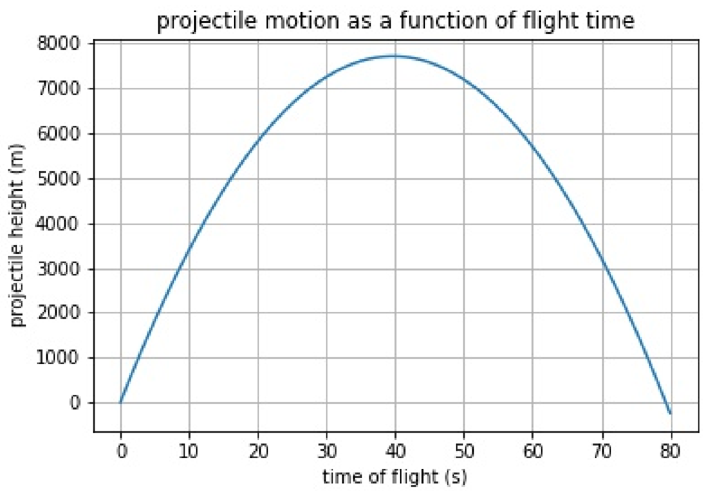

Figure 1: Projectile motion of a shell as a function of flight time

3.1 Calculating the rise time of a projectile

We know that to reach maximum height denoted s_{max}, the initial velocity in the y-direction i.e. $u_y = 389$ m/s has to decrease to a final velocity u_y of 0 m/s at a steady rate of 9.8 m/s every second i.e., g = -9.8 m/s^2. Therefore, we can use the equation

$$v = u + gt$$

To obtain the time to rise to maximum height, **rise time**, denoted, t_{rise}. We get

$$v_y = u_y + gt_{\text{rise}}$$

where

$v_y = 0$ m/s;

$u_y = 389$ m/s

$g = -9.8$ m/s^2

t_{rise} is the rise time.

Substituting these values in the above equation, we obtain

$$0 \text{ m/s} = 389 \text{ m/s} + (-9.8 \text{ m/s}^2)t_{\text{rise}}$$

Dropping the units, we get

$$0 = 389 + (-9.8)t_{\text{rise}}$$

$$0 = 389 - 9.8t_{\text{rise}}$$

$$9.8t_{\text{rise}} = 389$$

Dividing both sides by 9.8, we get

$$t_{\text{rise}} = \frac{389}{9.8}$$

$$t_{\text{rise}} = 39.7 \text{ s}$$

Therefore, it takes the projectile $t_{\text{rise}} = 39.7$ s to reach the maximum height.

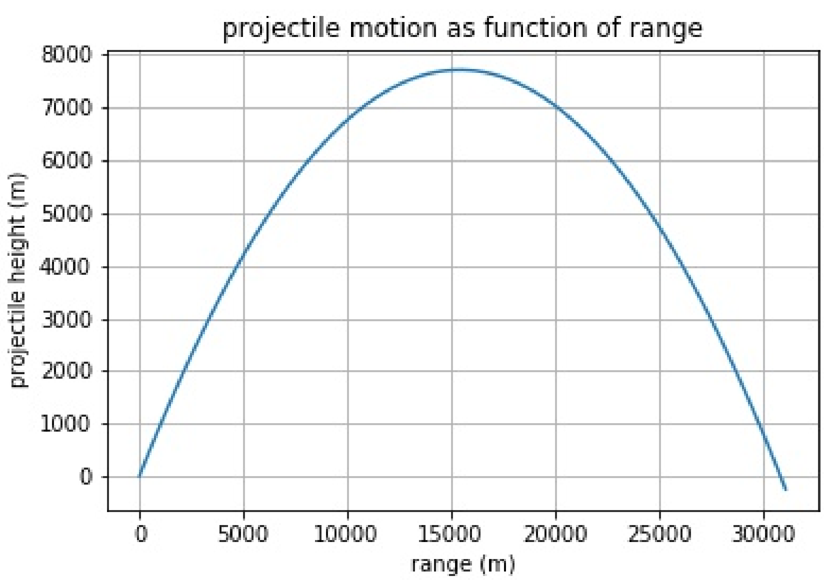

Figure 2: Projectile motion of a shell as a function of range.

3.2 Calculating maximum height reached by projectile

From the calculated rise time to reach maximum height $t_{\text{rise}} = 39.7$ s, we can calculate the maximum height, s_{max}, reached using the following equation

$$s = ut + \frac{1}{2}gt^2$$

We know $u_y = 389$ m/s, $g = -9.8$ m/s^2 and $t_{\text{rise}} = 39.7$ s.

$$s_{\text{max}} = u_y t_{\text{rise}} + \frac{1}{2} g t_{\text{rise}}^2$$

Substituting, we obtain

$$s_{\text{max}} = (389 \text{ m/s}) \times (39.7 \text{ s}) + \frac{1}{2} \times (-9.8 \text{ m/s}^2) \times (39.7 \text{ s})^2$$

Dropping off the units, we get

$$s_{\text{max}} = 389 \times 39.7 - 0.5 \times 9.8 \times (39.7)^2$$

We get the maximum height , s_{max} as

$$s_{\text{max}} = 7\ 720 \text{ m}.$$

The projectile rises to a maximum height of 7 720 m.

3.3 Calculating time of fall from maximum height

After obtaining the maximum height, we can proceed to calculate fall from maximum height, t_{fall}. We know that at maximum height, s_{max}, in the y-direction, the velocity of the projectile is 0 m/s and that the projectile will have to fall through a distance of 7 720 m to strike the ground. During this fall, the velocity of the projectile increases by 9.8 m/s every second. i.e. g = 9.8 m/s^2.

We can therefore, get the time of fall, t_{fall} from the equation

$$s = ut + \frac{1}{2} g t^2$$

In the y-direction, this equation becomes

$$s_{\text{max}} = u_y t_{\text{fall}} + \frac{1}{2} g\ t_{\text{fall}}^2$$

We know that $u_y = 0$ m/s, $g = 9.8$ m/s^2 and $s_{\text{max}} = 7\ 720$ m. We are looking for t_{fall}. Substituting, we obtain

$$7\ 720 \text{ m} = (0 \text{ m/s}) \times t_{\text{fall}} + \frac{1}{2} \times (9.8 \text{ m/s}^2) \times t_{\text{fall}}^2$$

Dropping off the units, we get

$$7\ 720 = 0 \times t_{\text{fall}} + \frac{1}{2} \times (9.8) \times t_{\text{fall}}^2$$

$$7\ 720 = 0 + 4.9\ t_{\text{fall}}^2$$

$$7\ 720 = 4.9\ t_{\text{fall}}^2$$

$$4.9\ t_{\text{fall}}^2 = 7\ 720$$

Dividing both sides of the equation by 4.9, we obtain

$$t_{\text{fall}}^2 = \frac{7\ 720}{4.9} = 1\ 575.5$$

Square rooting both sides of the equation, we obtain

$$\sqrt{t_{\text{fall}}^2} = \sqrt{1\ 575.5}$$

$$t_{\text{fall}} = 39.7 \text{ s}$$

The fall time ,t_{fall}, is 39.7 s. as can be seen in this case, the time it takes the projectile to rise to its maximum height and the time it takes to fall are the same.

$$t_{\text{rise}} = t_{\text{fall}} = 39.7 \text{ s}$$

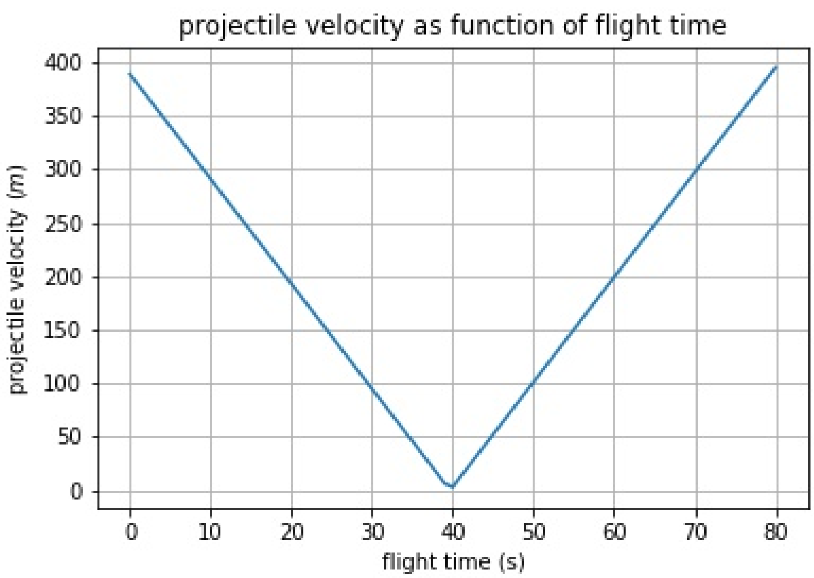

Figure 3: The projectile velocity of a shell as a function of flight time.

3.4 Calculating flight time of a projectile

We can obtain the flight time denoted t_{f} as a sum of the time to rise to maximum height,t_{rise}, and the time of fall,t_{fall}.

$$t_{\mathrm{f}} = t_{\mathrm{rise}} + t_{\mathrm{fall}}$$

$$t_{\mathrm{f}} = 39.7 \text{ s} + 39.7 \text{ s}$$

$$t_{\mathrm{f}} = 79.4 \text{ s}$$

3.5 Calculating vertical velocity with which projectile strikes the ground

We can also calculate the vertical final velocity v_y with which the projectile strikes the ground. This can be done using the time of fall, t_{fall}=39.7 s and the initial velocity of the projectile at maximum height, $u_y = 0$ m/s. We can obtain v_y from the equation

$$v = u + gt$$

In the y-direction, we get

$$v_y = u_y + gt_2$$

Substituting for $u_y = 0$ m/s and $t_2 = 39.7$ s, we get

$$v_y = (0 \text{ m/s}) + (9.8 \text{ m/s}^2) \times (39.7 \text{ s})$$

Dropping the units we get,

$$v_y = 0 + 9.8 \times 39.7$$

$$v_y = 389 \text{ m/s}$$

From this value of $v_y = 389$ m/s, we see that the projectile strikes the ground with the same velocity with which it rose up with.

$$u_y = v_y = 389 \text{ m/s}$$

We have found out all the key parameters about the motion of the projectile in the y-direction.

3.6 Calculating the range of the projectile

We can now consider motion in the x-direction. You need to be aware that there is no acceleration i.e. a_x=0 m/s in the x-direction. Therefore, the initial velocity of the projectile in the x-direction is the same as the final velocity of the projectile in the x-direction, i.e., $u_x = v_x = 389$ m/s. The time of flight of the projectile has been found to be 79.4 s. Therefore, we can calculate the range as

$$\text{range} = u_x \times t_\text{f}$$

$$\text{range} = 389 \text{ m/s} \times 79.4 \text{ s}$$

$$\text{range} = 30\ 886.6 \text{ m}$$

$$\text{range} \sim 30.9 \text{ km}$$

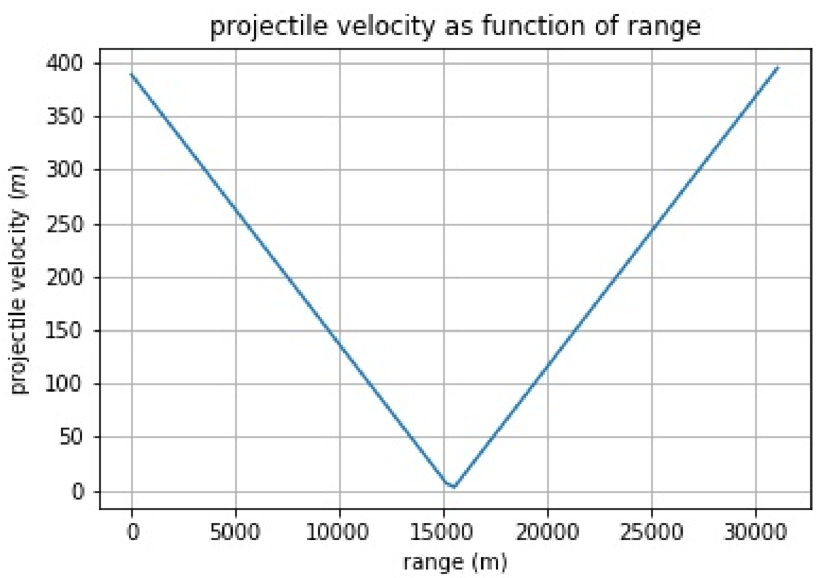

Figure 4: The projectile velocity of a shell as a function of flight time.

3.7 Calculating velocity with which projectile strikes the ground

When the projectile strikes the ground after flying for 79.4 s, it has a final velocity with $v_x = 389$ m/s and $v_y = 389$ m/s. The magnitude of the velocity with which the projectile strikes the ground is given by

$$v = \sqrt{v_x^2 + v_y^2}$$

.

Therefore, we get a magnitude of of

$$v = \sqrt{(389 \text{ m/s})^2 + (389 \text{ m/s})^2}$$

$$v = 550 \text{ m/s}$$

This projectile strike velocity is the same as the projectile muzzle velocity from the anti-aircraft gun. So we conclude that the strike velocity and muzzle velocity in this case are the same.

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Lecture 4A
Newton's Laws of Motion

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1 Introduction

The motion of any object is governed by forces that act on the object. This lecture introduces Newton's laws of motion, which are fundamental to understanding the connection between forces and motion.

2 Learning Outcomes

By the end of this lecture, the student should be able to:

1. understand that mass is a property of a body that resists change in motion;
2. state and apply each of Newtons's Laws of Motion;
3. recall the relationship $F = ma$ and solve problems using it, appreciating that the acceleration a and the resultant force F always are in the same direction;
4. describe and use the concept of weight as the effect of a gravitational field on a mass and recall that the weight of a body is equal to the product of its mass and the acceleration of free fall, $F_W = mg$;

3 Newton's Laws of Motion

Mass

The **mass** of an object is a measure of the inertia of the object. **Inertia** is the tendency of a body at rest to remain at rest, and of a body in motion to continue moving with unchanged velocity. The **kilogram** (kg) is the SI unit of mass. 1 kg is the mass of an object whose mass is equal to the **International Prototype Kilogram** or **standard kilogram** held at the International Bureau of Weights and Measures in Paris, France. The masses of all other objects are compared to the **standard kilogram**.

Force

In mechanics, **force** is that which changes the velocity of an object. Force is a vector quantity, having magnitude and direction. An **external force** is one whose source lies outside of the system (or body) being considered. The **net external force** acting on an object causes the object to accelerate in the direction of that force. The acceleration is proportional to the force and inversely proportional to the mass of the object. The **newton** (N) is the SI unit of force. 1 N is that resultant force which will give a 1 kg mass an acceleration of 1 m/s^2 .

Newton's First Law of motion

An object at rest will remain at rest; an object in motion will continue in motion with constant velocity until it is acted upon by an external force. Force is the changer of motion.

Newton's Second Law of Motion

If the resultant (or net), force $\mathbf{F}$ acting on an object of mass m is not zero, the object accelerates in the direction of the force. The acceleration a is proportional to the force F and inversely proportional to the mass of the object m.

$$a = \frac{F}{m}, \quad F = ma \tag{1}$$

The acceleration a has the same direction as the resultant force F.

The vector equation $F = ma$ can be written in terms of components as

$$\sum F_x = ma_x, \quad \sum F_y = ma_y \tag{2}$$

where the $\sum F_x$ and $\sum F_y$ are the resultant forces are the components of the external forces acting on the object in the x and y-directions. The resulting accelerations in the x and y-directions are given by a_x and a_y.

$$a = \sqrt{a_x^2 + a_y^2} \tag{3}$$

Newton's Third Law of Motion

For each force exerted on one body, there is an equal, but oppositely directed, force on some other body interacting with it. This is often called **the Law of Action and Reaction**. Notice that the action and reaction forces act on the two different interacting objects.

4 The Law of Universal Gravitation

When two masses m_1 and m_2 gravitationally interact, they attract each other with forces of equal magnitude. For point masses (or spherically symmetric bodies), the attractive force F_G is given by

$$F_{\text{G}} = G\frac{m_1 m_2}{r^2} \tag{4}$$

where r is the distance between mass centers, and where $G = 6.67 \times 10^{-11}$ N m^2/kg, where F_G is in newtons, m_1 and m_2 are in kilograms, and r is in meters.

To calculate the force of gravity, experienced here on Earth, we let m_1 be the mass of the Earth and denote it as $m_E = 5.97219 \times 10^{24}$ kg. We also let m_2 be the masss of an object on the surface of the Earth and denote it as just m. In this cases, the distance between the centre of the Earth and the object on the surface of the Earth is just the radius of the Earth denoted $r_E = 6\ 371\ 000$ m. Re-writing the Law of Universal Gravitation, we get

$$F_{\text{G}} = G\frac{m m_E}{r_E^2} \tag{5}$$

$$F_{\text{G}} = m\left(G\frac{m_E}{r_E^2}\right) \tag{6}$$

The term $\left(G\frac{m_E}{r_E^2}\right)$ is a constant on Earth and is denoted as g

$$g = G\frac{m_E}{r_E^2} \tag{7}$$

$$g = (6.67 \times 10^{-11}\ \text{N m}^2/\text{kg}^2)\left(\frac{5.97219 \times 10^{24}\ \text{kg}}{(6\ 371\ 000\ \text{m})^2}\right)$$

$$g \simeq 9.81\ \text{m/s}^2 \tag{8}$$

The g is our acceleration due to gravity here on Earth.

Therefore, the Law of Universal Gravitation can be expressed as

$$F_{\text{G}} = mg \tag{9}$$

Gravity on Earth causes objects to accelerate downwards towards the center of Earth by 9.8 m/s^2.

5 Types of forces

5.1 Weight

The weight of an object is the gravitational force acting downward on the object. On the Earth, weight is the gravitational force exerted on the object by the planet. Its SI units of weight are newtons

An object of mass m falling freely toward the Earth is subject to only one force, the pull of gravity, which we call the weight F_W of the object. The object's acceleration due to F_W is the free-fall acceleration g. Therefore, $F = ma$ provides us with the relation between $F = F_W$, $a = g$, and m; it is $F_W = mg$. Because, on average, $g = 9.8$ m s^{-2} on Earth, a 1.0 kg object weighs 9.8 N at the Earth's surface.

5.2 Normal force

The Normal force F_N on an object that is being supported by a surface is the component of the supporting force that is perpendicular to the surface.

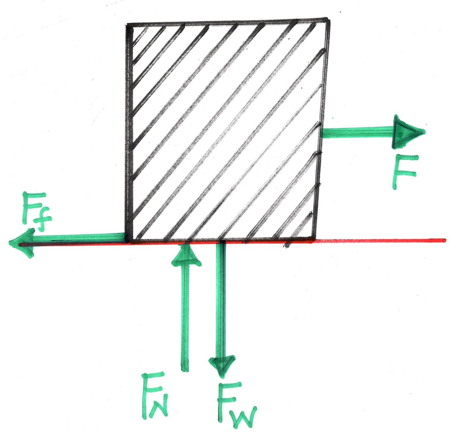

Figure 1: Forces acting on a box on a horizontal surface.

5.3 Tensile Force

The tensile force acting on a string or chain or tendon is the applied force tending to stretch it. The magnitude of the tensile force is the tension denoted F_T.

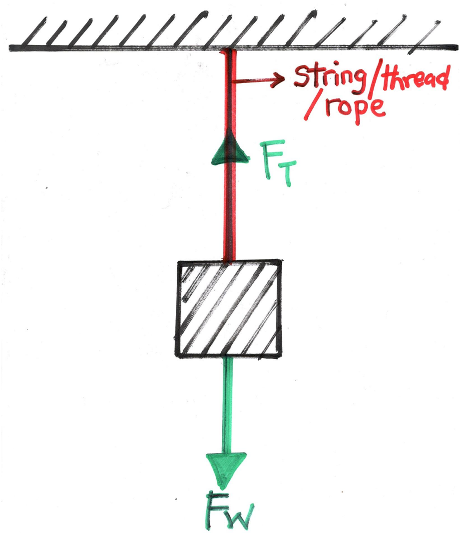

Figure 2: Forces acting on a box supported by a rope

5.4 Friction Force

The friction force F_f is a tangential force acting on an object that opposes the sliding of that object on an adjacent surface with which it is in contact. The friction force is parallel to the surface and opposite to the direction of motion or of impending motion. Only when the applied force exceeds the maximum static friction force will an object begin to slide.

6 Coefficients of Friction

6.1 Coefficient of static friction

The coefficient of static friction μ_s is defined for the case in which one surface is just on the verge of sliding across another surface. It is

$$\text{coefficient of static friction} = \frac{\text{static friction force}}{\text{normal force}}$$

$$\mu_s = \frac{F_f(\text{max})}{F_N} \tag{10}$$

$$\text{static friction force} = \text{coefficient of static friction} \times \text{normal force}$$

$$F_f(\text{max}) = \mu_s \cdot F_N \tag{11}$$

where the maximum friction force occurs when the object is just on the verge of slipping but is nonetheless at rest.

6.2 Coefficient of kinetic friction

The Coefficient of kinetic friction μ_k is defined for the case in which one surface is sliding across another at constant speed. It is

$$coefficient\ of\ kinetic\ friction = \frac{\text{kinetic friction force}}{\text{normal force}}$$

$$\mu_k = \frac{F_f}{F_N} \tag{12}$$

$$\text{kinetic friction force} = \text{coefficient of kinetic friction} \times \text{normal force}$$

$$F_f = \mu_k \cdot F_N \tag{13}$$

7 Calculation of the resultant force for simple physical systems

7.1 Motion of a box on a flat horizontal surface

The acceleration a of an object with mass m is dependent on the presence or absence of a resultant force $\sum F$ in a specific direction of interest. For example, if we consider the motion of the box shown in Figure 1, we are particularly interested in the motion of the box parallel to and perpendicular to the flat horizontal surface. Before we proceed any further, we chose to designate the direction parallel to the flat surface as our x-direction and that which is perpendicular to the flat surface as our y-direction. Furthermore, the y-direction above the flat surface is designated as the positive y-direction while that below the surface is designated as the negative y-direction. Along the x-direction, we assume that the box will tend to move to the right side hence this direction becomes our positive x-direction and the direction opposite the motion of the box i.e. the left side will become our negative x-direction.

With the issue of direction resolved, we can now shift our attention to whether the box is motionless or moving in a particular direction. Naturally, a box like the one shown in Figure 1 shows no tendency to change its velocity or position along the y-direction when its lying on a flat surface, therefore, has no acceleration along the y-direction i.e. $a_y = 0$ m/s^2. This shows that the two forces acting on the box in the y-direction i.e. F_N and F_W do not produce a resultant force. Since F_N points in a direction above the horizontal surface, we take it to be a positive force. Similarly, the weight of the box F_W which always points downwards is taken to be a negative force. Therefore, we can get the resultant of these two forces in the y-direction as

$$\sum F_y = +F_N + (-F_W)$$

$$\sum F_y = F_N - F_W \tag{14}$$

With this expression of resultant force along the y-direction, we can now write Newton's Second Law of Motion as follows:

$$\sum F_y = ma_y$$

where

$\sum F_y$ is the resultant force along the y-direction

m is the mass of the box

a_y is the acceleration of the box along the y-direction.

Substituting for the expression of $\sum F_y$, we get

$$F_N - F_W = ma_y \tag{15}$$

Since the box shows no tendency to move along the y-direction, the acceleration is zero along this direction i.e. $a_y = 0$ m/s^2. Substituting this value of acceleration we get

$$F_N - F_W = m(0 \text{ m/s}^2)$$

$$F_N - F_W = 0$$

$$F_N = F_W \tag{16}$$

We therefore conclude that the normal force F_N exerted by the flat surface on the box is always equal to the weight of the box F_W but acts in the opposite direction. The flat horizontal surface whenever it "feels" the weight of the box tends to push up against the surface of the box it is in contact with. This is a consequence of Newton's third law of motion.

Next, we shift our attention to the motion of the box along the x-direction. A close inspection of Figure 1 shows that there are two forces acting on it along the x-direction. The first force is the force F which could easily be a tensile force if a rope is attached to the box. The second force is the friction force F_f which always tends to oppose the motion of the box and thus always points in a direction opposite the motion. With these two forces, we can get an expresion for resultant force along the x-direction as follows:

$$\sum F_x = +F + (-F_f)$$

$$\sum F_x = F - F_f \tag{17}$$

Along the x-direction, the box can behave in a number of ways. The box can chose to remain motionless, move at constant velocity, move faster or slow down depending on the presence or absence of a resultant force along the x-direction.

If the box chooses to remain motionless, then its velocity or position will not change. In this case, the acceleration of the box along the x-direction will be zero i.e. $a_x = 0 \text{ m/s}^2$. We write Newton's second law of motion for a motionless box as:

$$\sum F_x = ma_x \tag{18}$$

Subsituting for the expression of resultant force and acceleration along the x-direction, we get

$$F - F_f = m(0 \text{ m/s}^2)$$

$$F - F_f = 0$$

$$F = F_f \tag{19}$$

where

F is the force pulling the box

F_f is the static friction force.

Recall that the static friction force is given by

$$F_f = \mu_s F_N \tag{20}$$

and that the normal force F_N is equal to

$$F_N = F_W = mg \tag{21}$$

Therefore, we get

$$F = \mu_s mg \tag{22}$$

If the box chooses to move with constant velocity, its velocity will remain unchanged but its position will change. Also, in this case, the acceleration of the box along the x-direction will be zero i.e. $a_x = 0 \text{ m/s}^2$. We write Newton's second law of motion for a box moving with constant velocity as:

$$\sum F_x = ma_x \tag{23}$$

The expression of resultant force remains unchanged and acceleration along the x-direction is zero. Substituting, we get

$$F - F_f = m(0 \text{ m/s}^2)$$

$$F - F_f = 0$$

$$F = F_f \tag{24}$$

where

F is the force pulling the box

F_f is the kinetic friction force.

Recall that the kinetic friction force is given by

$$F_f = \mu_k F_N \tag{25}$$

and that the normal force F_N is equal to

$$F_N = F_W = mg \tag{26}$$

Therefore, we get

$$F = \mu_k mg \tag{27}$$

If the box chooses to move faster or slower, undergoes an acceleration a_x which has to be worked out. The acceleration is important because it connects our equations of motion with the resultant force. In this case, we write Newton's laws of motion as

$$\sum F_x = ma_x \tag{28}$$

Since the resultant force remains unchanged and the acceleration is not equal to zero, we get

$$F - F_f = ma_x \tag{29}$$

Dividing both sides by the mass of the box, we get the acceleration in the x-direction as

$$a_x = \frac{F - F_f}{m} \tag{30}$$

If the pulling force F is greater than the friction force F_f, then the box moves faster and faster as the acceleration in the x-direction is positive. Similarly, if the pulling force F is less than the friction force F_f, then the box is slow down to a halt.

If the pulling force is absent i.e. $F = 0$ N, then the acceleration of the box alonng the x-direction is negative and will cause the box to come to a stop. This process is called "sliding" and the acceleration is given by

$$a_x = -\frac{F_f}{m} \tag{31}$$

7.2 Motion of a box supported by a rope

Let us now turn our attention to the motion of a mass m with weight F_W supported by a rope through a tensile force F_T as shown in Figure 2. The mass can only move along one direction i.e. up and down. We chose to designate this direction as the y-direction. Since the mass supported by the rope will naturally tend to move downwards if the rope was not present or got cut, we choose the downward direction as our positive y-direction and the upward direction as the negative y-direction. A short inspection of Figure 2 shows that only two forces are acting on the mass. Since the weight of the mass F_W is pointing in the positive y-direction we designate it as a positive force. The tensile force F_T which "lives" inside the rope will be a negative force. Thus, we can get resultant force on the mass along the y-direction as follows:

$$\sum F_y = +F_W + (-F_T)$$

$$\sum F_y = F_W - F_T \tag{32}$$

Next, we can write Newton's second law of motion for the mass supported by the rope as follows:

$$\sum F_y = ma_y \tag{33}$$

where

$\sum F_y$ is the resultant force on along the y-direction

m is the mass supported by the rope

a_y is the acceleration of the mass along the y-direction

The mass supported by the rope can behave in a variety of ways. The mass can chose to remain motionless, move downwards at constant velocity or accelerate towards downards.

If the mass chooses to remain motionless, then the velocity and position of the mass will remain unchanged. This shows that the mass undergoes zero acceleration i.e. $a_y = 0$ m/s^2. Substituting the expression for resultant force and the value of acceleration, we get

$$F_W - F_T = m(0 \text{ m/s}^2)$$

$$F_W - F_T = 0$$

$$F_T = F_W \tag{34}$$

We conclude that the tensile force F_T generated by the rope is equal in magnitude to the weight of the supported mass F_W.

If the mass supported by the rope is lowered towards the ground at constant velocity, the position of the mass will change but its velocity will remain unchanged. The acceleration of the mass along the y-direction will still be equal to zero i.e. $a_y = 0$ m/s^2. We can write Newton's second law of motion for this scenario as follows

$$\sum F_y = ma_y$$

Substituting for the resultant force and the acceleration, we obtain

$$F_W - F_T = m(0 \text{ m/s}^2)$$

$$F_W - F_T = 0$$

$$F_T = F_W \tag{35}$$

Again, we conclude that the tensile force F_T generated by the rope as the mass is lowered towards the ground is equal in magnitude to the weight of the supported mass F_W.

If the mass supported by the rope accelerates downwards or upwards, then we can write Newtons second law of motion as

$$\sum F_y = ma_y$$

Substituting for the resultant force and the acceleration, we obtain

$$F_W - F_T = ma_y$$

Making the acceleration a_y the subject of the formula, we get

$$a_y = \frac{F_W - F_T}{m} \tag{36}$$

The University of Zambia
School of Natural Sciences
Department of Physics
PHY 1010
Lecture 4B
Motion of a box on an inclined plane

Mr. Gift L. Sichone

April 27, 2021

Direction on an inclined plane

An inclined plane is a titled surface that makes an oblique angle (i.e. acute or obtuse angle) with the plane of the horizontal. Consider the incline plane shown in Fig. 1 that makes an angle θ with the horizontal. In this case, the x and y-directions of the inclined plane and those of the horizontal surface are not the same. For an inclined plane, we choose the x-direction to be the direction parallel to the surface of incline plane. On the other hand, the y-direction is chosen to be perpendicular or at 90° to the surface of the inclined plane.

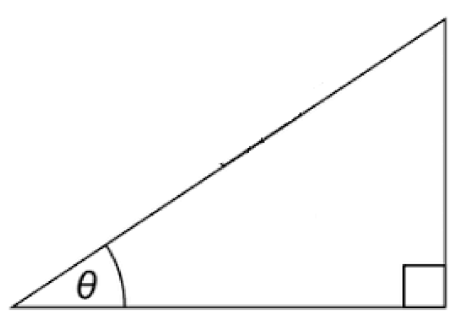

Figure 1: An inclinced plane that makes an angle θ with the horizontal surface.

Box on an inclined plane

A box of mass m on an inclined plane exerts a weight $F_W = mg$ that is directed downwards as shown in Fig. 2. The direction of the weight of the box $F_W = mg$ lies neither in the x nor y-direction of the surface of the incline. Thus to find the x and y-components of the weight F_W along the x and y-direction of the inclined plane, we have to resolve the weight F_W. Fig. 2 also shows the resolution of the weight of the box $F_W = mg$ into a x-component parallel to the surface of the incline plane and a y-component perpendicular to the surface of the incline. As can be seen from Fig. 2, the x-component of the weight of the box is equal to $F_W \sin\theta = mg\sin\theta$ while the y-component is equal to $F_W \cos\theta = mg\cos\theta$.

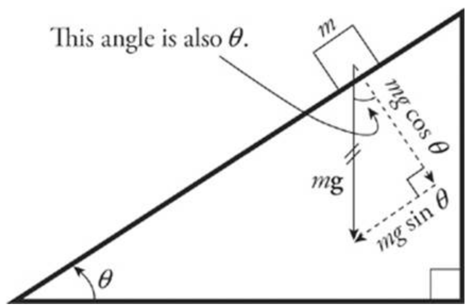

Figure 2: Resolution of weight F_W of a box on an inclinced plane

Thus, a box of weight $F_W = mg$ tends to exert a force of $F_W \cos\theta = mg\cos\theta$ perpendicular to the surface of the incline. As shown in Fig. 3, the incline is "not happy" about the box exerting a force of $mg\cos\theta$ on its surface, thus according to Newton's Third Law of Motion, the inclined plane exerts an equal but opposite directed force on the box. This reaction force exerted by the incline on the box is perpendicular to the surface of the incline and is referred to as the **normal force** denoted F_N. Thus, by choosing the upward direction to be *positive* and the downward direction to be *negative*, we can write the net force in the y-direction, $\sum F_y$, as

$$\sum F_y = F_N + (-mg\cos\theta) \tag{1}$$

$$\sum F_y = F_N - mg\cos\theta \tag{2}$$

The box on the incline does not accelerate in the y direction i.e. the box does not fly off from the inclined plane. Therefore, the acceleration of the box in the y-direction a_y is zero. We can therefore write Newton's Second Law of Motion for the box in the y-direction as

$$\sum F_y = ma_y = 0 \tag{3}$$

$$F_N - mg\cos\theta = 0 \tag{4}$$

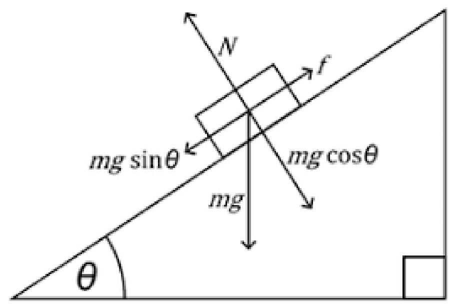

Figure 3: Reaction force from inclined plane on box.

