DIPLOMA AGRIBUSINESS MANAGEMENT

APPLIED MATHEMATICS

COOPERATIVE COLLEGE
RESEARCH AND CONSULTANCY DEPARTMENT

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Directorate of Open and Distance Education	Facilitation services
Chikiuswe. D.	Course Developer
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Cooperative College Management	For facilitation and guidance on development of manuals and guidelines

Contents

About this APPLIED MATHEMATICS

DIPLOMA AGRIBUSINESS MANAGEMENT Applied Mathematics has been produced by COOPERATIVE COLLEGE. All APPLIED MATHEMATICS modules produced by COOPERATIVE COLLEGE are structured in the same way, as outlined below.

How this APPLIED MATHEMATICS is structured

The course overview

The course overview gives you a general introduction to the course. Information contained in the course overview will help you determine:

- If the course is suitable for you.
- What you will already need to know.
- What you can expect from the course.
- How much time you will need to invest to complete the course.

The overview also provides guidance on:

- Study skills.
- Where to get help.
- Course assignments and assessments.
- Activity icons.
- Units.

We strongly recommend that you read the overview *carefully* before starting your study.

The course content

The course is broken down into units. Each unit comprises:

- An introduction to the unit content.
- Unit outcomes.
- New terminology.
- Core content of the unit with a variety of learning activities.
- A unit summary.
- Assignments and/or assessments, as applicable.

Resources

For those interested in learning more on this subject, we provide you with a list of additional resources at the end of this APPLIED MATHEMATICS module; these may be books, articles or web sites.

Your comments

After completing DIPLOMA INDIPLOMA AGRIBUSINESS MANAGEMENT we would appreciate it if you would take a few moments to give us your feedback on any aspect of this course. Your feedback might include comments on:

- Course content and structure.
- Course reading materials and resources.
- Course assignments.
- Course assessments.
- Course duration.
- Course support (assigned tutors, technical help, etc.)

Your constructive feedback will help us to improve and enhance this course.

Course overview

Welcome to DIPLOMA AGRIBUSINESS MANAGEMENT Applied Mathematics

This subject aims to equip trainees with knowledge, skills and attitudes in mathematical tichniques to enable them make informed decisions as they solve problems encountered in their operations in Agribusiness Management.

DIPLOMA AGRIBUSINESS MANAGEMENT Applied Mathematics —is this course for you?

This course is intended for people who need to apply basic mathematical techniques to solve every day problems and make well informed decisions.

This course is especially helpful for people who want to study Agribusiness Management outside the classroom environment. It is ideal for people in employment who cannot easily get away from workplaces to study on full time basis.

To qualify for this course one must have have a minimum of grade twelve school certificate with a minimum of credits in maths and English.

Course outcomes

Upon completion of DIPLOMA AGRIBUSINESS MANAGEMENT Applied Mathyematics you will be able to:

Outcomes

including

- *Use* Number and Algebraic operations correctly
- *Use* Financial Mathematics correctly
- *Use* Statistical Methods correctly
- *Apply* Basic Linear programming techniques correctly.
- *Use* Matrices correctly.
- *Apply* the concept of Probability correctly.
- *Use* Basic Differential calculus correctly.
- *Use* Basic Integral Calculus correctly.

Timeframe

How long?

The expected duration of this course is three (3) years

You will be expected to spend atleast twelve (12) weeks in residential school every year

You will be expected to spend at least four (04) hours on self- study every week

Study skills

As an adult learner your approach to learning will be different to that from your school days: you will choose what you want to study, you will have professional and/or personal motivation for doing so and you will most likely be fitting your study activities around other professional or domestic responsibilities.

Essentially you will be taking control of your learning environment. As a consequence, you will need to consider performance issues related to time management, goal setting, stress management, etc. Perhaps you will also need to reacquaint yourself in areas such as essay planning, coping with exams and using the web as a learning resource.

Your most significant considerations will be *time* and *space* i.e. the time you dedicate to your learning and the environment in which you engage in that learning.

We recommend that you take time now—before starting your self-study—to familiarize yourself with these issues. There are a number of excellent resources on the web. A few suggested links are:

- http://www.how-to-study.com/

The "How to study" web site is dedicated to study skills resources. You will find links to study preparation (a list of nine essentials for a good study place), taking notes, strategies for reading text books, using reference sources, test anxiety.

- http://www.ucc.vt.edu/stdysk/stdyhlp.html

  This is the web site of the Virginia Tech, Division of Student Affairs. You will find links to time scheduling (including a "where does time go?" link), a study skill checklist, basic concentration techniques, control of the study environment, note taking, how to read essays for analysis, memory skills ("remembering").

- http://www.howtostudy.org/resources.php

  Another "How to study" web site with useful links to time management, efficient reading, questioning/listening/observing skills, getting the most out of doing ("hands-on" learning), memory building, tips for staying motivated, developing a learning plan.

The above links are our suggestions to start you on your way. At the time of writing these web links were active. If you want to look for more go to www.google.com and type "self-study basics", "self-study tips", "self-study skills" or similar.

Need help?

Help

For updates and other information relating to the course, contact us by email on info@cooperativecollege.ac.zm or by phone on (+260)-211-264374. Visit the College website on: www.cooperativecollege.ac.zm

Specific information pertaining to this module may be obtained from the Department of Research and Consultancy on (+260)-211-264374

For routine enquiries regarding the Open and Distance Learning (ODL) programme, please contact the ODL Coordinator on (+260)-211-264374 or by email on odl@cooperativecollege.ac.zm

The College Librarian can be contacted on the above phone number or by email on librarian@cooperativecollege.ac.zm

For information on technical issues such as access to websites and other electronic resources contact the Computer Programmer on computerprogrammer@cooperativecollege.ac.zm

Assignments

Assignments

There are a total of three (03) assignments per year in this course

The assignments should be submitted by registered post or delivered by hand. Alternatively, assignments can be submitted by email to the Open and Distance Learning Coordinator on the following address: odl@cooperativecollege.ac.zm

Please note that specific instructions may be issued by Course Trainers and Coordinators regarding submission of assignments. Please abide by those instructions.

Generally, a schedule for submission of assignments will be issued to you at the beginning of the academic year. However, assignments should, in general be completed and submitted in the order in which they were given.

Assessments

Assessments

There will be two (2) mandatory continuous assessment tests which will be conducted in February and August. Students will be required to submit three (3) continuous assessment assignments in January, April/May, and September/October.

Within this module there also self-assessment tests, meant for you to gauge your progress at each stage of your study. The length and structure of the assignments will depend on the topic and type of material covered. However, the Course Trainer will advise you on the specific requirements for each assignment. Generally, a set of three (03) to four (04) assignments will be given, to be completed in a period of three (03) months. The Course Trainer will mark the assessments (assignments and examinations) within a period of one (01) month.

Release and dissemination of results will be done through the ODL coordination office. Results will also be communicated via email subject to payment of prescribed College fees.

Getting around this APPLIED MATHEMATICS

Margin icons

While working through this APPLIED MATHEMATICS you will notice the frequent use of margin icons. These icons serve to “signpost” a particular piece of text, a new task or change in activity; they have been included to help you to find your way around this APPLIED MATHEMATICS.

A complete icon set is shown below. We suggest that you familiarize yourself with the icons and their meaning before starting your study.

Activity	Assessment	Assignment	Case study
Discussion	Group activity	Help	Note it!
Outcomes	Reading	Reflection	Study skills
Summary	Terminology	Time	Tip

Unit 1

USING NUMBER AND ARITHMETIC OPERATIONS

outcomes

Introduction

This Unit introduces the number system which consists of all the types of numbers that we use in our mathematical operations. It also talks about the basic arithmetic and algebraic operations on real numbers.

Upon completion of this unit you will be able to:

Outcomes

- *Use* number systems without difficult
- *Apply* arithmetic operations on rational and real numbers without difficulty
- *Evaluate* indices and logarithms correctly
- *Evaluate* basic algebraic operations correctly
- *Solve* equations without difficulty

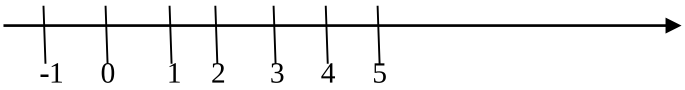
Terminology

Adding extra rows to the Table graphicRemoving rows from the table graphic

Number Line:	A number line is a picture that shows numerals in order of value
Modulus:	The Modulus or Absolute value is the positive quantity of any value
Closure Law:	This is a law that states that if any two numbers from a given set are combined by any operation and the result is still a member of that set, then the given set is closed under that operation
Indices:	This is another way of writing numbers that are multiplied by themselves more several times
Logarithms:	Given any two real numbers a and b there exists a third number such that $a^c = b$. the number is known as the Logarithm of b to the base a.
Simultaneous Equations:	These are equations that have a common solution
Quadratic equations:	These are equations that can be written as $ax^2 + bx + c = 0$ where a, b and c are constants and $a \neq 0$

THE NUMBER SYSTEM

The umber line can be used to show how numbers are added and consists of all the numbers that we use in our mathematical operations. The diagram below shows an example of a number line:



Types of Numbers

Natural or Counting Numbers

It is helpful to classify the various kinds of numbers that we deal with as sets. The **Counting numbers,** or **natural numbers,** are the numbers in the set {1, 2, 3, …} .(The three dots, called an **ellipsis,** indicate that the pattern continues indefinitely.) As their name implies, these numbers are often used to count things. This set of Natural numbers is given by the symbol **N**. Thus N = {1, 2, 3, 4, …}.

Whole Numbers

The whole numbers are the numbers in the set {0, 1, 2, 3, …}, that is, the counting numbers together with 0.The set of counting numbers is a subset of the set of whole numbers.

Integers

The **integers** are the set of numbers {…, -3, -2, -1, 0, 1, 2, 3,…}. This set is given by the symbol **Z.** Hence the set of integrs is Z ={…, -3, -2, -1, 0, 1, 2, 3, …}. These numbers are useful in many situations but But integers alone are not sufficient for all problems. For example, they do not answer the question “What part of a Kwacha is 38 Ngwee?” To answer such a question, we enlarge our number system to include rational numbers. For example, 38/100 answers the question “What part of a Kwacha is 38 Ngwee?”

Rational Numbers

A **rational number** is a number that can be expressed as a quotient $\frac{a}{b}$of two integers.The integer a is called the **numerator,** and the integer b, which cannot be 0, is called the **denominator.** The rational numbers are the numbers in the set $\left\{x|x=\frac{a}{b}, where\ a, b\ are\ Integers\ and\ b\neq 0\right\}$. Examples of Rational numbers include: $\frac{3}{4}, \frac{5}{2}. -\frac{2}{3}, \frac{0}{4}, 6\ and\ \frac{100}{3}$. Since $\frac{a}{1}=a$ for every integer a, it follows that every integer is a Rational number. Thus the set of Integers is a subset of rational numbers. It follows that the set of integers is a subset of the set of rational numbers. Rational numbers may be represented as **decimals.** For example, the rational numbers $\frac{3}{4}, \frac{5}{2}. -\frac{2}{3}$ and $\frac{7}{66}$ may be represented as decimals as follows: $\frac{3}{4}=0.75, \frac{5}{2}=2.5, -\frac{2}{3}=-0.666 \ldots=-0.\overline{6}$ and $\frac{7}{66}=0.1060606 \ldots=0.1\overline{06}$.

Notice that the decimal representations of $\frac{3}{4}$ and $\frac{5}{2}$ terminate, or end. The decimal representations of $-\frac{2}{3}$ and $\frac{7}{66}$ do not terminate, but they do exhibit a pattern of repetition. For the $-\frac{2}{3}$, the 6 repeats indefinitely, as indicated by the bar over the 6; for $\frac{7}{66}$ the block 06 repeats indefinitely, as indicated by the bar over the 06

Example

Write the following rational numbers in their fractional form:

a. $0.\overline{3}$

b. $5.\overline{432}$

Soltion

a. Let x = $0.\overline{3}$ { multiply both sides by 10}

10x = $3.\overline{3}$ { subtract the first equation from the second}

10x = $3.\overline{3}$

- x = $0.\overline{3}$

9x = 3 { divide both sides by 9}

$x = \frac{3}{9} = \frac{1}{3}$

b. Let x = $5.\overline{432}$ { multiply both sides by 1000}

1000x = $5432.\overline{432}$ {subtract the first equation from the second}

1000x = $5432.\overline{432}$

- x = $5.\overline{432}$

999x = 5427 {Divide both sides by 999 and reduce to lowest term}

$$x = \frac{5427}{999} = \frac{201}{37}$$

For the first example, the decimal was multiplied by 10 and for the second example, the decimal was multiplied by 1000. This is because for the first example there was only one digit (i.e. 3) recurring, while for the second example there were three digits (i.e. 432) recurring. In general, if you have one digit recurring, then multiply by 10. If you have two digits recurring, then multiply by 100. If you have three digits recurring, then multiply by 1000.

Irrational Numbers

On the other hand, some decimals do not fit into either of these categories. Such decimals represent **irrational numbers.** Every irrational number may be represented by a decimal that neither repeats nor terminates. In other words, irrational numbers cannot be written in the form $\frac{a}{b}$ where a a, b are integers and b $\neq 0$. Examples of Irrational numbers include: $\sqrt{2}$ and the number π.

Real Numbers

The set of **real numbers** is the union of the set of rational numbers with the set of irrational numbers. This set is represented by the symbol R.

Exercise

Given the following set of numbers $\left\{-3, \frac{4}{3}, 0.12, \sqrt{2}, \pi, 10, 2.\overline{15}\right\}$. List the numbers that are:

a. Natural numbers

b. Rational numbers

c. Irrational numbers

d. Integers

e. Real numbers

ARITHMETIC OPERATIONS ON RATIONAL AND REAL NUMBERS

Arithmetic is the simplest form of mathematics and is used everyday to solve most of the common problems encountered in work, play, and living. Basic arithmetic includes addition, subtraction, multiplication, and division. The four different operations of arithmetic are easy to recognize because of the use of signs to indicate the type of operation being performed.

1. + plus sign

2. - minus sign

3. x multiplication sign

4. ÷ division sign

The equal sign (=) is used to show equal or even values. For example, two plus two equals four, or stated another way 2 + 2 = 4. The values on each side of the equal sign are equal. The addition of two or more numbers gives a result called the SUM. The subtraction of one number from another results in an answer called the DIFFERENCE. When two numbers are multiplied the result is called a PRODUCT. The division of one number by another gives a result called a QUOTIENT**.**

Mathematical Symbols

X_i - a symbol meaning the individual values of a variable X.

Y_i – an alternative symbol, used to denote another set of values where two sets of values are involved.

ΣX – 'Sigma X' means the sum of the individual values of the variable X

ΣX^2 - 'Sigma X squared' means the sum of the squares of the individual X values.

$(\Sigma X)^2$ – 'the square of sigma X' i.e $(\Sigma X)^2 = (X_1 + X_2 + \ldots. + X\ )^2$

Σ(X + Y) – 'sigma X plus Y' means the sum of the sums of corresponding X and Y values.

Σ(X - Y) – 'sigma X minus Y' means the sum of the differences between corresponding X and Y values

Statements And Relationships

$\cong$	approximately equal to
$X \le Y$	X is less than or equal to Y
$X > Y$	X is greater than Y
$X \ne Y$	X is not equal to Y
$Z \le X \le Y$	X is greater than or equal to Z, but less than or equal to Y
$Z < X < Y$	X is greater than Z but less than Y
$\sqrt{X}$	square root of X
$X \subset Z$	X is a member of or a subset of Z

Basic Properties of Real Numbers

Let *a*, *b*, and *c* be arbitrary elements in the set of real numbers *R*.

Associative Property states that:

i. a + (b +c) = (a + b) + c
ii. a(bc) = (ab)c

Commutative Property states that:

i. a + b = b + a
ii. ab = ba

Distributive Property states that:

i. a(b + c) = ab + bc
ii. (a + b)c = ac + bc

The Closure Law of Real numbers

This law states that if any two real number from any given set A are combined by an operation * and the result is still a member of the given set A, then the set A is said to be closed under the operation *.

For instance the set of integers is closed under the operation of addition because when you add any two integers the result will always be an integer. However this set is not closed under division because you can get 3 and 0 which are both integers but when you divide 3 by 0 you get a number which is not defined and is not part of the set of integers.

