VECTORS

Coordinate Systems and Frames of Reference.

- measurements are made relative to a chosen frame of reference.

- we always note point of view when making observations. To be used in future predictions.

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- For observer car & occupants are moving relative to observer.

- But driver & occupants may not consider themselves relative to one another as moving.

- If observer is in a car moving ll^{el} to the other car at same speed, there would be no relative motion.

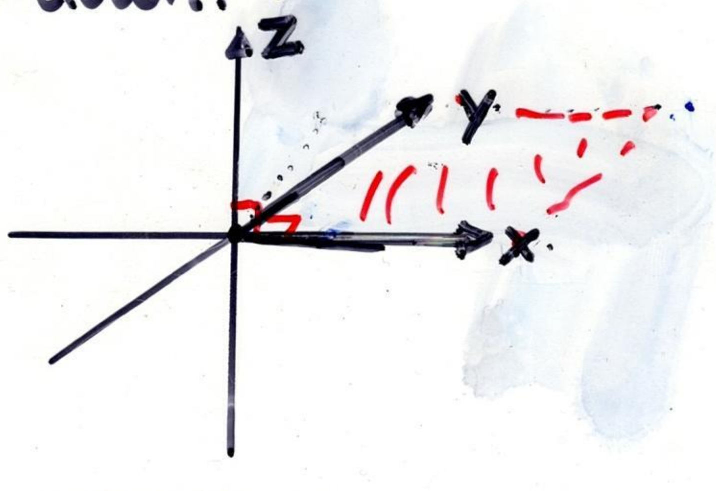
These different points of view are known as FRAMES OF REFERENCE

COORDINATE SYSTEM

- within a frame of reference we have a set of reference lines that intersect an arbitrarily chosen fixed point of reference called ORIGIN

Cartesian coordinate system

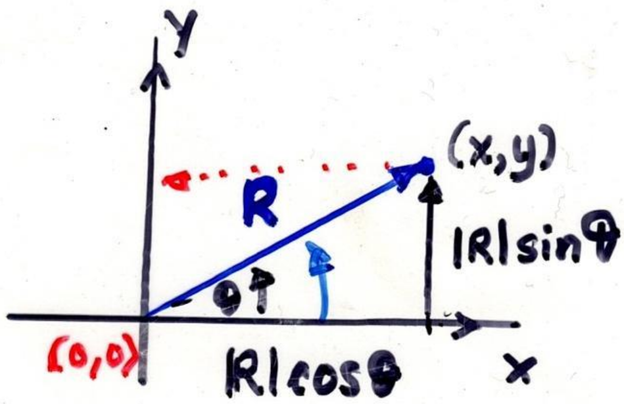
- lines mutually $\perp^{ar}$ designated X, Y, Z
- system has set of rules ~~for~~ locating objects within ~~the~~ frame of reference.
- 3-D Cart. coord. system
  x, y plane considered say as horizontal plane
  - then z axis specifies a direction up & down.



- Often useful to compare observations made in two different frames of reference.

Vectors

- position of object in Cartesian system given by a directed line segment or a VECTOR



R $\vec{R}$

- R or $\vec{R}$ is a position vector whose magnitude is given by |R| = linear distance between origin and point (x,y)

- Vector direction is given by the acute ∠ line (R) makes with x-axis. (∠ θ) with respect to positive x-axis.

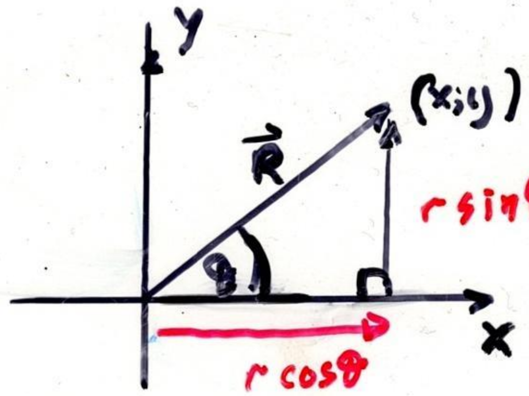
- ANY QUANTITY HAVING MAGNITUDE AND DIRECTION IS CALLED A VECTOR QUANTITY
- QUANTITY HAVING MAGNITUDE ONLY IS A SCALAR QUANTITY

EXAMPES:

vector quantities – force, velocity, weight & displacement.

scalar quantities – mass, distance, speed & energy.