Therefore, we get

$$F_N = mg\cos\theta \tag{5}$$

This equation $F_N = mg\cos\theta$ shows that the inclined plane only generates a normal force F_N that is equal in magnitude to the y-component of the weight of box i.e. $mg\cos\theta$.

In the x-direction, the x-compoment of the weight of the box $mg\sin\theta$ always tends to pull the box downwards along the incline. This force is only countered by the **friction force** denoted F_f between the incline plane and the box. Since the box will naturally tend to slid down, the friction force tends to oppose this downward motion and tries to pull the box upwards. If we choose the downward direction in which the box will naturally tend to slide as the *positive* direction and the upward direction as the *negative* direction, we can get the net force on the box in the x-direction as

$$\sum F_x = mg\sin\theta + (-F_f) \tag{6}$$

$$\sum F_x = mg\sin\theta - F_f \tag{7}$$

We know that the friction force F_f can be expressed in terms of the coefficient of friction μ and the normal force F_N as

$$F_f = \mu F_N \tag{8}$$

Therefore, the net force in the x-direction can also be expressed as

$$\sum F_x = mg\sin\theta - \mu F_N \tag{9}$$

If the box remains motionless and does not slid off the incline, then the acceleration of the box along the x-direction a_x is zero. Thus , we can write Newton's Second Law of Motion in the x-direction as

$$\sum F_x = ma_x \tag{10}$$

$$\sum F_x = m(0\ \mathrm{m/s^2}) = 0 \tag{11}$$

$$mg\sin\theta - \mu_s F_N = 0 \tag{12}$$

where μ_s denotes the coefficient of static friction.

$$mg\sin\theta = \mu_s F_N \tag{13}$$

Recall that $F_N = mg\cos\theta$, thus substituting for F_N, we get

$$mg\sin\theta = \mu_s mg\cos\theta \tag{14}$$

from this expession, we can obtain the μ_s as

$$\mu_s = \frac{mg\sin\theta}{mg\cos\theta} = \frac{\sin\theta}{\cos\theta}$$

$$\mu_s = \tan\theta \tag{15}$$

This angle θ is the maximum angle at which the box on an inclinced plane remains motionless without sliding. It is called the **angle of friction**. The **angle of friction** is equal to the arctangent of the coefficient of static friction between the surface of the inclined plane and the box.

$$\theta = \tan^{-1}(\mu_s) = \arctan(\mu_s) \tag{16}$$

Similarly, if the box begins to slide and accelerates down the incline, then the acceleration of the box along the x-direction a_x is nonzero. Thus , we can write Newton's Second Law of Motion in the x-direction as

$$\sum F_x = ma_x \tag{17}$$

$$mg\sin\theta - \mu_k F_N = ma_x \tag{18}$$

where μ_k denotes the coefficient of kinetic friction.

Again, recall that $F_N = mg\cos\theta$, thus substituting for F_N, we get

$$mg\sin\theta - \mu_k mg\cos\theta = ma_x \tag{19}$$

from this expession, we the acceleration a_x as

$$a_x = \frac{mg\sin\theta - \mu_k mg\cos\theta}{m} \tag{20}$$

$$a_x = \frac{mg\sin\theta}{m} - \frac{\mu_k mg\cos\theta}{m} \tag{21}$$

$$a_x = g\sin\theta - \mu_k g\cos\theta \tag{22}$$

Factoring out the g, we get the acceleration the the x-direction as

$$a_x = g(\sin\theta - \mu_k \cos\theta) \tag{23}$$

if the inclined surface is frictionless, then $\mu_k = 0$, we get the acceleration along the x-direction as

$$a_x = g\sin\theta \tag{24}$$

If the angle of the incline is set at $\theta = 0°$, then we get the acceleration along the x-direction as

$$a_x = g(\sin 0° - \mu_k \cos 0°) \tag{25}$$

$$a_x = -\mu_k g \tag{26}$$

If the angle of the incline is set at $\theta = 90°$, then we get the acceleration along the x-direction as

$$a_x = g(\sin 90° - \mu_k \cos 90°) \tag{27}$$

$$a_x = g \tag{28}$$

The University of Zambia
School of Natural Sciences
PHY 1010
Lecture
Work, Energy and Power

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May 29, 2020

Introduction

This topic introduces the concept of work, energy, the work-energy theorem, and the conservation of energy and power.

Learning Outcomes

The student should be able to:

1. understand the concept of work in terms of the product of a force and displacement in the direction of the force;
2. calculate the work done in a number of situations;
3. recall and apply the formula for kinetic energy $KE = \frac{1}{2}mv^2$
4. derive, from the defining equation $W = Fs$, the formula $PE_{\mathrm{G}} = mgh$ for gravitational potential energy changes near the Earths surface
5. recall and use the formula $PE_{\mathrm{G}} = mgh$ for gravitational potential energy changes near the Earths surface;
6. define power as work done per unit time and derive power as the product of force and velocity.
7. solve problems using the relationships $P = \frac{W}{t}$ and $P = Fv$.

8. recall and apply the principle of conservation of energy to simple examples.

Work

The **Work** done by a force F is defined as the product of the force and the displacement s in the direction of the force. In other word, work is the transfer of energy from one body to another by way of the action of a force applied over a distance. The object on which the force is applied must over if any work is to be done.

Consider a situation where a force F acts on a body and the body undergoes a displacement s. The component of the force F in the direction of displacement s is given by $F\cos\theta$ where θ is the angle between the Force F and the displacement s. Therefore, the work done by the force is $F\cos\theta$ multiplied by the displacement s.

$$W = (F\cos\theta)(s) = Fs\cos\theta$$

Work is a scalar quantity. It only has size and has no direction.

The SI unit for work is the *newton-meter* called the *Joule*. 1 Joule of work is the work done by a force of 1 N when it displaces an object 1 m in the direction of the force.

If F and s are in the same direction, the angle θ between F and s is $\theta = 0°$, we get

$$W = Fs\cos\theta = Fs\cos 0° = Fs$$

If the Force F is in the opposite direction to the displacement s e.g frictional force, then the angle $\theta = 180°$, we get

$$W = Fs\cos\theta = Fs\cos 180° = -Fs$$

This is negative work.

Example 1

A force of 3.0 N acts through a distance of 12 m in the same direction of the force. Find the work done.

The work done W is given by

$$W = Fs\cos\theta$$

where F is applied force, s is the displacement and θ is the angle between F and s.

$$W = (3.0 \text{ N})(12 \text{ m})\cos 0 = 36 \text{ J}$$

Example 2

An object is being pulled along the ground by a 75 N force directed at $28°$ above the horizontal. How much work does the force do in pulling the object 8.0 m.

The work done W is given by

$$W = Fs\cos\theta$$

where F is applied force, s is the displacement and θ is the angle between F and s.

$$W = (75\ \text{N})(8.0\ \text{m})\cos 28° = 530\ \text{J}$$

Example 3

A 0.3 kg object slides 0.80 m along the horizontal table. How much work is done overcoming friction between the object and the table if the coefficient of friction is 0.20.

The displacement $s = 0.80$ m, mass of object $m = 0.3$ kg, coefficient of friction $\mu = 0.20$

Friction force F_f is given by

$$F_f = \mu F_N$$

The normal force F_N is given by

$$F_N = mg = (0.3\ \text{kg})(9.8\ \text{m s}^{-2}) = 2.94\ \text{N}$$

Therefore, we get friction force as

$$F_f = \mu F_N = (0.20)(2.94\ \text{N}) = 0.588\ \text{N}$$

The friction force always acts in a direction that is opposite the motion. The angle between the friction force F_f and the displacement s is $180°$.

The work done W is given by

$$W = F_f s\cos\theta$$

where F_f is friction force, s is the displacement and θ is the angle between F and s.

$$W = (0.588\ \text{N})(0.8\ \text{m})\cos 180° = -0.47\ \text{J}$$

Example 4

> **How much work is done against gravity in lifting a 3.0 kg object through a distance of 0.4 m**
>
> To lift an object, an external force has to be applied on the object being lifted. If we assume that the object is lifted at a constant velocity i.e the object is not accelerating, then the pulling force will be equal to the weight of the object.
>
> $$F_{\text{pull}} = mg = (3\text{ kg})(9.8\text{ m s}^{-2}) = 29.4\text{ N}$$
>
> The pulling force F_{pull} and the displacement $s = 0.4$ m are in the same direction. Therefore, $\theta = 0°$.
>
> The work done W is given by
>
> $$W = F_{\text{pull}} s \cos\theta$$
>
> where F_{pull} is pulling force, s is the displacement and θ is the angle between F and s.
>
> $$W = (29.4\text{ N})(0.4\text{ m})\cos 0° = 11.8\text{ J}$$

Energy

Energy is a measure of the change imparted on a object. Energy is given to an object when a force does work on an object. The amount of energy transferred to the object is equal to the work done. When an object does work, it losses an amount of energy equal to the work done. Energy like work is a scalar and is measured in Joules. An object that is capable of doing work possesss energy.

Kinetic Energy

Kinetic Energy (K.E) is the energy possessed by an object because of its motion. If an object of mass m is moving with a speed v, it has translational energy KE given by

$$KE = \frac{1}{2}mv^2$$

where m is in kg and v in $m\ s^{-1}$, the units of KE are Joules.

Since a body with KE has velocity v and mass m, then the body has linear momentum p.

$$p = mv$$

Squaring both sides of we obtain

$$p^2 = (mv)^2 = m^2v^2$$

Dividing both sides by m, we obtain

$$\frac{p^2}{m} = \frac{m^2v^2}{m} = mv^2$$

Multiplying both sides by $\frac{1}{2}$, we obtain

$$\frac{1}{2}\frac{p^2}{m} = \frac{1}{2}mv^2$$

Thus, we see a relationship between linear momentum p and kinetic energy KE

$$KE = \frac{mv^2}{2} = \frac{p^2}{2m}$$

Gravitational Potential Energy (GPE)

GPE is the energy possessed by an object because of its position in a gravitational field. A gravitational field is a region in which the force of attraction due to gravity is felt. If an object with mass m falls through a vertical distance h, it does an amount of work equal to mgh. The GPE is always given relative to the Earth's surface. If an object is at a height h above the surface of the Earth, then its GPE is given by

$$PE_G = mgh$$

where g is the accleration due to gravity, m is mass in kg and h is height in m. Notice that mg is the weight of the object F_W. The units of GPE are Joules.

Example 5

A 2.0 kg mass falls through a distance of 4.0 m. (a) How much work is done on it by gravitational force (b) How much gravitational potential energy did it lose?

The object has mass $m = 2.0$ kg, thus has a weight F_W given by

$$F_W = mg = (2.0 \text{ kg})(9.8 \text{ m s}^{-2}) = 19.6 \text{ N}$$

The gravitation force is the weight of the object. The work done W is given by

$$W = F_W s \cos\theta$$

where F_W is the gravitaional force, s is the height and θ is the angle between F and s.

$$W = (19.6 \text{ N})(4 \text{ m})\cos 0^\circ = 78.4 \text{ J}$$

The gravitational potential energy GPE lost is given by

$$GPE = mgh = (2.0 \text{ kg})(9.8 \text{ m s}^{-2})(4.0 \text{ m}) = 78.4 \text{ J}$$

The Work-Energy Theorem

The Work-Energy Theorem is also known as the Principle of Work and Kinetic Energy. It states that when work is done on rigid body of mass m and there is no change in GPE, the energy imparted on the rigid body can only appear as KE.

$$\text{Work done} = \text{change in KE}$$

Example 6

A 0.5 kg block slides across a table with an initial velocity of 0.20 m s^{-1} and comes to rest at a distance of 0.70 m. Find the average frictional force that retards its motion

Mass of block, $m = 0.5$ kg, displacement of block $s = 0.70$ m.

The initial velocity of block $u = 0.20$ m s^{-1}, therefore block had initial Kinetic energy given by

$$KE_i = \frac{1}{2}mu^2 = \frac{1}{2}(0.5 \text{ kg})(0.20 \text{ m s}^{-1})^2 = 0.01 \text{ J}$$

When the block comes to rest, its final kinetic energy $v = 0$ m s^{-1}, therefore block has final kenetic energy given by

$$KE_f = \frac{1}{2}mv^2 = \frac{1}{2}(0.5 \text{ kg})(0 \text{ m s}^{-1})^2 = 0 \text{ J}$$

Change in kinetic energy , ΔKE, is given by

$$\Delta KE = KE_f - KE_i = 0 \text{ J} - 0.01 \text{ J} = -0.01 \text{ J}$$

The minus sign shows that the block lost kinetic energy.

We know use the principle of work and kinetic energy also known as the work energy theorem, which states that the work done is equal to the change in kinetic energy. Therefore, we obtain the work dones as

$$W = \Delta KE = -0.01 \text{ J}$$

But friction force F_f is always directed opposite the direction of motion. Therefore, the angle between the friction force and the motion is $\theta = 180°$.

$$W = F_f s \cos\theta = F_f(0.70 \text{ m})\cos 180° = -0.01J$$

$$-F_f(0.70 \text{ m}) = -0.01J$$

Dividing both side by -0.70 m, we obtain

$$F_f = \frac{-0.01 \text{ J}}{-0.7 \text{ m}} = 0.014 \text{ N}$$

The friction force F_f is found to be 0.014 N.

Example 7

The coefficient of friction between a 900 kg car and the pavement is 0.80. If a car is moving at 25 m s^{-1} along the level pavement when it begins to skid to stop, how far will it go before stopping?

The mass of the car $m = 900$ kg, the coefficient of friction $\mu = 0.80$. The initial velocity of the car is $u = 25$ m s^{-1} and since the car eventually comes to a stop, its final velocity is $v = 0$ m s^{-1}. The change in kinetic energy of the car is given by

$$\Delta KE = \frac{1}{2}m(v^2 - u^2) = \frac{1}{2}(900\,\text{kg})((0\,\text{m s}^{-1})^2 - (25\,\text{m s}^{-1})^2) = -281\,250\text{ J}$$

The friction force the car experiences as its skids is given by

$$F_f = \mu F_N = \mu mg = 0.80(900\text{ kg})(9.8\text{ m s}^{-2}) = 7\ 056\text{ N}$$

The change in kinetic energy is equal to work done by friction force in bring the car to a stop. The friction force is always directed opposite the direction of motion. Thus by applying the work - energy theorem, we have

$$W = F_f s\cos\theta = (7\ 056\text{ N})(s)\cos 180^\circ = -281\ 250\text{ J}$$

We obtain the displacement s as

$$s = \frac{-281\ 250\text{ J}}{-7\ 056\text{ N}} \approx 40\text{ m}.$$

The Principle of Conservation of Energy

The Principle of Conservation of Energy states that energy can neither be created or destroyed, but is only transferred from one form to another. In other words, the total energy of an object remains constant and is said to be conserved over time. For an object with KE and GPE, the sum of its KE and GPE is always constant.

$$\text{K.E} + \text{GPE} = \text{constant}$$

Example 8

A 1 kg projectile is shot upwards from earth with a speed of 20 m s^{-1}. How high is it when the speed is 8.0 m s^{-1} .

Mass of projectile, $m = 1$ kg. The initial velocity of projectile $u = 20$ m s^{-1} and the final velocity $v = 8.0$ m s^{-1}. The change in the kinetic energy of the projectile ΔKE is given by

$$\Delta KE = \frac{1}{2}m(v^2 - u^2) = \frac{1}{2}(1\text{ kg})((8.0\text{ m s}^{-1})^2 - (20\text{ m s}^{-1})^2) = -168\text{ J}$$

The negative sign in ΔKE shows that the projectile lost kinetic energy. This lost kinetic energy according to the principle of conservation of energy cannot be destroyed. Instead, it is only changed to a different from of energy. In this case, the lost kinetic energy is changed to gravitation potential energy. Thus, we have

$$\Delta GPE + \Delta KE = 0$$

$$\Delta GPE + (-168 \text{ J}) = 0$$

$$\Delta GPE - 168 \text{ J} = 0$$

$$\Delta GPE = 168 \text{ J}$$

The change in GPE is given by

$$\Delta GPE = mgh = (1 \text{ kg})(9.8 \text{ m s}^{-2})h = 168 \text{ J}$$

we obtain the height h the projectile rises to as

$$h = \frac{168 \text{ J}}{(1 \text{ kg})(9.8 \text{ m s}^{-2})} = 17.1 \text{ m}$$

The projectile rises to a height of 17.1 m.

Power

Power denoted P is defined as the rate of doing work. More, generally, Power is how measure of how fast energy is transferred to an object.

$$\text{Average Power} = \frac{\text{Work done by force}}{\text{time taken to do this work}}$$

$$P = \frac{W}{t} = \frac{Fs}{t} = F(\frac{s}{t}) = Fv$$

$$P = \frac{W}{t} = Fv$$

The velocity v is in the same direction as the applied force F on the object.

The SI unit for Power is the *Watt* (W) and 1 W is equal to 1 J/s or 1 N m/s.

The other useful unit of Power is the **kilowatt-hour**(kW·h). If a force is doing work at a rate of 1 kW (i.e 1000 J/s), then in 1 hour, it will do 1 kW·h of work. The kW·h is a measure of how much work a force does in 1 hour.

$$1 \text{ kW} \cdot \text{h} = 1000 \text{ J s}^{-1} \times 1 \text{ h} = 1000 \text{ J s}^{-1} \times 3\ 600 \text{ s} = 3\ 600000 \text{ J} = 3.6 \text{ MJ}$$

Example 9

How large a force is required to accelerate a 1 300 kg car from rest to a speed of 20 m s^{-1} in distance of 80 m

The car has a mass $m = 1300$ kg and increases speed from rest i.e. $u =$ 0 m s^{-1} to final velocity i.e. $v = 20$ m s^{-1} in a distance $s = 80$ m.

The change in kinetic energy ,ΔKE, of the car is given by

$$\Delta KE = \frac{1}{2}m(v^2 - u^2) = \frac{1}{2}(1300 \text{ kg})((20 \text{ m s}^{-1})^2 - (0 \text{ m s}^{-1})^2) = 260\ 000 \text{ J}$$

Using the work-energy theorem, the work done by is equal to the change in kinetic energy

$$W = Fs\cos\theta = F(80 \text{ m})\cos 0^\circ = 260\ 000 \text{ J}$$

The force provided by the car engine causing the acceleration is given by

$$F = \frac{260\ 000 \text{ J}}{80 \text{ m}} = 3\ 250 \text{ N}$$

Example 10

Calculate the average power required to raise a 150 kg drum to a height of 20 m in a time of 60 s.

The weight of the drum, F_W is given by

$$F_W = mg = (150 \text{ kg})(9.8\ 9.8 \text{ m s}^{-2}) = 1\ 470 \text{ N}$$

If we assume, that there is no acceleration when lifting the drum, i.e. lifting force is equal to the weight of the drum, then our lifting force F becomes

$$F = F_W = 1\ 470 \text{ N}$$

The work done in lifting the drum through a height of 20 m i.e. $s = 20$ m is given by

$$W = Fs\cos\theta = (1\ 470 \text{ N})(20 \text{ m})\cos 0^\circ = 29\ 400 \text{ J}$$

The average power, P, is defined as the amount of work,W, divided by the time to do the said work, t. We obtain the average power P as

$$P = \frac{W}{t} = \frac{29\ 400 \text{ J}}{60 \text{ s}} = 490 \text{ W}$$

The average power required is 490 W.

The University of Zambia
School of Natural Sciences
Department of Physics
PHY 1010
Lecture 9
Density, Pressure and Buoyancy
Supplementary Study Materials

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August 19, 2020

Introduction

This lecture introduces the concepts of density and pressure. It also introduces a special force called buoyant force that fluids always exert on objects. Archinmedes' principle of buoyancy is introduced and we show how it relates to the buoyant force. We use Archimedes's principle to explain why some objects sink or float in a fluid.

Learning Outcomes

By the end of this lesson, the student should be able to:

1. define and use density;
2. define and use pressure;
3. derive the weight of a fluid;
4. derive, from the definitions of density and pressure, the equation $P = \rho g h$;
5. use the equation $P = \rho g h$;
6. understand the origin of the buoyant Force acting on a body in a fluid;

Table 1: The density of some substances at room temperature and pressure

substance	ρ (kg m^{-3})
water	1 000
seawater	1 030
oil	900
petrol	680
glycerine	1 260
ethanol	790
mercury	13 600
iron	7 900
gold	19 320
soldium chloride	2 160
aluminium	2 700
copper	8 920
magnesium	1 740
lead	11 340

7. state Archimede's principle;
8. apply Archimede's principle to solve simple problems involving sinking and floating of objects;

Density

The **density** of a substance denoted ρ is defined as the mass of a substance in a unit volume. A unit volume of any substance is a volume of cubic metre i.e. 1 m^3. In other words, density of a substance refers to the mass of substance that can be contained in one cubic metre.

Mathematically, the density of a substance is defined as a ratio of mass of substance to the volume of substance and is given by

$$\rho = \frac{M}{V}$$

where M is the mass of a substance in kg and V is the volume of a substance in m^3. The SI unit for density is kg/m^3 or kg m^{-3}.

Table 1 shows the densities of some common substances at room temperature and pressure in kg m^{-3}. The density of a substance is an **intrinsic property** i.e. does not depend on the size or shape of a substance. Properties such as mass and volume that depend on size and shape are referred to as **extrinsic properties** of substances.

Table 2: The specific gravity some substances

substance	sp gr
seawater	1.03
oil	0.9
petrol	0.68
glycerine	1.26
ethanol	0.79
mercury	13.6
iron	7.9
gold	19.32
soldium chloride	2.16
aluminium	2.7
copper	8.92
magnesium	1.74
lead	11.34

Often, it is more convenient to express the volume of liquids in litres. The litre is a metric unit of volume and is denoted as L or l. The cubic metre and the litre are related as follows

$$1 \text{ m}^3 = 1\ 000\ l$$

$$1\ l = \frac{1}{1000} \text{ m}^3 = 10^{-3} \text{ m}^3$$

Sometimes, its is necessary to express the density of a substance in terms of a dimensionless ratio called **specific gravity**. The **specific gravity** of substance denoted **sp gr** is the ratio of the density of a substance to the ratio of the density of some standard substance. This standard substance is usually water at $4°C$ for liquids and solids, while for gases it is usually air.

$$\text{sp gr} = \frac{\rho}{\rho_{\text{standard}}}$$

Since **sp gr** is a dimensionless ratio has shown in Table 2, it has the same value for all systems of units.

Density, Mass and Weight of Substances

Sometimes, the masses of substances such as liquids and gases can be expressed more conveniently in terms of the density and volume of the substances under investigation. Mathematically, the mass of a substance M can be given in terms of the density ρ and volume V of the substance as

$$M = \rho V$$

where ρ is the density of the substance and V is the volume of the substance.

Since the substance under investigation has a mass M and "feels" the gravitational pull of the Earth, we can obtain the weight of the substance F_W as

$$F_W = Mg$$

where $g = 9.8$ m/s^2 is the acceleration due to gravity on Earth.

Substituting for $M = \rho V$, we get the weight of the substance F_W as

$$F_W = \rho V g$$

where ρ is density of the substance, V is the volume of the substance and g is the acceleration due to gravity.

Density and Pressure

Consider a beaker that is partially filled with a fluid to a height h. The beaker has a base with a circular cross section area A. We can get the volume of the fluid in the beaker V from the height of the fluid h and the cross section area of the beaker A as

$$V = Ah$$

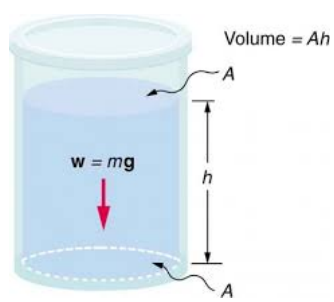

Figure 1: A fluid at rest in a partially filled beaker.

Next, we can get the mass of the fluid in the beaker M in terms of the density of the fluid ρ and volume of the fluid V as

$$M = \rho V$$

Substituting for $V = Ah$, we get the mass of the fluid in the beaker as

$$M = \rho Ah$$

Next, since the fluid in the beaker "feels" the pull of gravity due the Earth i.e. fluids tends to pour downwards if the base of the beaker is opened, we can get the weight of the fluid in the beaker F_W as

$$F_W = Mg$$

Substituting for $M = \rho Ah$, we get the weight of the fluid in the beaker as

$$F_W = \rho Ahg$$

Next, we seek to find the weight "felt" by a unit cross section area of the base of the beaker. This ratio of the weight of the fluid to the cross section area of the beaker or container is referred to as **Pressure** in Physics and is denoted by P. We get the pressure P as

$$P = \frac{F_W}{A} = \frac{\rho Ahg}{A}$$

$$P = \rho gh$$

where ρ is the density of the fluid measured in kg/m^3, $g = 9.8 \text{ m/s}^2$ is the acceleration due to gravity in and h is the height of the fluid column in metres. Since the weight of the fluid F_W is measured in Newtons (N) and the cross sectional area of the beaker is measured in m^2, we get SI units of Pressure as N/m^2. 1 N/m^2 is also known as 1 Pascal (Pa).

$$1 \text{ Pa} = 1 \text{ N m}^2$$

The idea of Pressure is not restricted to fluids only. Any substance that has weight F_W can exert Pressure on a surfaces underneath the substance.

Pressure is a **vector**. This is because to obtain Pressure, we have to divide a force (i.e. a vector quantity) by the cross section area (i.e. scalar quantity). Therefore, pressure has both magnitude and direction. The pressure and the force point in the same direction. In the case where the force is the weight of a substance, the pressure will be directed downwards towards the centre of the Earth.

Therefore, we can more generally redefine pressure as ratio of the applied Force F to the cross section area A. Mathematically, this idea of pressure can be expressed as

$$P = \frac{F}{A}$$

where F is the applied force in Newtons and A is the cross section area in m^2.

Buoyant Force

Fill up an overflow can shown in Figure ?? with water. Next hang a spring balance on a clamp as shown and attach a small solid metal bob to the end of the spring balance. Take note of the weight of the metal bob from the reading on the spring balance. Next, put the overflow can on the stand and slowly submerge the bob into the water. You will notice that as you immerse the bob in the overflow can, the level of water in the jug rises and even overflows into the beaker where it is collected. The water which as been "evicted" from the overflow can by the submerged bob is said to have been "displaced".

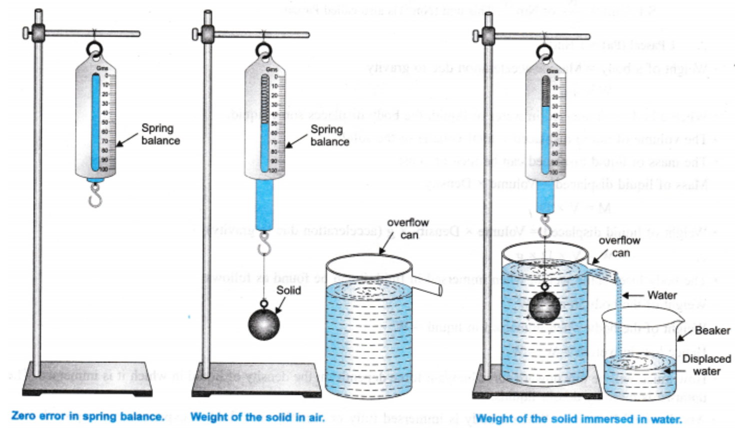

Figure 2: A bob suspended in a fluid.

The volume of the displaced water collected in the beaker is related to the volume of the solid bob submerged in the water. If the solid bob is completely submerged, then the volume of the displaced water is equal to the volume of the solid i.e.

$$V_w = V_{bob}$$

where V_w is the volume of the displaced water collected in the beaker and V_{bob} is the volume of the solid bob submerged in the water.

We denote the density of water as ρ_w. Therefore, we get the mass of the displaced water M_w as

$$M_w = \rho_w V_w$$

Since the displaced water of mass M_w "feels" the pull of gravity due to the Earth, we proceed to get the weight of the displaced water collected in the beaker F_W as

$$F_W = M_w g$$

Substituting for M_w, we get the weight F_W as

$$F_W = \rho_w g V_w$$

In this case, where the solid bob is completely submerged, we also the weight of the displaced water F_W as

$$F_W = \rho_w g V_{bob} \quad \text{since} \quad V_w = V_{bob}$$

This weight of the displaced water $F_W = \rho_w g V_w$ is known as the **buoyant force** denoted F_B. The **buoyant force** comes from the pressure exerted by any fluid on solid objects that is either partially or completely immersed in the fluid. Bacause the pressure increases with depth, the pressure at the bottom of the object is always greater than at the top, hence the net force is an **upward force**. Bouyant force is not always present in fluids. It is only present when an object sinks or floats on a fluid.

To demostrate the existence of this buoyant force, take a small wooden block and forcebly submerge it in jug filled with water.Immediately, after letting go of the wooden block, the wooden block will be pushed out of the water and accelerates towards the surface of the jug. The velocity of the wooden block increases until it reaches the surface of the water. Since the wooden block has mass and accelerates towards the surface of the jug, then there must be force causing this acceleration accroding to Netwon's second law of motion i.e. $F_B = ma$. This force is the buoyant force F_B and it always tries to pushes submerged objects out of any fluid. The greater the volume of the submerged object, the more the bouyant force a fluid will generate to try to push out the object from the fluid.

$$F_B = \rho_f \ g \ V_s$$

where ρ_f is the density of the fluid, g is acceleration due to gravity and V_s is the volume of the submerged solid.

The buoyant force F_B is always an **upward force** and constantly counteracts the weight of any object in a fluid. This is the reason why solid object submerged in a fluid always "feels lighter" in a fluid. A liquid will only generate a buoyant force F_B once an object submerged in it. It is this buoyant force F_B that enables us to swin in water, that keeps wooden logs, boats and ships afloat on a river, lake or at sea.

Archimedes' Principle of Bouyancy

Archimedes' Principle of Bouyancy states that the **buoyant force** F_B that is exerted on a body submerged in a fluid, whether fully or partially submerged, is equal to the weight of the fluid displaced by that body.

$$F_B = \rho_f V_f g$$

where the subscript f denotes the fluid.

If the weight of the fluid displaced by a body is less than the weight of the object, the object will sink. Otherwise, the object will float, with the weight of the displaced fluid being equal to the weight of the submerged object.

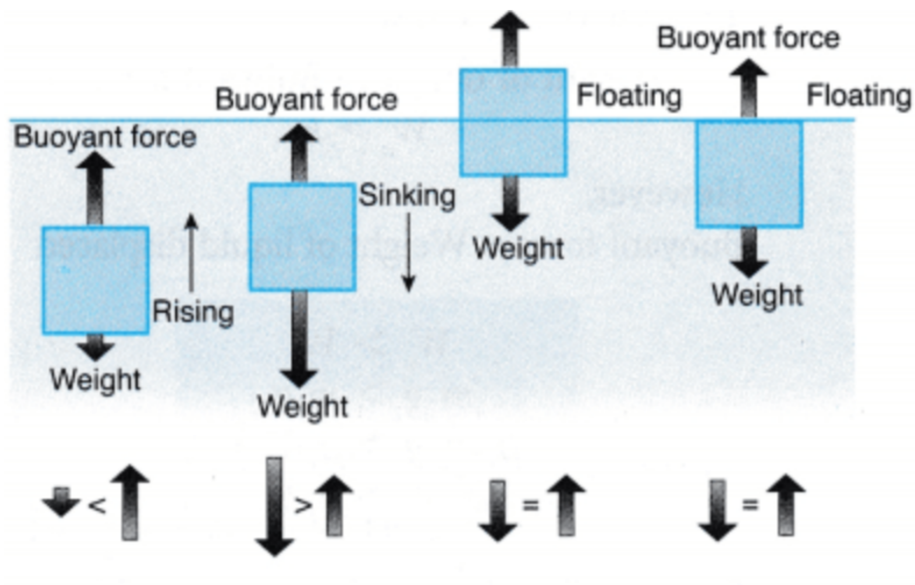

Figure 3: Why some submerged objects float, sink or remain motionless and partially submerged.