Fractional Properties of Real numbers

For all real numbers *a*, *b*, *c*, *d*, and *k* (division by 0 excluded):

i. $\frac{a}{b} = \frac{c}{d}$ if and only if ad = bc E.g $\frac{4}{6} = \frac{6}{9}$ since (4)(9) = (6)(6)
ii. $\frac{ka}{kb} = \frac{a}{b}$ E.g $\frac{7\cdot 3}{7\cdot 5} = \frac{3}{5}$
iii. $\frac{a}{b}\cdot\frac{c}{d} = \frac{ac}{bd}$ E.g $\frac{3}{5}\cdot\frac{7}{8} = \frac{7\cdot 3}{5\cdot 8} = \frac{21}{40}$
iv. $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b}\cdot\frac{d}{c}$ E.g $\frac{2}{3} \div \frac{5}{7} = \frac{2}{3}\cdot\frac{7}{5} = \frac{14}{15}$

v. $\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b}$ E.g $\frac{3}{6}+\frac{5}{6}=\frac{3+5}{6}=\frac{8}{6}$

vi. $\frac{a}{b}-\frac{c}{b}=\frac{a-c}{b}$ E.g $\frac{7}{8}-\frac{3}{8}=\frac{7-3}{8}=\frac{4}{8}$

vii. $\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$ E.g $\frac{2}{3}+\frac{3}{5}=\frac{2\cdot5+3\cdot3}{3\cdot5}=\frac{10+9}{15}=\frac{19}{15}$

Modulus or Absolute value of Real numbers

The **absolute value** or **magnitude** of a real number a is denoted by $|a|$ and is defined by $|a| = a$ if $a \geq 0$ and $|a| = -a$ if $a < 0$

Example

i. $|5| = 5$ since $5 > 0$

ii. $|-4/7| = -(-4/7) = 4/7$ since $-4/7 < 0$

iii. $|0| = 0$ since $0 \geq 0$

Note that the effect of taking the absolute value of a number is to strip away the minus sign if the number is negative and to leave the number unchanged if it is nonnegative.

Properties of Absolute values

If a and b are any real numbers, then:

i. $|a| = |-a|$

ii. $|ab| = |a||b|$

iii. $|a/b| = |a|/|b|$

Note that $|x| < a$ implies that x must lie between +a and –a, i.e $x > -a$ AND $x < a$

Whereas $|x| > a$ implies that x must either be greater than +a or must be less than –a, i.e $x > a$ OR $x < -a$

Example

Express the following as inqualities of x:

a. $|2x - 3| \leq 7$

b. $|3x + 1| > 8$

Solution

a. $|2x - 3| \leq 7$, then from the Note above:

$2x - 3 \leq 7$ and $2x - 3 \geq -7$

$2x \leq 10$ and $2x \geq -4$

$x \leq 5$ and $x \geq -2$

Hence $-2 \leq x \leq 5$

b. $|3x + 1| > 8$ implies:

$3x + 1 > 8$ or $3x + 1 < -8$

$3x > 7$ or $3x < -9$

$x > 7/3$ or $x < -3$

Order of operations in Real numbers

Numbers are often combined in a series of arithmetical operations. When this happens a definite sequence must be observed.

i. Brackets are used if there is any danger of ambiguity. The contents of the bracket must be evaluated before performing any other operation.
   Thus:
   25 - (14 - 5) = 25 - 9 = 16
   8 × (5 + 3) = 8 × 8 = 64

ii. Multiplication and division must be done before addition and subtraction.
   Thus:

   6 × 12 + 11= 72 + 11 = 83

   25 ÷ 5 – 2 = 5 - 2 = 3

   14 × 3 - 6 ÷ 2 = 42 - 3 + 5 = 39 +5 = 4

Proportions

A ratio is a comparison between two similar quantities. If the length of a certain road is 34km and a model of it is 1 centimeter long, then the length of the model is $\frac{1}{34}$ of the length of the road. This can also be written as 1 to 34 and usually represented as 1:34. The ratio has no units, it is dimensionless.

Example

The lengths are in the ratio 4:3. If the first length is 34km, what is the second length?

Solution

The second length = $\frac{3}{4}$ of the first length = (3/4) * 34 = 25.5 km

Direct Proportion

Two quantities are said to be in direct proportion if they increase or decrease at the same rate. If we buy 2kg packets of sugar at K7 000 then we suggest to pay K10 500 for 3kg sugar and K3 500 for 1 kilogram packet. That is if we double the amount bought, we double the costs; if we halve the amount bought, we halve the cost.

Inverse Proportion

If 10 men can dig a trench in 8 hours, how long would 20 men take to dig the same
trench, working at the same rate? If we double the number of men then we should halve
the time taken. If we halve the number of men, then the job will probably take twice as
long. This is an example of inverse proportion.

Foreign Currency Conversion

Every country has its own monetary system. If there is to be trade and travel between any two countries, there must be a rate at which the money of one country can be converted into money of the other country. This rate is called the **rate of exchange.** The methods used for direct proportion are applicable to problems in foreign exchange.

Example

A tourist changes traveller's cheques for \$500 into South African rands at 6.37 rands to the dollars. How many rands does she get?

Solution

\$ 500 = 500 × 6.37 rands = 3 185 rands.

Functions

Let X and Y be two nonempty sets. A **function** from X into Y is a relation that associates with each element of X exactly one element of Y.

The set X is called the **domain** of the function. For each element x in X, the corresponding element y in Y is called the **value** of the function at x, or the **image** of x. The set of all images of the elements in the domain is called the **range** of the function. Since there may be some elements in Y that are not the image of some x in X, it follows that the range of a function may be a subset of Y.

Finding the Value of a Function

Functions are often denoted by letters such as f, F, g, G, and others. If f is a function, then for each number x in its domain the corresponding image in the range is designated by the symbol $f(x)$read as " f of x" or as " f at x."We refer to $f(x)$as the **value of f at the number x;** $f(x)$ is the number that results when x is given and the function f is applied; $f(x)$ is the output corresponding to x or the image of x, $f(x)$ does *not* mean "f times x."

Example

Given the function $f(x) = 3x – 2$. Find the value of the following:

i. $f(5)$

ii. $f(x – 3)$

Solution

i. $f(x) = 3x – 2$ then

$f(5) = 3(5) – 2 = 15 – 2 = 13$

ii. $f(x) = 3x – 2$ then

$f(x – 3) = 3(x -3) – 2 = 3x – 9 – 2 = 3x – 11$

Types Of Functions

i. One-To-One Functions: In a One-To- One function every element in the domain is mapped onto a unique element in the range.

ii. Many-To-One: In a Many-To-One function you have more than one elemnt in the domain being mapped onto one elemnt in the range.

Graph of a function

Not every collection of points in the *xy*-plane represents the graph of a function. Remember, for a function, each number *x* in the domain has exactly one image *y* in the range. This means that the graph of a function cannot contain two points with the same *x*-coordinate and different *y*-coordinates.Therefore, the graph of a function must satisfy the following **vertical-line test:**

Vertical-lineTest
A set of points in the *xy*-plane is the graph of a function if and only if every vertical line intersects the graph in at most one point.

Note: If (x,y) is a point on the graph of a function *f*, then *y* is the value of *f* at *x*; that is, y = *f*(x). Also if y = *f*(x) then (x,y) is a point on the graph of *f*. For example, if (-2,7) is on the graph of *f*, then *f*(-2) = 7 and if f(5) = 8, then the point (5,8) is on the graph of y = *f*(x) .

The Inverse function

For every fuction y = f(x) there is a function that maps back the f(x) value in the range back to x in the domain. This function is known as the Inverse function and is denoted as $f^{-1}(x)$.

Note that only One-To-One functions have an inverse function. The following example illustrates how to find the inverse of a function:

Example

Given the functions below, find their inverse functions:

i. f(x) = 5x – 2

ii. $f(x) = \frac{x+1}{x-1}$

Solution

i. f(x) = 5x - 2 {first step is to convert f(x) to y}

y = 5x – 2 {make x the subject of the formula}

5x = y + 2

$x = \frac{y+2}{5}$ {next interchange the values of y and x to get the inverse}

$y = \frac{x+2}{5}$ {write y as $f^{-1}(x)$ to get inverse function}

$f^{-1}(x) = \frac{x+2}{5}$

ii. $f(x) = \frac{x+1}{x-1}$ {follow the steps outlined above}

$y = \frac{x+1}{x-1}$

y(x – 1) = x + 1

yx – y = x + 1

yx – x = y + 1

x(y – 1) = y + 1

$x = \frac{y+1}{y-1}$

$y = \frac{x+1}{x-1}$

$f\text{-}1(x) = \frac{x+1}{x-1}$

Even and Odd Functions

i. A function is **even** if and only if f(-x) = f(x)

ii. A function is **odd** if and only if f(-x) = -f(x)

Example

Determine whether each of the following functions is even, odd, or neither:

i. $f(x) = x^2 – 5$

ii. $g(x) = x^3 – 1$

iii. $h(x) = 5x^3 – x$

Solution

i. To determine whether f is even, odd, or neither, replace x by -x in $f(x) = x^2 – 5$, then

$f(-x) = (-x)^2 – 5 = x^2 – 5 = f(x)$

since f(-x) = f(x) we conclude that f is an even function.

ii. Replace x by –x in $g(x) = x^3 – 1$, then

$g(-x) = (-x)^3 – 1= -x^3 – 1$

since $g(-x) \neq g(x)$ and $g(-x) \neq -g(x)$ we conclude that g is neither even nor odd.

iii. Replace x by –x in $h(x) = 5x^3$ -1, then

$h(-x) = 5(-x)^3 – x = -5x^3 – (-x) = -5x^3 + x = -(5x^3- x) = -h(x)$

since h(-x) = -h(x) we conclude that h is an odd function.

INDICES AND LOGARITHMS

Exponents

Powers, indices all mean exponents, when *a*.. *a*. *a*. *a*. *a*. *a*.is abbreviated to a^5 , *a* is called the base and 5 is called the exponent. An exponent is then a positive integer, written to the right and slightly above the base, which indicates the number of times the base is to appear as a factor.

Example

a. $a^3 = a \,.\, a \,..\, a$

b. $16 = 2 . 2 . 2 . 2 = 2^4$

c. $216 = 2^3 . 3^3$

Laws of Exponents

If *m* and *n* are positive integers and $a \neq 0$, we have

a. $a^m \times a^n = a^{m+n}$

b. $\frac{a^m}{a^n} = a^{m-n}$ where m > n

c. $(a^m)^n = a^{mn}$

d. $(ab)^n = a^n b^n$

e. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

Zero, Negative and Fractional Exponents

The extension of notion of an exponent to include any rational number (i.e, zero, positive and negative integers and common fractions) is made by the additional definitions.

a. $a^0 = 1$; $a \neq 0$

b. $a^n = \frac{1}{a^n}$; $a \neq 0$ and n is positive

c. $a^{\frac{1}{n}} = \sqrt[n]{a}$; n is a positive integer

Logarithms

The logarithm, base b, of a positive number N (written) log*b N* is the exponent *x* such that $b^x = N$

Example

a. $Log_2 16 = 4$, since $2^4 = 16$

b. $Log_3 81 = 4$, since $3^4 = 81$

Rules for Logarithms

i. $Log_b MN = Log_b M + Log_b N$

ii. $Log_b(M/N) = Log_b M - Log_b N$

iii. $Log_b(M^r) = rLog_b M$

Example

Given Log2 = 0.3 and Log3 = 0.4; then

a. $Log\ 12 = Log\ (3 \times 4) = Log\ (3 \times 2^2) = Log\ 3 + Log\ 2^2$

$= Log\ 3 + 2Log2$

$= 0.4 + 2(0.3)$

$= 1.0$

b. $Log\ 120 = Log\ (3 \times 4 \times 10) = Log\ (3 \times 2^2 \times 10)$

$= Log\ 3 + Log 2^2 + Log10$

$= 0.4 + 0.6 + 1$

$=2$

c. Log 0.12 = Log (12/100) = Log12 – Log 100 = 1.0 – 2

= -1

Equations

An **equation in one variable** is a statement in which two expressions, at least one containing the variable, are equal. The expressions are called the **sides** of the equation. Since an equation is a statement, it may be true or false, depending on the value of the variable.

Solving equations in a single variable

Solve the equation: $3x - 5 = 4$

Solution

Place the variables on one side of the equation and all the constants on the other side of the equation:

$3x - 5 = 5$

$3x = 5 + 5 = 10$

$3x = 10$ {Divide both sides by 10}

$x = (10/3)$

Solving simultaneous Equations

Equations that have a common solution are known as simultaneous equations. Simultaneous equations in two variables can be solved either by substitution method or Elimination method. The example below illustrates how to use the to methods.

Example

Solve the following simultaneous equations using both the Elimination metho and the Substitution method

$x + 5y = 16$

$x + 2y = 7$

Solution by Elimination method: Just as the name suggests this method involves eliminating one of the variables and remain with an equation in one variable which can then be solved as illustrated earlier. To eliminate the variable x from the two equations subtract the first from the second equation. Thus:

$x + 5y = 16$

$- \underline{x + 2y = 7}$

$3y = 9$ {divide both sides by 3}

$y = 3$ {Use this value of y in any of the two equations to get x}

$x + 5(3) = 16$ {from first equation}

$x = 16 - 15 = 1$

Hence solution is $x = 1$ and $y = 3$

Solution by Substitution method: This method involves expressing one of the variables in one equation in terms of the other and substituting it in the other equation. We shall express x in terms of y in the first equation and substitute the this value of x in the second equation. Thus:

x + 5y = 16

x + 2y = 7

{expressing x in terms of y in first equation}

x = 16 – 5y

{substituting this value of x in the second equation}

16 – 5y + 2y = 7

16 – 3y = 7

-3y = 7 – 16

-3y = -9 { divide throughout by -3}

y = 3 {Use this value of y in the earlier expression for x}

x = 16 – 5(3) = 16 – 15 = 1

Hence solution is x = 1 and y = 3

Solving simultaneous Equations in Three variables

To solve equations in three variables, we use elimination method to get two equations in two variables which we can the solve either using elimination or substitution method depending on which is appropriate. We can then use the two variable in any of the three original equation to get the value for the other variable.

The example below illustrates this procedure:

Solve the following simultaneous equations in three variables:

3x + y – z = 2……………(1)

x + 2y – z = 2……………(2)

5x + 3y + z = 14…………(3)

Solution

We shall first add equation(2) and equation(3) to eliminate the variable z. Thus:

x + 2y – z = 2

<u>5x + 3y + z = 14</u> {add these two}

6x + 5y = 16……………..(4)

Next we add equation(3) to equation(1) to eliminate z and get another equation in x and y.Thus:

3x + y – z = 2

<u>5x + 3y + z = 14</u> {add these}

8x + 4y = 16……………..(5)

We can now solve equation(4) and equation(5) with which ever method is appropriate to find x and y:

4 × eq(4): $24x + 20y = 64$

5 × eq(5): $\underline{40x + 20y = 80}$ {subtract the two}

$-16x = -16$ {divide through by -16}

$x = 1$ {ues this value of x in eq(4) to get y}

$6(1) + 5y = 16$

$5y = 16 – 5$

$5y = 10$

$y = 2$ {use the values of x and y in eq(1) to get z}

Thus:

$3(1) + 2 – z = 2$

$-z = 2 – 3 – 2$

$-z = -3$ {divide through by -1}

$z = 3$

Hence the solution is x = 1, y = 2 and z = 3

Quadratic Equations

A **quadratic equation** is an equation equivalent to one of the form:

$ax^2 + bx + c = 0$ where *a*, *b*, and *c* are real numbers and $a \neq 0$. If a quadratic equation can be written as the product of two factors, then the solution can be written down immediately.

Example: Solve the equation: $9x^2 - 6x + 1 = 0$

Solution

$9x^2 - 6x + 1 = 0$

$(3x – 1) (3x – 1) = 0$, then

$x = 1/3$ or $x = 1/3$

If the factors are not obvious or there are no easy factors the Quadratic formula method is used. The solution to a quadratic equation $ax^2 + bx + c =$ is given by the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The quantity $b^2 - 4ac$ is called the **discriminant** of the quadratic equation, because its value tells us whether the equation has real solutions. In fact, it also tells us how many solutions to expect.

Discriminant of a Quadratic Equation

For a quadratic equation if:

i. $b^2 - 4ac > 0$, there are two unequal real solutions

ii. $b^2 - 4ac = 0$, there is a reapeted solution

iii. $b^2 - 4ac < 0$, there is no real solution

When asked to find the real solutions, of a quadratic equation, always evaluate the discriminant first to see if there are any real solutions.