Polar coordinate system.



magnitude is r

- If magnitude of vector $\vec{R}$ is r & direction is given by θ, then we have another system called the polar coordinate system.

- In 2-D location (x, y) is given by going x-units horizontally & y-units vertically.

- We can show that: (r, θ)

$$x = r\cos\theta$$
$$y = r\sin\theta$$

- having established this polar form of vectors, we easily show that r & θ can be expressed in terms of x & y as:

$$r^2 = x^2 + y^2$$
$$\& \quad \tan\theta = \frac{y}{x}$$

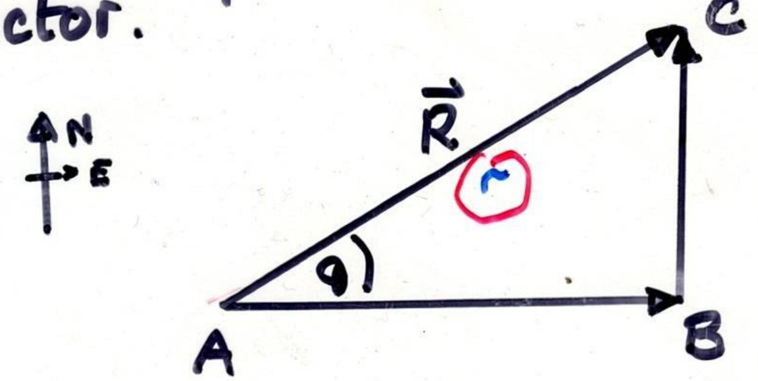
Magnitude of $\vec{R}$ is given by:

$|\vec{R}| = r = \sqrt{x^2+y^2}$, direction $\theta = \arctan\left(\frac{y}{x}\right)$

ADDITION OF VECTORS

- Scalar quantities added simply:
  e.g.
  $$7 + 3 = 10$$
  i.e. we simply sum magnitudes.

- But displacement from A to C is a vector.



- magnitude of $\vec{R}$ (AC) given length of straight line, direction by arrow.

e.g. suppose a car travels 30 km east, (A to B), then 10 km north. What is the resultant displacement and direction?

SOLUTION

From Polar coordinate sytem:

$$|\vec{R}| = r = \sqrt{x^2+y^2}$$

$$\text{or } \sqrt{(AB)^2+(BC)^2} = \sqrt{(30)^2+(10)^2}$$
$$= 31.12 \text{ km}$$

&

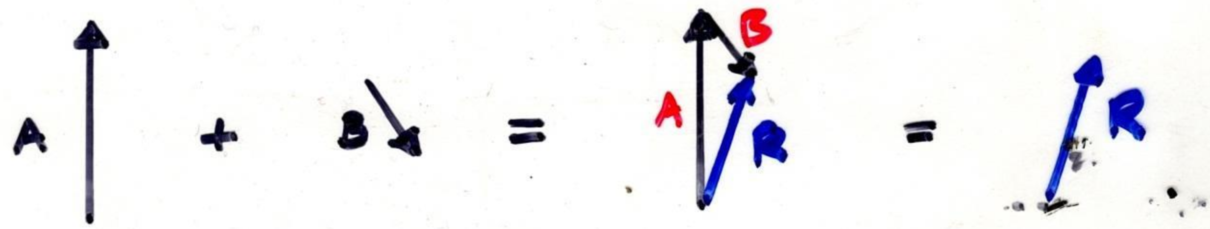
direction is $\theta = \arctan\left(\frac{y}{x}\right)$ $\tan^{-1}(y/x)$ $= \arctan\left(\frac{BC}{AB}\right)$

$= \arctan\left(\frac{10}{30}\right) = \arctan(0.333) \Rightarrow \theta = 18.4^\circ$

i.e. we added vector AB to vector BC getting result AC = $\vec{R}$

It is clear that vector addition is different from scalar addition.

- NB ANGLES ARE MEASURED RELATIVE TO POSITIVE X-AXIS IN THE ANTI-CLOCK-WISE DIRECTION.



Step i) shift B $\parallel^{el}$ to self until its tail is at head of A. In its new position B must maintain original length & direction.