Why Some Objects Float?

An object immersed in a fluid will float if the buoyant force F_B is greater than the weight of the object F_W.

$$F_B > F_W$$

From Newton's second law of motion, there is a net upward force that causes the object to accelerate upwards. Taking the upward direction as positive and the downward direction as negative, we get the net force F_{net} as

$$\sum F_{\text{net}} = F_B + (-F_W) = F_B - F_W > 0$$

where F_B is the buoyant force and F_W is the weight of the submerged solid. The net force is positive and is directed upward. Therefore, the submerged solid of mass m_s accelerates upwards in the direction of the net force F_{net}

$$\sum F_{\text{net}} = m_s a_y$$

where m_s is the mass of the submerged object and a_y is the acceleration of the submerged object in the upward direction.

Why Some Objects Sink?

An object submerged in a fluid will sink if the buoyant force F_B is less than the weight of the submerged object F_W.

$$F_B < F_W$$

From Newton's second law of motion, there is a net downward force that causes the object to accelerate downwards. Taking the upward direction as positive and the downward direction as negative, we get the net force F_{net} as

$$\sum F_{\text{net}} = F_B + (-F_W) = F_B - F_W < 0$$

where F_B is the buoyant force and F_W is the weight of the submerged solid. The net force is negative and is directed downwards.

Therefore, the submerged solid of mass m_s accelerates downwards in the direction of the net force.

$$\sum F_{\text{net}} = m_s a_y$$

where m_s is the mass of the submerged object and a_y is the acceleration of the submerged object in the downward direction.

Why Some Objects Partially Sink and Remain Motionless?

An object will be partially immersed and motionless if the buoyant force F_B is equal to the weight of the submerged object F_W.

$$F_B = F_W$$

Taking the upward direction as positive and the downward direction as negative, we get no net force F_{net}.

$$\sum F_{\text{net}} = F_B + (-F_W) = F_B - F_W = 0$$

where F_B is the buoyant force and F_W is the weight of the submerged solid. The net force is negative and is directed downwards. From Newton's second law of motion, since the $F_{\text{net}} = 0$, there submerged object does not experience any acceleration i.e. $a_y = 0$. It therefore, remains motionless and partially submerged in the fluid.

$$\sum F_{\text{net}} = m_s a_y = 0$$

where $m_s \neq 0$ and $a_y = 0$

The University of Zambia
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PHY 1010
Lecture 6
Angular Motion in a Plane

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Introduction to Angular or Circular Motion

Many useful everyday activities that make modern life meaningful require the use and application of circular motion. Our cars once in a while have to pass through a roundabout or bend so that we can change direction. Life on Earth as we know it would not be possible, if Earth was not constantly orbiting the Sun and spining on its own axis at the same time. The Moon orbits Earth once every day. Our artifical satellites which bring television to our living rooms are constantly orbiting the Earth in circular orbits. Our computers have hard drives which spin on their own axis in order to read or store data. In hospitals, circular motion is used in centrifuges and MRI scanning machines which are used for diagnosing diseases. In our kitchens, we use mixers and blenders to prepare food that keeps us health. Without mankind exploiting and harnessing the powers of angular motion, life as we know it would not be possible.

Learning Outcomes

The student should be able to:

1. define and use **angular displacement**, **angular velocity** and **angular acceleration**;
2. solve problems using equations of uniformly accelerated motion in a circular path;

3. describe qualitatively motion in a curved path due to a perpendicular force, and understand the centripetal acceleration in the case of uniform motion in a circle
4. recall and use centripetal acceleration equations $a_c = \frac{v^2}{r}$ and $a_c = r\omega^2$
5. recall and use centripetal force equations $F_c = mr\omega^2$ and $F_c = \frac{mv^2}{r}$

Description of Circular Motion

Angular Displacement

Consider a car going around a circular round about of length s and radius r. The length of the round about s and the radius r are related as follow:

$$s = 2\pi r$$

where s is the linear displacement of the round about if it was stretched out in a straight line and r is the radius of the round about. The SI units of both s and r are meters. On the other hand, 2π is an angle measured in radians and in circular motion, is referred to as **angular displacement**. The angular displacement is denoted θ.

Thus, we can rewrite the above relation between s and r as

$$s = \theta r$$

Thus we get angular displacement θ in terms of linear displacement s and radius of a circle r as

$$\theta = \frac{s}{r}$$

The SI units for θ are radians (rad in short).

The angular displacement θ can also be expressed in terms of degrees(°) and revolutions (rev in short).

$$1 \text{ rev} = 360^\circ = 2\pi \text{ rad.}$$

Example 1

The bob of a pendulum 90 cm long swings through a 15 cm arc, as shown in Fig. 1. Find the angle θ, in radians and in degrees, through which it swings. We can get the angular displacement, θ, as

$$\theta = \frac{s}{r} = \frac{15 \text{ cm}}{90 \text{ cm}} = 0.17 \text{ rad}$$

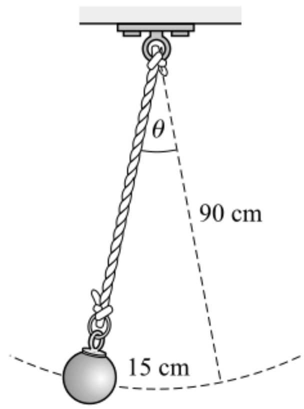

Figure 1: A bob swinging from a simple pendulum

Angular velocity

The angular velocity denoted ω refers to how fast an object is moving round a circular track like a roundabout or spining about its own axis. Angular velocity ω is the rate of change angular displacement θ. Mathematically, the angular velocity is given by

$$\omega = \frac{\theta}{t}$$

where θ is angular displacement and t is the time. The SI units for angular velocity are rad/s. If an object covers an agular displacement of $\theta = 2\pi$ in time T, then T is referred to as the period. This measures how fast the object goes round a circular path or spins. In Physics, this is reffered to as frequency and is denoted f. The frequency f is given by

$$f = \frac{1}{T}$$

and were T is the time required to complete one revolution. The period T has SI units of seconds but frequency f is measured in Hertz (Hz). 1 Hz $= 1$ s^{-1}.

Therefore, if an objects covers an angular displacement of $\theta = 2\pi$ in time T, we can express our angular velocity ω as

$$\omega = \frac{\theta}{t} = \frac{2\pi}{T} = 2\pi\frac{1}{T}$$

$$\omega = 2\pi f$$

If θ of an object spining on its axis changes from initial angular displacement θ_i to a final angular displacement θ_f in a time t, then the average angular velocity

ω_{av} is given by

$$\omega_{av} = \frac{\theta_f - \theta_i}{t}$$

Example 2

A fywheel turns at 480 revolutiopns per minute (rpm). Compute the angular speed at any point on the wheel and the tangential speed 30.0 cm from the center.

The angular speed ω in radians per second is

$$\omega = 480 \text{ rpm} = \frac{480 \text{ rev}}{1 \text{ min}} = \frac{480 \text{ rev} \times 2\pi \text{ rad/rev}}{60 \text{ s}} = 50.3 \text{ rad/s}$$

The radius of the flywheel is $r = 30.0$ m $= 0.30$ m.

We can get the tangential velocity v as

$$v = r\omega = (0.30 \text{ m})50.3 \text{ rad/s} = 15.09 \text{ m/s}$$

$$v \approx 15 \text{ m/s}$$

Example 3

A fan turns at a rate of 900 rpm (i.e., rev/min).

(a) **Find the angular speed of any point on one of the fan blades.**
The angular speed or velocity ω in rad/s is

$$\omega = 900 \text{ rpm} = \frac{900 \text{ rev}}{1 \text{ min}} = \frac{900 \text{ rev} \times 2\pi \text{ rad/rev}}{60 \text{ s}} = 94.2 \text{ rad/s}$$

(b) **Find the tangential speed of the tip of a blade if the distance from the center to the tip is 20.0 cm.**
The tangential velocity or linear velocity v is related to ω by

$$v = r\omega$$

where r is distance from center of rotation in meters.
Therefore, we get v as

$$v = (0.20 \text{ m})(94.2 \text{ rad/s}) = 18.8 \text{ m/s}.$$

Angular Acceleration

The angular acceleration (α) of an object whose axis of rotation is fixed is the rate at which its angular speed changes with time. If the angular speed changes uniformly from ω_i to ω_f in a time t, then we get angular acceleration α as

$$\alpha = \frac{\omega_f - \omega_i}{t}$$

The SI units of α is rad/s^2.

Example 4

> **A wheel of 40 cm radius rotates on a stationary axle. It is uniformly speeded up from rest to a speed of 900 rpm in a time of 20 s. Find**

(a) **the constant angular acceleration α of the wheel**

The radius of the wheel $r = 40$ cm $= 0.40$ m. The rest angular velocity is $\omega_i = 0$ rad/s. The final angular velocity ω_f in rad/s is

$$\omega_f = 900 \text{ rpm} = \frac{900 \text{ rev}}{1 \text{ min}} = \frac{900 \text{ rev} \times 2\pi \text{ rad/rev}}{60 \text{ s}} = 94.2 \text{ rad/s}$$

For a time $t = 20$ s, we get an angular acceleration of

$$\alpha = \frac{\omega_f - \omega_i}{t} = \frac{94.2 \text{ rad/s} - 0 \text{ rad/s}}{20 \text{ s}}$$

$$\alpha = 4.71 \text{ rad/s}^2$$

(b) **the tangential acceleration a of a point on its rim**

$$a = r\alpha = (0.40 \text{ m})(4.71 \text{ rad/s}^2) \approx 1.9 \text{ m/s}^2$$

.

Deriving Equations for Uniformly Accelerated Angular Motion

There is a connection between linear physical quantities and angular counterparts. This connection extends even to the equation of motions (EOM) used to describe circular motion. The EOMs for circular motion can easiy be derived from the EOM for linear motion by using the following substitions

relation between s and θ

$$s = r\theta$$

relationship between v and ω

$$v = r\omega$$

relationship between a and α

$$a = r\alpha$$

Starting with the formula for linear average velocity v_{av} given below

$$v_{av} = \frac{1}{2}(v_f + v_i)$$

Substituting for $v = r\omega$, we get

$$r\omega_{av} = \frac{1}{2}(r\omega_f + r\omega_i)$$

Cancelling out r on both side, we obtain

$$\omega_{av} = \frac{1}{2}(\omega_i + \omega_f)$$

In linear motion, we known that the linear displacement s is related to the linear average velocity v_{av} and time t as follows

$$s = v_{av}t$$

We can obtain an expression that shows how angular displacement θ relates to average angular velocity ω_{av} and time t, by substiting $s = r\theta$ and $v_{av} = r\omega_{av}$. We obtain

$$r\theta = r\omega_{av}t$$

Cancelling out the r on both side, we get

$$\theta = \omega_{av}t$$

Similarly, we can also obtain angular counterparts of the rest of the equations of motion by making suitable substitutings.

For example, in the case of the following equation below, we substitute $v = r\omega$ and $a = r\alpha$

$$v_f = v_i + at$$

$$r\omega_f = r\omega_i + r\alpha t$$

Cancelling out the r on both side, we get

$$\omega_f = \omega_i + \alpha t$$

Likewise, for the following linear equation below, we substitute $s = r\theta$, $v_i = r\omega_i$ and $a = r\alpha$. We get

$$s = v_i t + \frac{1}{2}at^2$$

$$r\theta = r\omega_i t + \frac{1}{2}r\alpha t^2$$

Table 1: Relations between Linear and Angular EOM

Linear	Angular
$v_{av} = \frac{1}{2}(v_i + v_f)$	$\omega_{av} = \frac{1}{2}(\omega_i + \omega_f)$
$s = v_{av}t$	$\theta = \omega_{av}t$
$v_f = v_i + at$	$\omega_f = \omega_i + \alpha t$
$s = v_it + \frac{1}{2}at^2$	$\theta = \omega_it + \frac{1}{2}\alpha t^2$
$v_f^2 = v_i^2 + 2as$	$\omega_f^2 = \omega_i^2 + 2\alpha\theta$

Cancelling out the r on both side, we get

$$\theta = \omega_it + \frac{1}{2}\alpha t^2$$

And finally, for the linear equation below, we substitute $v = r\omega$, $a = r\alpha$ and $s = r\theta$. We get

$$v_f^2 = v_i^2 + 2as$$

$$(r\omega_f)^2 = (r\omega_i)^2 + 2(r\alpha)(r\theta)$$

Expanding and simplifying, we get

$$r^2\omega_f^2 = r^2\omega_i^2 + 2r^2\alpha\theta$$

Cancelling out the r^2 on both side, we get

$$\omega_f^2 = \omega_i^2 + 2\alpha\theta$$

We have seen that each equation of linear motion, has an structurally identical counterpart in circular motion as again show in Table

Example 5

A belt passes over a wheel of radius 25 cm, as shown in Fig. 2. If a point on the belt has a speed of 5.0 m/s, how fast is the wheel turning?

The angular velocity ω is a measure of how fast an object is turning. It is related to the tangential velocity v and radius of the wheel r by

$$v = r\omega$$

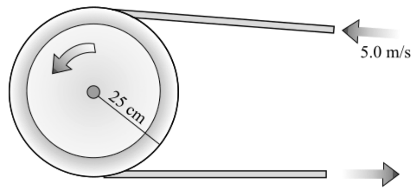

Figure 2: A fly wheel turned by belt

Therefore, we can get ω in turns of v and r as

$$\omega = \frac{v}{r}$$

$$\omega = \frac{5.0 \text{ m/s}}{0.25 \text{ m}} = 20 \text{ rad/s}$$

Example 6

A pulley of 5.0 cm radius, on a motor, is turning at 30 rev/s and slows down uniformly to 20 rev/s in 2.0 s. Calculate

(a) **the angular acceleration of the motor,**
initial angular velocity $\omega_i = 30$ rev/s $= 188.5$ rad/s
final angular velocity $\omega_f = 20$ rev/s $= 125.7$ rad/s
time taken $t = 2.0$ s
therefore, we get the angular acceleration α as

$$\alpha = \frac{\omega_f - \omega i}{t}$$

$$\alpha = \frac{125.7 \text{ rad/s} - 188.5 \text{ rad/s}}{2.0 \text{ s}}$$

$$\alpha = -3.14 \text{ rad/s}^2$$

(b) **the number of revolutions it makes in this time, and**
to get the number of revolutions, we first need to find the angular displacement θ and then divide it by 2π rad since each revolution is equal to an angular displacement of 2π rad. We get θ as

$$\theta = \omega_i t + \frac{1}{2}\alpha t^2$$

$$\theta = (188.5 \text{ rad/s})(2.0 \text{ s}) + \frac{1}{2}(-3.14 \text{ rad/s}^2)(2.0 \text{ s})^2$$

$$\theta = 370.7 \text{ rad}$$

$$\text{Number of revolutions} = \frac{\theta}{2\pi \text{ rad/rev}} = \frac{370.7 \text{ rad}}{2\pi \text{ rad/rev}} = 59 \text{ rev}$$

(c) **the length of belt it winds in this time.**

The linear displacement s is related to θ by

$$s = r\theta$$

where r is the radius of the pulley in meters.

$$s = (0.05 \text{ m})(370.7 \text{ rad/s}) \approx 18.5 \text{ m}$$

Example 7

A car has wheels of radius 30 cm. It starts from rest and accelerates uniformly to a speed of 15 m/s in a time of 8.0 s. Find the angular acceleration of its wheels and the number of rotations one wheel makes in this time.

Initial linear velocity $u = 0$ m/s

Final linear velocity $v = 15$ m/s

Time taken $t = 8.0$ s

We get the linear acceleration a as

$$a = \frac{v - u}{t}$$

$$a = \frac{15 \text{ m/s} - 0 \text{ m/s}}{8.0 \text{ s}} = 1.9 \text{ m/s}^2$$

The tangential acceleration a is related to angular acceleration α and radius of wheel r is given by

$$a = r\alpha$$

We get the angular acceleration a as

$$\alpha = \frac{a}{r}$$

$$\alpha = \frac{1.9 \text{ m/s}^2}{0.30 \text{ m}} \approx 6.3 \text{ rad/s}^2$$

The linear distance the wheel travels in this time is given by

$$s = \frac{1}{2}(u + v)t$$

Therefore, we get s as

$$s = \frac{1}{2}(0 \text{ m/s} + 15 \text{ m/s})(8.0 \text{ s}) = 60 \text{ m}$$

The angular displacement θ covered by a car wheel of radius $r = 0.30$ m from $s = 60$ m is

$$\theta = \frac{s}{r}$$

$$\theta = \frac{60 \text{ m}}{0.30 \text{ m}} = 200 \text{ rad}$$

The number of revolutions is given by

$$\text{Number of revolutions} = \frac{\theta}{2\pi \text{ rad/rev}}$$

$$\text{Number of revolutions} = \frac{200 \text{ rad}}{2\pi \text{ rad/rev}} = 31.8 \text{ rev}$$

Centripetal Acceleration

A point mass m moving with constant speed v around a circle of radius r is undergoing acceleration. Although the magnitude of its linear velocity is not changing, the direction of the velocity is continually changing. This change in velocity gives rise to an acceleration a_c of the mass, directed toward the center of the circle. We call this acceleration the **centripetal acceleration** and its magnitude is given by

$$a_c = \frac{v^2}{r}$$

where v is the speed of the mass around the perimeter of the circle.

Recall that $v = r\omega$, thus we can also rewrite the **centripetal acceleration** as

$$a_c = \frac{v^2}{r} = \frac{(r\omega)^2}{r} = \frac{r^2\omega^2}{r} = r\omega^2$$

$$a_c = \frac{v^2}{r} = r\omega^2$$

Centripetal Force

Objects do not like moving in a circular path. Given a choice, any object prefers to move in a straight line. For an object of mass m to keep moving in a circular path of radius r with velocity v, it must be forced. The force required to keep an object in a circular path is referred to as **centripetal force** and is given by

$$F_c = ma_c = m\frac{v^2}{r} = mr\omega^2$$

The **centripetal force** has many "faces". For a car going round a roundabout, the *friction force* provides the centripetal force. In this case, the centripetal force is the friction force.

$$F_c = F_f$$

In the instance of an mass attached to the end of a rope and being pulled round in a circular path by a child, the centripetal force is provided by the tension generated in the cord.

$$F_c = F_T$$

For planets orbiting the Sun or satellites orbiting our planet Earth, the centripetal force is provided by the pull of gravity.

$$F_c = F_G$$

Example 8

A 200 g object is tied to the end of a cord and whirled in a horizontal circle of radius 1.20 m at a constant 3.0 rev/s. Assume that the cord is horizontal, i.e., that gravity can be neglected. Determine

(a) **the acceleration of the object**

The mass of object $m = 200.0$ g $= 0.2$ m.

The radius of circle $r = 1.20$ m.

The angular velocity ω in rad/s is

$$\omega = 3.0 \text{ rev/s} = \frac{3.0 \text{ rev} \times 2\pi \text{ rad/rev}}{1 \text{ s}} = 18.8 \text{ rad/s}$$

The acceleration a_c is given by

$$a_c = r\omega^2 = (1.20 \text{ m})(18.8 \text{ rad/s})^2 \approx 424 \text{ m/s}^2.$$

(b) **the tension in the cord.**

The tension F_T in the cord provides the centripetal force. Thus the tension is equal to the centripetal force

$$F_T = F_c = ma_c = (0.20 \text{ kg})(424 \text{ m/s}^2) \approx 85 \text{ N}$$

Example 9

What is the maximum speed at which a car can round a curve of 25 m radius on a level road if the coefficient of static friction between the tires and road is 0.80?

A car going round a curve on a level road is kept in its circular motion by a centripetal force F_c. In this case, this centripetal force is provided by friction force F_f between between the car tires and the road.

The centripetal force F_c is given by

$$F_c = m\frac{v^2}{r}$$

where m is the mass of the object undergoing circular motion, v is the tangential velocity in m/s and r is the radius of the circular road in meters.

This centripetal force F_c is provided by friction force F_c which is given by

$$F_f = \mu_s F_N$$

where μ is the coefficient of static friction μ_s and F_N is the normal force. But F_N is also equal to mg, thus we get

$$F_f = \mu_s mg$$

Therefore, we get

$$F_c = F_f$$

$$m\frac{v^2}{r} = \mu_s mg$$

Cancelling out the m on both side, we get

$$\frac{v^2}{r} = \mu_s g$$

Multiplying both side by r, we get

$$v^2 = \mu_s gr$$

Taking the square root on both side, we get

$$v = \sqrt{\mu_s gr}$$

$$v = \sqrt{(0.8)(9.8\ \text{m/s}^2)(25\ \text{m})}$$

$$v \approx 14\ \text{m/s}$$

The maximum speed is 14 m/s.

Example 10

A spaceship orbits the Moon at a height of 20 000 m. Assuming it to be subject only to the gravitational pull of the Moon, find its speed and the time it takes for one orbit. For the Moon, $m_m = 7.34 \times 10^{22}$ kg **and** $r = 1.738 \times 10^6$ m**.**

A spacecraft orbit is circular and has a radius equal to the radius of the moon plus the height of the spacecraft above the moon. Thus we get the radius of the spacecraft orbit R as

$$R = r + h$$

where r is the radius of the moon and h is the height of the spacecraft above the moon.

$$R = r + s = 1.738 \times 10^6 \text{ m} + 20\ 000 \text{ m}$$

$$R = 1.758 \times 10^6 \text{ m}$$

The spacecraft is kept in its circular orbit by a centripetal force F_c that keeps pulling the spacecraft towards the center of the orbit. This centripetal force F_c is

$$F_c = m_s \frac{v^2}{R}$$

where m_s is the mass of the spacecraft, v is the tangential velocity of the spacecraft and R is radius of the spacecraft circular orbit.

The centripetal force F_c is provided by the gravitational pull of the moon on the spacecraft F_G. This F_G is given by

$$F_G = G \frac{m_s m_m}{R^2}$$

where $G = 6.674 \times 10^{-11}$ m$^3 \cdot$ kg$^{-1} \cdot$ s^{-2} is the gravitational constant, m_s is the mass of the spacecraft, m_m is the mass of the moon, and R is the radius of the spacecraft circular orbit.

Thus we have

$$F_c = F_G$$

$$m_s \frac{v^2}{R} = G \frac{m_s m_m}{R^2}$$

Cancelling out the m_s and R on both side , we get

$$v^2 = G \frac{m_m}{R}$$

Taking the square root on both side, we end up with

$$v = \sqrt{G\frac{m_m}{R}}$$

$$v = \sqrt{\frac{(6.674 \times 10^{-11}\ \mathrm{m}^3 \cdot \mathrm{kg}^{-1} \cdot \mathrm{s}^{-2})(7.34 \times 10^{22}\ \mathrm{kg})}{1.758 \times 10^{6}\ \mathrm{m}}}$$

$$v = 1\ 669\ \mathrm{m/s}$$

The spacecraft has a tangential speed of 1 669 m/s.

The average angular velocity of the spacecraft ω is

$$\omega = \frac{v}{R}$$

we get ω as

$$\omega = \frac{1\ 669\ \mathrm{m/s}}{1.758 \times 10^{6}\ \mathrm{m}} = 9.5 \times 10^{-4}\ \mathrm{rad/s}$$

To obtain the time to complete one revolution also known as period T, we divide the angular displacement for one revolution $\theta = 2\pi$ rad by the angular velocity ω of the spacecraft. We get the period T as

$$T = \frac{2\pi\ \mathrm{rad}}{\omega}$$

$$T = \frac{2\pi\ \mathrm{rad}}{9.5 \times 10^{-4}\ \mathrm{rad/s}} = 6618\ \mathrm{s}$$

The period T in minutes is given by

$$T = \frac{6618\ \mathrm{s}}{60\ \mathrm{s/min}} \approx 110.3\ \mathrm{min}$$

$$T \approx 110\ \mathrm{min}\ \ 18\ \mathrm{s}$$

The time it takes the spacecraft to complete one orbit is 110 minutes 18 seconds.

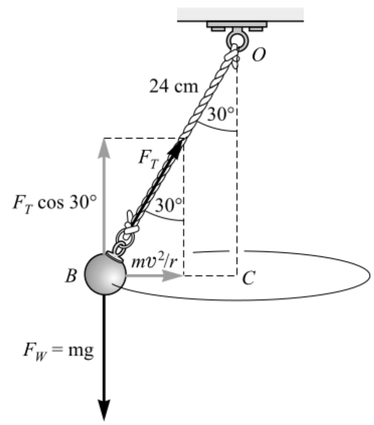

Figure 3: A bob attached to a roof through a rope

Example 11

As shown in Fig. 3, a ball B is fastened to one end of a 24 cm string, and the other end is held fixed at point Q. The ball whirls in the horizontal circle shown. Find the speed of the ball in its circular path if the string makes an angle of $30°$ to the vertical.

The ball is kept moving in a circle of radius $r = (0.24\text{ m})\sin 30° = 0.12$ m by a centripetal force F_c provided by the x-component of the tension F_T.

The centripetal force F_c is given by

$$F_c = m\frac{v^2}{r}$$

where m is the mass of the ball, v is the tangential velocity of the ball and r is the radius of the ball's circular path.

As earlier stated, the centripetal force F_c is provided by the x-componebt of the tension F_T thus, we get

$$F_c = F_T \sin 30°$$

As the ball is whirled round in a circe, it does not go up or down. Therefore, the ball experiences no acceleration in the y-direction. The ball has weight $F_W = mg$ which should pull the ball down. However, the y- component of the tension F_T i.e. $F_T \cos 30°$ pull the ball upward. Applying the first conditin of equilibrium, we obtain

$$\sum F_y = 0$$

Choosing the upward direction to be positive and the downward direction to be negative, we obtain

$$F_T \cos 30^\circ - F_W = 0$$

$$F_T \cos 30^\circ = F_W = mg$$

Therefore, we obtain the tension F_T as

$$F_T = \frac{mg}{\cos 30^\circ}$$

Substituting this expression of F_T into the centripetal force provided by the x-component of the tension F_T, we obtain

$$F_c = F_T \sin 30^\circ = (\frac{mg}{\cos 30^\circ})(\sin 30^\circ) = mg \tan 30^\circ$$

Therefore, we now get

$$mg \tan 30^\circ = m\frac{v^2}{r}$$

Cancelling out m on both sides, we obtain

$$g \tan 30^\circ = \frac{v^2}{r}$$

Multiplying both sides by r, we get

$$v^2 = gr \tan 30^\circ$$

Taking the square root on both sides, we obtain

$$v = \sqrt{gr \tan 30^\circ}$$

$$v = \sqrt{(9.8 \text{ m/s}^2)(0.12 \text{ m})(\tan 30^\circ)}$$

$$v = 0.82 \text{ m/s}$$

The speed of the ball is 0.82 m/s.

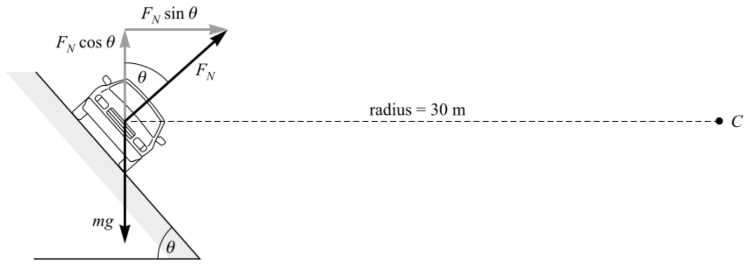

Figure 4: A car going round on a road banking

Example 12

A curve of radius 30 m is to be banked so that a car may make the turn at a speed of 13 m/s without depending on friction. What must be the slope of the curve (the banking angle)?

For a car to move along a banked curve of radius $r = 30$ m, there must be a centripetal force F_c to keep the car along its circular path. This centripetal force F_c is given by

$$F_c = m\frac{v^2}{r}$$

The centripetal force is provided by $F_N \sin\theta$. Thus we have

$$F_c = F_N \sin\theta$$

However, a closer look at the provided figure shows that

$$F_N \cos\theta = mg$$

Dividing the equation for F_c by $F_N \cos\theta$, we obtain

$$\frac{F_N \sin\theta}{F_N \cos\theta} = \frac{F_c}{mg}$$

$$\tan\theta = \frac{F_c}{mg}$$

$$\tan\theta = \frac{mv^2}{r} \times \frac{1}{mg}$$

$$\tan\theta = \frac{v^2}{gr}$$

$$\tan\theta = \frac{(13 \text{ m/s})^2}{(9.8 \text{ m/s}^2)(30 \text{ m})}$$

$$\tan\theta = 0.57$$

$$\theta = \arctan(0.57) \approx 30°$$

The banking angle is 30°.

Example 13

As shown in Fig. 5, a cylindrical shell of inner radius r rotates at angular speed ω. A wooden block rests against the inner surface and rotates with it. If the coefficient of static friction between block and surface is μ_s , how fast must the shell be rotating if the block is not to slip and fall? Assume $r = 150$ cm and $\mu_s = 0.30$.

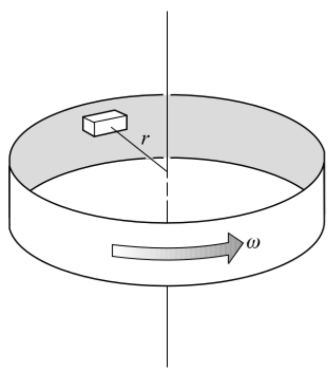

Figure 5: A bob stuck rotating ring

The wooden block remains at rest as long as the the ring is rotating at a particular value of ω. The wooden block is being kept in position by friction force F_f between the wooden block and the ring. If the ring stops rotating, the friction force F_f disappears and the wooden block falls under the weight of the wooden block.

The centripetal force F_c keeping the wooden block in position is given by

$$F_c = m\frac{v^2}{r} = mr\omega^2$$

The centripetal force F_c is provided by friction force

$$F_c = F_f = \mu_s F_N$$

But the the normal force $F_N = mg$, thus we get

$$F_c = \mu_s mg$$

Therefore, we get

$$\mu_s mg = mr\omega^2$$

Cancelling out m on both sides, we obtain

$$\mu_s g = r\omega^2$$

Dividing both sides by r, we get

$$\omega^2 = \frac{\mu_s g}{r}$$

Taking the square root on both sides, we get

$$\omega = \sqrt{\frac{\mu_s g}{r}}$$

We get

$$\omega = \sqrt{\frac{(0.30)(9.8 \text{ m/s}^2)}{1.5 \text{ m}}} = 1.4 \text{ rad/s}$$

The ring should be rotating at an angular velocity of 1.4 rad/s.

The University of Zambia
School of Natural Sciences
Department of Physics
PHY 1010
Lecture 9B
Introduction to Mechanical Properties of Matter
Supplementary Study Materials

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1 Introduction

This lesson will introduce idea of deformation of solids and the types of deformations a solid can undergo under the action of an applied force. Under this lesson, you will also be introduced to the concept of stress and strain together with the types of stresses and strains a solid can experience.

2 Learning Outcomes

By the end of this lesson, the student should be able to:

1. Define deformation of solids;
2. Distinguish between elastic and plastic deformation of solids;
3. Define and state the four types of stresses and strains that exit in elastic solids;

3 Definition of Deformation of Solids

Have you ever stretched a spring or bent a piece of wire? If you done these or similar activities, then you definitely changed the original size of these solids. For instance, if you stretched a spring without damaging it, you increased its length during this time you applied the stretching force. Once you removed the stretching force you had applied on the spring, it returned to its original shape and size. On the other hand, if you stretched the spring with a large force such that it got damaged, then upon removal of the stretching force the spring did not return exactly to its original shape and size. For a piece of wire, your experience of bending it was different. Once you had bent the wire and given it some form, the wire permanently keep this new form or shape and did not try to get back to its original form. These activities highlighted in this paragraph are excellent examples of deformation of solids.

Deformation of a solid refers to change in length, volume or shape of that solid as a result of an applied force. The force you apply on a solid causing it to deform is referred to as the deforming force. The amount of deformation you can cause on a solid depends on a number of factors. First and foremost, the amount of deformation depends on the magnitude (or size) of the force you are applying on the solid. If you apply a large stretching force on a spring, the spring will undergo a large increase in length and vice versa. The amount of deformation will also depends on how the force is being applied on the solid. A stretching force applied on a spring will increase its length while compressing force will decrease its length. Furthermore, imagine trying to stretch spring and a piece of rubber. How much deformation do you think you will cause for the same size of stretching force? Of course, your stretch will be more for the piece of rubber compared to the spring. This shows that the amount of deformation also hugely depends on the kind of substance the solid is made of e.g. rubber or steel.