Example: Use the quadratic formula to find the real solutions, if any, of the equation $3x^2 - 5x + 1 = 0$

Solution

Since the equation is in standard form we find the values of a, b and c: If we compare with the standard form a = 3, b = -5 and c = 1.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(3)(1)}}{2(3)} = \frac{5 \pm \sqrt{13}}{6}$$

The solution is $x = \frac{5+\sqrt{13}}{6}$ or $x = \frac{5-\sqrt{13}}{6}$

Unit summary

Summary

In this unit you learned about the different types of numbers in the number system and how touse arithmetic and algebraic operations on rational and real numbers. You also learnt how to evaluate indices and logarithms as well as how to solve different types of equations.

Assignment

Assignment

Assignment 1

i. If 20 men can dig in 5 hours, how long would it take 25 men to dig the same trench?

ii. If 5 litres of semi-sweet wine cost K75 000 , how much does 50 litres cost?

iii. Use the order of operations to find evaluate the following:

  a. $6 - [3\times5 + 4\div2 + 2(8 - 3)]$

  b. $10 - [6 - 2 \times 2 + (5 - 2)]$

iv. For the function defined by $f(x) = 2x^2 - 3$, evaluate:

  a. $f(3x)$

  b. $f(x + 3)$

  c. $f(-x)$

Unit 2

FINANCIAL MATHEMATICS

Introduction

This unit introduces learners to the various financial tools available for them to use as they operate their daily work.

Upon completion of this unit you will be able to:

Outcomes

- *Calculate* simple and compound interest .
- *Calculate* depreciation and discounting .
- *Evaluate* arithmetic and geometric progress .
- *Use* loan amortisation.
- *Evaluate* annuities.
- *Apply* capital investment appraisal.

Terminology

Interest:	This is the amount that is paid for the use of money lent or borrowed
Depreciation:	Loss of value of an asset of time due to tear and wear
Effective interest rate:	This is the interest rate that is compounded in time periods of less than a year.
Amortisation:	This is a way of repaying a loan where each payment goes partly to the capital and partly to the interest
Annuities:	These are payment or receipts made over regular time intervals
Progressions:	These are sequences of numbers where each number is derived by applying some rule to the previous number
Appraisal:	This is the term used to assess the viability of a project

SIMPLE AND COMPOUND INTEREST

Interest is the amount paid for the use of money that has been borrowed or invested. The easiest type of interest to calculate is called **simple interest.** The calculations are the same for both a loan and a purchase on credit. The interest rate is a percent of the principal for the period of the loan or credit. The quoted percent usually is an *annual* (yearly) rate. A rate of 10% means that the interest payment for 1 year will be 10% of the principal.

To compute the amount of simple interest on a 1-year loan, simply multiply the Principal by the Rate.

Interest wether simple or compound if found from the formula:

Interest = Principal × Rate × Time {abbreviated as below}

I = PRT

Accrued Amount for Simple Interest (A_n)

The accrued amount is the principal amount to which interest has been added after a certain period of time. For Simple Interest the accrued amount is given by the formula:

$A_n = P(1 + rn)$

Where A_n = the accrued amount after some time has elapsed

P = the principal amount

r = proportional interest rate

n = time (normally in years)

Example

Find the amount that will be in Mr. Banda's account if he invests K5,650 in an account that pays simple interest of 7% p.a. for 5 years.

Solution

$A_n = P(1 + rn)$; P = K5,650, r = 0.07, n = 5 years, therefore:

$A_n = P(1 + rn)$ = K5,650[1 + 5(0.07)] = K5,650(1.35)

= K627.50

Compound Interest

Compound interest is different frm simple interest in that in this case interest is also charged on the interest and not only on the principal amount.

Accrued amount for Compound Interest

The accrued amount for compound interest is given by the formula below:

$A_n = P(1 + r)^n$

Where An = Accrued amount

P = the principal amount

r = proportional interest rate

n = time (normally in years)

Example

Find the accrued amount if K15,400 is invested in an account that pay 10% compound interest for 5 years.

Solution

$A_n = P(1 + r)^n$; P = K15,400 , r = 0.1 and n = 5 years. Therefore:

$A_n = P(1 + r)^n = K15,400(1 + 0.1)^5 = K24,801.85$

Effective Interest Rate

In most cases interest is calculated in time periods of less than a year. When that happens we say interest is compounded and the compounding period may be a day, monthly, quarterly etc. This interest that is compounded is known as the **Effective Interest Rate or The Actual Percentage Rate (APR)** and the one calculated annually is known as the **Nominal rate.** The nominal rate is always slightly lower than the Actaul Percentage Rate.

Example: Calculate the accrued amount if K2,500 is invested for 3 years in an account that pays 12% nominal interest if the interest is compounded:

i. Quarterly
ii. Semi-annually

Solution

i. If interest is compounded quarterly then the Actual Percentage Rate = 12% divided by the number ofquarters in a year. Thus:

   APR = (12%)/4 = 3% = 0.03

   Since the investment is for three years we multiply the three years by the number of quarters in a year. Our value for n will then be n = 3 × 4 = 12 Quarters.

   The accrued amount will then become:

   $A_n = P(1 + r)^n = K2,500(1 + 0.03)^{12} = K2,500(1.03)^{12}$

   $= K3,564.40$

ii. If interest is compounded semi-annually, then APR = (12%)/2 = 6% = 0.06 and n = 3 × 2 = 6

   The accrued amount will then become:

   $A_n = P(1 + r)^n = K2,500(1 + 0.06)^6 = K2,500(1.06)^6$

   $= K3,546.30$

Given the nominal rate, one should be able to calculate the Actual Percentage Rate from the formula below:

$$\text{Actual Percentage Rate(APR)} = \left[1 + \frac{i}{n}\right]^n - 1$$

where i = nominal rate

n = number of equal compounding periods in a year

Example

Calculate the Actual Percentage Rate (APR) for a bank that charges a nominal rate of 10% compounded quarterly.

Solution

$$\text{Actual Percentage Rate(APR)} = \left[1 + \frac{i}{n}\right]^n - 1 \quad ; \; i = 10\% = 0.1 \text{ and } n = 4$$

$$\text{Actual Percentage Rate(APR)} = \left[1 + \frac{0.1}{4}\right]^4 - 1 = [1.025]^4 - 1$$

$$= 1.1038 - 1$$

$$= 0.1038$$

∴ Actual Percentage Rate (APR) = 10.38%

DEPRECIATION AND DISCOUNTING

Depreciation

Depreciation is the loss of value in an asset over time due to tear and wear. The two types of depreciation discussed in this module are:

i. Straight line depreciation

ii. Reducing Balance depreciation

Straight- line depreciation

In Straight-line depreciation an asset loses value by a constant amount each year. This constant amount is known as the Annual depreciation. We use the formula below to calculate the annual depreciation:

$$\text{Annual depreciation} = \frac{\text{Original value} - \text{Scrap value}}{\text{Years to get to scrap}}$$

Example

A computer that is worth K5,600 depreciates to a scrap value of K600 in 12 years. Using the Straight-line method of depreciation calculate the value of the after eight (8) years.

Solution

We first calculate the annual depreciation. Thus:

$$\text{Annual depreciation} = \frac{\text{Original value} - \text{Scrap value}}{\text{Years to get to scrap}} = \frac{\text{K5,600} - \text{K600}}{12}$$

= K 416.67

The value of the computer after eight (8) years is:

= Original value – 8(K416.67) = K5,600 – K3,333.36 = **K2,266.64**

Reducing Balance depreciation

In reducing balance depreciation an asset loses value by a constant percentage each year. In reducing balance depreciation the value of an asset after losing value is known as the depreciated value and this is given by the formula below:

$D = B(1 - r)^n$

where D = the depreciated value

B = the original value of an asset

r = the depreciation rate

n = the number of years

Example

Using reducing balance depreciation calculate the depreciated value of a machine after 12 years if it depreciates at a rate of 8% and had an original book value of K8,900.

Solution

$D = B(1 - r)^n$; B = K8,900, n = 12 years and r = 8% = 0.08.

The value of the machine after 12 years will be:

$D = B(1 - r)^n = K8,900(1 - 0.08)^{12} = K8,900(1.08)^{12} =$ **K3,272.23**

Exercise

i. Using Straight-line depreciation determine the value of a vehicle after 7 years if it depreciates from K35,500 to a Scrap value of K1,500 in 15 years.

ii. Calculate the rate of depreciation that would halve the value of an asset in 5 years.

PRESENT VALUES AND DISCOUNTING

In discussing Interest, we talked about the principal and accrued amount which respectively represent the value we have now to invest so that after some years a certain amount. So the principal amount we can look at it as the value we have presently (Present Value) and the accrued amount as the value we want to have in the future (Future Value).

We are really talking about the same thing here, only that now we have some amount in mind that we want to after a certain period of time and we want to determine how much we should invest now so that after this period of time we have that amount.

Suppose we want after n years to have the amount A for some project, the **Present value** of this amount A is what we should invest now at a certain interest rate (r) so that after n years we have the amount A. The amount A is what is known as the **Future value.**

In short we are just playing around with the formula for accrued amount so that we make the principal the subject of the formula. The Present Value (PV) of any amount A is given by the formula below:

$$PV = \frac{A}{(1 + r)^n}$$

where PV = Present Value

A = Future Value

r = Interest rate (Discount rate)

n = Number of years

The interest rate is now known as the discount rate because we are discounting the future value to its present value.

Example

Dertime the Present value of K12,000 in five (5) years' time if the discount rate is 10%.

Solution

$PV = \frac{A}{(1+r)^n}$; A = K12,000 , r = 10% = 0.1 and n = 5 years.

$$PV = \frac{A}{(1+r)^n} = \frac{K12,000}{(1+0.1)^5} = \frac{K12,000}{(1.1)^5} = K7,451.06$$

What this means is that K12,000 in five (5) years time is worth this amount now if the interest rate(discount rate) does not change.

Exercise

Mr Phiri has just won a lottery which gives him an option of receiving K20,000 now or get K35,000 in five(5) yeas' time. If the interest rate is at 7% and assuming it does not change for the next five(5) years advise Mr. Phiri on the course of action to take.

ARITHMETIC AND GEOMETRIC PROGRESS

Arithmetic Progressions

An arithmetic progression (A.P) is a sequence of numbers (Terms), in which any term after the first can be obtained from its immediate predecessor by adding a fixed number, called the common difference (d).

For example the sequence of numbers: 3, 7, 11, 19, etc is an A.P., since the common difference between one term and the next is 4 (the value of d).

The first term of an arithmetic progression is usually denoted by **a**. An A.P has the general form: **a, a + d, a + 2d, a + 3d, ….etc**.

The n-th Term of an A.P

If an a.p. has starting term **a** and common difference **d,** then: The n^{th} term of that progression is:

$$T_n = a + (n - 1)d$$

Where **a** is the first term, **(n – 1)** is the value of **n** minus 1, and **d** is the common difference.

Example: Find the 21st term of the following progression: 3, 5, 7,…

Solution

The first term, a = 3; common difference, d = 2; the number of terms, n = 21. The 21st term in the series is :

$T_{21} = 3 + (21 – 1)2 = 3 + (20)2 = 3 + 40 = 43$

Example:

A new company makes 250 products in the first week. If the rate at which these are produced increases by 6 each week, find how many will be produced in the 40th week of manufacturing.

Solution

The first term, a = 250; d = 6; n = 40

$T_{40} = 250 + (40 – 1)6 = \underline{484}$

The Sum Of An Arithmetic Progression

The sum of the first n terms of an a.p., (S_n), is given by the formula:

$$S_n = \frac{n}{2}[a + [(a + (n - 1)d)]]$$

$$= \frac{n}{2}[\text{First term} + n - \text{th term}]$$

Example

Find the sum of the first 12 terms of the progression 6, 11, 16, 21,….

Solution

$$S_n = \frac{n}{2}[2a + (n – 1)d]$$

$$S_{12} = \frac{12}{2}[2(6) + (12 – 1)5]$$

$$= 6 (12 + 55)$$

$$= 402$$

Example

The first term of an a.p. is 7 and the fourth term is 16. Find:

i. The sum of the first four terms
ii. The sum of the first ten terms

Solution

i. n = 4; a = 7; $T_4 = 16$

$$S_4 = \frac{n}{2}(\text{First term} + \text{Fourth term})$$

$$= \frac{4 (7 + 16)}{2}$$

$$= 2(23)$$

$$= 46$$

ii. 4th term = a + 3d
16 = 7 + 3d
9 = 3d
Therefore d = 3

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$S_{10} = \frac{10}{2}[2(7) + (9)3]$$

$= 5(14 + 27)$

$= 205$

Geometric Progressions

A geometric progression (g.p.) is a sequence of numbers, called **Terms**, in which any term after the first can be obtained from its immediate predecessor by multiplying by a fixed number, called the **common ratio (R).**

A geometric progression takes the general form:

$A, AR, AR^2, AR^3, \ldots$ etc

If a g.p. has starting term A and common ratio R, then the n^{th} term is given by:

$T_n = AR^{n-1}$

Sum of the First n terms of a g.p

The sum of the first n terms of a g.p. is given as

$$S_n = \frac{A(1-R^n)}{1-R} \text{ when } R < 1 \text{ and}$$

$$S_n = \frac{A(R^n - 1)}{R-1} \text{ when } R > 1$$

Example

Find the 11^{th} term and the sum of the first 20 terms of the g.p. below:

4, 8, 16, 32, 64, 128,…..

Solution

i. $T_{11} = AR^{11-1} = AR^{10} = (4)(2^{10}) = 4(1024) = 4096$

ii. $S_{20} = \frac{A(R^{20}-1)}{R-1} = \frac{4(2^{20}-1)}{2-1} = 4(2^{20}-1) = 4\ 194\ 300$

Example

Find the 7^{th} term and the sum of the first 4 terms for the g.p.: 3, 12, 48, 192….

Solution

A = 3 and R = 4.

i. The 7th term, $T_7 = 3(4^{7-1}) = 3(4^6) = 12288$
ii. The sum of the first 4 terms, $S_4 = \frac{3(4^4 - 1)}{4 - 1} = 255$

Sum of an infinite g.p

Even if a sequence is infinite it may have a finite sum. This might occur only if $-1 < R < +1$

$$S_n = \frac{A(1 - R^n)}{1 - R} \quad \text{when } R < 1$$

If $n \Rightarrow$ infinity (∞), then $S_\infty = \frac{A(1 - R^\infty)}{1 - R}$

If R lies between –1 and +1, as n gets larger, R^n becomes smaller, and as n tends to infinity R^∞ becomes zero.

Thus $S_\infty = \frac{A}{1-R}$

Example

Calculate the sum to infinity of the following g.p.: 4, 2, 1, 0.5, ….

Solution

A = 4 and R = 0.5

The sum to infinity, $S_\infty = A/(1 - R) = 4/(1 - 0.5) = 8$

Example

Find the sum to infinity of the g.p. ; 16, 4, 1, 0.25,…

Solution

A = 16 and R = 0.25

$S_\infty = A/(1 - R) = 16/(1 - 0.25) = 21.33$

ANNUITIES

An annuity is a sequence of fixed equal payments or receipts made over regular time intervals. Some common examples of annuities are: weekly wages or monthly salaries, insurance premiums, hire purchase payments etc.

Annuities may be paid:

i. At the end of payment intervals (an ordinary annuity)
ii. At the beginning of payment intervals (a due annuity)

The term of an annuity may:

i. Begin and end on fixed dates (a certain annuity),

ii. Depend on some event that cannot be fixed (a contingent annuity)

iii. Carry on indefinitely (a perpetual annuity)

Accrued value of an annuity

We shall use an example below to illustrate how the accrued value of an annuity is calculated.

Example

Suppose an annuity of three annual payments of K12 000 is invested in a fund that pays 12%. What will be the total value at the end of the three year period, if the money is invested at the beginning of each year?

Solution

The first payment, made at the beginning of year 1, will be invested for three years and thus will amount to K12 000 $(1.12)^3$.

The second payment, made at the beginning of year 2, will be invested for only two years and thus will amount to K12 000 $(1.12)^2$.

The third payment, made at the beginning of year 3, will be invested for only one year and will amount to K12 000 (1.12).