Step ii) Draw vector R (resultant) from tail of A to head of B.

i.e. vector $\vec{R} = \vec{A} + \vec{B}$

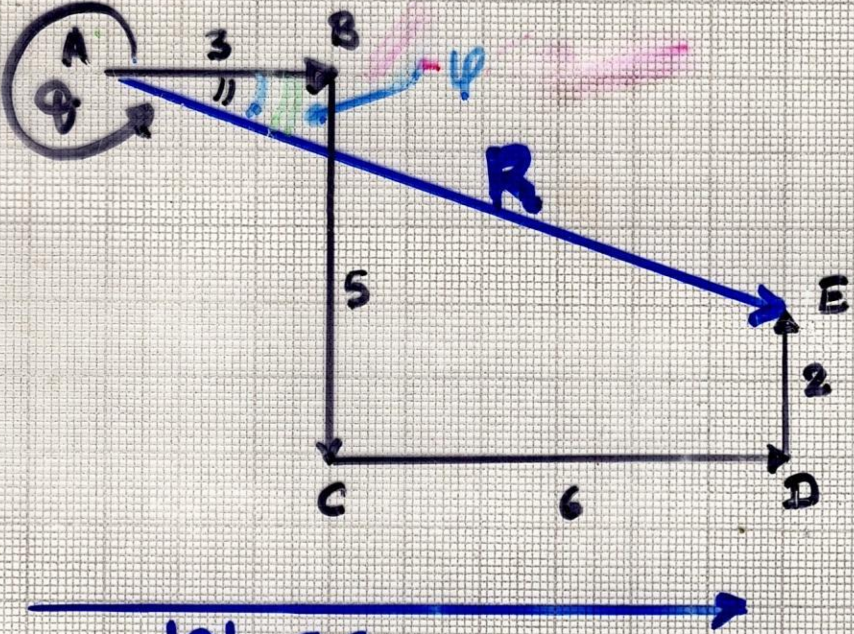
- Vector is represented by bold-face letter in books. **R**
- In writing $\vec{R}$
- 'displacement' from one point to another is a vector. (Direction given)
- 'resultant displament' maybe addition of a series of movements. [vector has tail at origin of motion and head at head of last]

Graphical Addition of Vectors

Finding the resultant:

$$\vec{AB} + \vec{BC} + \vec{CD} + \vec{DE} = \vec{AE}$$

$\vec{AB}$ = 3m East, $\vec{BC}$ = 5m South, $\vec{CD}$ = 6m East, $\vec{DE}$ = 2m North



movements at rt. ∠s.