There are two main types of deformations in solids. As you are aware from the preceding paragraphs, there is deformation where a solid is deformed and once the deforming force is removed, the solid returns to its original shape or size. This kind of deformation is referred to as elastic deformation. Elastic deformation refers to temporary (non-permanent) deformation of a solid that is recoverable (or self-reversing) once the deforming force is removed. An example of elastic deformation is stretching a spring without damaging it such that the spring returns to its original size and shape upon removal of the stretching force. If you have a solid that completely recovers its original shape and size once you remove the deforming force, then that solid is said to be perfectly elastic. No real solids are perfectly elastic.

There is another kind of deformation called plastic deformation. This is the kind of deformation that occurs when you bend a wire. As you are aware, the wire you have bent will permanently keep its newly acquired form or shape. Plastic deformation refers to a permanent deformation of a solid without fracture under the action of a sustained deforming force. If the solid you have deformed completely keeps its altered shape and size then that solid is said to be perfectly plastic. You also need to be aware that no real solid is perfectly plastic. As we delve deeper, you always need to keep in mind that perfect elastic or plastic deformation does not occur in real solids. Real solids will always undergo a mixture of both elastic and plastic deformation.

As you are now aware, for deformation of a solid to occur, a deforming force has to be present and act on the solid in some way. Thereafter, depending on the nature of the solid you are considering, the solid will experience a change in length, volume, shape or all. For your deforming force to cause an increase in length or volume of the solid, it has to increase the relative distance between atoms of the affected solid without causing the solid to fracture. For a change of shape to occur, the deforming force you have applied has to displace some atoms more relative to others. You also need to be aware that, before you applied a deforming force on the solid, the solid is at equilibrium and has a specific distance between the atoms (inter-atomic distance). The deforming force you apply on the solid, increases this inter-atomic distance yet your solid always prefers to have the equilibrium interatomic distance. To achieve its desire of having an equilibrium interatomic distance, your solid will setup a force internally (inside of itself) that will pull back the atoms making up the solid in a direction opposite to the direction of your deforming force. This force which is created internally in the solid to oppose the deforming force is referred to as the restoring (or recovery) force.

According to Newton's third law of motion : For every acting force, there is an equal but opposite reaction force, the restoring force (reaction force) is equal in magnitude to the deforming force (acting force) but acts in direction opposite. Due to this restoring force created internally in a solid as a result of your applied deforming force, solids will tend to regain their original length, volume or shape. This tendency of solids you have deformed to regain their original length, volume or shape after the removal of the deforming force is called elasticity.

4 Stress and Strain

4.1 Stress

You are now aware that a solid subjected to a deforming force experiences a force within its interior structure that tends to balance the deforming force and tries to restore the solid to its original condition. This restoring force created inside the solid will be distributed across various interior surfaces of the solid.

Consider the two rods shown in Figure 1 below. These two rods have the same original length $l_o = 1$ m, are made of the same substance but have different radii 5 cm and 10 cm respectively. From the formula of cross section area A given by $A = \pi r^2$, where r is the radius of the rod, you can calculate the cross section areas of the rods to be 0.0079 m^2 and 0.0314 m^2 respectively. If you subject each rod to a deforming force of 450 N , an overall restoring force of 450 N is setup internally from our use of Newtons third law of motion. This, however, does not give you a clear picture of how the rods are "feeling" internally as a result of this restoring force of 450N .

For you to have a clear picture of how these rods are "feeling" due to this restoring force of 450 N , you need to calculate the restoring force per unit cross section area of the rods . For the rod with radius of 5 cm, this restoring force

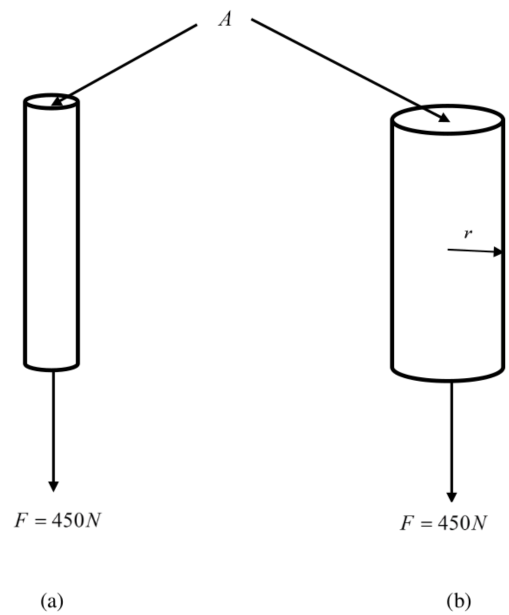

Figure 1: Schematic illustrations of two rods with different radii (a) 5 cm (b) 10 cm subjected to deforming force of 450 N.

per unit cross section area is 56 962 N/m^2 (i.e. 450 N divided by 0.0079 m^2). For the rod with radius of 10 cm, the same restoring force per unit cross section area is 1 433 N/m^2 (i.e. 450 N divided by 0.0314 m^2). You can clearly see that the rod of radius 5 cm experiences or "feels" a much greater "restoring force per unit cross section area" compared to the rod with 10 cm. This internal "restoring force per unit cross section area" is termed as **stress**.

4.1.1 Definition of Stress

Stress refers to the force F per unit cross sectional area A of a solid due to the presence of a deforming force. The symbol for stress is the Greek letter σ (lower-case sigma).

$$\sigma = \frac{F}{A} \tag{1}$$

where F is the deforming force in Netwons (N) and A is the cross section area in m^2. The SI units for stress are N/m^2 or Pascals (Pa).

$$1 \text{ Pa} = 1 \text{ N/m}^2$$

4.2 Strain

From our discussion of stress, you are aware that this deforming force of 450 N will subject these two rods to different values of deforming stresses, 56 962 N/m^2 for the rod with radius 5 cm and 1 433 N/m^2 for the rod with radius 10 cm. This causes the rods to undergo differing amounts of deformations with the rod subjected to a greater deforming stress expected to undergo a larger increase in actual length. If the rod with radius of 5 cm undergoes an increase in length of let's say 15 cm and the rod with radius of 10 cm undergoes an increase in length of say 2 cm, it will quickly become difficult for you to determine which rod has experienced the greater deformation.

You can see that the rod with the radius of 5 cm experiences a larger increase in length because its subjected to a larger deforming stress. By the same token, you can also see that the rod with radius of 10 cm undergoes a smaller increase in length because its subjected to a lesser deforming stress. You can clearly see that the actual increase in length or actual amount of deformation is not a very suitable measure of deformation of solids. But if you compare the amount of deformation (e.g. increase in length) to the original dimensions (e.g. original length) of the solid, then you obtain a ratio that is independent of the shape or form of the solid. This ratio of amount of deformation to the original dimensions of the solid is referred to as **strain**.

4.2.1 Definition of Strain

Strain refers to the ratio of the amount of deformation (increase in length, change in volume or shape) of a solid to its original dimensions (length, volume or shape) of the solid. The symbol for strain is the Greek letter ϵ (lower-case epsilon). If you consider any of the rods in Figure 1 the amount of deformation will be the increase in length (extension) while the original dimensions of the rod will be the rods original length l_o. Therefore, for a rod or a wire strain can be defined as extension per unit length and is given by

$$\text{strain} = \frac{\text{extension}}{\text{original length}}$$

We will denote the extension (i.e. increase or decrease in length) as Δl where Δ is the Greek capital letter Delta and l is the length. The final length of the rod after deformation is denoted as l while its original length is denoted as l_o . Therefore, strain can be written more compactly as

$$\epsilon = \frac{\Delta l}{l_o} = \frac{l - l_o}{l_o} \tag{2}$$

Where $\Delta l = l - l_o$ is the extension in metres, l is the final length of the rod after being deformed and l_o is the original length of the rod in metres. From Equation (2), you can clearly see that strain has no units.

5 Types of Stresses and Strains

Depending on the form of the solid in your possession (i.e. a rod or a cube) and how you apply the deforming force on the solid, you will obtain different types of stresses and strains.

Depending on the orientation of the deforming force, the deforming force you apply on a solid can cause it to come under tension (i.e. be stretched). This kind of deforming force is referred to as a tensile force and causes **tensile stress** and **tensile strain** in the solid. If the deforming force causes the solid to come under compression, then the deforming force is a compressive force. There are three main types of stresses you will consider in this lesson as you study the nature of deformation of solids. There is **tensile stress**, **bulk stress** and **shear stress**. These stresses will correspondingly induce **tensile strain**, **bulk strain** and **shear strain** the affected solids.

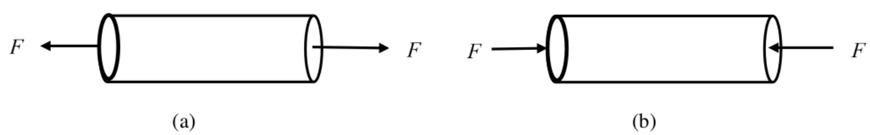

Figure 2: Schematic illustration of (a) a rod under the action of a tensile force (b) a rod under the action of a compressive force

5.1 Tensile Stress and Compressive Stress

Tensile stress is the kind of stress setup in solids (i.e., rod or wire) under tension while **compressive stress** builds up in solids under compression. Both the tensile and compressive forces acting on the solid (i.e. the rod) will deform the solid along the direction of the tensile or compressive force. The cross section area of the rod being acted upon will be perpendicular ($\perp$) to the direction in which you are applying the tensile or compressive force $F_\perp$ causing the deformation. Tensile stress,σ_{tensile} , or compressive stress, $\sigma_{\text{compressive}}$ is equal to the magnitude of the tensile or compressive force, $F_\perp$, per unit cross section area, A .

$$\text{tensile stress} = \frac{\text{tensile force}}{\text{cross section area}}$$

$$\text{compressive stress} = \frac{\text{compressive force}}{\text{cross section area}}$$

$$\sigma_{\text{tensile}} = \sigma_{\text{compressive}} = \frac{F_\perp}{A} \tag{3}$$

Where $F_{\perp}$ is the tensile or compressive force and A is the cross section area of the rod. The SI units of tensile or compressive stress are $\mathrm{N/m^2}$ or Pa.

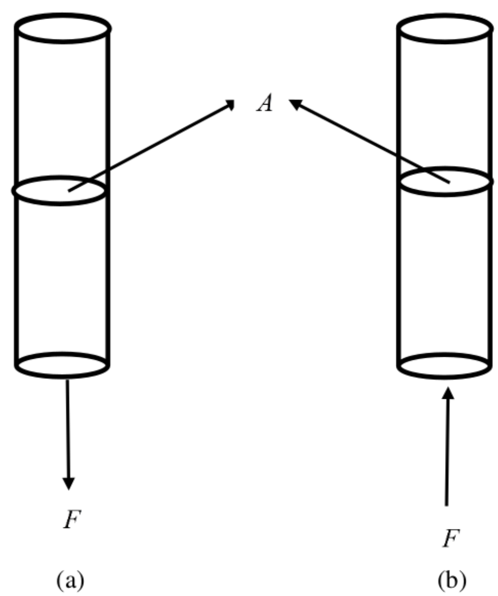

Figure 3: Schematic illustration of a rod under the the action of (a) a tensile stress (b) a compressive stress

5.2 Tensile Stress and Compressive Strain

Consider a rod fixed at one end and acted upon on the other end by a deforming force $F_{\perp}$ which brings about a change in its length as shown in the Figure 4. If your deforming force is a tensile force , the rod undergoes an extension (an increase in length), otherwise a compressive force will cause the rod to undergo a contraction (a decrease in length).

If the initial length of the rod before you extended or compressed it is l_o and its final length is l after its been extended or compressed, then the deformation is the extension, Δl , of the rod is given by

$$\Delta l = l - l_o \tag{4}$$

If the rod is under tension (i.e. being acted upon by a tensile force) as shown in Figure 4 (a) , then it will be stretched such that , $l > l_o$, and the extension Δl will be positive (i.e. the rod is elongated). On the other hand, if the rod is under compression (i.e. being acted upon by a compressive force), as shown in Figure 4 (b) then is will be compressed (shortened) such that , $l < l_o$. In this case the extension will be negative valued.

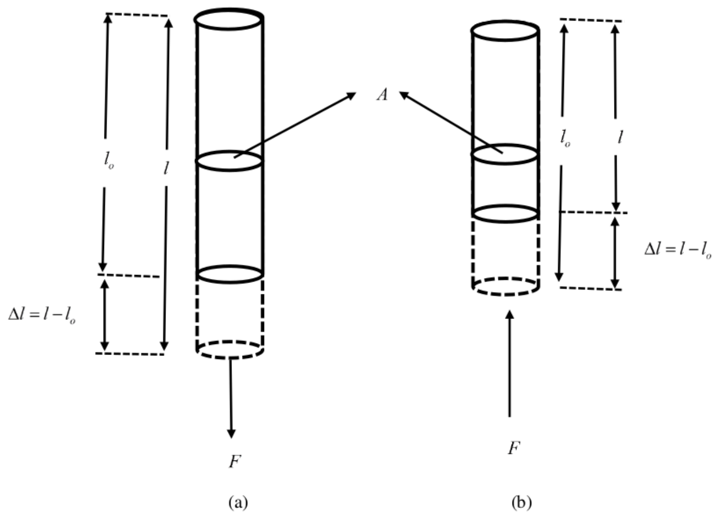

Figure 4: Schematic illustration of a rod undergoing (a) a tensile strain (b) a compressive strain

5.2.1 Definition of Tensile Strain and Compressive Strain

Tensile strain or compressive strain is the ratio of extension , Δl , to the original length l_o .

$$\text{tensile strain} = \frac{\text{elongation}}{\text{original length}}$$

$$\text{compressive strain} = \frac{\text{elongation}}{\text{original length}}$$

$$\epsilon_{\text{tensile}} = \epsilon_{\text{compressive}} = \frac{\Delta l}{l_o} = \frac{l - l_o}{l_o} = \frac{l}{l_o} - \frac{l_o}{l_o} = \frac{l}{l_o} - 1 \tag{5}$$

The ratio $\lambda = \dfrac{l}{l_o}$ is called the stretch ratio. $\lambda$is greater than one for tensile strain and less than one for compressive strain.

Equation (5) can be expressed in terms of the stretch ratio as

$$\epsilon_{\text{tensile}} = \epsilon_{\text{compressive}} = \lambda - 1 \tag{6}$$

5.3 Bulk Stress.

Consider the cube shown in Figure 5, this cube is subjected to compressive forces $F_{\perp}$ of equal magnitude which act perpendicularly on all eight sides of the cube. These compressive forces $F_{\perp}$ as can be seen in Figure 5 act perpendicularly to each side of the cube. The compressive force $F_{\perp}$ per unit area A exerted on each surface of the cube is known as **pressure**. As this pressure is being exerted on the cube, the interatomic distance between atoms making up the cube reduces. This results in an internal force per unit area called **bulk stress** or **volume stress** being set up in cube to oppose the effect of applied external pressure. According to Newton's third law, this bulk stress is equal but opposite to the direction of the applied pressure. The applied pressure is an external force per unit area while the bulk stress is the internal force per unit area. A solid subjected to these external pressures will therefore undergo decrease in volume without a distortion in its shape. However, once the externally applied pressure is removed, the bulk stress will attempt to restore the solid to its original volume.

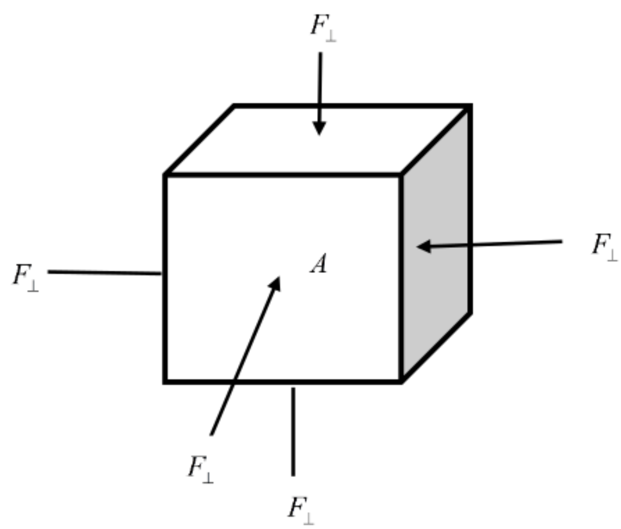

Figure 5: Schematic illustrations of cube under the action of equal compressive forces which are perpendicular to the cubes surfaces.

5.3.1 Definition of Bulk Stress

Bulk Stress is the kind of stress setup inside a solid when compressive forces $F_\perp$ are applied equally and perpendicularly to all sides (or surfaces) of the solid. Bulk stress denoted , σ_{bulk} , is given by

$$\sigma_{\text{bulk}} = \frac{F_\perp}{A} = \Delta P \tag{7}$$

Where A is the area on which the perpendicular force $F_\perp$ is being applied, and ΔP is the change in externally applied pressure. Bulk stress has units of $\mathrm{N/m^2}$ or Pa.

5.4 Bulk Strain

If the cube in Figure 5 had an initial volume of V_o , then after you apply these compressive forces $F_\perp$ equally on all faces of the cube, its volume will decrease to V .The cube retains its shape but just undergoes a decrease in volume due to the externally applied pressure which is also equal to the bulk stress. This change in volume, ΔV , is given by

$$\Delta V = V - V_o \tag{8}$$

If $V < V_o$ then $\Delta V < 0$ and the solid undergoes compression. On the other hand, if $V > V_o$, then $\Delta V > 0$ and the solid undergoes expansion.

5.4.1 Definition of Bulk Strain

Bulk strain is defined as the ratio of the change in volume of a solid, ΔV , to the initial volume V_o of the solid.

$$\epsilon_{\text{bulk}} = \frac{\Delta V}{V_o} = \frac{V - V_o}{V} = \frac{V}{V_o} - \frac{V_o}{V_o} = \frac{V}{V_o} - 1 \tag{9}$$

Where ΔV is the change in volume of the solid, V is final volume of the solid and V_o is the initial volume of the solid. Bulk strain has no units.

5.5 Shear Stress

Consider the cube shown in Figure 6, the face of this cube on which the side CD lies is acted upon by a deforming force $F_{||}$ applied tangentially (or parallel) to this face of the cube. Each face of the cube has area A . This tangential force $F_{||}$ applied in such a manner causes layers of atoms in the cube to be relatively displaced with respect to each other but the cube as a whole remains in the same position. This kind of deformation referred to as **shearing**. During shearing, the layers of atoms of the cube closer to the face CD experience the most displacement while those close to AB undergo no deformation. This results in the cube undergoing only a change in shape but its volume remains the same during the time the tangential force $F_{||}$ is applied on the face CD of the cube. As has been depicted in Figure 6, the point C will move to C' while D will move to $D6\prime$ under the action of $F_{||}$.

The tangential force $F_{||}$ per unit area A applied on the face CD of the cube causes the cube to shear resulting in a change in shape. Like before, through the action of Newton's third law of motion, this shearing will be opposed by the cube. A shearing stress equal the tangential force $F_{||}$ per unit area A in magnitude will be setup inside the cube. This shearing stress will to oppose the shearing effects of the externally applied tangential force $F_{||}$ per unit area A on the cube once the tangential force $F_{||}$ has been removed so that the cube can regain its original shape.

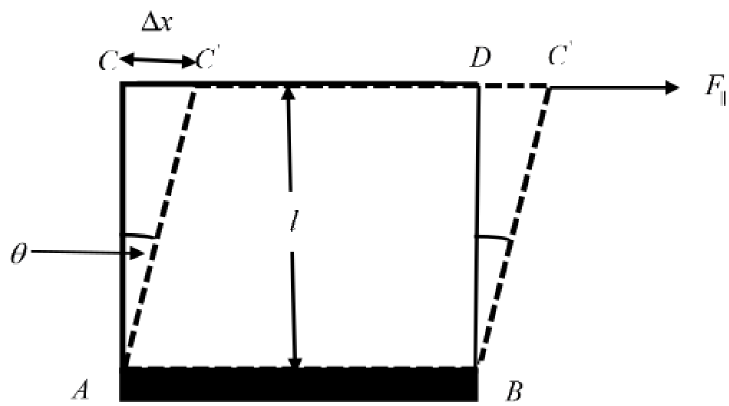

Figure 6: Schematic illustration of a cube being acted upon by a tangential force $F_{||}$.

5.5.1 Definition of Shear Stress

Shear stress is defined as the tangential force $F_{||}$ per unit area A . Shear stress is not the same as pressure since the tangential force $F_{||}$ is not perpendicular to the area A on which it is acting. The shear stress denoted σ_{shear} is given by

$$\text{shear stress} = \frac{\text{tangential force}}{\text{cross section area}}$$

$$\sigma_{\text{shear}} = \frac{F_{||}}{A} \tag{10}$$

The SI units of shear stress are N/m^2 or Pa.

5.6 Shear Strain

When the tangential force $F_{||}$ is applied on the cube shown in Figure 6 the maximum deformation occurs when the point C moves to C' and D moves D' on the surface CD . This length of CC' and DD' is the same and is in the direction of the applied tangential force $F_{||}$. This magnitude of CC' and DD' will be denoted by Δx while the length CD and AC will be denoted by l . The shear strain can thus be defined as the ratio of the maximum length of deformation Δx along the direction of the applied tangential force $F_{||}$ to the perpendicular length l in the plane of application of tangential force $F_{||}$.

5.6.1 Definition of Shear Strain

Shear strain is the ratio of the maximum length of deformation Δx to the perpendicular length l in the plane of action of the force.

$$\epsilon_{\text{shear}} = \frac{\Delta x}{l} = \tan\theta \tag{11}$$

In this definition of shear strain in term of tangent, the angle θ is in radians. The conversion factor for angles in degrees to radians is $360^\circ = 2\pi$ rad. Given any angle of θ in degrees you can convert it to radians as follows:

$$\theta(\text{rad}) = \theta(\text{deg}) \times \frac{2\pi}{360^\circ}$$

For $|\theta << \frac{\pi}{2}|$, then $\tan\theta \simeq \theta$, and Equation(11) becomes

$$\epsilon_{\text{shear}} = \tan\theta \simeq \theta \tag{12}$$

The University of Zambia
School of Natural Sciences
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PHY 1010
Lecture Notes
Temperature and Kinetic Theory of Gases

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1 Temperature and Thermometers

Temperature is one of the seven fundamental physical quantities in Physics. In a nutshell, temperature can be defined as the degreee of "hotness" or "coldness" of an object. Temperature affects our day to day lives in many ways. Normally, we are interested in the temperature of some specific object. The finite portion of matter whose temperature we want to measure is referred to as the **system**. Everything outside of this system that has a bearing on the behaviour of the system is referred to as the **surrounding**. Any physical system can be defined in terms of physical quantities related to the behaviour of the system such as the amount of substance, temperature, pressure and volume.

The temperature of any system can be measured using a device called a **thermometer**. Thermometers exploit various physical properties that depend on temperature such as thermal expansion of liquids, temperature dependent voltages at the junction of two different metals and the temperature dependence of electric resistance. The most common type of thermometer is the **Liquid in Glass thermometer** which exploits the thermal expansion of liquids. A liquid usually mercury or a coloured alcohol is sealed in a thin glass tube that has a bulb at one end which serves as a reserve of liquid. Because the liquid expands as the temperature rises, the liquid level in the thin capillary tube rises as the temperature increases. The glass also expands but much less than the liquid. Figure **??** shows a typical liquid in glass thermometer filled with a coloured alcohol.

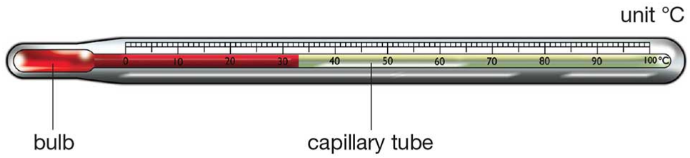

Figure 1: A liquid in glass thermometer filled with coloured alcohol

Gas thermometers are another commonly used type of thermometer in the physical sciences. As shown in Figure **??**, gas thermometers consist of a gas filled bulb that measure the temperature of their surrounding by measuring the pressure exerted by the dilute gas enclosed in a constant volume bulb and connected with thin tube to a mercury manometer. In such gas thermometers, the change in temperature of the the gas enclosed in the bottle is proportional to the pressure exerted by the gas. Unlike, liquid in glass thermometers, gas thermometers can accurately measure temperatures over very wide intervals.

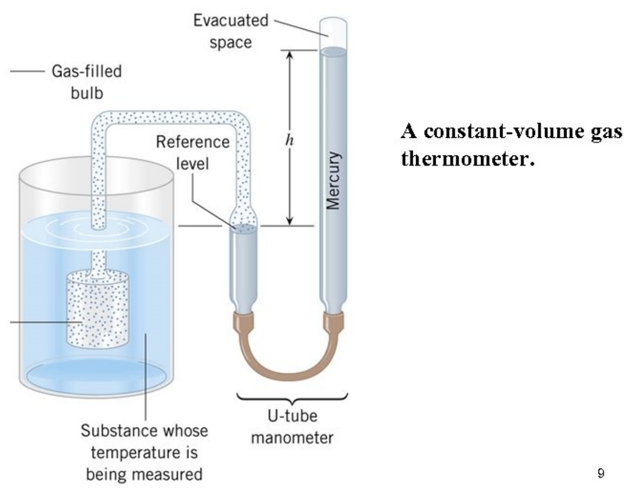

Figure 2: A constant volume gas thermometer immersed in a liquid bath whose temperature is to be measured

Due to the advert of the coronavirus disease also known as **Covid-19**, it has become increasingly fashionable and acceptable to constantly measure the temperature of individuals especially at points of entry in airports, hospitals, malls and other public buildings without the need to make contact with the person whose temperature is being measured. The fast spread of the **Covid-19** across the globe has necessiated and pushed to the fore the use of contactless thermocouple thermometers like the one shown in Figure **??** . Other types of thermometers which are used in industries are ordinary thermocouples and resistance thermometers.

Figure 3: A typical contactless thermocouple thermometer used to measure the body temperature at points of entry in airports, hospitals, malls and other public buildings

2 Thermometers and the Zeroth Law of Thermodynamics

Thermometers give quantitative and precise values of temperature. The numbers and units assigned to temperature are completely arbitrarily and need an arbitrarily chosen standard system that is easily reproducible called a **fixed point**. To precisely **calibrate** (i.e. to mark graduations) a thermometer, generally two fixed points that are globally easily reproducible are required depending on the type of thermometer.

We use thermometers to first check if objects have **the same temperature** and secondly to obtain accurate readings of **the temperature of objects** of interest. Every thermometer uses a fundamental law of Physics called the **Zeroth Law of Thermodynamics**. When you place a thermometer in intimate or close contact with an object whose temperature you want to measure, the thermometer soon reaches a steady or constant reading called the **temperature of the object**, and we say that the object and the thermometer are in **thermal equilibrium** with each other. If you now place this thermometer in intimate contact with another object, the thermometer will similarly reach a steady reading which will be the temperature of this object and will be in thermal equilibrium with this object.

Now, suppose the thermometer reads the **same temperature** for the two objects when they are in intimate contact with each other. The temperatures of these two objects will not change but will remain constant. We say that the two objects are in **thermal equilibrium** with each other or **are at the same temperature**. Objects that have the same temperature are in thermal equilibrium with each other. This in a nutshell is the essence of the Zeroth Law of Thermodynamics.

Formally, the **Zeroth Law of Thermodynamics** can be stated as follows: **Two objects that are in thermal equilibrium with a third object are in thermal equilibrium with each other**. In other words, objects that are in thermal equilibrium with each other have the same temperature.

3 Temperature Scales and Thermometers

3.1 Liquid in Glass Thermometers

A liquid in glass thermometer consists of a glass bulb containing a liquid (mercury or a colored alcohol) attached to a fine glass tube with a very thin bore evacuated of air and sealed. The glass tube has a numbered scale used to precisely track the rise and fall of the liquid column with temperature. If the liquid in the bulb of a liquid in glass thermometer expands, say by 1%, this 1% extra volume is forced to climb the evacuated narrow tube to quite a bit of height thereby magnifying the expansion of the fluid. If the liquid column in a liquid in glass thermometer expands to the same height when it is placed in intimate contact with two different objects of systems separated by a special wall called an **adiabatic wall**, then the two objects are at the same temperature using the Zeroth law of thermodynamics. A good experimental approximation of an adiabatic wall is **thick wood**, **concrete**, **absestos** or **styrofoam**. The other kind of wall that you will encounter during temperature studies is a **diathermic wall**. A good example of a diathermic wall is a thin sheet of metal.

To precisely calibrate a liquid in glass thermometer so as to create a temperature scale, two fixed points that are global and easily reproducible have to be found and marked on the thermometer. Next, the region between these two fixed points is then divided into some number of equal steps. The first fixed point called the **freezing point of water** is obtained or found by dipping the thermometer in a bucket containing a mixture of ice and water. This point is also called the **melting point of ice**. The liquid in bulb of the thermometer will expand to a certain height and this height is marked and postulated to be $0°C$. This temperature reading of $0°C$ is completely arbitrarily.

Next, we need to obtain the second fixed point. This is done by putting water on a stove, heating it up under standard atmospheric conditions and when the water begins to bubble, boil and evaporate, we dip the thermometer in the pot of boiling water. The liquid in the bulb of the liquid in glass thermometer will now be forced to climb up the bore of the glass tube to a much greater height and will finally settle at a certain height. This height is marked and is postulated to be $100°C$. Once again, this temperature reading of $100°C$ is completely arbitrarily.

Next the region on the thermometer capillary glass tube between the 0°C mark and 100°C mark is divided into 100 equal parts. This is postulated to be the temperature anywhere between 0°C and 100°C. Thus if the liquid in glass thermometer is put in intimate contact with an object or a system and the liquid column in the bore climbs 79% of the way to the top, then the temperature is read as 79°C. This temperature scale is referred to as the **centigrade scale** because there are 100 equal divisions between the two fixed points. The centigrade temperature scale is also known as **Celsius scale** in honor of Swedish scientist **Anders Celsius** who first proposed its use in 1742.

The German physicist **Gabriel Fahrenheit** originally proposed a sort of centigrade scale which came to be known as the **Fahrenheit scale**. He referenced 0°F (degrees Fahrenheit) as the freezing point of a saline solution and 98.6°F as the temperature of the human body. On the Fahrenheit scale, the freezing point and boiling point of pure water at standard atmospheric pressure (1 atm) are 32°F and 212°F respectively.

The problem with liquid in glass thermometers lies in the fact that different liquids do not expand at the same rate. Various liquid in glass thermometers when being calibrated will agree at the fixed points because they have been rigged to do so but will not necessarily agree in the region between the fixed points.

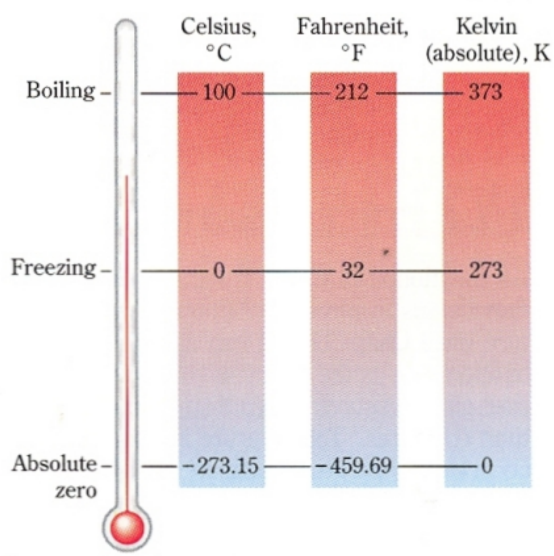

Figure 4: The Celsius and Fahrenheit scales are commonly used in liquid in glass thermometers

3.2 Constant Volume Gas Thermometers

The constant volume gas thermometer overcomes the limitation of the liquid in glass thermometer especially when it comes to the rate of expansions. All gases regardless of their nature expand at the same rate. To build a constant volume gas thermometer, take a very dilute fixed mass of gas of constant composition and fill in in bulk of constant volume. Connect the bulb using thin flexible tubes as shown In Figure to mercury manometer . The diluted the gas, the better it works in a constant volume gas thermometer.

Next, we submerge the bulb in a liquid bath sitted on a tripod stand under a bunsen burner which as been switched off. Once the system (i.e. gas) has settled down and there will be thermal equilibrium and the gas in the bulb will occupy a certain volume V and a corresponding pressure from the height of the liquid columns in the manometer liquid columns.