The total amount of the invested annuity can therefore be expressed in the form:

Total Value = K12 000 (1.12) + K12 000 $(1.12)^2$ + K12 000 $(1.12)^3$

= K12 000 [1.12 + $(1.12)^2$ + $(1.12)^3$]

= K12 000 (1.12 + 1.2544 + 1.4049)

= K12 000 (3.7793)

= K45 351.60

Present value of an annuity

If an annuity consists of payments of K A over n years subject to a discount rater, then the present values of the 1st , 2nd, 3rd, …..nth payments (at the end of the 1st , 2nd, 3rd, …..nth years) is given by:

$A/(1 + r)$, $A/(1 + r)^2$, $A/(1 + r)^3$,…….., $A/(1 + r)^n$.

Thus the present value of an ordinary annuity is:

$PV = A/(1 + r) + A/(1 + r)^2 + A/(1 + r)^3 + \ldots + A/(1 + r)^n$

$= A\,[1/(1 + r) + 1/(1 + r)^2 + 1/(1 + r)^3 + \ldots + 1/(1 + r)^n]$

The terms in the brackets can be shown to be equal to $1/r – 1/r(1 + r)^n$

This is known as the annuity discount formula.

Example

Find the present value of an annuity of K300 000 for 5 years, using compound interest at 4% pa, the first receipt being in one years time.

Solution

$PV = 300\,000 / (1.04) + 300\,000 / (1.04)^2 + 300\,000 / (1.04)^3 + 300\,000 / (1.04)^4$

$+ 300\,000\,(1.04)^5$

$= 300\,000\,[\,1 / (1.04) + 1 / (1.04)^2 + 1 / (1.04)^3 + 1 / (1.04)^4$
$+ 1 / (1.04)^5\,]$

$= 300\,000\,[\,1/ 0.04 – 1/ 0.04\,(1.04)^5\,]$

$= 300\,000\,(4.453)$

= 1 336 000 (Kwacha)

If the terms are not many the present value can be calculated by finding the values of the individual terms and adding them up.

Present Value of a Perpetual annuity

Example

What is the net present value of an ordinary annual annuity of K2500 000, indefinitely. given a discount rate of 11.5%.

Solution

$PV = 2500\,000\,[\,1/(1.115) + 1/(1.115)2 + \ldots\ldots.. + 1/(1.115)^{\infty}\,]$

The sum of a g.p. to infinity $S_{\infty} = A/(1 – R)$

R = A = 1/1.115 = 0.896861

$$S_{\infty} = \frac{0.896861}{1 - 0.896861} = \frac{0.896861}{0.103139} = \underline{8.6956523}$$

Thus PV = 2500 000 × 8.6956523

= <u>21 739 131</u>

The annuity discount factor for a perpetual annuity = 1/r. This is because the second term in $1/r – 1/(1 + r)^n r$ becomes very small as n gets very large. In fact $1/(1 + r)^n r \Rightarrow 0$ as $n \Rightarrow$ infinity.

AMORTISATION OF A LOAN

One way of repaying a debt is by paying an amortization annuity. This consists of a regular annuity in which each payment accounts for both repayment of capital and interest. The debt is said to be amortized if this method is used.

Amortisation Annuity payments

The amount borrowed , P, is equal to the net present value of the annuity payments of A over the period of the debt.

$P = A /(1 + r) + A /(1 + r)^2 + A /(1 + r)^3 + \ldots\ldots.. + A /(1 + r)^n$

Example

A man negotiates a loan of K20 000 000 over 15 years at 10.5% pa. Calculate the annual payment necessary to amortize the debt.

Solution

P = 20 000 000; r = 0.105; 1 + r = 1.105; n = 15, A is the annual payment to be found.

$20\,000\,000 = A\,\{1/ 1.105 + 1/(1.105)^2 + 1/(1.105)^3 + \ldots.. + 1/(1.105)^{15}\,\}$

The terms in the brackets are a g.p. with A = R = 1/ 1.105

The sum of the g.p. to 15 terms is:

$$S_{15} = \frac{0.905\,[1-(0.905)^{15}]}{1 - 0.905} = \frac{0.7025}{0.095} = 7.3950$$

Thus, 20 000 000 = A (7.395)

$$A = \frac{20\,000\,000}{7.395} = \underline{2\,704\,530\ (K)}$$

The annual payment necessary to amortize the debt is K2 704 530.

Example

A debt of K500 000 with interest at 5% compounded 6-monthly is amortized by equal semi-annual payments over the next three years.

(a) Find the value of each payment (to the nearest kwacha)
(b) Construct an amortization table (schedule)

Solution

(a) Standard time (period) = 6 months

Interest rate = 2.5% per 6 months

Time periods (n) = 6

Thus P = 500 000; n = 6; r = 0.025; 1 + r = 1.025 and

$500\,000 = A\,[\,1/1.025 + 1/(1.025)^2 + 1/(1.025)^3 + 1/(1.025)^4 + 1/(1.025)^5 + 1/(1.025)^6\,]$

$= A\,(0.9756 + 0.9518 + 0.9286 + 0.9060 + 0.8838 + 0.8623)$

$= A\,(5.5081)$

$$\text{Therefore } A = \frac{500\,000}{5.5081} = 90\,775.40\ (K)$$

(b) Amortisation Schedule

6-month period	Outstanding debt	Interest paid	Payment made	Principal repaid
1	500 000	12 500	90 775	78 275
2	421 725	10 543	90 775	80 232
3	341 493	8 537	90 775	82 238
4	259 255	6 481	90 775	84 294
5	174 961	4 374	90 775	86 401
6	88 560	2 214	90 775	88 561

Note:

Principal repaid = payment made – interest paid

Outstanding debt(current year) = Outstanding debt (previous year) – Principal repaid

CAPITAL INVESTMENTS APPRAISAL

Aspects of a capital investment include

(a) An initial outlay of capital
(b) A set of estimated cash inflows and outflows over the life of the project
(c) In some cases, a resettlement figure which might be caused by resale of plant or shares, or a cash settlement to clear any liabilities incurred

Methods of appraising capital investments

i. Discounted cash flow (Net Present Value)
ii. Internal Rate of Return

Net Present Value or Discounted Cashflow Technique

This involves calculating the sum of the present values of ALL cash flows associated with a project. This sum is known as the Net Present Value (NPV) of the project

Interpretation of Net Present Value

NPV can be interpreted in the following way:

NPV > 0 ⇨ Project is in profit (worthwhile) i.e project earns more than the discount rate.

NPV = 0 ⇨ Project breaks even i.e project earns the same as the discount rate.

NPV < 0 ⇨ Project makes a loss (not worthwhile) i.e project earns less than the discount rate.

Example

It is estimated that an investment in a new process will cause the following cash flows:

End year	0	1	2	3	4	5	6
Cash flow			1500000	2000000	2000000	2000000	2000000
Cash outflow	6000000	1000000					

The firm wishes to earn at least 15% per annum on its projects. Calculate the NPV of the project and comment on the course of action to be taken.

Solution

With respect to a discount rate of 15%, if the project earned exactly 15%, the NPV would be zero. Thus a positive NPV(obtained using a discount rate of 15%) would signify that the return on the project is higher than 15% and , on the criterion of present value, the firm should initiate the project. However, if the NPV is negative, the firm should not proceed with the project.

Year	**Net cash flow**	**Discount factor at 15%**	**Present value**
0	(6000 000)	1.0000	(6000 000)
1	(1000 000)	0.8696	(869 6000
2	1 500 000	0.7561	1 134 150
3	2 000 000	0.6575	1 315 000
4	2 000 000	0.5718	1 143 600
5	2 000 000	0.4972	994 400
6	2 000 000	0.4323	864 600
Total	-	-	(1 417 850)

Since the NPV is negative and comparatively large, the project clearly earns much less than 15% and hence would not be worthwhile.

Internal Rate Return (IRR) method

The IRR of a project is the value of the discount rate that gives an NPV of zero. Alternatively, it can be defined as the rate that a project earns.

Estimation of IRR

Although we can estimate the IRR using the graphical method and the formula we shall only make mention of the graphical method but restrict ourselves to the formula method in this module.

To estimate the IRR graphically:

(a) Scale the vertical axis to include both NPVs;
(b) Scale the horizontal axis to include both discount rates;
(c) Plot the two points on the graph and join them with a straight line;
(d) Identify the estimate of the IRR where this line crosses the horizontal (discount rate) axis.

Estimating IRR using the Formula method

The exact formula equivalent to the graphical interpolation method is given by:

$$IRR = \frac{N_1 I_2 - N_2 I_1}{N_1 - N_2}$$

Where discount rate I_1 gives NPV = N_1 and discount rate I_2 gives NPV = N_2

Example

A project has the following Net Present Values at orrespondig discount rates. Calculate its IRR

$I_1 = 0.15$, and N_1 = K14 000 000

$I_2 = 0.17$, and N_2 = -K7000 000

Solution

$$IRR = \frac{N_1 I_2 - N_2 I_1}{N_1 - N_2}$$

$$= \frac{(14\ 000\ 000)(0.17) - (-7000\ 000)(0.15)}{14000\ 000 - (-7000\ 000)} = \frac{2\ 380\ 000 + 1\ 050\ 000}{21\ 000\ 000}$$

$$= 0.1633\ (16.33\%)$$

Unit summary

Summary

In this unit you learned how to calculate:

- Simple and Compound interest
- Depreciation
- Accrued amounts for simple and compound interest
- Acrrued amounts for annuities
- Present values
- Presnt values for annuities

You have also learned how to amortise loans and the different ways in which you can appraise a capital investment.

Assignment

Assignment

1. A firm wishes to earn at least 15% pa on a certain project. If the NPVs corresponding to each of two discount rates of 6% and 9% are K5 596 000 and (K2 125 000) respectively, estimate the IRR for the project and interpret the result.
2. A business project is considered. It has K12 000 000 initial costs and estimated revenues (inflows) over the following four years of K8000 000, K12 000 000, K10 000 000, and K6 500 000 respectively. If the project costs (outflows) over the four years are estimated at K8 500 000, K3000 000, K1 500 000, and K1 500 000 respectively and the discount rate is 18.5%, evaluate the project's NPV.

Assessment

Assessment

1. Find the maturity value of an annuity of K500 000 invested at 8% compound interest over 5 years if the annuity is:
    a. Ordinary
    b. Due
2. W hat is the net present value of an ordinary annuity of K2 500 per year:
    a. Over 5 years
    b. Over 10 years
    c. Indefinitely

Unit 3

STATISTICAL METHODS

Introduction

This unity gives the learners an opportunity to appreciate the various statistical tools that they can use to better their decision making as they do their work in whatever environment.

Upon completion of this unit you will be able to:

Outcomes

- *Examine* sources of statistics .
- *Examine* collection and sampling techniques of data .
- *Demonstrate* presentation of data .
- *Construct* frequency distributions.
- *Use* statistical measures.
- *Calculate* simple index numbers.

Terminology

Statistics:	A collection of techniques and methods used to describe and analyse data.
Descriptive Statistics:	Descriptive statistics refers to the organization, presentation and summarizing of data.
Inferential Statistics:	statistics or statistical inference is concerned with analysis and generalization of results, which are obtained from sample
Population:	This consists of all those items falling into a defined category.
Sample:	This is a (hopefully) representative subset of the population
Parameter:	This is a characteristic of a population
Statistic:	This is a characteristic of a sample
Sampling frame:	This is a structure which lists or identifies the members of a population.

SOURCES OF STATISTICS

In many applications of statistics, businesses use internal data – that is data arising from bookkeeping practices, standard operating business procedures, or planned experiments by research divisions with the company. Examples are profit and loss statements, employee salary information, production data and economic forecasts. In many applications of statistics, businesses use internal data – that is data arising from bookkeeping practices, standard operating business procedures, or planned experiments by research divisions with the company. Examples are profit and loss statements, employee salary information, production data and economic forecasts. The data sourced from outside the firm is called external data. Internal data may be of two types. Primary data and Secondary data. By primary data, we obtain data from the organization that originally collected them. An example is the population data collected by and made available from the Central Statistical Office (CSO) Zambia. Secondary data come from a source other than the one that originally collected them. Users of secondary data cannot have a clear understanding of the background as the original investigator, and so may be unaware of the limitations of the data at hand. There are many excellent sources of published (Primary and secondary) data compiled by the state, by business and economic associations, and by commercial sources (periodicals).

TYPES OF STATISTICS

The body of knowledge called statistics is sometimes divided into two main areas, depending on how data are used. The two areas are:

i. Descriptive Statistics
ii. Inferential Statistics

Descriptive Statistics

Descriptive statistics consists of the collection, organization, summarization, and presentation of data.

In *descriptive statistics* the statistician tries to describe a situation. Consider the national census conducted by the Zambian government every 10 years. Results of this census give you the average age, income, and other characteristics of the Zambian population. To obtain this information, the Central Statistical Office (CSO) must have some means to collect relevant data. Once data are collected, the Office must organize and summarize them. Finally, the CSO needs a means of presenting the data in some meaningful form, such as charts, graphs, or tables.

Inferential Statistics

Inferential statistics consists of generalizing from samples to populations, performing estimations and hypothesis tests, determining relationships among variables, and making predictions.

Here, the statistician tries to make inferences from *samples* to *populations.* Inferential statistics uses **probability,** i.e., the chance of an event occurring. You may be familiar with the concepts of probability through various forms of gambling. If you play cards,dice, bingo, and lotteries, you win or lose according to the laws of probability. Probability theory is also used in the insurance industry and other areas.

Populations and Samples

It is important to distinguish between a sample and a population. A **population** consists of all subjects (human or otherwise) that are being studied whereas a **sample** is a group of subjects selected from a population.

If the subjects of a sample are properly selected, most of the time they should possess the same or similar characteristics as the subjects in the population. The techniques used to properly select a sample will be explained in in a later section.

VARIABLES AND TYPES OF DATA

A variable is defined as a characteristic or attribute that can assume different values. **Data** are the values (measurements or observations) that the variables can assume.

Variables can be classified as qualitative or quantitative. **Qualitative variables** are variables that can be placed into distinct categories, according to some characteristic or attribute. For example, if subjects are classified according to gender (male or female), then the variable *gender* is qualitative. Other examples of qualitative variables are religious preference and geographic locations.

Quantitative variables are numerical and can be ordered or ranked. For example, the variable *age* is numerical, and people can be ranked in order according to the value of their ages. Other examples of quantitative variables are heights, weights, and body temperatures. Quantitative variables can be further classified into two groups: discrete and continuous.

Discrete variables can be assigned values such as 0, 1, 2, 3 and are said to be countable. Examples of discrete variables are the number ofchildren in a family, the number of students in a classroom, and the number of calls received by a switchboard operator each day for a month. **Discrete variables** assume values that can be counted.

Continuous variables, by comparison, can assume an infinite number of values in an interval between any two specific values. Temperature, for example, is a continuous variable, since the variable can assume an infinite number of values between any two given temperatures. **Continuous variables** can assume an infinite number of values between any two specific values. They are obtained by measuring. They often include fractions and decimals.

MEASUREMENT SCALES OF DATA

In addition to being classified as qualitative or quantitative, variables can be classified by how they are categorized, counted, or measured. This type of classification—i.e., how variables are categorized, counted, or measured—uses **measurement scales,** and four common types of scales are used: nominal, ordinal, interval, and ratio

- Nominal Scale

  The **nominal level of measurement** classifies data into mutually exclusive (nonoverlapping), exhausting categories in which no order or ranking can be imposed on the data.

- Ordinal scale

  The **ordinal level of measurement** classifies data into categories that can be ranked; however, precise differences between the ranks do not exist.

- Interval Scale

  The **interval level of measurement** ranks data, and precise differences between units of measure do exist; however, there is no meaningful zero.

- Ratio Scale

  The **ratio level of measurement** possesses all the characteristics of interval measurement, and there exists a true zero. In addition, true ratios exist when the same variable is measured on two different members of the population.

COLLECTION AND SAMPLING OF DATA

COLLECTION OF DATA

Data can be collected in a variety of ways. One of the most common methods is through the use of surveys. Surveys can be done by using a variety of methods. Three of the most common methods are the telephone survey, the mailed questionnaire, and thepersonal interview.

Telephone surveys

These have an advantage over personal interview surveys in that they are less costly. Also, people may be more candid in their opinions since there is no faceto-face contact. A major drawback to the telephone survey is that some people in the population will not have phones or will not answer when the calls are made; hence, not all people have a chance of being surveyed. Also, many people now have unlisted numbers and cell phones, so they cannot be surveyed. Finally, even the tone of the voice of the interviewer might influence the response of the person who is being interviewed.