$\theta = 360^\circ - \varphi$

$= 340^\circ$

$|R| = 9.5\text{ m}$

$\varphi = 20^\circ$

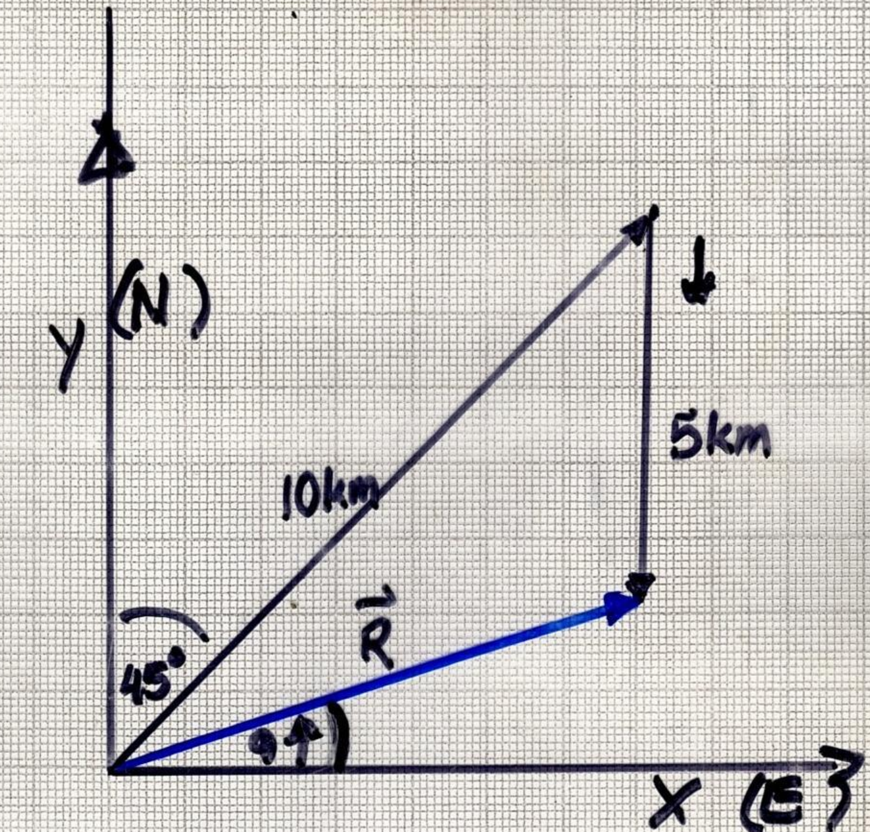
Resultant $\vec{R}$ = AE

Angle θ can be measured or ∠ BAE [φ] then subtracting it from 360° to get θ

Magnitude of $\vec{R}$ obtained by some linear scale eg a metre rule

EXAMPLE

Suppose a person moves 10km north-east, followed by 5km south. Find the resultant of these motions.



10km north-east, followed by 5km south

Measuring resultant we get ≈ 7.3km & we can measure θ ≈ 17° wrt to positive x-axis.

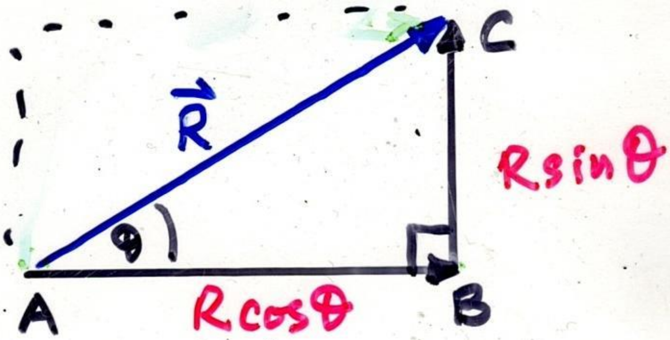
N.B. Graphical Method is:

i) Cumbersome

ii) Only as accurate as scale drawings.

∴ We need a more convinient method.

RECTANGULAR COMPONENTS OF VECTORS



- Vector AC is resultant, while $\vec{AB}$ & $\vec{BC}$ are the rectangular components (ie at rt ∠s to each other)

- Any vector can be resolved into its components provided component vectors add up to original vector.

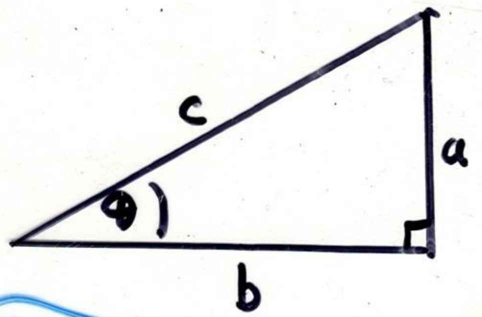
- Above example $\vec{R}$ can be resolved into vectors $\vec{AB}$ & $\vec{BC}$

  recall that $|\vec{AB}| = |\vec{R}| \cos\theta$ $\overset{R\cos\theta}{} \equiv x$

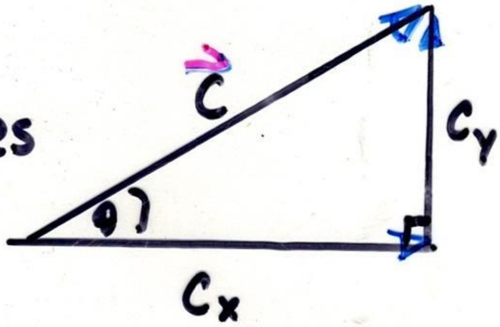
  & $|\vec{BC}| = |\vec{R}| \sin\theta \equiv y$

- Useful way of adding vectors is to take component vectors along mutually $\perp^{ar}$ directions. Then to add components in respective directions.

REMINDERS



becomes



$\boxed{\sin\theta = \frac{a}{c}}$, $\cos\theta = \frac{b}{c}$, $\tan\theta = \frac{a}{b} = \frac{\sin\theta}{\cos\theta}$

$c^2 = a^2 + b^2$ becomes $C^2 = C_x^2 + C_y^2$.