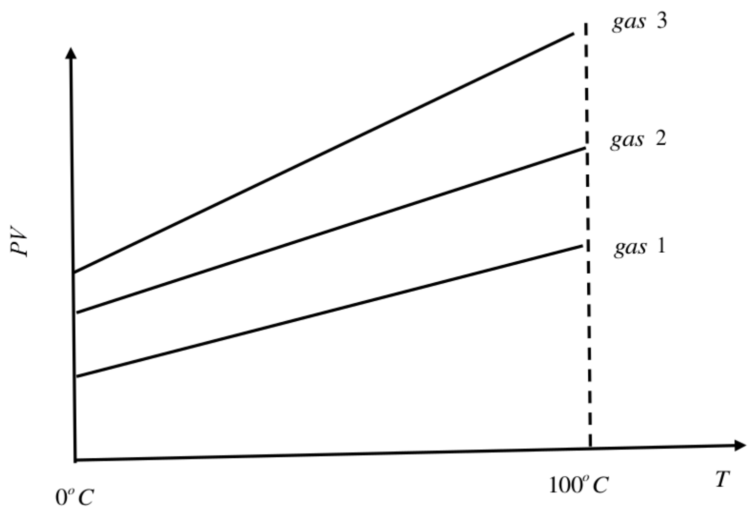

Figure 5: Schematic graphs of PV against T three gases used to calibrate a constant volume gas thermometer on Celsius scale using the **triple point of pure water** and the **boiling point of pure water** as fixed points

To calibrate any constant volume gas thermometer, we put the bath in which the bulb is submerged on different surfaces like a hot plate or stove whose temperature is known from a standard method like using the mercury liquid in gas thermometer. On each surface, P and V for a sample of a gas used in a constant volume gas thermometer is measured precisely at a known temperature T and the product PV is taken and plotted on a graph of PV against T . In an ice and water mixture, PV is obtained and the temperature T is postulated to be 0°C . Similarly, on a surface of boiling water PV is again obtained and the temperature T is postulated to be 100°C .

These two calibration points when connected by a straight line and divided

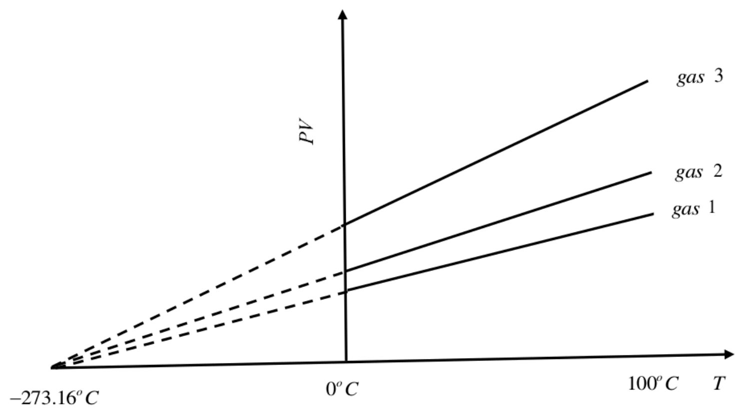

Figure 6: Schematic graphs of PV against T of three gases used to calibrate a constant volume gas thermometer on Celsius scale extrapolated beyond 0°C

equally along the length of the straight line lead to an equal increase in the product PV . The product PV for a gas is better than the volume of a liquid e.g. mercury or alcohol because it does not depend on the nature of the gas. If different samples of gas are used in the bulb and their PV against T graphs are obtained, the graphs produced are straight lines but with different slopes. The constant volume gas thermometer not only agrees between the fixed points but also agrees all the way in between the fixed points. All gases have the property that if they are calibrated between 0°C and 100°C , they agree in between.

If any constant volume gas thermometer is cooled below 0°C , the product PV will eventually vanish i.e., become zero at a temperature of −273.15°C . All other gases making a gas thermometer when cooled or when their PV against T graphs are extrapolated backwards vanish at a temperature of −273.15°C . The pressure due to a gas P vanishes at this temperature. This temperature is called **Absolute Zero Temperature** because it is the lowest possible temperature and no further cooling is possible beyond this temperature.

The 0°C is based on human obsession with water and is tied to life on planet Earth, its not universal and natural. For any gas, the product PV vanishing at −273.15°C is both natural and universal. Thus, on the **Kelvin scale**, 0K temperature is set at −273.15°C as the first fixed point. The second fixed point used on the Kelvin scale is the **triple point of water**. Water and ice can co-exist at 0°C while water and steam can coexist at 100°C . By varying the P , T and V in a unique way, a standard fixed point at which pure water, ice and steam coexist simultaneously is found. This point is referred to as the **triple point of water** and is assigned an arbitrarily value of 273.16 Kelvins, abbreviated 273.15K . The region between the absolute zero temperature mark 0K and the triple point of water 273.15K is divided into 273.15 equal parts and the magnitude of each division is defined as 1 Kelvin abbreviated 1K.

Since the triple point of water is a better standard fixed point than the

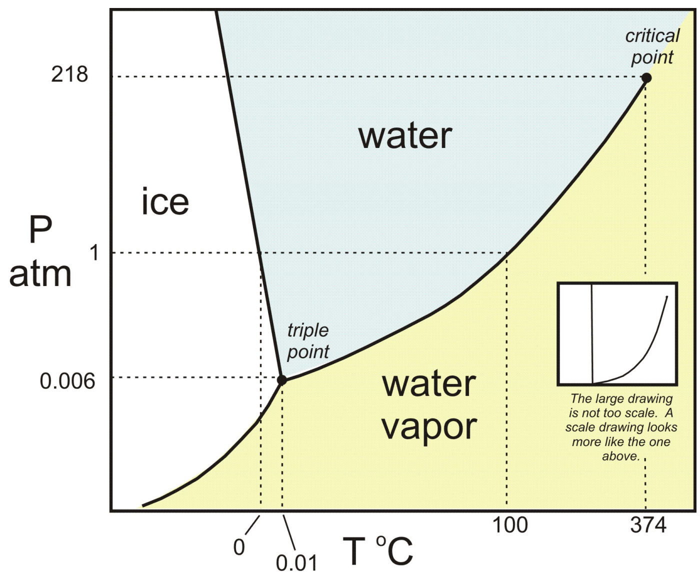

Figure 7: A phase diagram of pure water showing the position of the triple point of pure water

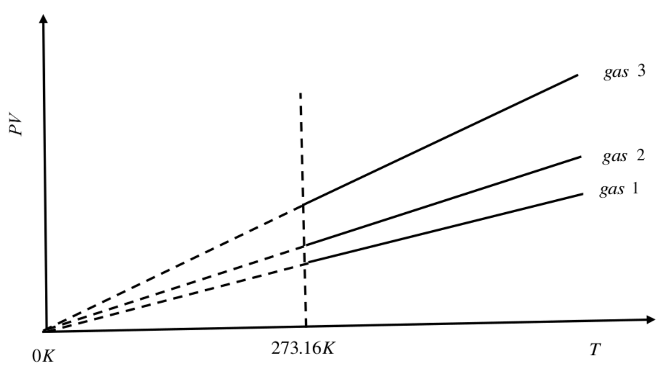

Figure 8: Schematic graphs of PV against T of three gases used to calibrate a constant volume gas thermometer on the Kelvin scale

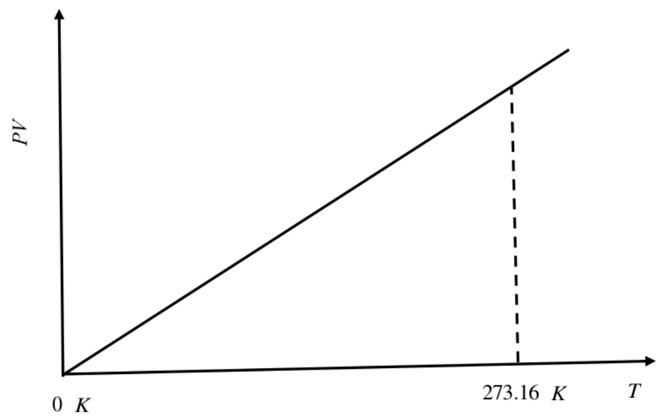

Figure 9: A schematic graph of PV against T of a gas used to calibrate a constant volume gas thermometer on the Kelvin scale

melting point of ice, the Celsius temperature scale has its zero point set to Celsius temperature of the triple point of water at 0.01°C . The Kelvin scale is like the Celsius scale except that the zero point has been shifted to 273.15°C.

The conversion between the Celsius temperatures T_c and Kelvin temperatures T is

$$T_{\mathrm{K}} = T_{\circ\mathrm{C}} + 273.15^{\circ}\ \mathrm{C} \tag{1}$$

$$T_{\circ\mathrm{C}} = T_{\mathrm{K}} - 273.15\ \mathrm{K} \tag{2}$$

4 The Ideal Gas Law

The graph of PV against T on the Kelvin scale shows a linear relationship that can be expressed as follows:

$$PV = cT \tag{3}$$

where

P is the pressure of the gas used the constant volume gas thermomter

V is the volume of the gas of the constant volume gas thermometer

T is the absolute temperature of the gas in Kelvins

c is a constant of proportionality.

But for a gas that is sufficiently diluted and from experiments, it is found that PV and T are precisely related as follows:

$$PV = Nk_BT \tag{4}$$

where

N is is the number of atoms of a given kind

$k_B = 1.4 \times 10^{-23}$ J K^{-1} is a constant called **Boltzmann's constant**.

Notice that both N and k_B are huge numbers. There is a natural number called **Avogadro's number** denoted N_A which defines a mole of a substance as 6.022×10^{23} atoms/mole. Thus the **number of moles** denoted n in N atoms is given by

$$n = \frac{N}{N_A} \tag{5}$$

Making N the subject of the formula, we get N in terms of n and N_A as

$$N = nN_A \tag{6}$$

Substituting Equation (6) back into Equation (4), we get

$$PV = nN_Ak_BT \tag{7}$$

Next, we define a constant called the **gas constant** denoted R as follows

$$R = N_Ak_B \tag{8}$$

$$R = 6.022 \times 10^{23} \text{ atoms/mol} \times 1.4 \times 10^{-23} \text{ J K}^{-1}$$

$$R = 8.314 \text{ J/mol K}$$

Substituting Equation (8) into Equation (7), we get

$$PV = nRT \tag{9}$$

Equation (9) is known as the **ideal gas law**, and gases that obey it are called **ideal gases**. An **ideal gas** is a very dilute gas such that the atoms or molecules in the gas are so far apart hence they do not feel any forces between each other unless they collide. All gases that are far removed from conditions under which they condense show nearly ideal behaviour.

5 Kinetic Theory of Gases

We know that a gas is composed of atoms or molecules of a substance (or a mixture of substances) which are free to fill any volume that contains them. We are interested in studying an ideal gas for which the following conditions hold:

(i) The atoms or molecules composing the gas are very small and take up negligible volume compared their container's volume V.

(ii) The small gas molecules are quite far apart from each other and their is no significant force acting between the atoms or molecules except when they collide with each other and the boundaries of the container. There collission are all assummed to be **perfectly elastic** i.e. the linear momentum and kinetic energy of the colliding atoms are both conserved.

(iii) The gas molecules are in constant radom motion.

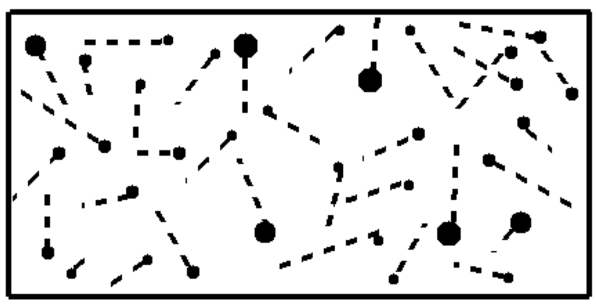

Figure 10: A model of the motion of ideal gas atoms according to the kinetic theory of gases

Consider a cube with sides $L \times L \times L$ which has an ideal gas inside. The ideal gas inside the cube has some pressure P. If the faces of the cube are not nailed down to each other, then they will just fly off. We want to find the value of this pressure and get reasons why there is pressure inside the cube. We also want to get the connection between the temperature of the gas T and the mechanical properties of the ideal gas atoms or molecules.

The ideal gas atoms or molecules are constantly bouncing off the wall and every time each one of them bounces of the wall, its linear momentum changes because the direction in which it moves changes. The wall changes the linear momentum of the atoms or molecules colliding with it. The atoms or molecules exert some force F on the wall and the wall pushes back with the same force F according to Newton's Third Law of Motion.

We are interested in finding the value of the force F exerted by these atoms or molecules on the wall. To do this , we exploit the definition of force F as the rate of change of linear momentum. Starting with Newton's Second Law of Motion , we get the resulatnt force F as

$$F = ma \tag{10}$$

where

F is the resultant force

m is the mass of the ideal gas atom or molecule

a is the linear acceleration of the the atom as it bounces of the wall

We know from our previous studies of motion that linear acceleration a is given by

$$a = \frac{v - u}{t} \tag{11}$$

Substituting Equation (11) into Equation (10), we get

$$F = m\left(\frac{v - u}{t}\right) = \frac{mv - mu}{t} = \frac{p_f - p_i}{t} = \frac{\Delta p}{t} \tag{12}$$

If we suppose that the ideal gas in the cube has N atoms or molecules randomly moving in the cube, each with its own direction colliding against the wall, the problem of calculating the force F becomes complicated. We simplify the problem as follows:

(i) We approximate that one third of the N atoms are moving from left to right, another one third of the N atoms are moving up or down and the final one third of the N atoms are moving in and out of the board. Our simplification assumes that atoms or molecules are moving only in these 3 primary directions. In reality atoms or molecules move in all directions and even rotate. In our simplification, $\frac{N}{3}$ atoms are assumed to be going back and forth between the two walls of the cube.

(ii) Next, we assume that all the atoms have the same speed v. This is a gross and crude description of the problem.

Now we take one atom which hits the wall of a cube with initial linear momentum $-mv$ and bounces back with final linear momentum mv. The negative sign in the initial linear momentum is because after colliding with the wall, the direction of motion of the atom will be opposite. However, the magnitude of the velocity will still be the same as we have assummed that all the atoms move with a constant velocity v. We get the change in linear momentum aka impulse of the atom after colliding with the wall as

$$\Delta p = p_f - p_i = mv - (-mv) = 2mv \tag{13}$$

The atom experiences this change its linear momentum after travelling a distance of $2L$ at constant speed v . Therefore, we get atom's the time of travel Δt as

$$\Delta t = \frac{2L}{v} \tag{14}$$

Next, we can calculate the average force exerted by one atom on the wall during a single collision F_{one} as

$$F_{\text{one}} = \frac{\Delta p}{\Delta t} = \frac{2mv}{\left(\dfrac{2L}{v}\right)} = \frac{2mv^2}{2L} = \frac{mv^2}{L} \tag{15}$$

But there is a large number of atoms colliding with the wall. Thus even if each atom collides with a wall after travelling a distance of $2L$, the force exerted by the atoms on th ewall appears to be steady or constant. The average force $\bar{F}$ due to all $\dfrac{N}{3}$ atoms colliding with a wall of area L^2 is given by

$$\bar{F} = \frac{N}{3} \times F_{\text{one}} = \frac{N}{3} \times \frac{mv^2}{L} = \frac{Nmv^2}{3L} \tag{16}$$

Only $\dfrac{N}{3}$ atoms apply this average force $\bar{F}$ on the wall. The other two directions are parallel to the wall, so the atoms do not apply any force on those walls. The average pressure $\bar{P}$ due to an average force of $\bar{F}$ acting on a wall of area $A = L^2$ is given by

$$\bar{P} = \frac{\bar{P}}{A} = \frac{\left(\dfrac{Nmv^2}{3L}\right)}{(L^2)} = \frac{Nmv^2}{3L^3} \tag{17}$$

Now, setting $V = L^3$, we get the average pressure $\bar{P}$ exerted on the wall as

$$P = \frac{Nmv^2}{3V} \tag{18}$$

Multiplying on both sides of Equation (18) by V, we get

$$\bar{P}V = \frac{Nmv^2}{3} \tag{19}$$

Now, recall from the ideal gas law, we have

$$PV = nRT \tag{20}$$

These two equations (19) and (20) are equivalent, so we can write them as

$$PV = \frac{Nmv^2}{3} = nRT \tag{21}$$

Therefore, we can then proceed to write

$$\frac{Nmv^2}{3} = nRT \tag{22}$$

Multiplying on both sides of Equation (22) by 3, we get

$$Nmv^3 = 3nRT \tag{23}$$

Dividing both sides of Equation (23) by 2, we get

$$\frac{Nmv^2}{2} = \frac{3nRT}{2} \tag{24}$$

Next recall that $n = \frac{N}{N_A}$ and $R = N_A k_B$, thus substituting into Equation (24), we have

$$\frac{Nmv^2}{2} = \frac{3\left(\frac{N}{N_A}\right)(N_A k_B)\,T}{2} = \frac{3Nk_BT}{2} \tag{25}$$

Cancelling out the N on both sides, we have

$$\frac{mv^2}{2} = \frac{3k_BT}{2} \tag{26}$$

From our previous studies on work, energy and power, you should be able to notice and recall that the expression $\frac{mv^2}{2}$ refers to the average kinetic energy of the atoms. Therefore we can write

$$K\bar{E} = \frac{mv^2}{2} = \frac{3}{2}k_BT \tag{27}$$

Equation (27) shows that the average kinetic energy of the atoms in the cube is related to the absolute temperature as follows

$$K\bar{E} = \frac{3}{2}k_BT \tag{28}$$

Equation (28) is one very important result of the kinetic theory of gases. It gives the meaning of gas temperature as a measure of the average kinetic energy of the gas atoms or molecules. So we can now define the **absolute temperature** as a measure of of the average kinetic energy of the atoms or molecules in an ideal gas. Thus, at **absolute zero** denoted 0K, the ideal gas atoms or molecules cease to have any kinetic energy. They become motionless.

It also important to notice that we could Equation (28) to explain the meaning of **thermal equilibrium**. Recall that substances in thermal equilbrium have the same temperature. Therefore, if two ideal gases are in thermal equilibrium with each other, then their average translational kinetic energy per atom or molecule is the same in both.

The expression for average kinetic energy of the ideal gas atoms or molecules can modified and rewritten as

$$\frac{mv^2}{2} = \frac{3}{2}k_B T \tag{29}$$

$$\frac{m}{2}(v_x^2 + v_y^2 + v_z^2) = \frac{3}{2}k_B T \tag{30}$$

where v_x, v_y and v_z are the components of the velocity of atoms in the 3 directions are atoms were allowed to move.

$$\frac{1}{2}mv_x^2 + \frac{1}{2}mv_y^2 + \frac{1}{2}mv_z^2 = \frac{1}{2}k_B T + \frac{1}{2}k_B T + \frac{1}{2}k_B T \tag{31}$$

Therefore, we get the average kinetic energy of the ideal gas atoms in each direction as

$$\frac{1}{2}mv_x^2 = \frac{1}{2}k_B T \tag{32}$$

$$\frac{1}{2}mv_y^2 = \frac{1}{2}k_B T \tag{33}$$

$$\frac{1}{2}mv_z^2 = \frac{1}{2}k_B T \tag{34}$$

Last but not least, we can obtain from Equation (29) a kind of average velocity of the atoms called the **root mean square speed** denoted v_{rms}

Cancelling the 2 on both sides, we get

$$mv^2 = 3k_B T \tag{35}$$

Next, dividing by m on both sides, we get

$$v^2 = \frac{3k_B T}{m} \tag{36}$$

Finally, taking the square root on both sides, we get the root mean square speed as

$$v = \sqrt{v^2} = \sqrt{\frac{3k_B T}{m}} \tag{37}$$

$$v_{rms} = \sqrt{\frac{3k_B T}{m}} \tag{38}$$

The **rms speed** not the same as the usual average speed or mean speed. Rather, it is the speed that an atom or molecule with average kinetic energy possesses.

The University of Zambia
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PHY 1010
Lecture Notes
Description of Thermodynamic Systems

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1 The state of a system

Consider an ideal gas sitting in a container with a movable piston as shown in Figure **??**. An ideal gas is a very dilute gas such that the atoms or molecules in the gas are so far apart hence they do not feel any forces between each other unless they collide. At equilibrium, the ideal gas in a container with a movable piston has a definite value of volume V , pressure P and absolute temperature T. These three measurable physical quantities in the case of an ideal gas are related via the ideal gas law as follows

$$PV = nRT \tag{1}$$

The ideal gas in the container makes up the **system** and the **state** of this system is a particular situation where the system has specified or definite values of P , V and T. The measurable physical quantities used to describe the state of a system are referred to as **macroscopic coordinates**. Since the pressure P , volume V and absolute temperature T of an ideal gas in a container are related by Equation (1), only two independent macroscopic coordinates P and V are needed to describe the state of an ideal gas. The absolute temperature T is not an independent macroscopic coordinate of a gas as it depends on P and V via the ideal gas law.

If the macroscopic coordinates P , V and T which are used to describe a system change in any way, then the system is said to undergo a **change of state**. When a system is not influenced in any way by its surroundings it is

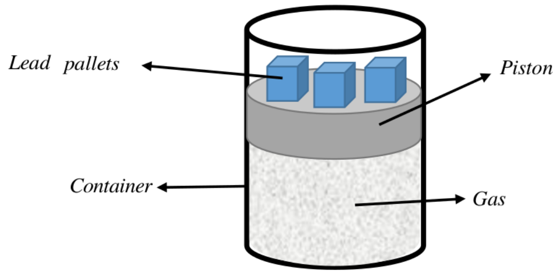


Figure 1: Schematic illustration of an ideal gas filled container with a movable piston weighed down by lead pallets

said to be **isolated**. In practice, a system will be affected by its surroundings. The surrounding may exert forces on the system or provide contact between the system and a body that is at some definite temperature.

2 Types of equilibrium and equation of state

A system that has no unbalanced forces in its interior or between the system and its surrounding is said to be in a **state of mechanical equilibrium**. If a system is not in a state of mechanical equilibrium, it will undergo a change of state which will only cease when mechanical equilibrium is restored. A system in a state of mechanical equilibrium that does not tend to undergo a chemical reaction or transfer of matter from one part of the system to another is said to be in a **state of chemical equilibrium**.

A system in mechanical and chemical equilibrium which does not undergo a spontaneous change in its macroscopic coordinates when it is separated from its surroundings by a diathermic wall (e.g., a thin metal wall) is said to be in a **state of thermal equilibrium**. In thermal equilibrium, all parts of the system have the same absolute temperature T and this temperature is the same as that of the surroundings. When these conditions are not reached, the system will undergo a change of state until thermal equilibrium is reached.

When a system has conditions for mechanical, chemical and thermal equilibriums, it is said to be in a **state of thermodynamic equilibrium**. In thermodynamic equilibrium, the system has no tendency whatsoever to undergo a change of state. States of thermodynamic equilibriums are the only ones described using macroscopic coordinates such as P , V and T . These macroscopic coordinates used to describe states of thermodynamic equilibrium are called **thermodynamic coordinates**. When any of three types of equilibriums which constitute thermodynamic equilibrium are not satisfied, the system is said to be in a **non equilibrium state** and this state cannot be described using thermodynamic coordinates because the macroscopic coordinates used to describe the system are still changing and have not yet become steady or constant.

For any system in a state of thermodynamic equilibrium, an equation called an **equation of state** exists for that particular system that connects the independent and dependent thermodynamic coordinates used to describe the system. In the case of a container with a movable piston filled with an ideal gas in a state of thermodynamic equilibrium, the independent thermodynamic coordinates are P and V while the dependent thermodynamic coordinate is T . The equation of state for an ideal gas connecting P , V and T is the ideal gas law given by Equation (1). Since P and V can adequately describe any state of an ideal gas, it is no longer necessary to use a PV against T diagram previously used to construct Kelvin temperature scale for a gas thermometer. Instead PV diagrams will be used going forward to describe any state of an ideal gas.

3 Intermediate states, state variables and internal energy of ideal gas

If three lead pallets are added on top of a movable piston as shown in Figure **??**, the piston will suddenly push down on the gas filled container due to the weight of the lead pallets and will bounce back and forth a few times then with time settle in a new location. After a while, the system will attain thermodynamic equilibrium and the thermodynamic coordinates P and V of the system will cease to change. This equilibrium state of the gas which we will refer to as **state 1** can be depicted with a dot on the PV diagram located at (P_1, V_1) as shown in Figure **??**.

When one lead pallet is suddenly pulled out from the top of the piston leaving behind two lead pallets, the gas inside the container suddenly expands and the piston shots up and bounce back and forth. After a while, the piston settles at another new location and the gas attains a new state of thermodynamic equilibrium we will refer to as **state 2** located at the dot on (P_2, V_2) on the PV diagram. By continuing to remove the reminder of the lead pallets in similar fashion, two other states of thermodynamic equilibrium denoted on the PV diagram as **state 3** located at (P_3, V_3) and **state 4** located at (P_4, V_4) are obtained.

In between each state of thermodynamic equilibrium dshown on the PV diagram shown in Figure **??**, the system is not in a state of thermodynamic equilibrium. There are no definite values of P and V that can be associated with the ideal gas as it moves from one equilibrium state to the next one. Different parts of the ideal gas will have different values of pressure P, thus cannot be depicted as states on the PV diagram. Only when the system has settled down and attained definite values of pressure P and volume V can that particular state be represented on a PV diagram.

To always keep the system on the PV diagram at all times as it transitions

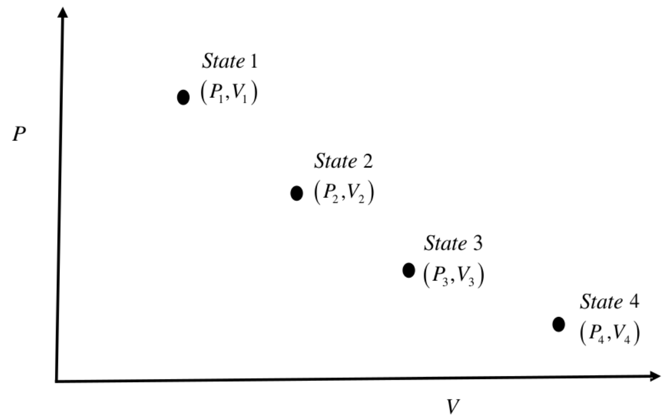


Figure 2: PV diagram depicting the different thermodynamic states an ideal gas in a container with a movable piston passes through as lead pallets are suddenly removed.

from one state to the next, the process of removing the lead pallets has to be done **quasistatically**. A **quasistatic process** is one that proceeds sufficiently slowly such that the system is always at equilibrium. In the case of the three lead pallets used in Figure **??**, they will have to be pulled very slowly from the piston such that the system always remains in equilibrium. This approach is impossible and not practical at all. A more practical approach is to replace the 3 lead pallets with grains of lead that produce the same amount of pressure as shown in Figure **??** .

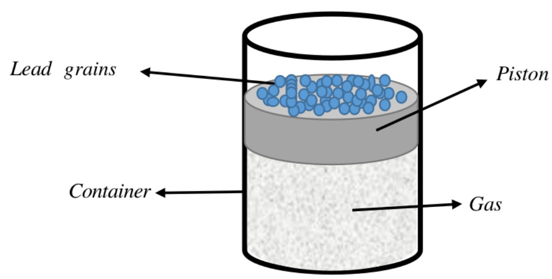


Figure 3: A schematic illustration of a ideal gas in container with movable piston weighed down by grains of lead

Now, removing the grains of lead one at a time from the piston, the system will quickly settle down to a definite value of P and V . This new state of the system will be much closer to the initial state. However, during each settling down process, the system has no definite value of P and V . The system only has definite values of P and V once it has settled and attained a state of thermodynamic equilibrium. Continuing to remove more grains of lead and allowing the system to settle, several new states called **intermediate states** which previously could not be obtained with the three lead pallets are obtained as depicted in the PV diagram in Figure. The equation of state for ideal gases can be applied to each intermediate state derived from this quasistatic process.

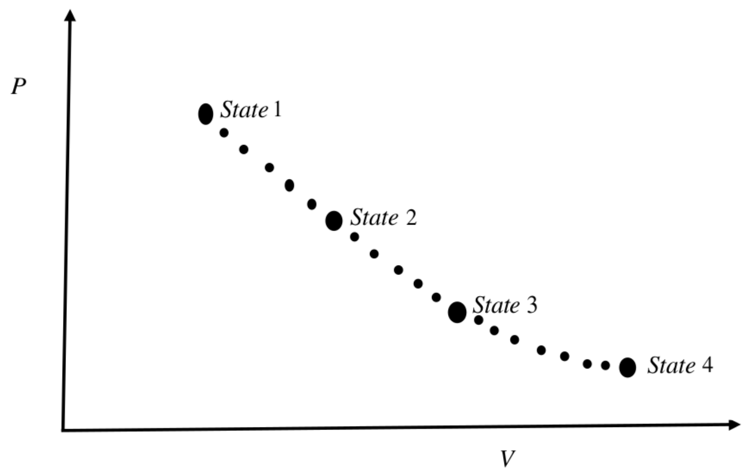


Figure 4: PV diagram depicting **intermediate states** found when the lead pallets are replaced with grains of leads.

If the size of the grains of lead is reduced further, more intermediate states are obtained between the already found intermediate states shown in the PV diagram in Figure **??** . Using calculus, as the size of the grains is reduced such that it approaches zero or vanishes, a continuous line is obtained showing the path taken by the quasistatic process as the system moved from its initial thermodynamic state to a final thermodynamic stat as shown in Fig. 4.5.

If all the grains of lead removed are returned on the piston one grain at a time and the system is allowed to settle down each time a grain is returned, the system ideally should return to its initial state following the same path. This is referred to as a **reversible process**. Most often, processes are not reversible when they are done quasistatically unless the system is completely frictionless. When a system is returned to a previous state, those thermodynamic coordinates that are always the same in that state are called **state variables**. Examples of state variables for a system of an ideal gas in a container with a movable piston are pressure P , volume V and absolute temperature T . State variables have the same values regardless of how the system reaches a particular thermodynamic state.

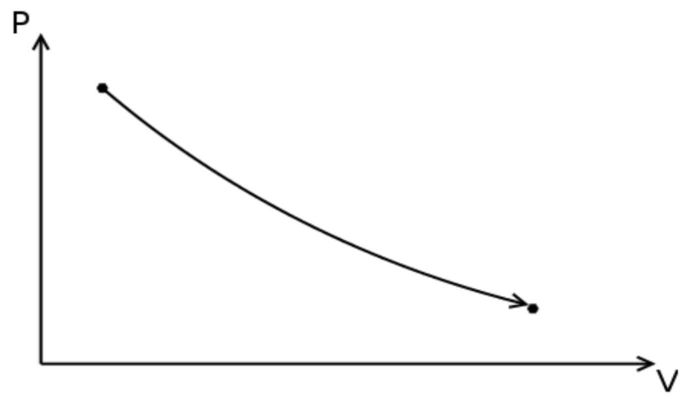


Figure 5: PV diagram showing a continuous path taken by the system as it moved from its initial state to its final state..

The system of a gas in a container with a movable piston also has another important state variable called **internal energy** of gas denoted U . In every thermodynamic state , the system has a well-defined value of internal energy. The internal energy U of a system is the total energy of every type possessed by the system. For a system of ideal gas containing N atoms or molecules that is allowed to move with f degrees of freedom , the Internal energy U of such as an ideal gas is given by

$$U = \frac{Nmv^2}{2} = \frac{f}{2}Nk_BT = \frac{f}{2}nRT = \frac{f}{2}PV \tag{2}$$

Every state on the PV diagram has a specific value of internal energy U associated with it and the value of U does not depend on how the state is reached. The internal energy U is a state variable. For a system composed of a monoatomic gas allowed to move in space only in 3 different directions (i.e. left to right, up and down , in and out of the page) without rotating or vibrating we have $f = 3$. In this case, the internal energy U of the ideal gas is given by

$$U = \frac{3}{2}Nk_BT = \frac{3}{2}nRT = \frac{3}{2}PV \tag{3}$$

For a system composed of a diatomic gas whose molecules in move in 3 directions in space and has rotational and vibrational motion, the degrees of freedom increase to five i.e $f = 5$. In this case, the internal energy U of the ideal gas is given by

$$U = \frac{5}{2}Nk_BT = \frac{5}{2}nRT = \frac{5}{2}PV \tag{4}$$

Chapter 12

12.0 Thermodynamics

12.1 Introduction

Thermodynamics is primarily associated with heat and temperature and their relation to energy and work. It was developed in a quest to improve the efficiency of the early steam engines. In general terms the basic concern is the transformation of thermal energy (heat) into mechanical energy. It follows that a device or a system that converts heat into mechanical energy is called a heat engine. Devices or systems may be simple or complex. The measurable quantities used to describe a system are normally temperature, pressure and volume. In addition quantities such as internal energy, heat, work and entropy are also used. For example, n moles of gas which is in equilibrium in closed container will have a definite temperature, pressure and volume, such a system will be said to be in a thermodynamic state. Thus the variables describing a given thermodynamic state are called state variables. The properties of a given thermodynamic state will remain the same as long as the state variables have the same values. An additional important quantity that characterizes a system is internal energy (U). The internal energy constitutes of the sum of all the kinetic and potential energies possessed by its atoms or molecules. A system's temperature is an indicator of its internal energy.