Mailed questionnaire surveys

These can be used to cover a wider geographic area than telephone surveys or personal interviews since mailed questionnaire surveys are less expensive to conduct. Also, respondents can remain anonymous if they desire. Disadvantages of mailed questionnaire surveys include a low number of responses and inappropriate answers to questions. Another drawback is that some people may have difficulty reading or understanding the questions.

Personal interview surveys

These have the advantage of obtaining in-depth responses to questions from the person being interviewed. One disadvantage is that interviewers must be trained in asking questions and recording responses, which makes the personal interview survey more costly than the other two survey methods. Another disadvantage is that the interviewer may be biased in his or her selection of respondents.

SAMPLING TECHNIQUES OF DATA

To obtain samples that are unbiased—i.e., that give each subject in the population an equally likely chance of being selected—statisticians use four basic methods of sampling: random, systematic, stratified, and cluster sampling.

Random samples are selected by using chance methods or random numbers. One such method is to number each subject in the population. Then place numbered cards in a bowl, mix them thoroughly, and select as many cards as needed. The subjects whose numbers are selected constitute the sample. Since it is difficult to mix the cards thoroughly, there is a chance of obtaining a biased sample. For this reason, statisticians use another method of obtaining numbers. They generate random numbers with a computer or calculator. Before the invention of computers, random numbers were obtained from tables.

Systematic Sampling

Researchers obtain **systematic samples** by numbering each subject of the population and then selecting every kth subject. For example, suppose there were 2000 subjects in the population and a sample of 50 subjects were needed. Since $2000 \div 50 = 40$, then $k = 40$ and every 40th subject would be selected; however, the first subject (numbered between 1 and 40) would be selected at random.

Stratified Sampling

Researchers obtain **stratified samples** by dividing the population into groups (called strata) according to some characteristic that is important to the study, then sampling from each group. Samples within the strata should be randomly selected. For example, suppose the president of a two-year college wants to learn how students feel about a certain issue. Furthermore, the president wishes to see if the opinions of the first-year students differ from those of the second-year students. The president will select students from each group to use in the sample.

Cluster Sampling

Researchers also use **cluster samples.** Here the population is divided into groups called clusters by some means such as geographic area or schools in a large school district, etc. Then the researcher randomly selects some of these clusters and uses all members of the selected clusters as the subjects of the samples. Suppose a researcher wishes to survey apartment dwellers in a large city. If there are 10 apartment buildings in the city, the researcher can select at random 2 buildings from the 10 and interview all the residents of these buildings. Cluster sampling is used when the population is large or when it involves subjects residing in a large geographic area.

PRESENTATION OF DATA IN FREQUENCY DISTRIBUTIONS

When conducting a statistical study, the researcher must gather data for the particular variable under study. For example, if a researcher wishes to study the number of people who were bitten by poisonous snakes in a specific geographic area over the past several years, he or she has to gather the data from various doctors, hospitals, or health departments. To describe situations, draw conclusions, or make inferences about events, the researcher must organize the data in some meaningful way. The most convenient method of organizing data is to construct a *frequency distribution.*

Two types of frequency distributions that are most often used are the *categorical frequency distribution* and the *grouped frequency distribution.*

Categorical frequency distribution

The example below illustrates how to construct a categorical frequency distribution.

Example

Twenty-five army inductees were given a blood test to determine their blood type. The data set is:

A	B	B	AB	O
O	O	B	AB	B
B	B	O	A	O
A	O	O	O	AB
AB	A	O	B	A

Solution

Since the data are categorical, discrete classes can be used. There are four blood types: A, B, O, and AB. These types will be used as the classes for the distribution:

Class	Frequency	Percent (%)	CumPerc.(%)
A	5	(5/25)*100 = 20	20
B	7	(7/25)*100 = 28	48
O	9	(9/25)*100 = 36	84
AB	4	(4/25)*100 = 16	100

Grouped Frequency Distribution

When the range of the data is large, the data must be grouped into classes that are more than one unit in width, in what is called a **grouped frequency distribution.**

Before constructing a grouped frequency distribution a few terms need to be explained. Consider the class 24 – 30. The **lower class limit** is 24; it represents the smallest data value that can be included in the class. The **upper class limit** is 30; it represents the largest data value that can be included in the class. The lower and upper class boundaries are 23.5 and 24.5 respectively. Note that:

Lower limit – 0.5 = Lower boundary

Upper limit + 0.5 = Upper boundary

Finally the class width is found by subtracting the lower boundary from the upper boundary.

To construct a grouped frequency distribution, follow these rules: **1.** *There should be between 5 and 20 classes.* Although there is no hard-and-fast rule for the number of classes contained in a frequency distribution, it is of the utmost importance to have enough classes to present a clear description of the collected data.

2.*It is preferable but not absolutely necessary that the class width be an odd number*.This ensures that the midpoint of each class has the same place value as the data.The **class midpoint** X_m is obtained by adding the lower and upper boundaries anddividing by 2, or adding the lower and upper limits and dividing by 2

3. *The classes must be mutually exclusive.* Mutually exclusive classes have nonoverlapping class limits so that data cannot be placed into two classes.

4.*The classes must be continuous.* Even if there are no values in a class, the classmust be included in the frequency distribution. There should be no gaps in afrequency distribution. The only exception occurs when the class with a zero frequency is the first or last class. A class with a zero frequency at either end can be omitted without affecting the distribution. **5.** *The classes must be exhaustive.* There should be enough classes to accommodate all the data.

6. *The classes must be equal in width.* This avoids a distorted view of the data. One exception occurs when a distribution has a class that is open-ended. That is, the class has no specific beginning value or no specific ending value. A frequency distribution with an open-ended class is called an **open-ended distribution.**

Example

These data represent the record high temperatures in degrees Fahrenheit (F) foreach of the 50 states. Construct a grouped frequency distribution for the data using7 classes.

112	100	127	120	134	118	105	110	109	112
110	118	117	116	118	122	114	114	105	109
107	112	114	115	118	117	118	122	106	110

116	108	110	121	113	120	119	111	104	111
120	113	120	117	105	110	118	112	114	114

Solution

Determine the Range R = 134 – 100 = 34. Select the number of classes desired (usually between 5 and 20). In this case, 7 is arbitrarily chosen.

Find the class width by dividing the Range by the number of classes.

Width = (34/7) = 4.9, Scale up this value to the next whole number in this case 5

The completed frequency distribution is:

Class limits	Class Mid-Point	Frequency
100 – 104	102	2
105 -109	107	8
110 – 114	112	18
115 – 119	117	13
120 – 124	122	7
125 – 129	127	1
130 – 134	132	1

STATISTICAL MEASURES

MEASURES OF CENTRAL TENDENCY

Measures of central tendency are averages, intended as representatives, which can characterize a whole group. There are different types of averages to suit different situations and requirements.

Arithmetic Mean

The arithmetic mean of a set of values = $\frac{\text{Sum of all the vales}}{\text{Number of values}}$

Sample mean (Ungrouped data)

If a sample has **n** items, then the sample mean is given as $\bar{X} = \sum X/n$

Example:

Find the mean for the following values; 10, 20, 15, 16, 12, 15, 6, 8, 5, 3

$X = \frac{10 + 20 + 15 + 16 + 12 + 15 + 6 + 8 + 5 + 3}{10} = \frac{110}{10} = 11$

Population mean (Ungrouped data)

If a population is of size N, then the mean of X in the population is given by:

$\mu = \sum X/N$

Sample Mean (Grouped data)

The sample mean from a grouped frequency distribution is given as:

$$\bar{X} = \frac{\sum X f}{\sum f}$$

where $\sum Xf$ is the total of the products of the individual values and their respective frequencies, and $\sum f$ is the total frequency.

Example

Calculate the mean of the following simple distribution:

Number of sales(X)	0	1	2	3	4	5
Number of salesmen(f)	1	14	23	21	15	6

Solution

Number of sales (X)	Number of salesmen (f)	Xf
0	1	0
1	14	14
2	23	46
3	21	63
4	15	60
5	6	30
Total	**80**	**213**

$\bar{X} = \sum Xf / \sum f = 213 / 80 = 2.66$

Example

Calculate the sample mean age of heads of household for the following data (assumed values):

Class (age of head of household)	20-24	25-29	30-44	45-59	60-64	65-69	70-79	80-84
Frequency (% of households)	4.2	6.9	25.3	29.2	10.2	8.8	11.7	3.7

Solution

Class (age of head of household)	Class mid-point (X)	Frequency (f)	Xf
20-24	22.0	4.2	92.4
25-29	27.0	6.9	186.3
30-44	37.0	25.3	936.1
45-59	52.0	29.2	1518.4
60-64	62.0	10.2	632.4
65-69	67.0	8.8	589.6
70-79	74.5	11.7	871.65
80-84	82.0	3.7	303.4
Total	**-**	**100.0**	**5130.25**

$\bar{X} = \sum Xf / \sum f = 5130.25/100 = 51.3025$ (years)

The Median

The median is the middle value of a set of data values when they are arranged in ascending order or descending order.

Median for Ungrouped data

i. An Odd Number Of Observations

If a set of **n** values (where n is odd) of a variable X is arranged in ascending order, then the median value is given by:

Median = the value of $X_{(n+1)/2}$

Example

If X represents the number of persons in each of seven households and these take the values:

1	1	2	2	3	3	4
X_1	X_2	X_3	X_4	X_5	X_6	X_7

Then n = 7 and (n+1)/2 = 8/2 = 4, and so $X_{(n=1)2} = X_4 = 2$

ii. An Even Number Of Observations

In this case the median is the mean of the two middle values

$$\text{Median} = \frac{X_{n/2} + X_{n/2+1}}{2}$$

Example

For the set of values:

1	1	2	2	3	3	4	5
X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8

Where n = 8 and n/2 = 4, and n/2 + 1 = 4 + 1 = 5

$$\text{Median} = \frac{X_4 + X_5}{2} = \frac{2 + 3}{2} = 2.5$$

Median for Single valued Distribution

To calculate the median for a single valued frequency distribution, the following procedure should be used.

- Calculate the value of $\frac{\Sigma f + 1}{2}$ (the central item)
- Form a cumulative frequency column (F)
- Find the F value that first exceeds $\frac{\Sigma f + 1}{2}$

Example

Calculate the median for the following distribution of delivery times of orders sent out from a firm:

Delivery time (days)	0	1	2	3	4	5	6	7	8	9	10	11
Number of orders	4	8	11	12	21	15	10	4	2	2	1	1

Solution

Delivery time in days (X)	**No of orders (f)**	**Cumulative frequency (F)**
0	4	4
1	8	12
2	11	23
3	12	35
4	21	56
5	15	71
6	10	81

7	4	85
8	2	87
9	2	89
10	1	90
11	1	91

To find the median, we need the $(92/2)^{th}$ = 46^{th} item (n + 1 = 91 + 1 = 92)

The first F value to exceed 46 is F = 56

Thus the median is 4 (days)

Median for a Grouped frequency Distribution

The procedure for estimating the median by formula for a grouped frequency distribution is:

- Form a cumulative frequency (F) column
- Find the value of $\Sigma f/2$
- Find the F value that first exceeds $\Sigma f/2$ which identifies the median class

The median is calculated using the formula below:

$$\text{Median} = L_M + \left[\frac{\Sigma f/2 - F_{M-1}}{f_M}\right] C_M$$

Where : L_M - lower boundary or limit of the median class

F_{M-1} – cumulative frequency of class immediately prior to the median class

C_M – median class width

And f_M – actual frequency of the median class

Example

Estimate the median for the following data, which represents the ages of a set of 130 people who took part in a statistical survey

Age (years)	20 and under 25	25 and under 30	30 and under 35	35 and under 40	40 and under 45	45 and under 50
Number of people (f)	2	14	29	43	33	9

Solution

Age (years)	Number of people (f)	Cum. Freq. (F)
20 and under 25	2	2
25 and under 30	14	16
30 and under 35	29	45
35 and under 40	43	88
40 and under 45	33	121
45 and under 50	9	130

$\Sigma f / 2 = 130$

Median class $\Rightarrow$ 35 – 40

$L_M = 35$, $F_{M-1} = 45$, $f_M = 43$, $C_M = 5$

$$\text{Median} = 35 + \left[\frac{65 - 45}{43}\right] 5$$

= 37.33 years

The Mode

The mode of a set of data is that value which occurs most often or, has the largest frequency.

Examples:

The mode of the set 2,1,3,3,1,1,2,4 is 1, since this value occurs most often

Example

For the following simple frequency distribution:

X	4	5	6	7	8	9	10
f	2	5	21	18	9	2	1

Mode = 6 (the value with the largest frequency of 21)

Mode for Grouped Frequency Distribution

For a grouped frequency distribution the mode can be estimated:

- Using an interpolation formula
- Graphically

The formula for estimating the mode is:

$$\text{Mode} = L + \left[\frac{D_1}{D_1+D_2}\right] C$$

Where L is the lower limit of modal class, C is the width of the modal class, D_1 is the difference between the modal class frequency (largest frequency) and the frequency immediately preceding it, and D_2 is the difference between the modal class frequency and the frequency immediately following it.

Example

Estimate the mode of the following distribution of ages:

Age (years)	Number of people (f)
20 and under 25	2
25 and under 30	14
30 and under 35	29
35 and under 40	43
40 and under 45	33
45 and under 50	9

Solution

L = 35

$D_1 = 43 - 29 = 14$

$D_2 = 43 - 33 = 10$

$$\text{Mode} = 35 + \left(\frac{14}{14+10}\right) 5 = 37.92 \text{ years}$$

MEASURES OF DISPERSION

Dispersion refers to the spread or variability of data. Measures of dispersion describe the how spread out or scattered a set or distribution of numeric data is.

There are different bases on which the spread of data can be measured. The most common are:

1. Range
2. Inter quartile Range
3. Semi Inter quartile Range
4. Mean deviation
5. Standard deviation

The Range

The range is defined as the numerical difference between the smallest and the largest values in a set or distribution.

Example:

The daily numbers of rejected items detected from the separate output of two industrial machines over fourteen days were:

Machine 1: 4, 7, 1, 2, 2, 6. 2, 3, 0, 4, 5, 3, 7, 4

Machine 2: 3, 2, 2, 3, 3, 2, 4, 1, 1, 3, 2, 4, 2, 2

The range of values for Machine 1 is 7 – 1 = 7

The range of values for Machine 2 is 4 – 1 = 3

Quartiles

The quartiles are the values which divide the data into four equal parts. These values are known as the first, second and third quartiles, are usually denoted as Q_1, Q_2, and Q_3, respectively.

Q_1 is the value below which one quarter of the observations lie and above which three-quarters lie; Q_2 is the median; Q_3 is the value below which three-quarters of the observations lie and above which one quarter lie.

Quartiles for Ungrouped data

For a set of data X_1 , X_2 , ……, X_n , on a variable X , the quartiles are calculated as follows:

If **n** is odd, Q_1 = the value of the $X_{(n+1)/4th}$ observation

If **n** is even, Q_1 = the value of the $(X_{n/4} + X_{1 + n/4})/ 2^{th}$ observation

If **n** is odd, Q_3 = the value of the $X_{3(n+1)/4th}$ observation

If **n** is even Q_3 = the value of $(X_{3n/4} + X_{1 + 3n/4})/ 2^{th}$ observation

For example, the number of children per household for a sample of 12 households arranged in ascending order is as follows:

1 1 2 3 3 3 3 3 4 4
4 6

This is an even number of values (n = 12), so:

$Q_1 = (X_{12/4} + X_{1 + 12/4})/ 2^{th}$ observation i.e. $(X_3 + X_4)/ 2 = (2 +3)/ 2 = 2.5$

$Q_3 = (X_9 + X_{10})/ 2 = (4 + 4) /2 = 4$

Suppose now that the number of households is 13 as follows:

1 1 2 3 3 3 3 3 4 4 4 6 7 ; then

$Q_1 = X_{(13+1)/4} = X_{14/4} = X_{3.5}$ i.e. the midpoint of X_3 and $X_4 = (2 + 3)/2 = 2.5$

$Q_3 = X_{3(13+1) /4} = X_{3(14) /4} = X_{42/4} = X_{10.5} = (X_{10} + X_{11})/2 = (4 + 4)/2 = 4$

The inter quartile range is the difference between the first and the third quartiles. It measures the range of the middle 50% of the data values and can be used when there are extreme values on each side of the data values or each tail of a distribution.

For the above example, **inter quartile range (IQR)** = $Q_3 – Q_1 = 4 – 2.5 = 1.5$

The **semi inter quartile range** can be defined as the length of one quartile.