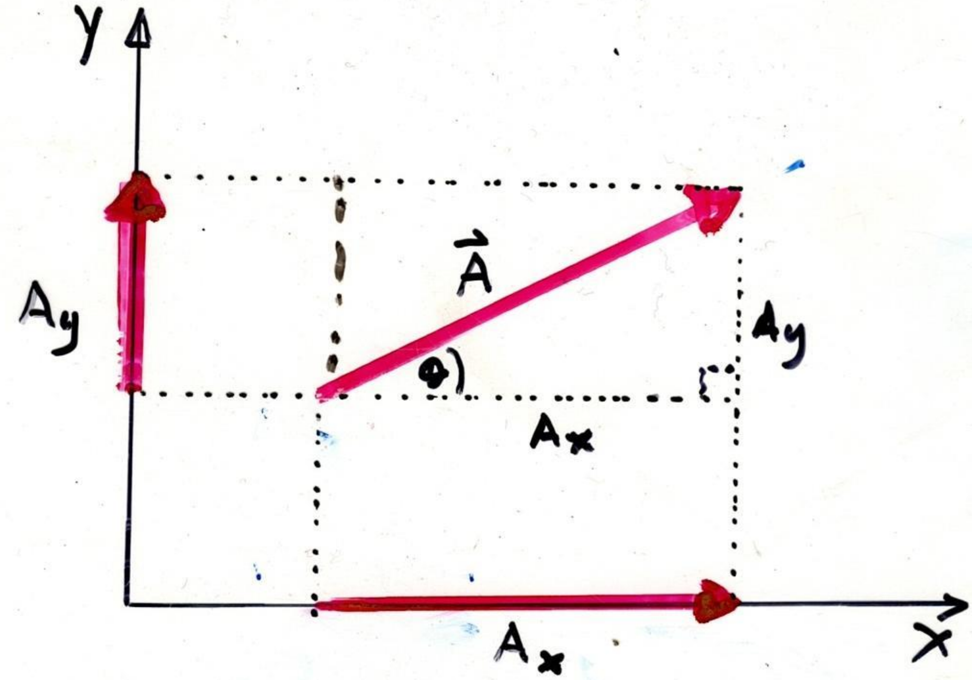
$a = c\sin\theta$ $\qquad$ $C_y = C\sin\theta$

$b = c\cos\theta$ $\qquad$ $C_x = C\cos\theta$

$\tan\theta = \frac{a}{b}$ $\qquad$ $\tan\theta = \frac{C_y}{C_x}$

also $(\sin^2\theta + \cos^2\theta) = 1$

RECTANGULAR COMPONENTS



Vector $\vec{A}$ in x-y plane.

i) From its ends drop lines $\perp^{ar}$ to x & y axes

ii) Projections A_x & A_y are rectangular components of $\vec{A}$

iii) Instead of vector specification being (A, θ) ie magnitude & direction

we can specify it by its components:

(A_x, A_y)

since $\cos\theta = \frac{A_x}{A}$ & $\sin\theta = \frac{A_y}{A}$

then: $A_x = A\cos\theta$

$A_y = A\sin\theta$

ie mag. & direction are: $A = \sqrt{A_x^2 + A_y^2}$, $\tan\theta = A_y/A_x$

Using the component method of addition of vectors along mutually $\perp^{ar}$ directions.

Add the following movements:

a = 3 m east

b has components 2 m east & 2 m north

c " " 3 m north & 4 m west

To add, we consider components in each direction

NORTH : $= 0 + 2 + 3 = 5\,m$

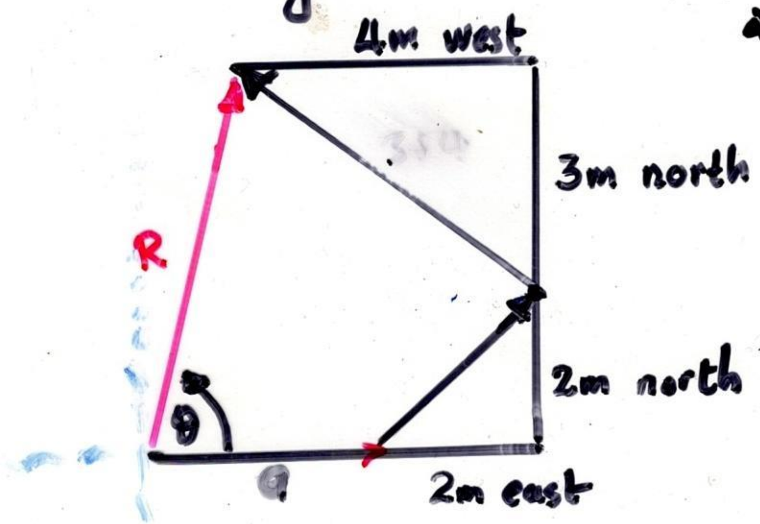
EAST-WEST $= 3 + 2 - 4 = 1$

$\therefore$ Resultant $\vec{R}$ has components:

5 m north & 1 m east

Magnitude $|\vec{R}| = \sqrt{1^2 + 5^2} = \sqrt{26} = 5.1\,m$

R_x R_y



$\tan^{-1}\left(\frac{R_y}{R_x}\right)$

Direction of $\vec{R}$: $\theta = \tan^{-1}\left(\frac{5}{1}\right)$

$= \tan^{-1}(5)$

$= 78.9°$

$R_x = 1\,m$, $R_y = 5\,m$

with $R = 5.1\,m$ at $\angle 78.9°$ wrt positive x-axis.

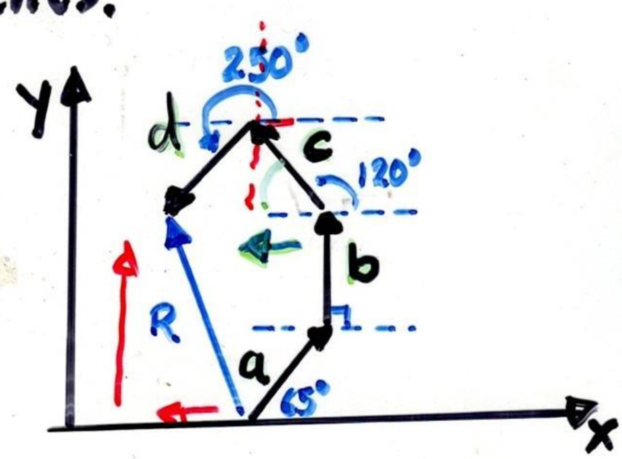
10(c)

Trigonometric Addition of Vectors.

Suppose we have the following vectors a, b, c, and d having displacements:

a) 30mm at 65° w.r.t. the +ve x-axis
b) 30mm at 90° " " " "
c) 30mm at 120° " " " "
d) 30 mm at 250° " " " "

Since we know how to find vector components, we can easily add the displacements.



We use x & y components of the vectors. Usually a table helps:

Vector	x-comp.	y-comp
a = 30 mm	30 cos 65° = 12.7mm	30 sin 65° = 27.2mm
b = 30mm	30 cos 90° = 0 "	30 sin 90° = 30 "
c = 30mm	30 cos 120° = −15 "	30 sin 120° = 26 "
d = 30mm	30 cos 250° = −10.3 "	30 sin 250° = −28.2 "
	R_x = −12.6mm	R_y = 55 mm

To find R_x & R_y add individual components of x comps. & y comps.

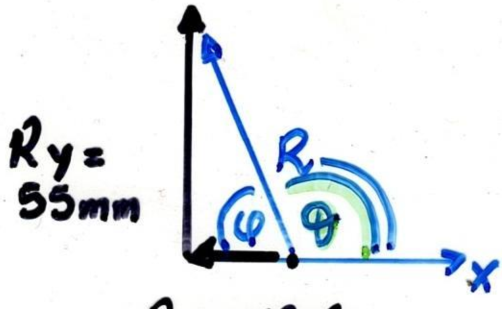
$$\therefore \quad R = \sqrt{R_x^2 + R_y^2}$$