12.2 Zeroth Law of Thermodynamics

When two systems are each in thermal equilibrium with a third, then they are also in thermal equilibrium with each other. The latter statement is called the zeroth law of thermodynamics. For example if the temperature of an object measured with a thermometer is found to be the same when a second object's temperature is measured, we can conclude that the objects are in thermal equilibrium with each other.

12.3 First Law of Thermodynamics

In discussing thermodynamics laws we consider heat as a form energy which it must conserved like its mechanical equivalent. If a system is in a given thermodynamic state then it can be stated that it has a definite amount of internal energy. When heat flows into a system it can do two things, one it can increase the internal energy of the system and secondly it can provide energy to the system allowing it to do work to the surroundings. This law is an extension of the law of conservation of energy that includes heat energy and internal energy of the system under consideration. The first law of thermodynamics says that the amount of energy Q going into a system is equal to the sum of the increase in internal energy ΔU and the external work done W.

$$Q = \Delta U + W \qquad 12.1$$

It follows that a system in a given state has a definite amount of internal energy. When heat flows into a system two are likely to happen:

i) The internal energy of the system increases or

ii) The heat may provide energy for the work to be done the surroundings.

For example, if we have a system comprising of a gas enclosed by a movable piston in cylindrical vessel with a closed end see figure 12.1, and then heat is added to the system. The expanding gas will move the piston and thus the gas does work on the piston. The piston moves against an external force. Suppose heat was removed from the enclosed gas we expect the piston to move inwards. The latter situation can be equated to the surroundings doing work on the system, in which case the work will be negative.

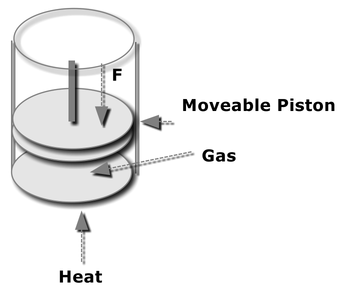


Figure 12.1 Gas tight moveable piston in a cylinder.

In general many other systems give us similar results, we can conclude that the heat added to the system should be equal to the increase in internal ΔU plus the external work done by the system *W*. It follows that the first law of thermodynamics is a law of conservation of energy which includes thermal energy. Convention has it that when heat flows into the system the quantity of energy *Q* is positive and when the heat flows out of the system then quantity of energy *Q is* negative. For example in figure 12.1 if the piston is pushed downwards by an outside force *F* the work done is negative. We can make an analogy with the definition of work in mechanics which states that:

$$\text{Work}=\text{Force}\times\text{Displacement}\times\cos\theta$$

Where the angle θ is the angle between the force vector and the displacement vector. Suppose that the piston is pushed by a force *F* downwards and it moves a distance Δy the work done *W* is:

$$W = F\Delta y\cos\theta = F\Delta y\cos 0^{o} = F\Delta y \qquad 12.2$$

This result is because cos0^{o} is equal to one as the force vector and displacement vector are in the same direction. In figure 12.2 the piston is displaced by an amount Δy and if the cross section area of the cylinder is *A* then the volume increase is the product of this area and Δy.

$$\Delta V = A\Delta y \qquad 12.3$$

The pressure generated by the force F (arising from the weight of the piston) is $P = F/A$. This equation can be written in terms of the force as $F = PA$ and substituting in equation 12.2 we get that work by the expanding gas is:

$$W = F\Delta y = PA\Delta y = P\Delta V \qquad 12.4$$

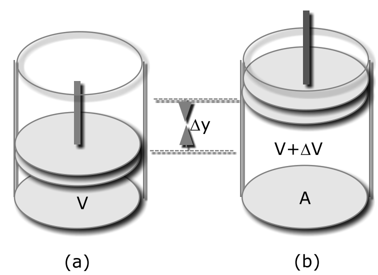


Figure 12.2 Gas expansion in a cylinder

During expansion the volume occupied by the enclosed gas increases and the work done is positive, whereas when the gas is compressed there is volume reduction i.e. ΔV is negative, hence the work done is negative. In the case of applying constant pressure on the gas, the work done is simply calculated from the equation:

$$W = P\Delta V \qquad 12.5$$

12.4 Constant Pressure

In the case where expansion takes place under constant pressure, namely, isobaric process the work done can be calculated using equation 12.5. This process can be plotted on graph of pressure versus volume (*P*-*V* diagram) as in figure 12.3.

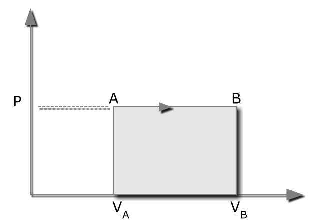


Figure 12.3 Pressure-volume plot at constant pressure.

The expansion takes place at a constant pressure P and the volume increases from V_A to V_B then the work done in this case is:

$$W = P\Delta V = P(V_B - V_A) \qquad 12.6$$

It should be noted that in equation 12.6 the term $P(V_B - V_A)$ is the area under the process path. In general the area under the process path is equal to the work done. In the case where the process is reversed (compression) the area remains the same but the work done and the work is taken to be negative.

12.5 Non-constant Pressure

The work done during a non-constant pressure process can also be worked out by finding the area under the process path. However, it is tedious for such a process as we may have to use some form of crude integration of the area on the *P-V* diagram as illustrated in figure 12.4.

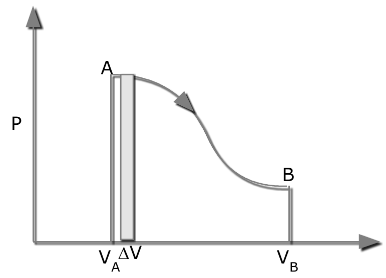


Figure 12.4 Pressure-volume plot for changing pressure.

Considering the small shaded strip in figure 12.4, we can assume that the pressure over ΔV is constant and the work done is:

$$W = P\Delta V$$

The total work done will be equal to the sum of several such strips taken over the interval V_A to $V_{B.}$ Taking much smaller strips will yield a more accurate answer of the work done from A to B.

Example 12.1

A gas undergoes a process in which it is expanded from point A to B as shown in Figure 12.5. Calculate the work done by the gas.

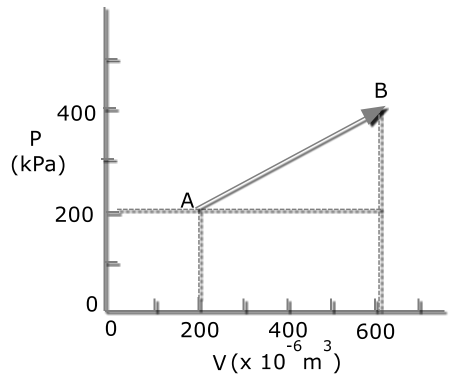


Figure 12.5 PV diagram

Solution

The work done by the gas is equal to the area under the process path *AB* and is equal to the sum of the area of the triangle and the rectangle in figure 12.5.

Area of triangle = ½ base x height

$$= \tfrac{1}{2}\left(600\times10^{-6} - 200\times10^{-6}\right)\text{m}^3 \times \left(400 - 200\right)\text{kPa}$$

= 40 J

Area of rectangle = $\left(600\times10^{-6} - 200\times10^{-6}\right)\text{m}^3 \times \left(200 - 0\right)\text{kPa}$

= 80 J

Total area under process path = 40 J +80 J = 120 J

The work done for this thermodynamic process is 120 J.

12.6 Internal Energy of an Ideal Gas

It was earlier on alluded to that the system's temperature is an indicator of its internal energy. The internal energy of a gas is mostly assigned to the kinetic energy possessed by the atoms/molecules of a gas. Thus we can see the effect of temperature on the internal energy of a monatomic gas from equation 12.7. This shows that the internal energy of an ideal gas depends only on its temperature.

$$KE = N(\overline{KE}) = \tfrac{3}{2}nRT \qquad 12.7$$

Where N is the number of molecules in a gas, n is the number moles, and R the gas constant. In the case of a monatomic gas the energy goes into mainly translational motion and its internal energy is assigned to the kinetic energy as expressed in equation 12.8.

$$U = KE_{trans.} = \tfrac{3}{2}nRT \qquad 12.8$$

It follows that the change in internal energy of a monatomic gas can be calculated from equation 12.9.

$$\Delta U = \tfrac{3}{2}nR\Delta T \qquad 12.9$$

It should be noted that for molecular gases the energy flowing into them is partitioned into other forms of motion such as rotation and vibration. These gases will have higher internal energy when at the same temperature as a monatomic gas. Equation 12.9 can be re-written in a more general form as:

$$U = K(\tfrac{1}{2}nRT) \qquad 12.10$$

Where K is an integer greater than or equal to 3. In the case of monatomic gases K is equal to 3.

12.7 Heat Transfer and Specific Heat Capacity of an Ideal Gas

The calculation of the amount of heat transferred into or out of an ideal gas depends on the process used. Two common processes used are the constant-volume process and the constant pressure process. These processes are discussed with reference to the first law of thermodynamics which is about the conservation of energy.

12.8 Constant–Volume Process

Recall, the first law of thermodynamics states that energy added to system goes into increasing the internal energy of the system plus work done, if any, by the system equation 12.1. In doing work we expect a volume change in the system. Thus in a constant volume process where there is no volume change $(\Delta V = 0)$ the work done is zero and all the energy goes into increasing the internal of the system. The first law of thermodynamic reduces to:

$$Q = \Delta U \quad \text{(Constant-volume)}$$

Referring to equation 12.9 for a monatomic gas we say that:

$$Q = \Delta U = \tfrac{3}{2} nRT \qquad 12.11$$

We know that the only quantity that relates Q and ΔT is the specific heat capacity of a substance in equation 11.21.

$$c = \frac{Q}{m\Delta T}$$

We can now define a new quantity the molar specific heat capacity C by replacing mass m in the last equation with the mole n.

$$C = \frac{Q}{n\Delta T} \qquad 12.12$$

To differentiate between the processes a subscript is attached to molar specific heat capacity term and for constant-volume process C_v is used. Substituting for Q in equation 12.12 term from equation 12.11 we get for a monatomic gas the following result:

$$C_v = \frac{\tfrac{3}{2} nR\Delta T}{n\Delta T} = \tfrac{3}{2} R \qquad 12.13$$

The above result can be generalized for molecular gases by appropriately changing the preceding fraction.

12.9 Constant-Pressure Process

The work done in the case of a constant-pressure is simply the product of the pressure and the change in volume ($W = P\Delta V$). It follows that the first law of thermodynamics can re-written as

$$Q = \Delta U + W = \Delta U + P\Delta V \qquad 12.4$$

At constant pressure the ideal-gas law equation 11.16 can be stated as

$$P\Delta V = nR\Delta T$$

The quantity of heat involved becomes

$$Q = \Delta U + nR\Delta T \quad \text{(Constant pressure)}$$

In a similar manner we defined C_v, the molar specific heat at constant pressure, C_p is defined as

$$C_p = \frac{Q}{n\Delta T} = \frac{\Delta U + P\Delta V}{n\Delta T} = \frac{\Delta U}{n\Delta T} + \frac{nR\Delta T}{n\Delta T}$$

At constant volume the ideal-gas law reduces to $Q = \Delta U$, hence if we substitute for ΔU the equation above becomes

$$C_p = C_v + R \qquad 12.5$$

This result tell us that the molar specific heat capacity at constant-pressure is always larger that the molar specific heat capacity at constant volume. This follows from the fact that in this process some heat goes into work. Recall the molar specific heat capacity at constant volume for a monatomic gas gave us the result that $C_v = \frac{3}{2}R$. It follows that if we substitute this value in equation 12.5 we get

$$C_p = \frac{3}{2}R + R = \frac{5}{2}R$$

$$C_p = \frac{5}{2}R \quad \text{(monatomic gas)} \qquad 12.6$$

The ratio of the two processes gives an approximately constant value γ. The value of this constant for monatomic gases is 1.67 and for diatomic gases 1.40. The values of γ for a few selected gases are given in table 12.1

$$\gamma = C_P / C_V \qquad 12.7$$

Table 12.1 Ratio of C_p to C_v of Some Gases

Gas	γ
Monatomic	
Helium (He)	1.67

Argon (Ar)	1.67
Krypton (Kr)	1.69
Diatomic	
Hydrogen (H_2)	1.41
Oxygen (O_2)	1.40
Carbon Monoxide (CO_2)	1.40
Polyatomic	
Steam (H_2O)	1.30
Sulphur Dioxide (SO_2)	1.29
Carbon Dioxide (CO_2)	1.30

12.10 Zero-Heat Transfer Process

In the zero-heat transfer process transfer no energy is moved into or out of the system and it is called an adiabatic process. For this to occur the process must be thermally isolated. In practice this is very difficult to attain, however, if a process is very fast then it can be considered to be adiabatic as there is very little time for the transfer of thermal energy? For this process Q = 0 and the first law of thermodynamics becomes

$$0 = \Delta U + W \text{ or } \Delta U = -W \text{ (adiabatic)}$$

This implies that work is done at the expense of internal energy of the system or that the internal energy of the system must decrease.

12.11 PV Diagram Representation of the Thermodynamic Processes

The first two processes, namely, the Constant-pressure and Constant-volume are simple as these are represented by straight lines, for example, as in figure 12.3. The other two processes are complex as for them there is an inverse relationship between the pressure and the volume arising out of the ideal gas law. It states that if the temperature T is held constant then PV is also constant. So that for the isothermal process the equation of the isotherm is

$$PV = \text{constant or } P = \frac{\text{constant}}{V} \qquad 12.8$$

A plot for an isothermal process involving compression at constant temperature is shown in figure 12.6.

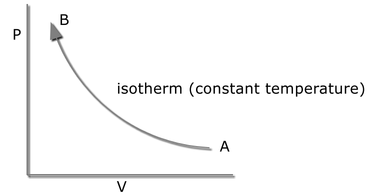


Figure 12.6 PV diagram for the isothermal process.

Note that work is done during is the isothermal compression that is given by

$$W = nRT \ln\left(\frac{V_f}{V_i}\right) \qquad 12.9$$

Where *T*, is the temperature at which compression takes place, V_f and V_i are the final and initial volumes respectively. The final volume during this process is less than the initial volume and we see that this is consistent with the fact that equation 12.9 will give us a negative number as the natural log of a number less than one is negative.

In the case of the last process the adiabatic process where work is done at the expense of internal energy the equation of the adiabatic is

$$PV^{\gamma} = \text{constant or } P = \frac{\text{constant}}{V^{\gamma}} \qquad 12.10$$

Where γ is as defined in equation 12.7. A plot of this process is in figure 12.7.

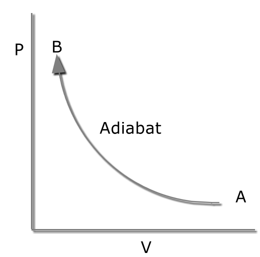


Figure 12.7 PV diagram for the adiabatic process

It should be noted that for adiabatic compression the pressure rises very rapidly when compared to the isothermal process.

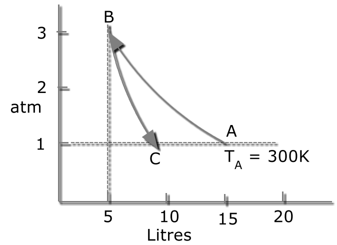
Example 12.2

A sample of air $(\gamma = 1.40)$ is slowly compressed from a pressure of 1 atmosphere to 3 atmospheres. The original volume v_1 of air is 15 litres and the temperature is 300 K. The temperature is constant during compression (isothermal process). Later on the air is suddenly (adiabatically) expanded back to its original pressure of 1 atmosphere.

i) Sketch a PV-diagram of these processes,
ii) find the final volume and temperature,
iii) find the ΔU, Q and W for each process, and
iv) number of moles of the air.

Solution

i)



ii) We can find the initial volume air by considering the isothermal part (AB) on the *PV*-diagram using the ideal gas law.

$$P_A V_A = P_B V_B$$

$$V_B = V_A\left(\frac{P_A}{P_B}\right) = 15\left(\frac{1}{3}\right) = 5\text{litres}$$

The final volume V_C at C has to be calculated using the adiabatic portion for which:

$$P_C V_C^{\gamma} = P_B V_B^{\gamma}$$

$$V_C^{\gamma} = V_C^{\gamma}\left(\frac{P_B}{P_C}\right) \text{ or } V_C = V_B\left(\frac{P_B}{P_C}\right)^{1/\gamma} = 5\left(\frac{3}{1}\right)^{1/1.4} = 10.96\text{litres}$$

To get the temperature at C we use the ideal gas law equation 11.14 from which we get that

$$\frac{P_B V_B}{T_B} = \frac{P_C V_C}{T_C}$$

$$T_C = T_B\left(\frac{P_C}{P_B}\right)\left(\frac{V_C}{V_B}\right) = 300\left(1/3\right)\left(10.96/5\right) = 219.1\text{K}$$

iii) For the isothermal process there is no change in temperature $(\Delta T = 0)$ and therefore there is no change in internal energy $(\Delta U = 0)$.

The work done is given by equation 12.9.

$$W_{AB} = nRT\ln\left(V_B/V_A\right) = P_A V_A \ln\left(V_B/V_A\right) = \left(1.01\times10^5\right)\left(15\times10^{-3}\right)\ln\left(5/15\right) = -1664.4\text{J}$$

We can get Q_{AB} from the first law of thermodynamics.

$$Q_{AB} = \Delta U + W = 0 + (-1664.4) = -1664.4\text{J}$$

For the adiabatic process Q_{BC} = 0 and the result from the first law of thermodynamics is that $W_{BC} = \Delta U_{BC}$.

iv) To find the number of moles of air the ideal gas law equation 11.16 re-written as,

$$n = \frac{P_A V_A}{RT_A} = \frac{(1.01\times10^5)(15\times10^{-3})}{(8.314)(300)} = 0.61\text{mol}$$

12.12 Second Law of Thermodynamics

The second law of thermodynamics is concerned with the conversion of heat energy into usable form i.e. into mechanical energy. Heat energy can be obtained from a variety of sources such as wood, coal, oil etc. In our earlier discussion it was pointed out that heat is governed by the random motion of atoms/molecules of a substance. To convert heat into a usable form we need to extract energy from these random motions of atoms/molecules. This law limits the choice of fuels for engines in general. The second law of thermodynamics is anchored on the flow of heat energy in a given system. In everyday situations we know that ice left in the sun will melt and the temperature of the ice can never increase when left in such an environment. In other words the heat flows into the ice and a reverse flow is physically impossible.

12.13 Statement of the Second Law of Thermodynamics

It states that heat flows spontaneously from a substance at a higher temperature to a substance at a lower temperature and cannot flow spontaneously in the opposite direction. This law is best illustrated in figure 12.8.

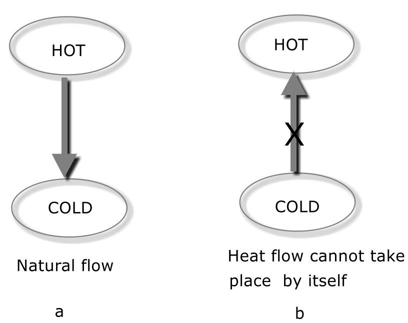


Figure 12.8

12.14 Entropy (Re-statement of the second law of thermodynamics)

The concept of entropy states that if an isolated system made up of many parts is allowed to undergo spontaneous change, it changes in such a way that the disorder increases or, at best, does not decrease. Usually entropy is interpreted in terms of order and disorder of a system. Irreversible processes lead to increase in the entropy of the universe, they lead to energy degradation meaning that part of the energy becomes unavailable to do work. Entropy S (ΔS denotes change in entropy) is a state variable whose change is defined for a reversible process at a temperature T in which a quantity of energy Q is the heat absorbed.

$$\Delta S = \frac{\Delta Q}{T} \qquad 12.11$$

Equation 12.11 is a measure of the amount of energy which is unavailable to do work or a measure of the disorder of a system. The quantity Q is the amount of heat added in a reversible manner to a system at temperature T. It should be noted that reversible processes do not change the total entropy of the universe.

Example 12.3

A 10 g mass of helium gas is expanded by the isothermal process at a temperature of -85 ^{0}C to 4 times its original volume. What is the change in entropy of the helium gas?

Solution

The process is isothermal so ΔT = 0 and ΔU = 0, the from the first law of thermodynamics we get that $Q = W$ and work for this process is

$$W = nRT\ln\left(\frac{V_f}{V_i}\right) = \left(\frac{0.01}{4}\right)(8314)(188)\ln\left(\frac{4}{1}\right) = 5417\text{J} .$$

$$\Delta S = \frac{Q}{T} = \frac{5417}{188} = 28.8\text{J/K}$$

Example 12.4

Find the change in entropy that occurs when 50 g of ice melts slowly at 0 ^{0}C. Consider that heat flows into the ice and regard it as an isolated system.

Solution

The ice is melting at a constant temperature hence the process is isothermal taking place at 273 K. The heat needed for this process can be calculated using equation for the heat of fusion.

$$\Delta Q = mH_f = (50\text{g})(80\text{cal/g})(4.184\text{J/cal}) = 16736\text{J}$$

Entropy can be calculated from equation 12.11.

$$\Delta S = \frac{\Delta Q}{T} = \frac{16736}{273} = 61.3\text{J/K}$$

12.15 Heat Engine

A heat engine is any device that converts heat energy into mechanical energy. We are familiar with various types of engines powered by fuels such petrol, diesel, kerosene and rarely these days by steam. All of the above have one thing in common in that they all transform heat energy into mechanical energy by employing a repetitive cycle. They employ a working substance that is returned to its initial state at the end of each cycle. Figure 12.9 is a general representation of a heat engine that operates between a hot reservoir at temperature T_h and a cold reservoir at temperature T_c. In each operation cycle the engine absorbs an amount of heat Q_h from the hot reservoir. The absorbed heat is partly used to perform work, W, and the remainder, Q_c, is transferred to the cold reservoir.

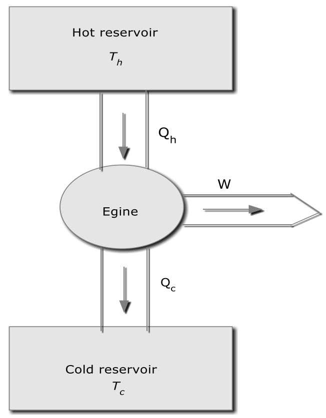


Figure 12.9Schematic heat engine.

A heat engine obeys the law of conservation of energy. It follows that the heat energy from the hot reservoir is portioned between doing work and being exhausted to the cold reservoir. In this case the cold reservoir is the surrounding area. Since a heat engine obeys the law of conservation of energy we can apply to it the first law of thermodynamics equation 12.5.

$Q = \Delta U + W$ or that

$$Q_{net} = Q_h - Q_c = W + \Delta U \qquad 12.12$$

Where W is the work output per cycle and ΔU is the change in the internal energy. A complete cycle results in no net change in internal energy so that ΔU is zero and equation 12.12 reduces to

$$W = Q_h - Q_c \qquad 12.13$$

To find the efficiency of a heat engine we use the definition of efficiency which states that the efficiency of a "machine" is the ratio of its output to its input energy.

$$\text{Efficiency}=\frac{\text{Output}}{\text{Input energy}} \qquad 12.14$$

Substituting for quantities in equation 12.4 and using the letter e for efficiency we get

$$e = \frac{W}{Q_h} = \frac{Q_h - Q_c}{Q_h} = 1 - \frac{Q_c}{Q_h} \qquad 12.15$$

The heat energy Q_c exhausted to the surroundings is the amount of energy that does not take part in doing work and is responsible for the inefficiency of a heat engine. If there was no exhausted energy (i.e. Q_c= 0) to the surroundings then the efficiency of an engine will be hundred percent. It is well known that this is a physical impossibility. We can conclude that there are definite efficiency limits for heat engines.

Example 12.5

A certain engine absorbs 2000 J of heat energy from the hot reservoir and exhausts 800 J to the surroundings during each cycle of operation.

i) What is the efficiency of this engine?
ii) How much work does it perform during each cycle?

Solution

i) The efficiency can be calculated using equation 12.15

$$e = 1 - \frac{Q_c}{Q_h} = 1 - \frac{800}{2000}$$

$$= 1 - 0.4 = 0.6$$

i.e. the engine is 60% efficient.

ii) The work performed during each cycle is equal to the difference between the heat input and the exhausted heat.

$$W = Q_h - Q_c$$

$$W = 2000\ \text{J} - 800\ \text{J} = 1200\ \text{J}$$

12.16 Carnot Engine

A Carnot engine is an idealized engine which sets the maximum efficiency that can be obtained by any engine operating at the same temperature extremes. No machine can be considered as ideal as some heat energy is lost during operation due to friction. The friction losses can be accounted for by the law of conservation of energy. A Carnot engine consists of a gas enclosed in a cylinder with a movable piston and its operation cycle consists of two isothermal processes and two adiabatic processes. In order that we approach Carnot efficiency, the processes involved must be reversible and involve no change in entropy. Such conditions cannot be obtained in reality as real engine processes are not reversible and physical processes lead to increase in entropy. The efficiency of this engine can be calculated using equation 12.15.

12.17 Carnot Cycle

A Carnot cycle is usually represented on a *PV* diagram as, as shown in figure 12.10. Work is done by the engine during the two expansions, and work is done on the engine during the two compressions. The net work done per cycle of operation is the area enclosed by the four processes.

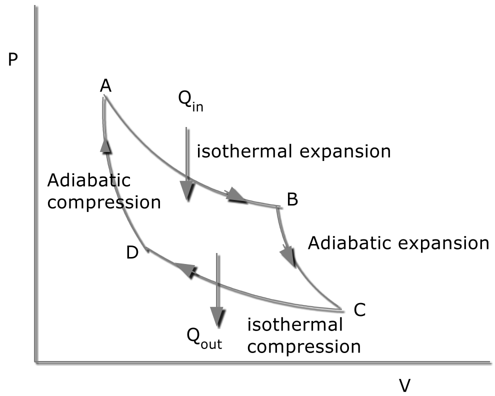


Figure 12.10 Carnot cycle.

The isothermal processes in the diagram are from *A* to *B* and from *C* to *D*. During isothermal expansion *A* to *B* at temperature T_A heat Q_{in} is absorbed from the reservoir and work is done. In the adiabatic expansion *BC* there is no further heat input but work is done at the expense internal energy and the temperature falls to T_C. During the isothermal compression *CD* the engine surroundings are at temperature T_C that is lower than T_A. In this compression heat Q_{out} is released and work is done. To complete the cycle an adiabatic compression *DA* returns the engine to its original state. The work done on the gas (compression) raises the temperature to T_A. The total work done during a cycle is equal to the area enclosed within the curves. The efficiency of a Carnot engine can be calculated from equation 12.15. We know that the net work done in a cycle is equal to the net heat transfer ($Q_{in} - Q_{out}$) as the change in internal energy is zero.

It is a fact that the amount of heat *Q* transferred from a Carnot engine is directly proportional to the absolute temperature *T*.

$$Q \propto T$$

$$\therefore \frac{Q}{T} = \text{constant i.e.}$$

$$\frac{|Q_c|}{|Q_t|} = \frac{T_c}{T_h} \qquad 12.16$$

It follows that the efficiency of a Carnot engine can be stated in terms of temperature by substituting 12.16 into 12.15. The temperature must be in Kelvin for the latter equation to give correct results.

$$e_{\text{Carnot}} = 1 - \frac{T_c}{T_h} \qquad 12.17$$

Example 12.6

A Carnot engine whose hot reservoir is at 550 ^{0}C takes 5000 J of heat per cycle and exhausts 2600 J of heat to the cold reservoir (surroundings).

i) Find the temperature of the cold reservoir, and
ii) Calculate the engine's efficiency by using the temperatures and the heat quantities.

Solution

i) We are given the following: T_h = 550 ^{0}C = 823 K, Q_h = 5000 J and Q_c= 2600 J. We can use equation 12.16 to get the temperature of the cold reservoir.

$$\frac{|Q_c|}{|Q_t|} = \frac{T_c}{T_h} \text{ i.e.}$$

$$T_c = \frac{Q_c}{Q_h}.T_h = \frac{2600}{5000}.823 = 428\text{K}$$

ii) Using equation 12.17

$$e_{\text{Carnot}} = 1 - \frac{T_c}{T_h}$$

$$e_{\text{Carnot}} = 1 - \frac{428}{823} = 0.48$$

Using equation 12.15

$$e = 1 - \frac{Q_c}{Q_h} = 1 - \frac{2600}{5000} = 0.48$$

Note that in both cases we get the same value of the efficiency, however this only holds for Carnot engine. Any other engine operating at the same temperature extremes will result in a lower value of the efficiency.

12.18 Refrigerators and Heat Pumps

A refrigerator operates in reverse fashion to a heat engine to extract heat from a low temperature reservoir and transfer it to the high temperature reservoir. The natural tendency of heat is to flow from a hot region to colder region. The operation of the refrigerator is contrary to what the second law of thermodynamics states that heat cannot flow from a cold region to hot region on its own accord. However if energy is supplied then heat can be forced to flow from a low temperature region to a high temperature region. Refrigerators, air conditioners and heat pumps are devices that do precisely that. Figure 12.11 is a schematic representation of a refrigerator.

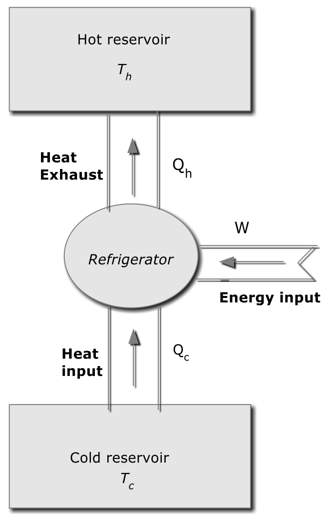


Figure 12.11 Refrigeration

The process in which work is done on the system to lower the temperature of a substance is known as a refrigeration cycle. In this case the energy flow as shown in figure 12.11 is the reverse of a heat engine. A refrigerator operates at temperatures T_c and T_h where the work input W allows heat at a low temperature Q_c to be moved to the high temperature region of the system. According to the first law of thermodynamics that is for energy to be conserved, the energy into the system must be equal to the energy out of the system.

Energy in = Energy out

$$Q_c + W = Q_h \qquad 12.18$$

In order for the refrigeration cycle to be very effective (efficient), it depends on removing large amounts of heat for very little input work.

In refrigeration we specify the effectiveness of the system in terms of the *coefficient of performance*, COP in place of the efficiency. This quantity is a ratio of the heat removed from the cold reservoir Q_c to the work input W.

$$\text{COP} = \frac{Q_c}{W} \qquad 12.19$$

If we re-arrange equation 12.18 so that $W = Q_h - Q_c$ and then we substitute this in equation 12.19 we get:

$$\text{COP}_{\text{refrigerator}} = \frac{Q_c}{Q_h - Q_c} \qquad 12.20$$

Referring to equation 12.16 where the quantity of heat energy involved is taken to be directly proportional to the absolute temperature in Kelvin, then we can re-state equation 12.20 in terms of absolute temperature.

$$\mathrm{COP_{Max}} = \frac{T_c}{T_h - T_c} \qquad 12.21$$

The latter equation implies that when the difference between T_c and T_h is small, less work is needed to extract heat from the cold reservoir that is exhausted into the hot reservoir. This equation also gives the maximum coefficient of performance.

12.19 Heat Pumps

Heat pumps such as air-conditioners operate in a similar manner to refrigerators. These can also be used to heat up the interior of buildings during the cold season in moderate climates, in designs that have a mechanism of reversing the flow of energy between the reservoirs. In this case the heat pump heats up the cold interior as opposed to cooling it. It follows that the coefficient of performance is the ratio of heat Q_h delivered into the interior to the work W needed to deliver it.

$$\text{Heat pump} \quad \mathrm{COP} = \frac{Q_h}{W} \qquad 12.22$$

Since the work done is $W = Q_h - Q_c$ we can substitute the term on the right side of this equation into equation 12.22.

$$\mathrm{COP_{Heat\ pump}} = \frac{Q_h}{Q_h - Q_c} \qquad 12.23$$

We can in a similar manner to a refrigerator define the maximum COP by substituting with temperatures of the reservoirs in equation 12.23. However, this is a theoretical COP that can never be achieved in reality.

Example 12.7

What mass of water at 0^0 C can a freezer with a coefficient of performance (COP) of 4.5 change into ice-cubes at 0^0 C with a work input of 3.0×10^6 J. The latent heat of fusion, H_f, is 335 kJ/kg.