For the example above it can be calculated as follows:

Semi inter quartile range = $(Q_3 – Q_1)/2 = (4 – 2.5)/2 = 1.5 / 2 = 0.75$

Quartiles for Grouped data

For grouped data:

Q_1 = the value of the ¼ n^{th} item

Q_3 = the value of the ¾ n^{th} item.

Quartile values are easily estimated using cumulative frequencies

Example

A frequency distribution of a sample of incomes is as follows:

Income (K'000)	60 and less than 80	80 and less than 100	100 and less than 120	120 and less than 140	140 and less than 160
No of people (frequency)	7	16	28	21	8

Determine the Quartiles for this distribution

Solution

Income (K'000)	No of people (frequency)	Cumulative Frequency
60 and less than 80	7	7
80 and less than 100	16	23
100 and less than 120	28	51
120 and less than 140	21	72
140 and less than 160	8	80
Total	80	

The quartiles can then be found as follows:

Q_1 is indicated by a cumulative frequency of $n/4 = 80/4 = 20$; Q_2 by $n/2 = 80/2 = 40$; Q_3 by ¾ n = ¾ x 80 = 240/4 = 60.

Standard Deviation

The standard deviation is the most commonly used measure of dispersion. It is calculated on the basis of deviations from the mean.

For a simple set:

Standard Deviation = $\sqrt{\frac{\sum(x-\bar{x})^2}{n}}$ where n is the number of X values of a variable X. The above formula can be transformed to the more convenient formula below:

Standard Deviation = $\sqrt{\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2}$

Example

Calculate the standard deviation for the following set of values: 2, 4, 6, and 8.

Solution

X	**X^2**
2	4
4	16
6	36
8	64
$\sum X = 20$	**$\sum X^2 = 120$**

Standard deviation = $\sqrt{\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2} = \sqrt{\frac{\sum 120^2}{4} - \left(\frac{\sum 20}{4}\right)^2} = 2.24$

Standard deviation for a Grouped frequency distribution

For large data sets, a frequency distribution is usually compiled by the formula

Standard Deviation = $\sqrt{\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2}$

Example

Calculate the standard deviation for the following frequency distribution of class marks out of a maximum of 15 points.

Class Mark (X)	5	6	7	8	9	10	11
Frequency (f)	3	2	5	6	8	5	1

Solution

X	F	X^2	fX	f X^2
5	3	25	15	75
6	2	36	12	72
7	5	49	35	245
8	6	64	48	384
9	8	81	72	648
10	5	100	50	500
11	1	121	11	121
Total	**30**	**-**	**243**	**2045**

Standard deviation = $\sqrt{\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2} = \sqrt{\frac{2045}{30} - \left(\frac{243}{30}\right)^2} = 1.60$

Note: When finding the standard deviation for a grouped frequency distribution the same formula works but the value of x becomes the class mid-point.

SIMPLE INDEX NUMBERS

An index number measures the percentage change in the value of some economic variable over a period of time.

Notation for Index Numbers

P_0 – price at base time point

P_n - price at some other time point

And q_0 – quantity at base time point, q_n – quantity at some other time point.

INDEX RELATIVES

An index relative is an index number which measures the change in a single distinct commodity.

Price relative: $I_p = p_n/p_0 \times 100$

Quantity relative: $I_q = q_n/q_0 \times 100$

Example

The following table gives details of prices and quantities sold of two particular items in a store over two years.

ITEM	1994		1995	
	PRICE	NUMBER SOLD	PRICE	NUMBER SOLD
	P_0	Q_0	P_n	Q_n
VIDEO RECORDER V90	K438 000	37	K462 000	18
TELEVISION TV 90	K322 000	26	K384 000	45

For the video:

Price relative = 462000/438000 × 100 = 105.5

Quantity relative = 18/37 × 100 = 48.6

For the television:

Price relative = 384000/322000 × 100 = 119.3

Quantity relative = 45/26 × 100 = 173.1.

Time Series Relatives

When values of some variable are given over time (a time series), there are two ways in which relatives can be calculated:

(a) Fixed base relatives: Here, each relative is calculated based on the same fixed time point.

(b) Chain base relatives: Here, each relative is calculated with respect to the immediately preceding time point

If p_1, p_2, p_3,….. represent prices during successive intervals of time 1,2,3,…,then $I_{p(1/2)}$, $I_{p(2/3)}$, $I_{P(3/4)}$, … represent price relatives of each time interval with respect to the preceding time interval and are called chain base relatives.

Example: If the average prices of a commodity during 1993, 1994, 1995, and 1996 were 800, 1200, 1500, and 1800 kwacha respectively, the chain base relatives are:

$I_{p(93/94)}$ = 1200/800 × 100 = 150

$I_{p(94/95)}$ = 1500/1200 × 100 = 125

$I_{p(95/96)}$ = 1800/1500 × 100 = 120

The price relative for a given period with respect to any other period taken as base can always be expressed in terms of chain relatives. This is due to the circular property of relatives.

Thus $I_{p(1993/1996)}$ = [1200/800 × 1500/1200 × 1800/1500] × 100 = 225%.

Unit summary

Summary

In this unit you learned about the various sources of Statistics, the different types of statistics and the various methods of collecting data. You have learned about the various way in which sample that represent a population can be collected. You have have learnt how to construct frequency distributions and the various statistical methods of measuring data as well as how to compute simple index numbers.

Assignment

Assignment

Assignment:

The table below shows the relatives for the quantity of potato crop harvested for a region ('000 tons).

YEAR	1991	1992	1993	1994	1995	1996	1997	1998
INDEX (1993=100)	97	102	100	112	128	121	145	149

(a) Convert the above information to show a set of chain base relatives.

(b) Given that the amount of potatoes harvested in 1994 was 587 000 tons, calculate the amount of potatoes harvested (to the nearest 1000 tons) each year from 1994 to 1997.

Unit 4

BASIC LINEAR PROGRAMMING TECHNIQUES

Introduction

This unit shows the learners how to allocate resources which are most of the time scarce by optimising an objective function

Upon completion of this unit you will be able to:

Outcomes

- *Construct* the objective function and the constraints .
- *Appreciate* the graphical solution to linear programming problems
- *Solve* Solve linear Programming problems algebraically.

Terminology

Objective function:	This is a mathematical statement of the objective of the linear programming problem
Constraints:	These are limitations placed on the objective function
Decision variables:	These are variables whose values must be found to meet the objective
Feasible Region:	This is the area in the XOY plane that satisfies all the constraint

OBJECTIVE FUNCTION OPTIMISATION

Linear programming (LP) is used to optimize the allocation of resources such as labour, materials, machines and money. Thus, the framework of a linear programming problem is to optimize (maximize or minimize) an **objective function**, subject to set of conditions or **constraints**.

A constraint that is a limiting factor in a linear programming problem is called an effective constraint. Constraints are usually expressed as **inequalities.**

Variables whose quantities or values are to be found by solving a linear programming problem are called **decision variables**.

The **feasible region** is the area of the X-Y plane that satisfies all the constraints of an LP problem.

The objective function is expressed in terms of the decision variables. For example, if a company makes two products A and B and it wishes to maximize contribution, the objective function will be expressed in terms of the quantities of A and B that are made and sold.

The variables in LP models should be non-negative in value i.e. 0 or positive.

GRAPHICAL SOLUTIONAL TO A LP PROBLEM

LP problems can be solved graphically. A graphical solution is, however only possible when there are only two decision variables. One variable is represented by the X-axis and the other by the Y-axis. However an understanding of inequalities is essential for graphical solutions.

To use the graphical method to solve LP problems you need to graph the constraints which are expressed as inequalities on the XOY plane to come up with the feasible region.The Optimal solution is usually found on the vertices of the feasible region.

In this module we will put more emphasis on solving LP problems using the algebraic method as gives a more precise solution.

FORMULATING AN LP PROBLEM

We shall use the example below to illustrate how we can formulate a Linear Programming problem and solve it algebraically:

Example

A small firm builds two types of garden sheds- Type A and Type B sheds. Type A requires 2 hours of machine time and 5 hours of craftman time. Type B requires 3 hours of machine time and 5 hours of of craftman time. Each day there are 30 hours of machine time available and 60 hours of craftman time. The profit on each type A shed is K60 and on each type B shed is K84.

Formulate an appropriate linear programming problem and determine the number of each shed to produce to maximise profit.

Solution

Step 1. Define the unknowns

Let x = number of type A sheds produced each day

y = number of type B sheds produced each day

Step 2. Define the objective function

Max Profit = K60x + K84y

Step 3. Construct the constraints

Machine Time: $2x + 3y \leq 30$

Craftman Time: $5x + 5y \leq 60$

Non Negative: $x, y \geq 0$

To find the optimal solution you solve the two constraints as equations. Thus 5*Eq1 : $10x + 15y = 150$

2*Eq2 : $\underline{10x + 10y = 120}$ {subtract the two}

$5x = 30$ { divide by 5}

$x = 6$ {replace in equation 1}

$2(6) + 3y = 30$

$3y = 30 – 12 = 18$ {divide by 3}

$y = 6$

Hence the optimal solution is that the firm should produce 6 of each type of sheds.

Unit summary

Summary

In this unit you learned to construct the object functions and the constraints of a linear programming problem and you have also learnt how to formulate LP and solve it algebraically

Unit 5

PROBABILITY

Introduction

This uint introduces probability in a general concept as the chance of an event occurring. Many people are familiar with probability from observing or playing games of chance, such as card games, slot machines, or lotteries. In addition to being used in games of chance, probability theory is used in the fields of insurance, investments, and weather forecasting and in various other areas. Finally probability is the basis of all Inferential Statistics.

Upon completion of this unit you will be able to:

Outcomes

- *Examine* Probability.
- *Identify* Mutually exclusive and Independent events.
- *Calculate* probability in given problems .
- *Use* probability in decision making ..

Terminology

Experiment: Any situation, specially set up or occurring naturally, which can be performed, enacted or otherwise considered in order to gain useful information.

Event: An **event** is defined as any subset of the given outcome set that is of interest.

Outcome: An outcome set for an experiment is a specification of all possible distinct results of the experiment when it is performed once.

Sample Space: The total number of all possible outcome

CONCEPT OF PROBABILITY

A statistical experiment can be described as any situation, specially set up or occurring naturally, which can be performed, enacted or otherwise considered in order to gain useful information.

Outcomes of Experiments

An outcome set for an experiment is a specification of all possible distinct results of the experiment when it is performed once.

Equally likely outcomes are defined when any one outcome of an experiment is no more likely to occur than any other when the experiment is performed.

A classic example of an equally likely outcome set is that obtained when an unbiased standard six-sided die is to be rolled. The outcome set for this experiment is { 1,2,3,4,5,6 }. As long as the die is 'fair' (unbiased) then the outcome '1' would be as likely as any other.

Statistical events

An **event** is defined as any subset of the given outcome set that is of interest. Thus an event consists of a set of outcomes, and the event is said to have occurred if the outcome of the experiment is contained in the event set.

Example: For a production line experiment, we might define the event 'fast production' as 15 or more products in five minutes. Thus the event set is { 15,16,17,18,… }. If the experiment is performed (i.e. the number of products coming off the line in a particular 5-minute period is determined) and the outcome is 23, then the event will have occurred, since 23 is an element of the event set.

For the die –rolling experiment, the event 'even number' is the set {2,4,6}.

If the die is rolled and a '5' occurs, then the event 'even number' WILL NOT have occurred since '5' is not an element of its event set.

DEFINITIONS OF PROBABILITY

Classical definition

Suppose an event E can happen in **h** ways out of a total of **n** possible equally likely ways. Then the probability of occurrence of the event (called its success) is denoted by $p = P(E) = h/n$..

The probability of non-occurrence of the event (called its failure) is denoted by $q = P(\text{not } E) = 1 - P(E)$.

Thus $p + q = 1$ or $P(E) + P(\text{not } E) = 1$.

Example: Let E be the event that the numbers 3 or 4 turn up in a single toss of a die. There are six ways in which the die can fall, resulting in the numbers 1,2,3,4,5,or 6 and if the die is fair we can assume these six ways to be equally likely.

Since E can occur in two of these ways, p = P(E) = 2/6 = 1/3. The probability of not getting a 3 or 4 (i.e. getting a 1,2,5,6) is q = P(E) =1-1/3 = 2/3.

The classical definition of probability is therefore :

P(E) = The number of different ways that the event can occur

The number of different outcomes of the experiment

Empirical Probability

Empirical probability is therefore defined as:

P(E) = the number of times the event occurred = f(E)/Σf

the no. of times the experiment was performed

Example

A number of families of a particular type were measured by the number of children they contain to give the following frequency distribution:

Number of children	0	1	2	3	4	5or more
Number of families	12	28	22	8	2	2

Use the information to calculate the (relative frequency) probability that another family of this type will have:

(a) 2
(b) 3 or more
(c) less than 2 children

Solution

Σf = 74 (the number of times the experiment 'determine the number of children in a single family' has been repeated)

(a) P(2 children) = f(2 children)/Σf = 22/74 = 0.294.
(b) P (3or more children) = 12/74 = 0.162
(c) P (less than 2 children) = 40/74 = 0.541

MUTUALLY EXCLUSIVE AND INDEPENDENT EVENTS

Mutually Exclusive events

Two events of the same experiment are said to be mutually exclusive if their respective event sets do not overlap. Thus when the experiment is performed events that are mutually exclusive cannot happen at the same time (i.e. cannot occur together).

Examples:

(a) when rolling a die, the two events 'Even number' and 'Odd number' would be mutually exclusive.
(b) For the length of time that completed jobs have taken in a factory, the events 'On time ' and 'Over 2 days late' would be mutually exclusive.

Independendent Events

Two events are said to be independent if the occurrence of (or not) of one of the events will not affect the occurrence (or not) of the other. Also, two events that are defined on two physically different experiments are said to be independent.

Note: Mutually exclusive events cannot be independent.

Examples:

(a) The events 'a particular supplier will deliver on time' and 'a particular customer's account will not be paid within a specified time' would be independent events, because they are based on physically different situations or experiments.
(b) Suppose two people had to be chosen randomly from 6 men and 3 women. The events 'a woman chosen as the first person' and 'a woman chosen as the second person' would **not** be independent, since if the first event did occur at the first selection, it would mean that there are less women to choose from for the second selection.

PROBABILITY RULES

1. Probability Limits: The probability of any event E occurring must lie between 0 and 1 inclusive. Thus $0 \leq P(E) \leq 1$

If P(E) = 0, then E is known as an **impossible event**

If P(E) = 1, then E is a **certain event**

2. Total Probability Rule: The sum of the probabilities of all possible outcomes of an experiment must total 1. Thus $\Sigma p = 1$

3. Complimentary Rule: P(E) = 1 – P (E)

4. Addition Rule: In general, the addition rule states that, for two events A and B, the probability of the union of these two events is given by:

$P(A\cup B) = P(A) + P(B) - P(A\cap B)$ **or** P(A or B) = P(A) + P(B) – P(A and B)

i.e. the union of two events is the sum of their individual probabilities minus the probability of the intersection of the two events.

Example: Suppose in a seminar group there are four male students, two of whom are badly prepared for the seminar topic, and three female students, one of whom is badly prepared. Suppose further that one student is always chosen at random to present a seminar paper (this is the statistical experim.).

The sample space for this experiment is:

$S = \{M_w, M_w, M_b, M_b, F_w F_w, F_b\}$

Where M = Male, F= Female, w= well-prepared, b= badly prepared. What is the probability of the union of the two events ($M\cup W$),i.e. students either being male or well prepared.

Solution

P(M) = 4/7 = 0.571

P(W) = 4/7 =0.571

$P(M \cap W)$ = 2/7 = 0.286

Therefore P(M or W) = 0.571 + 0.571- 0.286 = 0.856

Example

The purchase department of a company has analysed the number of orders placed by each of the 5 departments in the company by type for this financial year as given in the table below:

Dept. / Order Type	Sales	Purchase	Production	Accounts	Maintenance	Σ
Consumables	10	12	4	8	4	38
Equipment	1	3	9	1	1	15
Special	0	0	4	1	2	7
Σ	11	15	17	10	7	60

An error has been found in one of the orders. What is the probability that the incorrect order :

(a) was from consumables?
(b) Was not from consumables?
(c) Came from maintenance?
(d) Came from production?
(e) Came from maintenance or production?
(f) Came from neither maintenance nor production?
(g) Was an equipment order from purchase?