$$= \sqrt{(-12.6)^2 + (55)^2} = 56.4 \text{ mm}$$

& the direction : $\tan\varphi = |R_y| / |R_x|$

$$= \frac{55}{12.6} = 4.37$$

Hint, make a sketch of R_x & R_y



Thus the ∠ we want $\theta = 180° - \varphi$

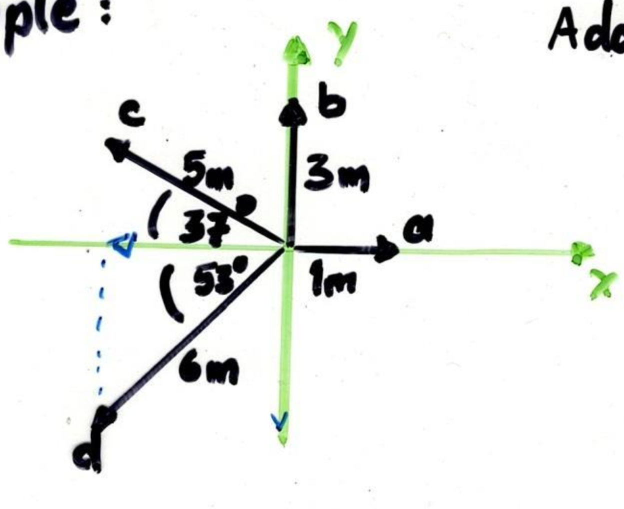
$$\varphi = \tan^{-1}(4.37) = 77°$$

$$\therefore \quad \theta = 180° - 77° = 103° \text{ wrt +ve x-axis}$$

N.B. a sketch confirms the actual direction.

Example :

Add vectors a, b, c & d



$a + b + c + d$

or

$a + c + d + b$

Remember order of addition does not matter. components can be added in any order
e.g. a_x, d_x, b_x, c_x

Tabulate components.

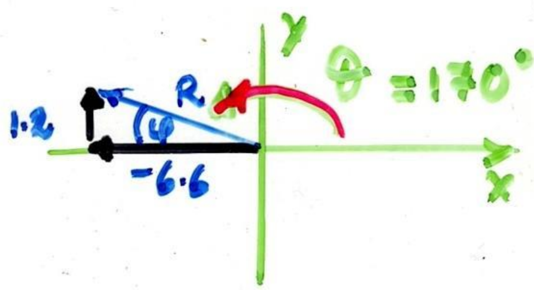
Vector	x-components	y-components
a	$1\cos 0° = 1.0$	$1\sin 0° = 0.0$
b	$3\cos 90° = 0.0$	$3\sin 90 = 3.0$
c	$5\cos 37° = -4.0$	$5\sin 37° = 3.0$
d	$6\cos 53° = -3.6$	$6\sin 53° = -4.8$
R	$R_x = -6.6$	$R_y = +1.2$

Magnitude $R = \sqrt{(-6.6)^2 + (1.2)^2}$

$= 6.7$ m

Direction: $\tan\varphi = \boxed{1.2/6.6} = 0.182$

from which $\tan^{-1}\varphi = 10°$

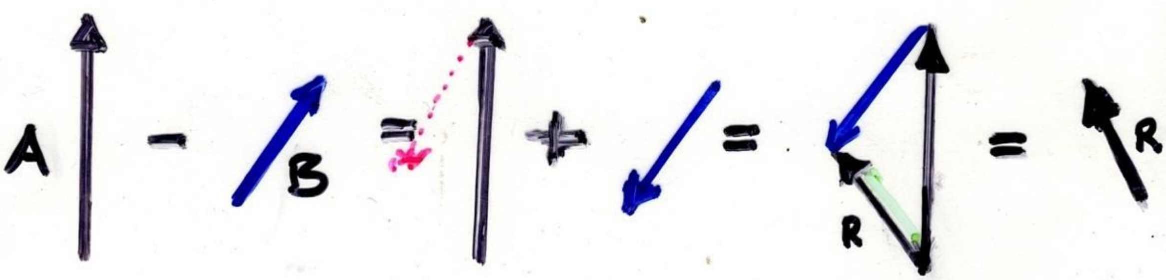


From sketch $\theta = 180° - 10° = 170°$ with +ve x-axis

SUBTRACTION OF VECTOR QUANTITIES

Subtraction of vector quantities is equivalent to the addition of the same vector, but with its <u>direction reversed</u>