Solution

From equation 12.19 $\mathrm{COP} = \frac{Q_c}{W}$, we need to calculate the energy used by the freezer using equation 11.23.

$$Q = mH_f$$

$$\text{COP}=\frac{mH_f}{W}=\frac{m\times 335\times 10^3}{3.0\times 10^6}$$

$$m=\frac{4.5\times 3.0\times 10^6}{335\times 10^3}=40.3\ \text{kg}$$

The University of Zambia
School of Natural Sciences
Department of Physics
PHY 1010
Lecture 12B
First Law of Thermodynamics

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October 2, 2020

1 First Law of Thermodynamics

The **first law of thermodynamics** states that the change in internal energy $\Delta U = U_2 - U_1$ of a system as it moves from one state to another on a PV diagram is equal to the difference between the heat supplied to (or removed from) the system ΔQ and the work done by (or done on) the system ΔW

$$\Delta U = \Delta Q - \Delta W \tag{1}$$

where

ΔU is the change in the internal energy of the system

ΔQ is the heat supplied to (or removed from) the system. If the heat is supplied to the system, ΔQ is taken to be positive. If the heat is removed from the system then ΔQ is taken to be negative.

ΔW is the work done by (or done on) the system. If the system expands and does work on the surrounding, then ΔW is taken to be positive. If the system contracts and the surrounding does work on the system, then ΔW is taken to be negative.

The change in internal energy ΔU depends only on the initial and final states of the system. For a system of an ideal gas in a cylinder with a movable piston, the internal energy of the system can be changed by

(a) putting the system on top of a hot reservoir (e.g. hot plate),

(b) allowing the system to do work on its surroundings or doing work on the system.

The internal energy of an ideal gas depends only on absolute temperature T
.

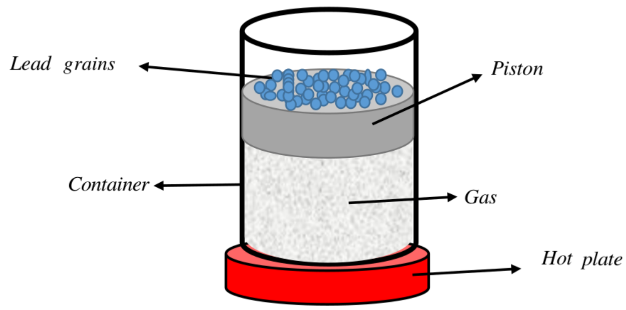

Figure 1: A schematic illustration of a ideal gas in cylinder with movable piston on a hot plate

2 First law of thermodynamics and work done by an ideal gas

Consider a system of an ideal gas in a cylinder with a movable piston allowed to do work on its surroundings as shown in Figure **??**. The movable piston has cross section area A and exerts pressure P on the gas. The force exerted by the ideal gas on the piston is $F = PA$. The system is put on a hot plate at absolute temperature T, causing thermal energy to be added to the system in the form of heat. The ideal gas expands and pushes the piston upwards by a distance dy
. The work done by the expanding ideal gas on its surroundings is given by

$$\Delta W = Fdy = PAdy = P\Delta V \tag{2}$$

where

P is the pressure exerted by the ideal gas on the movable piston

ΔV is the change in volume of the cylinder.

Substituting Equation (2) into Equation (1), the first law of thermodynamics becomes

$$\Delta U = \Delta Q - P\Delta V \tag{3}$$

If the ideal gas in the cylinder is put on a hot plate, heat is added to the system. If the ideal gas expands, pushing on the movable piston up, the ideal gas does work on its surroundings and the system loses energy. If the movable piston pushes down on the ideal gas, work is done of the gas by its surroundings and the system gains energy. The work done is positive if the ideal gas expands and negative if the ideal gas is compressed.

If the movable piston is nailed down so that it cannot move while the cylinder with ideal gas has been put on top of a hot plate, the term $P\Delta V$ in Equation (3) vanishes. In such as case, the first law of thermodynamics reduces to

$$\Delta U = \Delta Q \tag{4}$$

Equations (4) shows that the heat supplied by the hot plate into the system goes into changing the internal energy of the system.

If the system is thermally isolated so that no heat can flow in or out of the system from or to the surrounding, then the heat supplied or removed from the system ΔQ in Equation (3) vanishes. If the piston is free to move, you can either have a volume increase or a volume decrease. In this case, where the system is thermally isolated, the first law of thermodynamics as given by Equation (3) reduces to

$$\Delta U = -P\Delta V \tag{5}$$

If the ideal gas in the cylinder with a movable piston expands, the change in volume ΔV is positive and the work done by the system on its surroundings is positive. However, this positive work results in a negative change in internal energy ΔU. The ideal gas pays for doing positive work against the movable piston through a loss of internal energy U.

On the other hand, if the ideal gas is compressed by a movable piston, the change in volume ΔV is negative and the work done by the gas is negative. The surrounding does work on the system resulting in a positive change in internal energy ΔU. The ideal gas molecules colliding with the movabel piston during compression causes the ideal gas to gain kinetic energy.

3 Work done by an ideal gas under isothermal process

If an ideal gas in a cylinder with a movable piston is put on top of a hot plate at absolute temperature T. The absolute temperature of the system will remains constant while ideal gas undergoes quasistatic expansion. This is an **isothermal process** i.e., a constant temperature process. The PV diagram shown in Figure depicts an isotherm (i.e. a constant temperature line on a PV diagram) at absolute temperature T

The total work done by the ideal gas during the expansion is equal the total area under the PV diagram. without doing a lot of calculus, we give the work done ΔW by the ideal gas during an isothermal process as

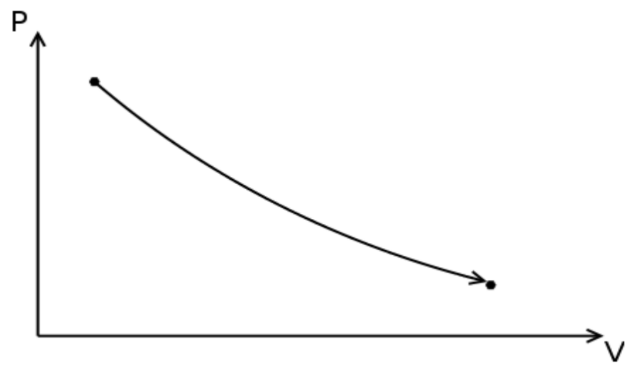

Figure 2: PV diagram for an isothermal expansion of an ideal on an isotherm at absolute temperature T.

$$\Delta W = nRT \ln \left(\frac{V_1}{V_2} \right) \tag{6}$$

where

n is the number of moles of the ideal gas

R is the gas constant

T is the absolute temperature of the ideal gas

V_1 is the initial volume of the ideal gas

V_2 is the final volume of the ideal gas after expansion or compression.

For an isothermal process, the ideal gas does not undergo a change in absolute temperature T. Therefore, the internal energy U of the initial state and final state which lie on the same isotherm is the same, thus there is no change in internal energy i.e. $\Delta U = 0$ The first law of thermodynamics given by Equation (3), for an isothermal process becomes

$$0 = \Delta Q - P\Delta V \tag{7}$$

Taking the $P\Delta V$ in Equation (7) to the left hand side, we get

$$\Delta Q = P\Delta V \tag{8}$$

where

ΔQ is the heat supplied into the system

$P\Delta V$ is the work done by the system on its surroundings during this isothermal process.

Since the work done by an ideal gas on its surroundings during an isothermal process is given by Equation (6), we substitute into Equation (8) and get

$$\Delta Q = nRT \ln\left(\frac{V_1}{V_2}\right) \tag{9}$$

Equation (9) shows that the heat supplied into the system from the hot plate during isothermal expansion is used to do work against the surrounding without changing the internal energy of the system.

4 Work done by an ideal gas under isochoric process

Under an **isochoric process** (i.e. constant volume process), the volume of the system is not allowed to change because the piston is nailed down. The system does not do any work on its surroundings nor can the surrounding do work on the system. If the movable piston is clamped and the cylinder put on a hot plate, the $P\Delta V$ term in the first law of thermodynamics given by Equation (3) vanishes since $\Delta V = 0$. The work done by an ideal gas under any isochoric process is always zero. The first law of thermodynamics given by Equation (3) reduces to

$$\Delta U = \Delta Q \tag{10}$$

The heat energy ΔQ supplied from the hot plate into the system goes into changing the internal energy ΔU of the system. The absolute temperature T of the system is raised.

5 Work done by an ideal gas under isobaric process

The total work done during an **isobaric process** (i.e. constant pressure process) as the system moves from an initial state at absolute temperature T_1 to a final state at absolute temperature T_2 is the total area under the PV diagram. Since the pressure P is constant, we get the work done by the gas from only the change in volume of the ideal gas ΔV. If the ideal gas changes volume from initial volume V_1 to final volume V_2, we get the work done by the ideal gas on its surrounding as

$$\Delta W = P\Delta V = P(V_2 - V_1) \tag{11}$$

where

P is the pressure the ideal gas exerts on the movable piston

V_1 is the initial volume of the ideal gas

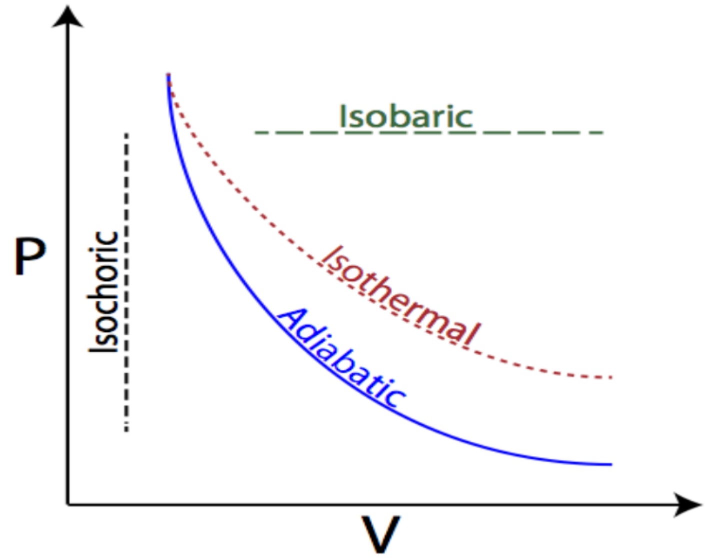

Figure 3: PV diagram for various processes : an **isochoric**, **isobaric**, **isothermal** and **adiabatic** processes

V_2 is the final volume of the ideal gas

The change in internal energy ΔU of the ideal gas exerting a pressure P on the movable piston and undergoing a change in volume from V_1 to V_2 is given by

$$\Delta U = \frac{f}{2} P \Delta V = \frac{f}{2} P (V_2 - V_1) \tag{12}$$

Therefore, the first law of thermodynamics given by Equation (3) becomes

$$\frac{f}{2} P \Delta V = \Delta Q - P \Delta V \tag{13}$$

Collecting the terms in Equation (13) with $P\Delta V$, we get

$$\Delta Q = \frac{f}{2} P \Delta V + P \Delta V \tag{14}$$

From Equation (14), we see that the heat supplied ΔQ into the system from the hot plate during an **isobaric procoess** is used to do work by the ideal gas on its surroundings and to increase the internal energy of the system.

Factoring out the $P\Delta V$ from Equation (14), we get

$$\Delta Q = \left(\frac{f}{2} + 1 \right) P \Delta V \tag{15}$$

Simplying the factor in the brackets, we get

$$\Delta Q = \left(\frac{f+2}{2}\right) P\Delta V \tag{16}$$

For a monoatomic gas , the degrees of freedom is $f = 3$, therefore we get

$$\Delta Q = \left(\frac{3+2}{2}\right) P\Delta V = \frac{5}{2} P\Delta V \tag{17}$$

For a diatomic gas , the degrees of freedom is $f = 5$, therefore we get

$$\Delta Q = \left(\frac{5+2}{2}\right) P\Delta V = \frac{7}{2} P\Delta V \tag{18}$$

6 Work done by an ideal gas under a cyclic process

A **cyclic process** is a process that starts and ends in the same state on a PV diagram. Since the initial and final state in a cyclic process are the same, the change in internal energy $\Delta U = 0$. For a cyclic process, the first law of thermodynamics given by Equation (3) reduces to

$$0 = \Delta Q - P\Delta V \tag{19}$$

Simplifying Equation (19), we get

$$\Delta Q = P\Delta V \tag{20}$$

The heat supplied ΔQ to the system in a cyclic process goes entirely into doing work on the surroundings. The work done is equal to the area enclosed by the path on the PV diagram.

The following convention is adopted for work done and heat supplied under cyclic process. For a clockwise path, ΔQ and ΔW are positive. The system absorbs heat from the hot plate and does work on its surroundings. For a counterclockwise path, ΔQ and ΔW are negative, work is done on the system and heat is removed from the system.

For a cyclic process composed of isothermal expansion process $A \rightarrow B$, isobaric compressive process $B \rightarrow C$and isochoric process $C \rightarrow A$, the area enclosed by the path shown in Figure is the work done by the ideal gas during this cyclic process.

The work done during **isothermal** expansion process $A \rightarrow B$ is positive since $V_B > V_A$ and is given by

$$\Delta W_{A\rightarrow B} = nRT \ln\left(\frac{V_B}{V_A}\right) \tag{21}$$

The work done under **isobaric** compressive process $B \rightarrow C$ is negative since $V_C < V_B$ and is given by

$$\Delta W_{B\rightarrow C} = P(V_C - V_B) \tag{22}$$

The work done during the isochoric process $C \rightarrow A$ is zero since ΔV and there is no area under the $C \rightarrow A$process. Thus we have

$$\Delta W_{C\rightarrow A} = 0 \tag{23}$$

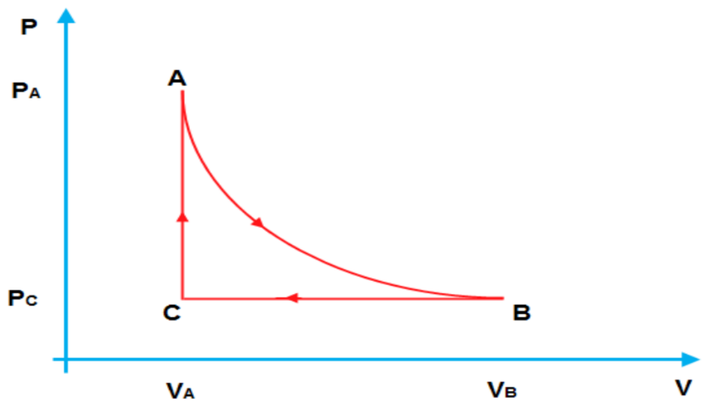

Figure 4: PV diagram showing a cyclic process composed of **isothermal** expansion process, **isobaric** compression process and an **isochoric** process

The **net work done** by the system during the entire cyclic process from $A \rightarrow B \rightarrow C \rightarrow A$ is positive and is given by

$$\Delta W = \Delta W_{A\rightarrow B} + \Delta W_{B\rightarrow C} + \Delta W_{C\rightarrow A} \tag{24}$$

$$\Delta W = nRT \ln\left(\frac{V_B}{V_A}\right) + P(V_C - V_B) + 0 \tag{25}$$

$$\Delta W = nRT \ln\left(\frac{V_B}{V_A}\right) + P(V_C - V_B) \tag{26}$$

An alternate cyclic path going in the opposite direction composed of **isochoric** process $A \rightarrow C$, **isobaric** expansion $C \rightarrow B$ and **isothermal** compression process $B \rightarrow A$. The work done during the isochoric process $A \rightarrow C$ is zero since $\Delta V = 0$ and there is no area under the $A \rightarrow C$ process. Thus we have

$$\Delta W_{A\rightarrow C} = 0 \tag{27}$$

The work done under **isobaric** expansion process $C \to B$ is positive since $V_B > V_C$ and is given by

$$\Delta W_{C \to B} = P(V_B - V_C) \tag{28}$$

The work done during **isothermal** compression $B \to A$ is negative since $V_A < V_B$ and is given by

$$\Delta W_{B \to A} = nRT \ln \left(\frac{V_A}{V_B} \right) \tag{29}$$

The **net work done** by the system during the entire cyclic process from $A \to B \to C \to A$ is positive and is given by

$$\Delta W = \Delta W_{A \to C} + \Delta W_{C \to B} + \Delta W_{B \to A} \tag{30}$$

$$\Delta W = 0 + P(V_B - V_C) + nRT \ln \left(\frac{V_A}{V_B} \right) \tag{31}$$

$$\Delta W = P(V_B - V_C) + nRT \ln \left(\frac{V_A}{V_B} \right) \tag{32}$$

The University of Zambia
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PHY 1010
Lecture 13
Simple Harmonic Motion and Forced Vibrations

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1 Introduction

This lesson introduces you to Simple Harmonic Motion (SHM) or Simple Harmonic Oscillations (SHO). SHM is of great importance in physics because it is exhibited by many physical systems. This lesson will show you what SHM is and pinpoint the causes SHM in the absence of friction for a mass attached to a spring. The lesson will also show you how to obtain the position, velocity and accceleration of a spring-mass system undergoing SHM in the absence of friction.

2 Learning Outcomes

By the end of this lesson, the student should be able to:

1. Define simple harmonic motion;
2. Calculate the periodic time, frequency, and natural frequency, maximum velocity of a spring-mass harmonic oscillator
3. Calculate the position, velocity and accceleration of a spring-mass harmonic oscillator;

3 Introduction to Simple Harmonic Motion

SHM in physics is very important because it is exhibited by many physical systems. SHM involves repetitive movement of an object back and forth about a position of equilibrium. During SHM, the displacement of the oscillating object about the position of equilibrium on one side is equal to the displacement on the other side.

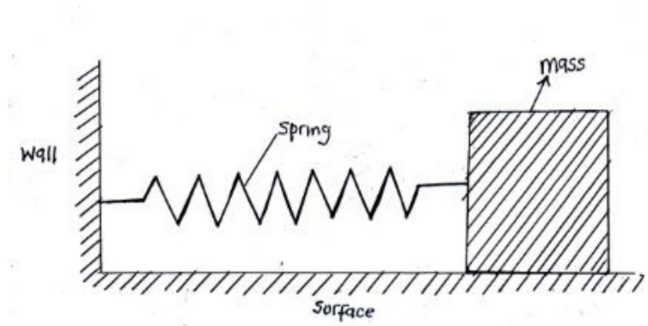

Figure 1: Schematic illustration of a spring-mass system on a frictionless surface.

One of the most common systems that exhibits SHM is a spring-mass system consisting of a mass hanging from a spring attached to a roof or a mass attached to a spring on a frictionless surface. Figure 1 depicts a mass attached to the spring that has come to rest after a long while moving back and forth on a frictionless surface. The spring- mass system remains motionless (stationary) in this position and is said to be in equilibrium because there is no net force causing it to move. If the spring-mass system remains undisturbed, it will continue to occupy this equilibrium position and remain stationary.

Let us consider a spring-mass system on in Figure 2 below in which a mass m is attached to a spring with spring constant k on a frictionless surface. Assume that the spring obeys Hooke's law and let it have a relaxed length or extension of $x = 0$ at stable equilibrium. If the mass is pulled by a deforming force F, it stretches the spring over a distance x . If the mass is suddenly released after being pulled, the mass attached to the spring will execute back and forth motions about the equilibrium point $x = 0$ (i.e. the mass-spring system executes oscillatory motions). The deforming force $F = kx$ causes a restoring force $F = -kx$ to be set up in the spring, causing the system to oscillate about $x = 0$.

From Newton's second law of motion, the motion of the oscillating spring-mass system is being caused by the restoring force $F = -kx$. Therefore, we can write Newton's Second Law of Motion in terms of the mass attached to the spring m, the acceleration of the mass a and the restoring force $F = -kx$ as

$$F = ma \tag{1}$$

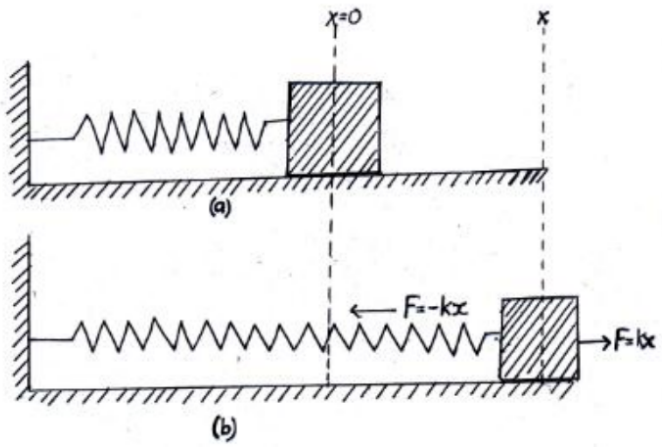

Figure 2: Schematic illustration of (a) a relaxed spring-mass system and (b) a stretched spring-mass system on a frictionless surface

$$-kx = ma \tag{2}$$

Making the acceleration a the subject of the formula, we get

$$a = -\frac{k}{m}x \tag{3}$$

So now we have the acceleration of the mass a in terms of the spring constant k and the mass m.

Next, we proceed to set $\frac{k}{m} = \omega^2$ so that we have

$$\omega = \sqrt{\frac{k}{m}} \tag{4}$$

This ω is called the **natural frequency** of the spring mass oscillator.

The **natural frequency** is related to the **frequency** f of a simple harmonic oscillator as follows:

$$\omega = 2\pi f \tag{5}$$

where f is the frequency measured in Hertz (Hz).

Making f the subject of the formula, we get

$$f = \frac{\omega}{2\pi} \tag{6}$$

The **periodic time** τ of a simple harmonic oscillator is related to the frequency f as follows:

$$\tau = \frac{1}{f} \tag{7}$$

Substituting for f, we get τ as

$$\tau = \frac{2\pi}{\omega} \tag{8}$$

Substituting for ω, we get the periodic time τ as

$$\tau = \frac{2\pi}{\sqrt{\frac{k}{m}}}$$

$$\tau = 2\pi\sqrt{\frac{m}{k}} \tag{9}$$

Substituting Equation (4) into Equation (3), we get the acceleration of the spring mass oscillator as

$$a = -\omega^2 x \tag{10}$$

Now, it is possible to write the acceleration a in terms of the displacement x as a second order derivative of time t

$$v = \frac{\Delta x}{\Delta t} \tag{11}$$

$$a = \frac{\Delta v}{\Delta t} = \frac{\Delta}{\Delta t}\left(\frac{\Delta x}{\Delta t}\right) = \frac{\Delta^2 x}{(\Delta t)^2} \tag{12}$$

So very very small changes in x and t, the acceleration a becomes

$$a = \frac{d^2 x}{dt^2} \tag{13}$$

Substituting back into the equation for acceleration we get,

$$\frac{d^2 x}{dt^2} = -\omega x \tag{14}$$

$$\frac{d^2 x}{dt^2} + \omega x = 0 \tag{15}$$

This Equation (15) is something called a **differential equation** in Advanced Mathematics. This is the kind of equation that describes all forms of simple harmonic equation.

The solution to Equation (15) is the function $x = f(t)$ and is given by

$$x = x_o \cos(\omega t + \epsilon) \tag{16}$$

where

x is the position or displacement of the spring - mass oscillator at time t from $x = 0$

x_o is the maximum displacement of the spring mass oscillator from stable equilibrium $x = 0$ called **amplitude**.

ω is the natural frequency of the spring mass oscillator

ϵ is called the **phase angle**. The phase angle depends on the **initial conditions** of the spring mass oscillator system. Often the phase angle is zero i.e $\epsilon = 0$.

$$x = x_o \cos(\omega t) \tag{17}$$

Equation (16) shows that the displacement x of a simple harmonic oscillator changes with time following the sine or cosine and is not constant. The sine function motion of the simple harmonic oscillator is also known as **sinusoidal motion**.

If the spring - mass system is oscillating in the y-direction, then its motion is given by

$$y = y_o \cos(\omega t) \tag{18}$$

The velocity of a simple harmonic oscillator at time t is obtained by differentiating the extension x with respect to time t and is given by

$$v = \frac{dx}{dt} = \frac{d}{dt}\left(x_o \cos(\omega t)\right)$$

$$v = -\omega x_o \sin(\omega t) \tag{19}$$

The acceleration of the simple harmonic oscillator at time t is obtained by differentiating the velocity v with respect to time t and is given by

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(-\omega x_o \sin(\omega t)\right)$$

$$a = -\omega^2 x_o \cos(\omega t) \tag{20}$$

From these above equations, we are specifically intrested in the maximum value of displacement, velocity and acceleration. These occur when the $\sin(\omega t)$ and $\cos(\omega t)$ are equal to 1.

Therefore, we can get the maximum displacement in the x or y-directions as

$$x_{\max} = x_o, \quad y_{\max} = y_o \tag{21}$$

The maximum velocity of the simple harmonic oscillator is given by

$$v_{\max} = -\omega x_o \tag{22}$$

The maximum acceleration of the simple harmonic oscillator is given by

$$a_{\max} = -\omega^2 x_o \tag{23}$$

Example 1

A particular spring stretches 0.2 m when a 0.5 kg mass is hung from it. Suppose that this mass is replaced by a 2.0 kg mass and the system is then vibrated horizontally as shown in Figure by displacing the mass 0.40 m from its equlibrium position and releasing it. Find

(i) the spring constant k

The spring constant k is obtained from Hooke's Law using the deforming force $F_\perp$ and the extension x. We get

$$k = \frac{F_\perp}{x}$$

where

$$F_\perp = (0.5 \text{ kg}) \times (9.8 \text{ m/s}^2) = 4.9 \text{ N}$$

and extension $x = 0.2$ m.

Therefore, we get spring constant k as

$$k = \frac{F_\perp}{x} = \frac{4.9 \text{ N}}{0.2 \text{ m}}$$

$$k = 24.5 \text{ N/m}$$

(ii) the natural frequency ω of the spring mass oscillator.

The simple harmonic oscillations of the spring mass oscillator are caused when the 2.0 kg mass is displaced from its equilibrium position and released. The spring- mass oscillator begins to go back and forth about its equilibrium position. The natural frequency ω of the oscillator is given

$$\omega = \sqrt{\frac{k}{m}}$$

$$\omega = \sqrt{\frac{24.5 \text{ N/m}}{2.0 \text{ kg}}} = \sqrt{\frac{24.5 \text{ kg m s}^{-2}\text{/m}}{2.0 \text{ kg}}} = \sqrt{12.25 \text{ / s}^2}$$

$$\omega = 3.5 \text{ rad/s}$$

(iii) the frequency of vibration of the system

The frequency of an oscillator f can be obtained from the natural frequency ω as follows

$$\omega = 2\pi f$$

Making f the subject of the formula, we get

$$f = \frac{\omega}{2\pi}$$

We get the frequency f as

$$f = \frac{3.5 \text{ rad/s}}{2\pi \text{ rad}} = 0.557 \text{ /s}$$

$$f \simeq 0.56 \text{ Hz}$$

(iv) the periodic time τ of the spring-mass oscillator

The periodic time τ is given by

$$\tau = \frac{1}{f}$$

Substituting, we get

$$\tau = \frac{1}{0.56 \text{ Hz}} \simeq 1.79 \text{ s}$$

(v) Plot the the position of the spring-mass oscillator over a time period of 5 seconds.

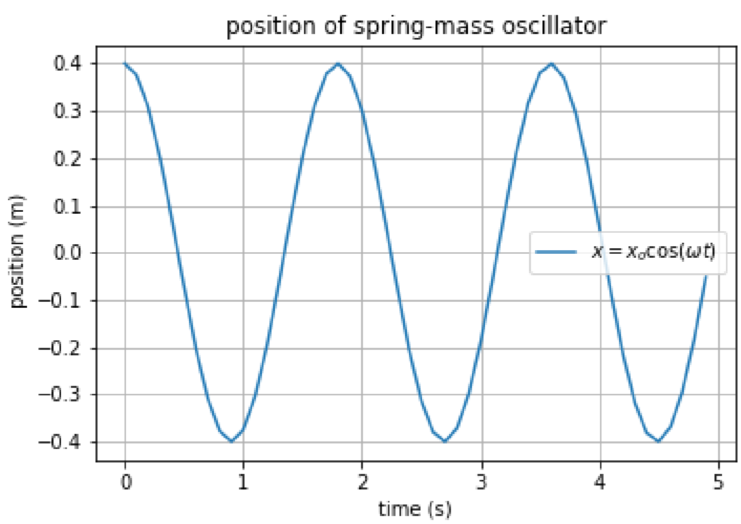

Figure 3: Position of spring mass oscillator over a time of 5 seconds

(vi) the maximum position of the mass,

We get the position of the spring mass oscillator from

$$x = x_o \cos(\omega t)$$

To get the maximum position $x_{\max}$, we set $\cos(\omega t) = 1$. Therefore, we get

$$x_{\max} = x_o = 0.4 \text{ m}$$

This value of $x_{\max}$ is matches the amplitude of the plot of position.

(vii) Plot the the velocity of the spring mass oscillator over a time period of 5 seconds.

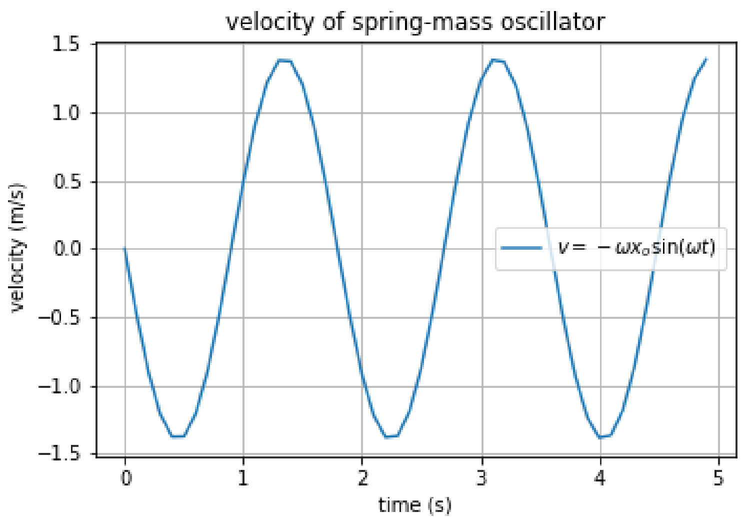

Figure 4: Velocity of spring-mass oscillator over a time of 5 seconds

(viii) The maximum velocity of the mass,

The velocity of the spring-mass oscillator is given by

$$v = -\omega x_o \sin(\omega t)$$

To get the maximum velocity $v_{\max}$, we set $\sin(\omega t) = 1$

$$v_{\max} = -\omega x_o$$

$$v_{\max} = -(3.5 \text{ rad/s})(0.4 \text{ m})$$

$$v_{\max} = -1.4 \text{ m/s}$$

(ix) Plot the the acceleration of the spring-mass oscillator over a time period of 5 seconds.

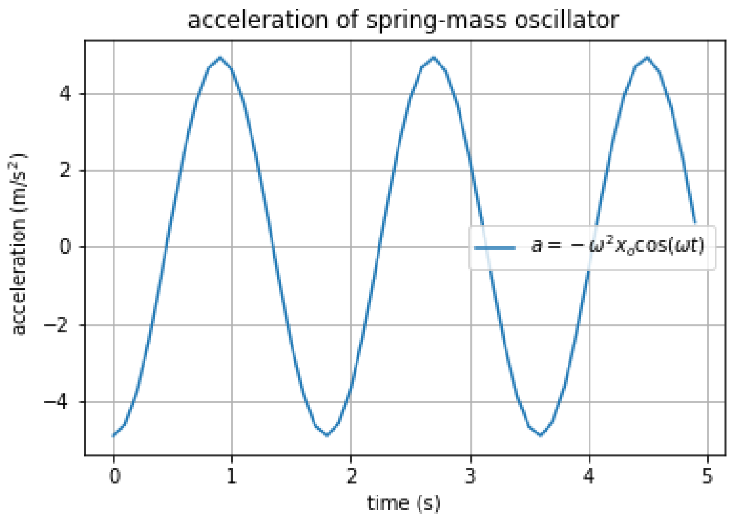

Figure 5: Acceleration of spring-mass oscillator over a time of 5 seconds

(x) its maximum acceleration,

We get the acceleration from the equation

$$a = -\omega^2 x_o \cos(\omega t)$$

To get the maximum acceleration, we set $\cos(\omega t) = 1$. Therefore, we get acceleration as

$$a_{\text{max}} = -\omega^2 x_o$$

$$a_{\text{max}} = -(3.5 \text{ rad/s})^2 (0.4 \text{ m})$$

$$a_{\text{max}} = -4.9 \text{ m/s}^2$$

(d) the position, velocity and acceleration at of the mass at $t = 1.0$ s

The position of the spring-mass oscillator after $t = 1$ s is

$$x = x_o \cos(\omega t)$$

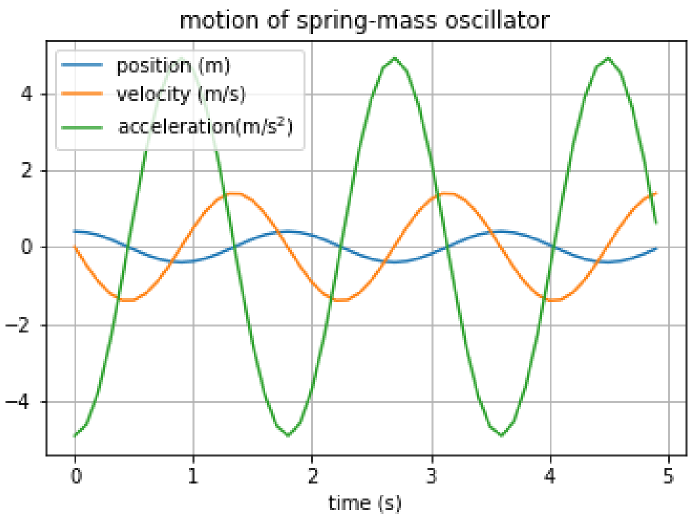

Figure 6: Motion of spring-mass oscillator over a time of 5 seconds

$$x(t = 1\text{ s}) = (0.4\text{ m})\cos(\ (3.5\text{ rad/s})\cdot(1\text{ s})\ )$$

$$x(t = 1\text{ s}) = -0.37\text{ m}$$

The velocity of the spring-mass oscillator after $t = 1$ s is

$$v = -\omega x_o \sin(\omega t)$$

$$v(t = 1\text{ s}) = -(3.5\text{ rad/s})(0.4\text{ m})\sin(\ (3.5\text{ rad/s})\cdot(1\text{ s})\ )$$

$$v(t = 1\text{ s}) = 0.49\text{ m/s}$$

The acceleration of the spring-mass oscillator after $t = 1$ s is

$$a = -\omega^2 x_o \cos(\omega t)$$

$$a(t = 1\text{ s}) = -(3.5\text{ rad/s})^2(0.4\text{ m})\cos(\ (3.5\text{ rad/s})\cdot(1\text{ s})\ )$$

$$a(t = 1\text{ s}) = 4.59\text{ m/s}^2$$

(e) its velocity and acceleration when $x = 0.10$ m.