Solution

(a) P(Consumables) = 38/60 = 0.633
(b) P(not consumables) = 1- P(consumables) = 1-0.633 = 0.367
(c) P(maintenance) = 7/60 = 0.117
(d) P(production) = 17/60 = 0.283
(e) P(maintenance or production) = P(maintenance) + P(production)

= 0.117 + 0.283

= 0.400

(f) P(neither maintenance nor production) = 1- P(maintenan or prod.)

= 1 – 0.400

= 0.600

CONDITIONAL PROBABILTY

Suppose a contract is put out to tender and four firms (A, B, C, & D) submit proposals. A, B, and C are local firms, D is a national firm and each firm has an equal chance of winning the contract. Given no further information P(A) =1/4, since each of the four firms has an equal chance of winning.

However given the information that a local firm has won the contract, we can remove firm D from the field.

Thus P(A wins the contract) = 1/3

This can be written as P(A wins contract/Local firm has won contract) and read as 'the probability that A wins the contract given that a local firm has won the contract'. This is known as conditional probability.

In general P(A/B) means 'the probability of A given (or conditional upon) B'.

If P(A/B) = P(A) then A and B are independent

If P(A/B) ≠ P(A) then A and B are dependent or conditional.

RULE: For two events A and B conditional probability $P(A/B) = \frac{P(A \cap B)}{P(B)}$

i.e. the probability of A given B is equal to the probability of the intersect of A and B, divided by the simple probability of B.

The multiplication rule enables us to calculate probabilities for the intersection of events and can be derived by rearranging the equation for conditional events:

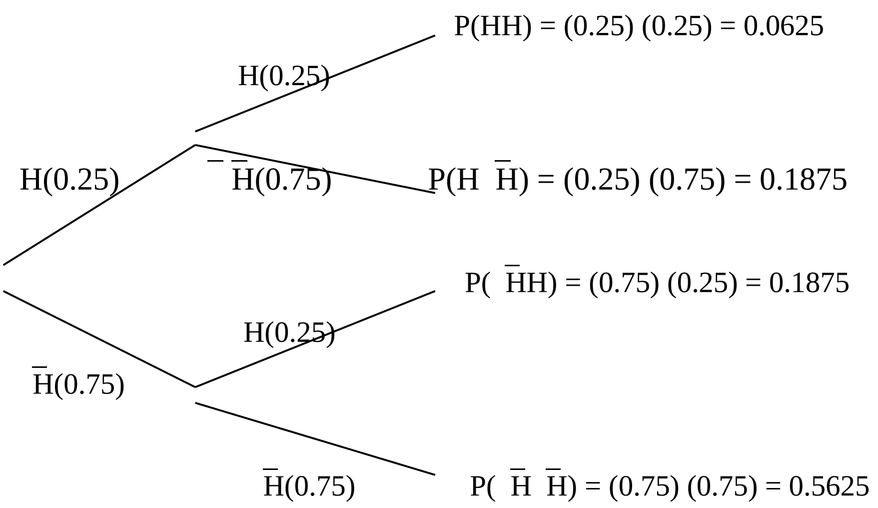
$P(A \cap B) = P(A) \times P(B/A)$ and $P(A \cap B) = P(B) \times P(A/B)$

PROBABILITY TREE DIAGRAMS FOR DECISION MAKING

Tree diagrams can be used in laying out alternatives when independent events are involved.

Example: A lorry makes just two trips per day and on each trip is likely to carry a heavy load (H) with a probability of 0.25. Assuming the load carried in the second part of the day is independent of the previous load , what is the probability that the lorry carries just one heavy load on a particular day?

POSSIBILITIES (TREE DIAGRAM)



Unit summary

Summary

In this unit you learned different defitions of probability, how to calculate probabilities in different given problems. You have also learned what mutually exclusive and independent events are and how you can identify them. We have also looked at how you can make use of tree diagrams to make important decisions

Assignment

Assignment

Assignment

A firm is independently working on two separate jobs. There is a probability of only 0.3 that either of the jobs will be finished on time.

Find the probability that:

a. both
b. neither
c. just one
d. at least one of the jobs will be finished on time

Unit 6

DIFFERENTIATION

Introduction

This unit discusses differentiation that can be used to find the maximum or minimum points of the curves of certain business functions, such as cost, revenue and profit functions.

Differentiation can also be used to measure "rate of change', which is usually applied to cost and revenue functions to obtain marginal cost or marginal revenue respectively.

Upon completion of this unit you will be able to:

Outcomes

- *Use* the basic notations for differentiation.
- *Use* the basic rules for differentiation .
- *Find* Maximum and Minimum points.
- *Determine* Cost, Revenue and Profit functions.

.

Terminology

Derivative:	This is the differential coefficient of y with respect to x
Gradient:	The ratio of the y-step to the x-step on the tangent of any curve

RULE AND NOTATION FOR DIFFERENTIATION

The simple function $y = ax^b$ (where a and b are any numbers) can be differentiated to give the new function:

$$\frac{dy}{dx} = abx^{b-1}$$

where $\frac{dy}{dx}$ is 'the derivative of y with respect to x' and is read as 'dee y by by dee x'. In particular: If y = ax, then $\frac{dy}{dx} = 0$

Examples

(1) If $y = 4x^3$, then $\frac{dy}{dx} = (4)(3)x^{3-1}$

(2) If $y = -3x^2$, then $\frac{dy}{dx} = (-3)(2)x^{2-1} = -6$

(3) If $y = -5x$, then $\frac{dy}{dx} = (-5)(1)x^{1-1} = -5x^0 = -5$

(4) ,, $y = 20x$, $\frac{dy}{dx} = (20)(1)x^{1-1} = 20x^0 = 20$

(5),, $y = 20$, $\frac{dy}{dx} = (20)(0)x^{0-1} = 0$

Thus, the derivative of a constant is always zero.

Derivative of a Product

If u(x) and v(x) are functions differentiable at a certain interval, then:

$$\frac{d(uv)}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

Example

Find the derivative of the following function: $y = (2x - 1)(x^2 - 2x)$

Solution

Let $u = 2x - 1$; $v = x^2 - 2x$

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dx}{dx} = (x^2 - 2x)2 + (2x - 1)(2x - 2)$$

$= (x^2 - 2x)2 + 2x(2x - 2) - 1(2x - 2$

$= 2x^2 - 4x + 4x^2 - 4x - 2x + 2$

$= \underline{6x^2 - 10x + 2}$

Derivatives of a quotient of functions

$$\frac{d(u/v)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Example

Find the derivative of the function: $y = \frac{2x - 1}{3 + x}$

Solution

$$\frac{dy}{dx} = \frac{2(3 + x) - (2x - 1)1}{(3 + x)^2} = \frac{6 + 2x - 2x + 1}{(3 + x)^2} = \frac{7}{(3 + x)}$$

The Chain rule of Differentiation of a composite function

If y = F(u), and u = u(x), then y(x) = F[u(x)] is a composite function and is differentiated as follows: $\frac{dy}{dx} = \frac{dF}{du} + \frac{du}{dx}$

Example

Find the derivative of the function: $y(x) = \sqrt{x^2 - 2x}$

Solution

Setting up $u(x) = x^2 - 2x$, then $y(x) = \sqrt{u(x)}$

Since $\sqrt{u} = u^{1/2}$, $\frac{d(u^{1/2})}{dx} = ½ \quad u^{-1/2} = \frac{1}{2\sqrt{u}}$

Then then $\frac{dy}{dx} = \frac{d(\sqrt{u})}{du} \times \frac{du}{dx} = \frac{1}{2\sqrt{u}} \times (2x - 2) = \frac{2x - 2}{\sqrt{2\ x^2 - 2x}} = \frac{x - 1}{\sqrt{x^2 - 2x}}$

Procedure for Identifying the Turning points of a curve

Differentiation can be used to determine the position of any turning points of the curve defined by a given function.

Example

For the function $y = 0.5x^2 - 8x + 60$, where y is the cost (K'm) of manufacturing x(hundred) items for some process.

The turning points are identified as follows:

(a) Obtain $\frac{dy}{dx} = x - 8$

(b) Solve the equation $\frac{dy}{dx} = 0$, which gives the x-coordinates of any

exist. $\frac{dy}{dx} = x - 8$, $x - 8 = 0$; $x = 8$ identifies a turning point.

(c) Evaluate $\frac{d^2 y}{dx^2}$ at each (and any) x-value found

$\frac{d^2 y}{dx^2} > 0$ indicates a minimum point; $\frac{d^2 y}{dx^2} < 0$ indicates a maximum point;

In our example $\frac{d^2 y}{dx^2} = 1$

The turning point at x = 8 is a minimum. Thus, 800 items must be manufactured in order to minimize total costs.

The minimum cost can be calculated by subtracting x = 8 into

$y = .5x^2 - 8x + 60$, which gives:

Minimum cost = $0.5(8)^2 - 8(8) + 60 = 32 - 64 + 60 = 28$ (K'm)

Cost, Revenue and Profit Functions

The gross profit can be considered as a simple function of the difference between revenue and costs.

Profit = Revenue – Costs (or P = R – C)

The Profit Function

Both cost and revenue can be considered as functions of the number of items demanded or supplied.

Profit function $P(x) = R(x) - C(x)$; Where x is the quantity of items demanded, supplied or produced;

P(x) is the profit function in terms of x;

R(x) is the revenue function in terms of x;

C(x) is the cost function in terms of x;

Note: Supply and demand of items are assumed to be identical.

The maximum profit point for some process can found by solving the equation:

$\frac{dP}{dx} = 0$, for x where: x is the quantity of items; P is the profit function in terms of x.

Example

A manufacturer knows that if x (hundred) items are demanded in a particular week:

The total cost function is $14 + 3x$ (K'm); The total revenue function is $19x - 2x^2$ (K'm)

i. Derive the total profit function
ii. Find the profit break-even points
iii. Calculate the level of demand that maximizes profit and the amount of profit obtained

Solution

i. $C(x) = 14 + 3x$; $R(x) = 19x - 2x^2$

$P(x) = R(x) - C(x)$

$= 19x - 2x^2 - (14 + 3x)$

$= 19x - 2x^2 - 14 - 3x$

$= 16x - 2x^2 - 14$

ii. The profit break-even points are the levels of demand that make $P(x) = 0$ i.e $16x - 2x^2 - 14 = 0 \Leftrightarrow x^2 - 8x + 7 = 0$
$(x - 7)(x - 1) = 0$ and $x = 7$ (hundred) or $x = 1$(hundred)

iii. Profit is maximized at $\frac{dP}{dx} = 0$

$\frac{dP}{dx} = 16 - 4x$; $16 - 4x = 0$; $4x = 16$; $x = 4$ (hundred)

$\frac{d^2P}{dx^2} = (-4)(1)x^{1-1} = -4x^0 = -4$; therefore the turning point at $x = 4$ is a max. when $x = 4$, $P = 16(4) - 2(4)^2 - 14 = 18$ (k'm)

Cost Function

The costs involved in standard processes are usually divided as follows:

(a) **Fixed Costs:** this type of cost is normally associated with the purchase, rent or lease of equipment and fixed overheads. In general, it can be considered as all those costs that need to be borne before production can physically begin (and thus independent of the number of items to be produced)
(b) **Variable Costs:** these are normally associated with wages and the supply of raw materials necessary to manufacture each product. Thus they depend on the number of items produced.
(c) **Special costs:** the optional cost factor is sometimes included in a total cost function and might cover costs relating to storage, maintenance or deterioration.

Form of a Cost Function

$\mathbf{C(x) = a + bx + cx^2}$

Where x is the quantity of items demanded or produced; a is the fixed cost associated with the product; b is the variable cost per item; c is the special (optional) cost factor.

Example

The variable costs associated with a certain process are K650 per item. The fixed costs per day have been calculated as K250 000 with special costs estimated as $K0.02x^2$, where x is the number of items produced.

- Derive a function to describe cost per item for a days production
- Calculate the size of the daily run that will minimize cost per item
- Find the cost of a day's production that for a run that minimizes cost per item

Solution

Total cost function = $C = 250\,000 + 650x + 0.02x^2$ for the production of x items

(a) Therefore cost per item, $C_x = \frac{C}{x} = \frac{250\,000 + 650x + 0.02x^2}{x} = \frac{250\,000}{x} + 650 + 0.02x$

(b) The cost per item will be minimized when $\frac{dC_x}{dx} = 0$

$$\frac{dC_x}{dx} = -250\,000x^{-2} + 0.02 = \frac{-250\,000}{x^2} + 0.02 = 0$$

Thus $x^2 = \frac{250\,000}{0.02} = 12\,500\,000$ and $x = 3536$

Thus $x^2 = \frac{250\,000}{0.02} = 12\,500\,000$ and $x = 3536$

The size of the daily run that minimizes cost per item is 3536

When the daily run is 3536 (the size of run which minimizes cost per item) the cost of the run is found by substituting x = 3536 into the total cost function:

$C = 250\,000 + 650x + 0.02x^2 = 250\,000 + 650(3536) + 0.02(3536)^2 = 2\,798\,466$ (K)

Thus the cost of a day's run that minimizes cost per item is 2 798 466 (K).

Thus $x^2 = \frac{250\,000}{0.02} = 12\,500\,000$ and $x = 3536$

The size of the daily run that minimizes cost per item is 3536 When the daily run is 3536 (the size of run which minimizes cost per item) the cost of the run is found by substituting x = 3536 into the total cost function:

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Thus the cost of a day's run that minimizes cost per item is 2 798 466 (K).

Revenue Function

An expression for the revenue obtained from the output of some process is given by:

$R(x) = x.Pr(x)$

Where: x is the quantity of items demanded or supplied; Pr(x) is the demand or price function.

Demand Price Function

Price or demand functions are used to vary the price of an item according to how many items are being considered.

Demand functions are normally **linear.**

Form of a demand function: $Pr(x) = a + bx$

Where: x is the quantity of items; a and b are coefficients that can take any numeric value.

For example if $Pr(x) = 25 - 2x$, the revenue function $R(x) = x(25 – 2x) = 25x – 2x^2$.

Identifying a Linear demand Function

Example

Given that the price of an item is K3500 when 250 items are demanded, but when the price rises to K5500 per item, only 50 are demanded, identify the linear demand function and calculate the price per item at a demand level of 115.

Solution

Putting $Pr = a + bx$:

When $Pr = 3500$, $x = 250$ and $3500 = a + 250b$

When $Pr = 5500$, $x = 50$ and $5500 = a + 50b$

Solving a system of simultaneous equations:

$3500 = a + 250b$

$\underline{5500 = a + 50b}$ {subtracting}

$-2000 = 200b$

$\therefore$ $b = -10$ and $a = 6000$.

$Pr(x) = 6000 – 10x$

At a demand level of 115 items, $Pr = 6000 – 10(115) = 6000 – 1150 = 4850$ (K)

Obtaining a Revenue function from a given Demand (Price)

Given the demand function: $Pr = 10.4 – 1.3x$ (where x is in hundreds), find the level of production (the value of x) which maximizes total revenue.

Solution

Revenue is found by multiplying the demand function (Pr) by the number of items demanded.

Thus, $R = x.Pr = x(10.4 – 1.3x) = 10.4x – 1.3x^2$

To maximize revenue, we need to solve the equation:

$\frac{dR}{dx} = 0$; $\frac{dR}{dx} = 10.4 – 2.6x$ and $\frac{d^2R}{dx^2} = -2.6$ (maximum)

$10.4 - 2.6x = 0$

$\therefore 2.6x = 10.4$

$\underline{x = 4}$

Thus, the level of production necessary to maximize revenue is 400 items.

Marginal Cost and Marginal Revenue Functions

If for some process, the total cost function, C and the revenue function, R are identified, where q is the level of activity:

1. $\frac{dC}{dq}$ is the marginal cost function and can be interpreted as the extra cost incurred of producing another item at activity level q. In other words, the marginal cost function gives the rate of change of total cost with quantity.
2. $\frac{dR}{dq}$ is the marginal revenue function and gives the extra revenue obtained from producing or supplying another item at activity level q. It can be interpreted as the rate of change of total revenue with quantity.
3. The maximum profit point can be found by solving the equation:

   $\frac{dR}{dx} = \frac{dC}{dx}$ which is equivalent to MR = MC

   (Note: Beyond this point, each additional unit will increase total cost by more than total revenue)

Example

A company has estimated that the demand curve for its product is Pr = 10 – 0.003q, where Pr is unit price and q is the quantity of sales.