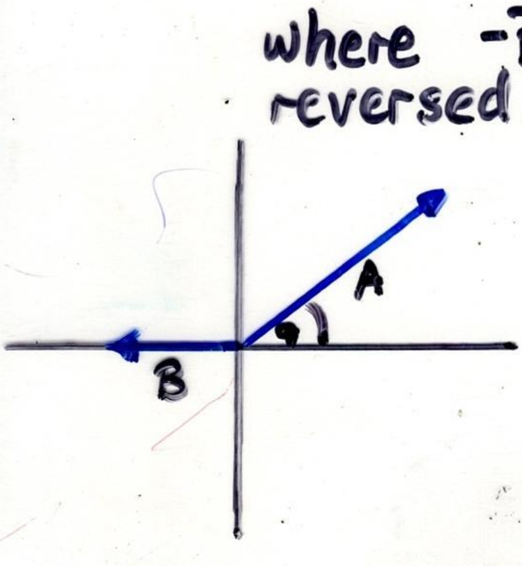
$10 - 4 = 10 + (-4) = 6$

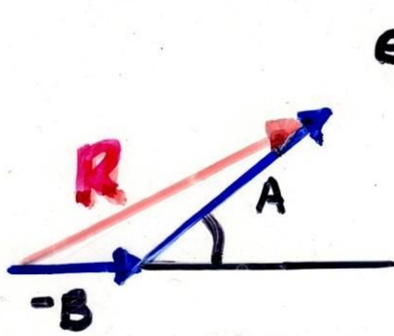


To subtract a vector, reverse direction and add it to the 1st vector.

i.e. $\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$

where $-\vec{B}$ is vector $\vec{B}$ with its direction reversed.

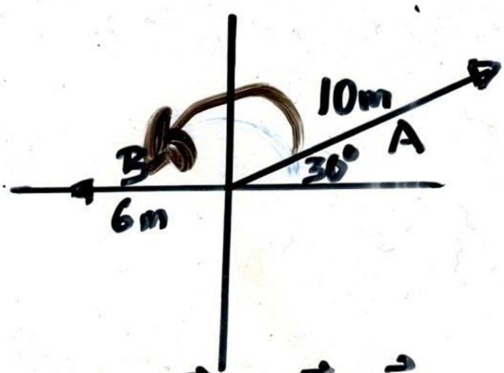




eg. $A = 10\,m$
$B = 6\,m$
$\theta = 30°$

reverse direction of $\vec{B}$ and add to

Find the resultant of $\vec{A} - \vec{B}$ in the diagram.



To compute $\vec{R} = \vec{A} - \vec{B} = \vec{A} + (-\vec{B})$

Components of $\vec{A}$: $A_x = 10\cos 30° = 8.66$
$A_y = 10\sin 30° = 5$

" " $\vec{B}$: $B_x = 6\cos 180° = -6$
$B_y = 6\sin 0° = 0$

In the table we changes signs on the components of $\vec{B}$ and add to $\vec{A}$

Vector	Comp. X	Comp Y
A	8.66	5
B	+6.0	0
R	14.66	5

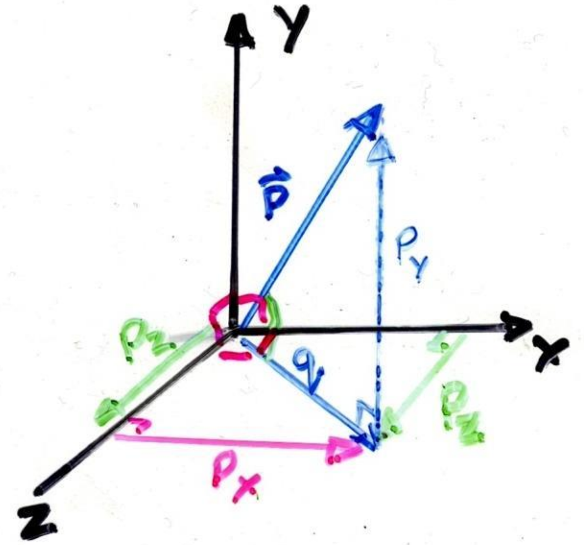
Magnitude $\vec{R} \Rightarrow R = \sqrt{R_x^2 + R_y^2}$
$= \sqrt{14.66^2 + 5^2} = 15.49\,m$

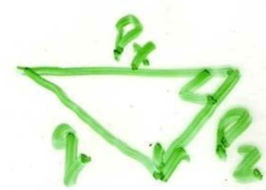
$\tan\theta = \dfrac{5}{14.66} = 0.341$

$\theta = \tan^{-1}(0.341) = 18.8°$

Resolving Vectors in 3D

Any vector can be resolved into rectangular components





P_y , P_x & P_z are components of $\vec{P}$

Vector sum $P_y + P_x + P_z = P$

Components are mutually $\perp^{ar}$ to each other.

$$\therefore \quad P^2 = P_y^2 + q^2 \quad \text{(i)}$$

$$q^2 = P_x^2 + P_z^2 \quad \text{(ii)}$$

Substituting for q^2 in (i)

$$P^2 = P_y^2 + P_x^2 + P_z^2$$

Magnitude $\therefore P = \sqrt{P_x^2 + P_y^2 + P_z^2}$