4 The Energy of a Simple Harmonic Oscillator

When the spring is stretched to maximum, work is done on the spring and it undergoes a tensile strain. The work done on the spring is stored as elastic potential energy. When the mass is released and its starts to move, the elastic potential energy stored in the spring is converted into kinetic energy of the moving mass. The total elastic potential energy U_T is given by

$$U_T = \frac{1}{2}Fx_o = \frac{1}{2}(kx_o)x_o = \frac{1}{2}kx_o^2 \tag{24}$$

This total elastic potential energy is converted in kinetic energy and some elastic potential energy at x.

$$U_T = U(x) + KE \tag{25}$$

$$\frac{1}{2}kx_o^2 = \frac{1}{2}kx^2 + \frac{1}{2}mv^2 \tag{26}$$

We proceed to cancel out the $\frac{1}{2}$ from both sides and get

$$kx_o^2 = kx^2 + mv^2$$

Collecting the terms with k on one side we get

$$mv^2 = kx_o^2 - kx^2$$

Factor out the k on the RHS, we get

$$mv^2 = k(x_o^2 - x^2)$$

Divide on both sides by m, we get

$$v^2 = \frac{k}{m}(x_o^2 - x^2)$$

Taking the square root on both sidem we get

$$v = \sqrt{\frac{k}{m}(x_o^2 - x^2)}$$

$$v = \sqrt{\frac{k}{m}}\ \sqrt{(x_o^2 - x^2)}$$

$$v = \omega\sqrt{(x_o^2 - x^2)} \tag{27}$$

$$v = \pm\ \omega\sqrt{(x_o^2 - x^2)} \tag{28}$$

5 Damped and Forced Vibrations

In any vibrating system, there is always some loss of energy due to friction. As a result, a mass attached to the end of a spring vibrates with constantly decreasing amplitude as time goes by. Figure 7 shows the position of a spring-mass oscillator that is vibrating in the ideal cases with no friction.

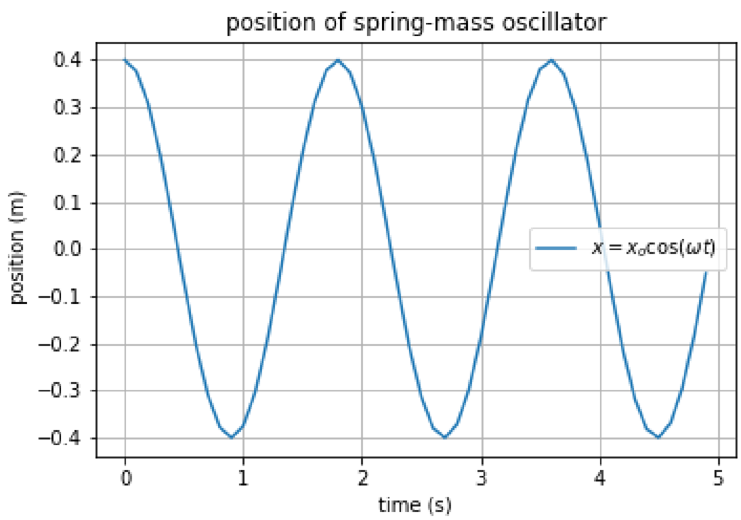

Figure 7: Position of spring mass oscillator over a time of 5 seconds

In a more realistic situation, the motion of our spring-mass oscillator is influencced by friction. if the spring-mass oscillator is hung from the roof and vibrates in air, then the friction is provided by the air. If the spring- mass oscillator is vibrating on a very smooth surface, then friction comes from the interaction between the horizontal surface and the mass. In either case, the amplitude of the spring-mass oscillator will be seen to decrease constantly with time until the oscillator becomes motionless. We say such as system is damped and, that in this case, the amplitude of of the vibrations damp down fairly quickly.

When the friction forces are very large, the spring-mass oscillator does not vibrate at all; instead it simply returns slowly to its equilibrium position as shown in Figure. Such a system is said to be **overdamped**. This situation exists, for example, when the mass at the end of the spring is immersed in a very viscous fluid. The mass does not move beyond the equilibrium position in such as case. When the friction forces are just large enough that the system returns to the equilibrium position without overshooting it, we say that the system is **critically damped**.

If any system is to vibrate for an extended time, energy must be added continuously to replace the energy lost doing work against friction forces. For

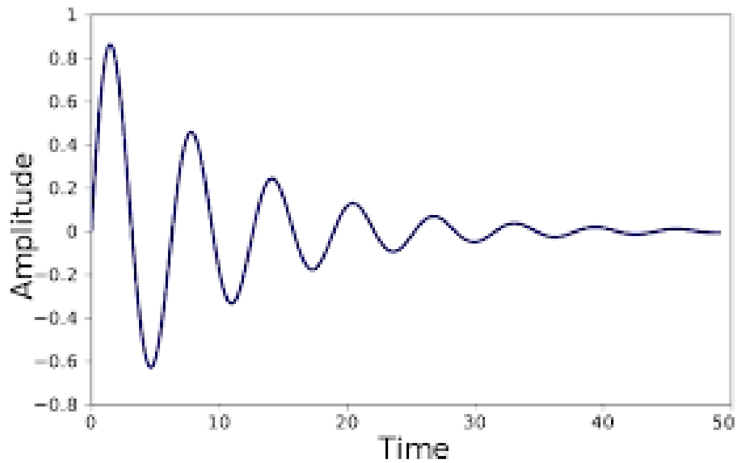

Figure 8: Damped oscillations or vibrations over a time in the presence of little friction

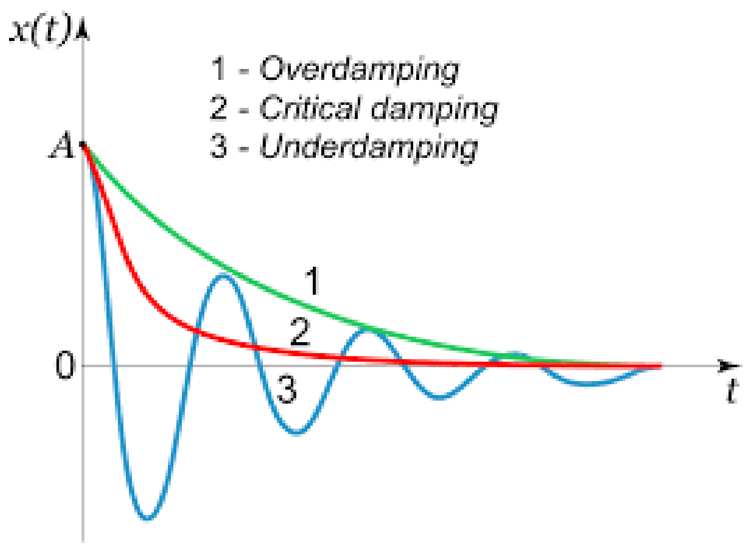

Figure 9: Types of damped vibrations in the presence of varying amounts of frictions

example, to keep a child swinging at constant amplitude on a swing, you must push the swing from time to time to add energy to the system (i.e. the child and swing). In addition, everyone knows that there is a right and wrong way to push a swing if it is to swing high. You must push with the motion of a the swing and not against it. Only in this way can energy be added to the system effectively. if you push against the motion, you can stop the vibration, since the vibrating object must then do work on you, the pushing agent. Pushing a swing is an example of a **driven vibration** or **forced vibration**.

In a driven system, the vibration is usually sustained by a repetitive driven force acting on the system. This force has a frequency f, which may or may not be the same as the natural frequency of the vibrating system f_o. When the $f = f_o$, the drving agent is most effective at adding energy to the system. At all other frequencies, the driving force is not quite in step with the motion of the system, so the action of the force is less effective in adding energy. The drving force is most effective when the its frequency f is equal to the natural frequency of the of the system f_o. In this case, we say that the force is in **resonance** with the system. We call the frequency f_o the **resonance frequency** of the system.

The University of Zambia
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PHY 1010
Lecture 14
Waves

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1 The Origin of Waves and Wave Terminology

Many vibrating objects act as sources of waves. For example, sound waves can originate from a vibrating tuning fork (see Figure 1) or from a vibrating guitar string (see Figure 2).

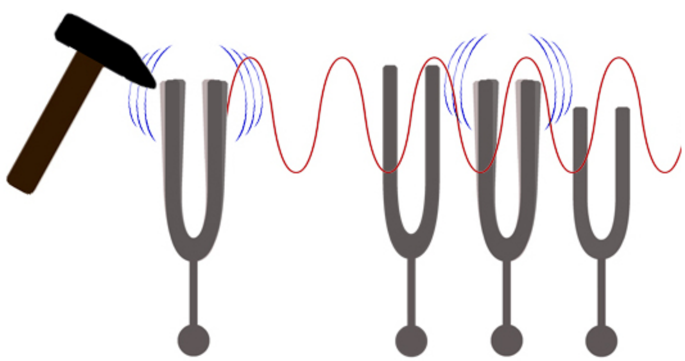

Figure 1: Schematic illustration of a vibrating tuning fork producing sound waves

In this lesson, we will first focus our attention on the study of waves on a string and later extend our study to waves on a spring. We consider a taut (or

Figure 2: Schematic illustration of a vibrating guitar strings producing sound waves

tight) string tied at one end and held by a hand on the free end as shown in Figure 3. Thereafter, a pulse (or disturbance) is sent down the string by the sudden up and down motion of the hand holding the string. The pulse travels with speed v along the string and carries energy down the string. When the pulse reaches a given point on the string, it causes that portion of the string to momentarily acquire both kinetic and elastic potential energy. This energy was given to the pulse by the source (i.e. up and down motion of hand) that initiated the pulse. The energy moves with speed v down the string along with the pulse. The pulse also serves as a record of what the source that initiated it did. If the pulse is moving with velocity v along the string then after time period t from the source, the distance travelled by the pulse will be given by

$$x = vt \tag{1}$$

where

x is the displacement covered by the pulse in time t

v is the velocity with which the pulse moves along the string

When the source of the pulse vibrates with simple harmonic motion as shown in Figure 4, the up and down motion of the string is transimitted down the string with speed v, which is referred to as the **wave speed**. As a result, the string has a sinusoidal shape at any given instant and this sinusoidal pattern travels with speed v. As it moves the pattern carries energy down the string obtained from the source of vibration.

The top of the wave marked A, C and E are called **wave crests**. The lowest parts of the wave marked B and D are called **wave troughs**. The maximum displacement of the string from its equilibrium position denoted y_o is called the **amplitude** of the wave. The distance between two successive wave crests or troughs is called **wavelength** and is denoted with the Greek letter λ.

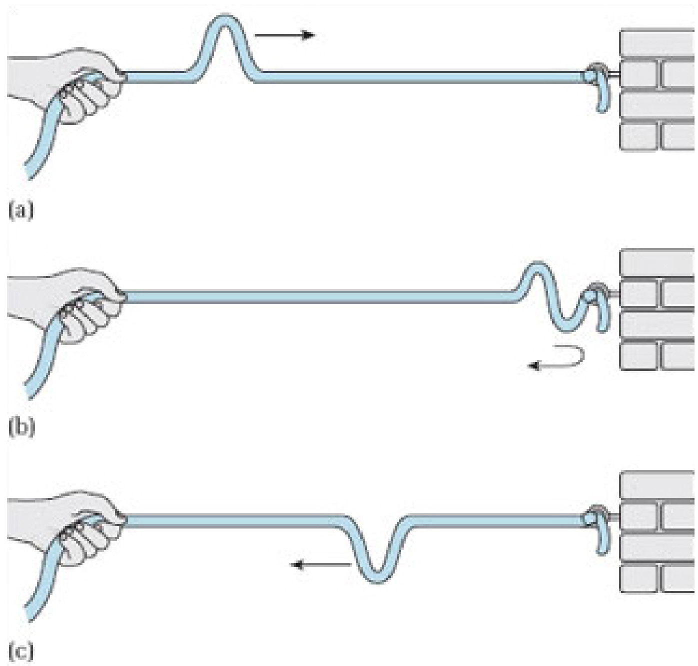

Figure 3: Schematic illustration of a pulse travelling on a string with speed v

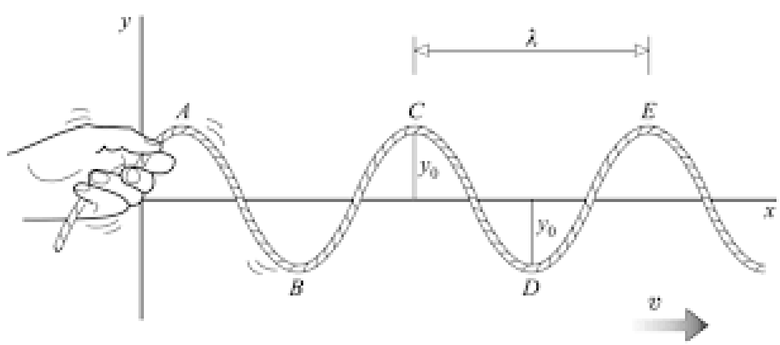

Figure 4: Schematic illustration of a wave travelling on a string with speed v generated by a human hand executing simple harmonic motion

One wavelength is sent out by the wave source as its executes one complete vibration. The time it takes the source to send out one wavelength is called the **periodic time** and is denoted τ. During the time it takes the source to send out one wave length, one wavelength passes through a point P. As a result, point P undergoes a complete cycle of motion in the same time it takes to complete one vibration. Thus the period of vibrating source τ_{source} is the same as the period of the vibration point P in the path of the wave,τ_{wave}. This time taken for a complete vibration of a point P on the wave is called the **period** of wave and is also denoted τ.

$$\tau_{\text{source}} = \tau_{\text{wave}} \tag{2}$$

Like for the oscillator, the period of the wave is related to the frequency of the wave denoted f as

$$f = \frac{1}{\tau} \tag{3}$$

The SI units for frequency are Hertz (Hz). Furthermore, the frequency of the wave can also be defined in terms of the number of wave crests that pass through a point P each second.

A very important relationship exist between the wavelength and the frequency of a wave. Note that in time period τ the wave travels a distance λ at a speed v. This distance travelled by the wave λ is related to the periodic time τ and wave speed v as follows

$$\lambda = v\tau \tag{4}$$

But recall that $\tau = \dfrac{1}{f}$, substituting we get

$$\lambda = v\frac{1}{f} = \frac{v}{f} \tag{5}$$

Making the wave speed v the subject of the formula, we get

$$v = f\lambda = \frac{\lambda}{\tau} \tag{6}$$

This relationship is true for all waves. The frequency of a wave is determined by the frequency of the source of the wave. The wave speed v is determined by the properties of the medium through which the wave passes.

The speed of a wave on a string is given by the following relationship without any derivation. If the tension in a string is T and the mass of length L is m, then the speed of the wave along the string is given by

$$v = \sqrt{\frac{T}{\mu}} \tag{7}$$

where $\mu = \dfrac{m}{L}$ is the mass per unit lengtrh of the string.

Equating Equation (7) and (7), we get the velocity of a wave travelling on a string as

$$v = f\lambda = \frac{\lambda}{\tau} = \sqrt{\frac{T}{\mu}} \tag{8}$$

The tension in the string T is responsible for the force that accelerates a piece of string as the pulse passes through the region. The greater the tension, the greater the acceleration, and so the motion of the pulse is faster if the tension is high.

On the other hand, the more massive the string, the more inertia it has. The mass per unit length μ therefore affects the wave speed v with which a pulse moves with. A massive string has larger inertia and the speed of the pulse on it is relatively low.

2 Reflection of Waves

In order for a wave to travel along a string, the string must be held taut at its right-hand end. A wave travelling to the right cannot continue beyond the support. So what happens to the kinetic energy and elastic potential energy carried by the wave since energy cannot be destroyed or just disappear? Two things may happen:

(i) some of the energy may be absorbed by the material of the support, and

(ii) some of the energy may be reflected back along the string. We assume that all the energy of the incident waves is reflected and this is a valid assumption for many cases.

To help us study the reflection of waves, we consider a single pulse propagating along as string as shown in Figure 3(a). Notice that as the pulse moves along the string, it exerts an upward force on small sections of string where it has reached displacing that section momentarily from its equilibrium position. When this pulse reaches the support, it exerts an upward force on the support. Since the support is fixed in place, it cannot move and therefore does not accelerate upwards. However, the support is “unhappy” by the pulse’s attempt to displace it upwards, therefore according to Newton’s third law of motion, the support exerts an equal but opposite force downwards on the string. This downward force exerted by the support on the string accelerates the string downwards to the extent that its linear momentum carries below the equalibrium position. The result is that the pulse is turned upside down (i.e. inverted) as it hits the support as shown in Figure 3(b), and the reflected pulse appears as shown in Figure 3(c). If the string had been completely free to move up and down at its right end, the pulse would not have been inverted, although it would still be reflected, since the energy could not just disappear at the end of the string. In summary, **a pulse is inverted by reflection at a fixed end, and it is reflected but not inverted at the free end**.

Next, we consider what happens when a reflected pulse travelling to the left along the string meets an incident pulse moving to the right along the same string as shown in Figure 5. If the reflected pulse is not inverted as shown in Figure 5(a), as the incident and reflected pulses begin to overlap, the displacements of the string at the point of meet is the individual displacement or amplitude of the pulses are added up which is a vector summation process. The net displacement is the vector sum of the individual pulse displacements. Similary, in shown in Figure 5(b), where the reflected pulse is inverted, the displacement of the inverted pulse is negative but equal in magnitude to that of the incident pulse. Therefore, as the inverted reflected pulse and the incident pulse overlap, the net displacement is zero. This kind of interference which occurs when an incident pulse and a reflected pulse interact is an example of the **principle of superposition** which states: **a point subjected to two or more wave pulses simultaneously is displaced an amount equal to the vector sum of the individual displacements**.

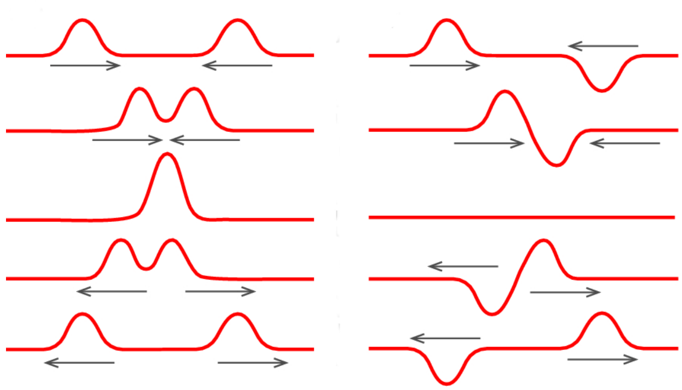

Figure 5: Interference of a reflected pulse moving to the left and incident pulse moving to the right along the same string

The principle of superposition can also be extended and applied to the study of sinusoidal waves travelling down along string held taut at its right-hand end and reflected by a rigid support at its end. Recall that sinusoidal waves consist of a series of pulses propagating along the string, thus whenever an incident pulse reaches the support, the pulse is inverted and reflected. Next the reflected pulses now moving to the left begin to interfer with the incident pulses moving to the right. At the support, the displacement will be zero, as it must always be.

As the inverted reflected pulses move to the left, there will be several points along the string where the net or resultant displacement will be zero at any instant. These points along the string that never move are called **nodes** and are denoted N. In between the nodes, different points will move with different displacements either upwards or downwards. However, exactly midway between the nodes there is one point that always moves the most i.e oscillates the most. This point is referred to as an **antinode** and is denoted A. This type of vibration in which a string vibrates back and forth within a well-defined envelope is called a **standing wave**. Standing waves on a string also referred to as **wave resonance** occur on a vibrating string when incident and reflected waves precisely reinforce one another. The combined incident and reflected waves give rise to nodes and antinodes on the string.

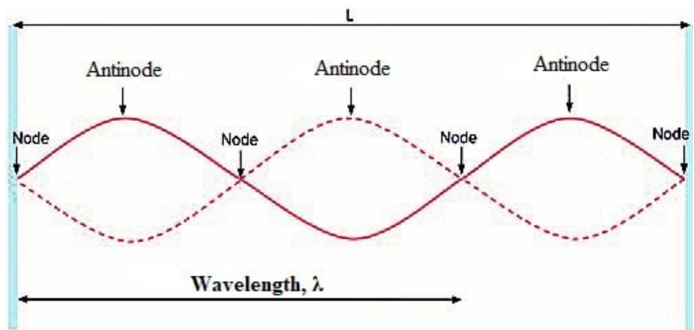

Figure 6: Standing waves occur when a vibrating string with incident and reflected waves precisely reinforce one another. The combined incident and reflected waves give rise to nodes and antinodes on the string.

If you look at the standing waves shown in Figure 6, you will see that the nodes are $\frac{\lambda}{2}$ apart. Similarly, the distance between two adjacent antimodes is $\frac{\lambda}{2}$. Note that **the distance between two adjacent nodes or antinodes in a standing wave is always** $\frac{\lambda}{2}$.

3 Wave Resonace : Standing Waves

Recall from our study of simple harmonic motion that any vibrating system has a natural frequency of vibration. If we force such a system to vibrate at its natural frequency then it undergoes resonance, that is, it vibrates most strongly when the frequency of the driving force matches the natural frequency of the system. Similarly, if you vibrate a string with too low frequency, the string vibrates so little that it appears motionless. If you slowly increase the frequency of vibration, the string begins to vibrate widely at a certain frequency. This frequency is called the **fundamental resonance frequency** and is denoted f_1. At this fundamental resonance frequency, the string vibrates widely and appears as a blur. Experiments on vibrations of strings show that a string can also resonate to other higher frequencies which are integer multiples of the fundamental resonance frequency such as $2f_1$, $3f_1$, $4f_1$ and so on.

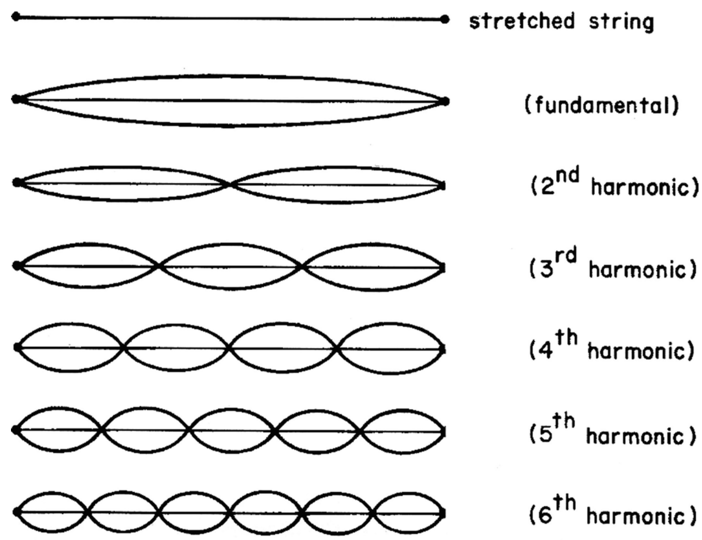

Figure 7: Schematic illustration of standing waves or wave resonance on a taut string

During resonance, the string always vibrates in whole **segments**, where a **segment** is defined as the distance between two adjacent nodes or antinodes. The fixed ends of the string are always nodes. Therefore, the string resonates only if it is one segment long, two segments long, three segments long and so on. Since the length of segment is $\frac{\lambda}{2}$, however, the string can resonate only if its is $\frac{\lambda}{2}$ long, $2\left(\frac{\lambda}{2}\right)$ long, or $3\left(\frac{\lambda}{2}\right)$ long and so on. In general, we can state that a string fastened firmly at its two ends resonates only if it is a whole number of $\frac{\lambda}{2}$ long. In other words, the string will resonate only if it is $\frac{\lambda}{2}$ long, $2\left(\frac{\lambda}{2}\right)$ long, or $3\left(\frac{\lambda}{2}\right)$ long and so on. Thus for resonance to occur on a string fastened at both ends, the following condition must be met:

$$L = n\left(\frac{\lambda_n}{2}\right), \quad \text{where} \quad n = 1, 2, 3, ... \tag{9}$$

Also, recall that the wavelength λ of a wave propagating along the string, the wave speed v and the frequency of the wave f are related as follows:

$$v = f\lambda \tag{10}$$

When we make the frequency f the subject of the formula, we obtain

$$f = \frac{v}{\lambda} \tag{11}$$

Thus, we see that if a string vibrates with n segments, its wavelength can be obtained as

$$\lambda_n = \frac{2L}{n}, \quad \text{where} \quad n = 1, 2, 3, ... \tag{12}$$

For $n = 1$, we get the wavelength λ_1 as

$$\lambda_1 = \frac{2L}{1} = 2L \tag{13}$$

The frequency corresponding to λ_1 is

$$f_1 = \frac{v}{\lambda_1} = \frac{v}{2L} \tag{14}$$

We see that each number of segments n has a corresponding value of frequency of vibration f_n.

$$f_n = \frac{v}{\lambda_n} = \frac{v}{\left(\frac{2L}{n}\right)} = \frac{nv}{2L} = n\left(\frac{v}{2L}\right) \tag{15}$$

where

$\lambda_1 = 2L$ is the wavelength

Substituting Equation (14) into Equation (15), we get the **allowed** frequencies for a string to vibrate with each of the n segments as

$$f_n = nf_1 \tag{16}$$

where

$f_1 = \frac{v}{2L}$ is the fundamental resonance frequency

We notice that the string vibrates at only very special frequencies. In Physics we say that the frequencies where resonance occurs are **quantized** meaning that they occur at discrete values, separated by frequency gaps. These discrete values of resonance frequency f_n are integer multiples of the fundamental resonance frequency.

$$f_n = nf_1, \quad \text{where} \quad n = 1, 2, 3, ... \tag{17}$$

Often the resonance frequency of a taut string is connected to the music of string instruments. The **fundamental resonance frequency** f_1 is referred to as the **first harmonic**, with f_2 as **second harmonic**, f_3 as **third harmonic**, f_4 as **fourth harmonic** and f_n as **nth harmonic**. The term **harmonic** is used to refer to a vibration of a single sinusoidal wave frequency and the term **simple harmonic motion** refers to periodic motion that can be described by sine and cosine functions of a single frequency.

4 Type of Waves

4.1 Transverse and Longitudinal Waves

Waves propagating in a medium can generally be divided into two categories: **transverse** and **longitudinal** waves. Waves of a string are examples of transverse waves. This is because the particles in the medium through which the wav passes e.g. the string move perpendicular (or tranverse) to the direction of wave propagation. As the wave on a string propagates from left to right, the string moves up and down.

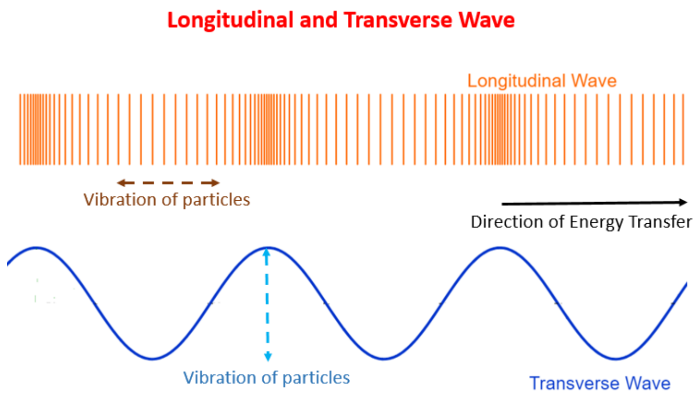

Figure 8: Schematic illustration of transverse waves on a string and longitudinal waves on a spring

Another type of wave called **longitudinal waves** are produces from during the force compression of a string tied on the right to the fixed end. If a compressive force is applied on the left side to the spring as shown, the loops near the end where the compressive force is applied and compressed before the rest of spring experiences the disturbance or pulse. The compressive loops in turn exerts a force on the loops to the right of them, and the compression travels down the spring.

$$L = n\left(\frac{\lambda_n}{2}\right), \quad \text{where} \quad n = 1, 2, 3, ... \tag{18}$$

When the compression reaches the fixed end at the right, the compression energy is reflected, thus the compression is reversed and ends up travelling to to the right. In this type of wave, the particles in the spring move back and forth in the same direction in which the wave is propagated along the spring. This kind of compressive waves in which the motion of particles is along the direction of wave propagation are called **longitudinal waves**.

When we create a continuous longitudinal wave by connecting the free end of the spring to a vibrating source that alternatively pushes and pulls the end with frequency f. Regions of closely spaced coils are sent down the spring attenating with regions of stretched out coils. If the vibrating sources acauses the end of the spring to move in simple harmonic motion, then we the variation in the extension (stretch out and compression loops of the coil follow a sinusoidal wave.

This pattern of compression and extension travels along the spring with velocity v determined by the properties of the spring, the wavelength λ of a wave i.e. the distance between two successive extension or compressions. The **amplitude** is the difference between two adjacent loops or coil distance of maximum compression (or maximum extension) and the adjacent coil distance of the undisturbed spring. The same relationship between between the wave speed v, frequency f and wavelength λ holds:

$$v = f\lambda \tag{19}$$

Sound waves are an important example of longitudinal waves.

4.2 Standing Compressional Waves on a Spring

A longitudinal wave has many features in common with transverse waves on a string. If a longitudinal wave is sent down a spring, the wave and its energy are reflected at the end of the spring. This reflected wave can interfere with later waves being sent down the spring by the source from the source of vibration. If a proper relation is maintained between the frequency of the source of vibration and the various parameters of the spring, then resonance occurs.

As with resonance on a string, the source of vibration in the spring system is usually close to a node since at resonance the spring moves much more than the source. Also, if the other end of the spring is held motionless, it too must be at a mode. The displacements on the spring along the x-axis vary sinusoidally with x. Nodes will occur along the spring at points where waves travelling to the right and to the left cancel out, leaving the spring neither compressed nor extended. Standing waves or resonance occurs when the length of the spring is an integral multiple of the distance between two successive nodes i.e. $\dfrac{\lambda}{2}$. The condition for resonance for longitudinal waves on a spring fixed at both ends is the same as for transverse waves and is given by

$$L = n\left(\frac{\lambda_n}{2}\right) \qquad \text{where} \quad n = 1, 2, 3, ... \tag{20}$$

We get the wavelength λ for n segments as

$$\lambda_n = \frac{2L}{n} \qquad \text{where} \quad n = 1, 2, 3, ... \tag{21}$$

Again, recall that the wavelength λ, wave speed v and frequency of any wave f are related as follows:

$$v = f\lambda \tag{22}$$

Making the frequency f the subject of the formula, we get

$$f = \frac{v}{\lambda} \tag{23}$$

Substituting λ_n, we get the frequency f as

$$f_n = \frac{v}{\left(\frac{2L}{n}\right)} \tag{24}$$

$$f_n = n\frac{v}{2L} \qquad \text{where} \quad n = 1, 2, 3, ... \tag{25}$$