The total cost function (in K'000) is $C = 1000 + 3q + 0.004q^2$

a) Calculate the level of output and the unit price at which profit will be maximized
b) Calculate the amount of profit at this level

Solution

If Pr = 10 – 0.003q, then total revenue , R = q.Pr. = q(10 – 0.003q) = $10q - 0.003q^2$

The marginal revenue function, $MR = \frac{dR}{dq} = 10 - 0.006q$

Since $C = 1000 + 3q + 0.004q^2$,

$MC = \frac{dC}{Dq} = 3 + 0.008q$

Profit is maximized at MC = MR

i.e. $3 + 0.008q = 10 - 0.006q$

$0.014q = 7$

$\underline{q = 500}$

The unit price is $Pr = 10 - 0.003q = 10 - 0.003(500) = 8.5$ (K'000)

Revenue = $q.Pr = 8.5 \times 500 = 4\ 250$ (K'000)

Costs = $1000 + 3(500) + 0.004(500)^2 = 3\ 500$ (K'000)

Profit = Revenue – Costs = 4 250 000 – 3 500 000 = 750 000 (K)

Example

A company has estimated that the demand curve for its product is $Pr = 8 - 0.05q$, where Pr is the unit price and q is the quantity of sales, in thousands. The total cost function is $C = 4 + 5q + 0.1q^2$ (C is in millions of kwacha). What value of q will maximize profits and what is the maximum profit?

Solution

$Pr = 8 - 0.05q$

$R = \text{Quantity} \times \text{Price} = q(8 - 0.05q) = 8q - 0.05q^2$

$MR = 8 - 0.1q$

$C = 4 + 5q + 0.1q^2$

$MC = 5 + 0.2q$

MC = MR at maximum profit point

Thus $5 + 0.2q = 8 - 0.1q$

$0.3q = 3$

$\underline{q = 10}$

The quantity of sales to maximize profit is therefore 10 000 units.

Profit, $P = R - C = 8q - 0.05q^2 - (4 + 5q + 0.1q^2) = 8q - 0.05q^2 - 4 - 5q - 0.1q^2$

$\therefore \quad P = 3q - 0.15q^2 - 4$

Amount of maximum profit = $3(10) - 0.15(10)^2 - 4 = 30 - 15 - 4 = \underline{9}$ (K'millions)

Unit summary

Summary

In this unit you learned the various notations used in differentiation as well as the techniques that are used to differentiate different types of functions. You also learned how to identify the turning points of a curve.Further you were also taught how to construct the cost, revenue and profit functions as well as how to formulate the marginal revenue and marginal cost functions

Assignment

Assignment

Assignment

A refrigerator manufacturer can sell all the refrigerators of a particular type that he can produce. The total cost of producing q refrigerators per week is given by 300q + 200 (K'000).

The demand function is estimated as 500 – 2q.

(a) Derive the revenue function, R
(b) Obtain the total profit function
(c) How many units per week should be produced in order to maximize profit
(d) What is the maximum profit available?

Unit 7

MATRICES

Introduction

This unit introduces matrices as structures within which numeric data can be stored and manipulated. It also explains how data is referenced within then and of their size and description. It also lists the rules for adding and subtracting matrices and describes special matrices which enable simultaneous equations to be represented and solved.

Upon completion of this unit you will be able to:

Outcomes

- *Evaluate* matrices and determinants .
- *Apply* basic operations .
- *Compute* inversions of 2 x 2 matrices .
- *Solve* simultaneous equations using matrices.

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Terminology

Matrix:	A simple mathematical structure that holds numeric information in a rectangular form
Square matrix:	A matrix that has the same number of rows as columns
Dimension:	The numer of rows and columns that a matrics has
Row matrix:	A matrix that contains only one row
Column matrix:	A matrix that contains only one column
Zero Matrix:	A matrix whose elements are all zeros
Determinants:	This is a real number that is associated with any square matrix of any dimension

DEFINITION AND DESCRIPTION

A matrix is a rectangular arrangement of elements (entries) presented between brackets [ ] or double lines || ||. A matrix can consist of any number of complete rows and columns and the value at the intersection of a row and a column is referred to as a cell, element or data value. If a matrix is labelled as A, then $A_{i,j}$ is the value held in the cell at the intersection of row *i* and column *j*. A matrix which has *a* rows and *b* columns is called an ‘ *a* by *b*’ matrix. In this case ‘ *a* by *b*’, which can also be written as $a \times b$, is said to be the size of the matrix. Examples of matrices include:

$\begin{pmatrix} 2 & 3 \end{pmatrix}$ which is a 1 by 2 matrix

$\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$ which is a 2 by 2 matrix

$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$ which is a 3 by 2 matrix

As can be seen from the above examples, the dimension of a matrix is always given by the numbers of rows first followed by the number of columns.

Types of Matrices

Row matrix: This is a matrix which contains only one row-$\begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$

Column matrix: This is a matrix with only one column-$\begin{bmatrix} 2 \\ 4 \end{bmatrix}$

Zero matrix: This is a matrix were all entris are zeros-$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

Square matrix: This is a matrix with the same number of rows as columns-$\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$

Transpose matrix: This is a matrix obtained by interchanging rows and columns. If A = $\begin{bmatrix} -2 & 5 \\ 4 & 7 \end{bmatrix}$ then the Transponse matrix of A which is written as A^T is $\begin{bmatrix} -2 & 4 \\ 5 & 7 \end{bmatrix}$

Addition and Subtraction of Matrices

Just as the sum or difference of two real numbers is a unique real number, the sum or difference of two matrices is a unique matrix. Only matrices of the same dimensions can be added or subtracted. If dimensions are different the addition or subtraction is undefined. Addition is performed by adding together corresponding elements. Similarly for subtraction.

Example

If A = $\begin{bmatrix} 2 & -1 & 3 \\ 5 & 4 & 4 \end{bmatrix}$ and B = $\begin{bmatrix} -4 & 0 & 8 \\ -3 & -7 & 3 \end{bmatrix}$ find the following matrices:

i. A + B

ii. A – B

Solution

i. $A + B = \begin{bmatrix} 2 & -1 & 3 \\ 5 & 4 & 4 \end{bmatrix} + \begin{bmatrix} -4 & 0 & 8 \\ -3 & -7 & 3 \end{bmatrix}$

$= \begin{bmatrix} 2+(-4) & -1+0 & 3+8 \\ 5+(-3) & 4+(-7) & 4+3 \end{bmatrix} = \begin{bmatrix} -2 & -1 & 11 \\ 2 & -3 & 7 \end{bmatrix}$

ii. $A - B = \begin{bmatrix} 2 & -1 & 3 \\ 5 & 4 & 4 \end{bmatrix} - \begin{bmatrix} -4 & 0 & 8 \\ -3 & -7 & 3 \end{bmatrix}$

$= \begin{bmatrix} 2-(-4) & -1-0 & 3-8 \\ 5-(-3) & 4-(-7) & 4-3 \end{bmatrix} = \begin{bmatrix} 6 & -1 & -5 \\ 8 & 11 & 1 \end{bmatrix}$

Scalar Multiplication

When we work with matrices we refer to any real number as a scalar. The product of a scalar k and a matrix A is the matrix kA each of whose entries are k times the corresponding entry in K.

Example

If $A = \begin{bmatrix} 4 & 3 \\ 7 & 11 \end{bmatrix}$, then $4A = \begin{bmatrix} 4 \times 4 & 4 \times 3 \\ 4 \times 7 & 4 \times 11 \end{bmatrix} = \begin{bmatrix} 16 & 12 \\ 28 & 44 \end{bmatrix}$

Matrix Multiplication

The process of adding the products obtained by multiplying the elements of a row in one matrix by the corresponding elements of a column in another matrix defines the process of matrix multiplication. It is important to note that for matrix multiplication to occur:

- The number of elements in a row in the first matrix must match the number of elements in a column in the second matrix.
- The resulting answer will always have dimensions defined by the number of rows in the first matrix and the number of columns in the second matrix.

The example below illustrates the multiplication of matrices:

Example

$\begin{bmatrix} 3 & -2 & 4 \end{bmatrix} * \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix} = [3 \times 1 + (-2) \times (-3) + 4 \times 5] = [29]$

$\begin{bmatrix} 4 & 5 \\ -2 & 3 \end{bmatrix} * \begin{bmatrix} 8 & -3 \\ -2 & 1 \end{bmatrix}$

$= \begin{bmatrix} (4 \times 8) + (5 \times (-2)) & (4 \times (-3)) + (5 \times 1) \\ (-2 \times 8) + (3 \times (-2)) & (-2 \times (-3)) + (3 \times 1) \end{bmatrix}$

$= \begin{bmatrix} 22 & -7 \\ -22 & 9 \end{bmatrix}$

Matrix multiplication differs from that of real numbers in that it is **not,** in general, commutative. Thus if A and B are matrices then in general:

$$AB \neq BA$$

Identity or Unit Matrix

If we have a square matrix whose main diagonal (from upper left to lower right) consists of entries of "1" and all other entries are "0", we refer to this as an identity matrix and is labeled “I”. The following are examples of identity matrices for 2 by 2 and 3 by 3 matrices:

- $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ {a 2 by 2 Identity Matrix}
- $\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$ {a 3 by 3 Identity Matrix}

Whether we use left or right multiplication, the identity matrix times a given matrix results in the given matrix. This is shown below:

$$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} * \begin{bmatrix}5 & 3\\ -2 & 4\end{bmatrix} = \begin{bmatrix}5 & 3\\ -2 & 4\end{bmatrix} = \begin{bmatrix}5 & 3\\ -2 & 4\end{bmatrix} * \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

Note:

The product of two matrices can give a zero matrix even though none of the two matrices is a zero matrix. This is illustrated below:

$$\begin{bmatrix}3 & 5\\ -6 & -10\end{bmatrix} * \begin{bmatrix}-5 & -10\\ 3 & 6\end{bmatrix}$$
$$= \begin{bmatrix}(3 \times (-5)) + (5 \times 3) & (3 \times (-10)) + (5 \times 6)\\ (-6 \times (-5)) + (-10 \times 3) & (-6 \times (-10)) + (-10 \times 6)\end{bmatrix}$$
$$= \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

Determinant of a 2 by 2 Matrix

The determinant is a real number that is associated with any square matrix of any order. The determinant of a 2 by 2 matrix A is denoted as |A| or Det A and is a real number defined as follows:

If A = $\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ then |A| = *ad* – *bc*

Determinants can be distinguished from a matrix because they are always enclosed within a single set of vertical lines “| |“.

Example

Given that A = $\begin{bmatrix}3 & 4\\ 1 & 2\end{bmatrix}$, find Det A or |A|.

Solution

Det A = (3)(2) – (1)(4) = 6 – 4 = 2

The Inverse Matrix

The Inverse of a square matrix is denoted by A^{-1} and satisfies the relationship: $A*A^{-1} = A^{-1}*A = I_2$ {2 by 2 unit matrix}

The inverse of a 2 by 2 matrix can be specifically calculated as follows:

If A = $\begin{bmatrix}a & b\\ c & d\end{bmatrix}$, then the inverse denoted as A^{-1} is given by:

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$$

Example

Find the inverse of the matrix A = $\begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}$ and check the result, using the relationship $A*A^{-1} = I$

Solution

First we find Det A

Det A = (4×3) – (2×5) = 12 – 10 = 2, That is:

Det A = |A| = 2

Thus $A^{-1} = \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}^{-1} = \frac{1}{|A|}\begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & \frac{-5}{2} \\ \frac{-2}{2} & \frac{4}{2} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & \frac{-5}{2} \\ -1 & 2 \end{bmatrix}$

For the check: A*A-1

$= \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} \frac{3}{2} & \frac{-5}{2} \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 4\frac{3}{2} + 5(-1) & 4\frac{(-5)}{2} + 5(2) \\ 2\frac{3}{2} + 3(-1) & 2\frac{(-5)}{2} + 3(2) \end{bmatrix}$

$= \begin{bmatrix} 6-5 & -10+10 \\ 3-3 & -5+6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Solving Simultaneous Equations

Matrices can be used to solve any size of systems of simultaneous equations but for the purposes of this manual we shall only consider 2 by 2 equations of the form:

$ax + by = c$

$dx + ey = f$

The following section gives the procedure to solve such equations:

STEP 1. Write the equation in the matrix form: $\begin{pmatrix} a & b \\ d & e \end{pmatrix} * \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c \\ f \end{pmatrix}$

{**Note:** The above matrix equation is exactly equivalent to the original pair of simultaneous equations.}

STEP 2. Solve for *x* and *y* using: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ d & e \end{pmatrix}^{-1} * \begin{pmatrix} c \\ f \end{pmatrix}$

The right-hand side of the above equation will reduce to a 2 by 1 matrix, containing the values of *x* and *y*

Example

Solve the simultaneous equations below using matrices:

5x + 9y = -30

6x – 2y = 28

Solution

STEP 1. Putting the equations into matrix form gives:

$$\begin{bmatrix} 5 & 9 \\ 6 & -2 \end{bmatrix} * \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -30 \\ 28 \end{bmatrix}$$

STEP 2. $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 & 9 \\ 6 & -2 \end{bmatrix}^{-1} * \begin{bmatrix} -30 \\ 28 \end{bmatrix} = \frac{1}{5(-2)-9(6)} \begin{bmatrix} -2 & -9 \\ -6 & 5 \end{bmatrix} * \begin{bmatrix} -30 \\ 28 \end{bmatrix} = \frac{1}{-64} \begin{bmatrix} -192 \\ 320 \end{bmatrix}$, That is: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$

Thus the required solutions are: $x = 3$ and $y = -5$

Unit summary

Summary

In this unit you learned about the various types of matrices and how to add and subtract them. You also learned not only how to multiply matrices by a scalar but how you can multiply two matrices together. You went on to learn about the determinant of a 2 by 2 matrix and how you can use it to find the inverse of a 2 by 2 matrix. Finally you learned how to use matrices to solve simultaneous equations in two variables.

Assignment

Assignment

1. Given that A = $\begin{bmatrix} 4 & -3 \\ 2 & 1 \end{bmatrix}$, B = $\begin{bmatrix} -3 & -1 \\ 9 & 5 \end{bmatrix}$ and C = $\begin{bmatrix} 5 & 3 \\ 2 & 4 \end{bmatrix}$.
   Determine the resultant matrix based on the following operations:

   (A + B) + C

   2A – 3C

2. Solve for a, b, c and din the following:

   $\begin{bmatrix} a & b \\ c & d \end{bmatrix} + 3 \begin{bmatrix} -2 & 4 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} -6 & 8 \\ 1 & 3 \end{bmatrix}$

3. Calculate the value of the determinant for the matrix given below:

   $\begin{bmatrix} -3 & 2 \\ 6 & 7 \end{bmatrix}$

4. Given that matrix A = $\begin{bmatrix} 4 & 2 \\ -3 & 5 \end{bmatrix}$, determine the inverse of A.

5. Find the solution to the following system of equations using matrices:

   3x – 5y = 7

   4x + 3y = -4

Unit 8

INTEGRATION

Introduction

When a function $f(x)$ is known we can differentiate it to obtain its derivative $\frac{dy}{dx}$. The reverse process is to obtain the function $f(x)$ from knowledge of its derivative. This process is called **integration**

Upon completion of this unit you will be able to:

Outcomes

- *Identify* the two types of integrals.
- *Identify* the rules of Integration.

Terminology

Integration: Reverse process of Differentiation

Integrand: The function being integrated

INTEGRATION AS REVERSE OF DIFFEREENTIATION

Suppose we differentiate the function $y = x^2$. We obtain $\frac{dy}{dx} = 2x$. Integration reverses this process and we say that the integral of $2x$ is x^2. The situation is just a little more complicated because there are lots of functions we can differentiate to give $2x$. Here are some of them: $x^2 + 4$, $x^2 - 15$, $x^2 + 0.5$

All these functions have the same derivative, $2x$, because when we differentiate the constant term we obtain zero. Consequently, when we reverse the process, we have no idea what the original constant term might have been. So we include in our answer an unknown constant, *c* say, called the **constant of integration**. We state that the integral of $2x$ is $x^2 + c$

Notation and Rule for Integration

The symbol for integration, $\int$ is known as an **integral sign**. To integrate $2x$ we write:

$$\int 2xdx = x^2 + C$$

Thus the integral of a simple function ax^b is given by:

i. $\int ax^b dx = \frac{a}{b+1}x^{b+1} + C$

ii. $\int adx = ax + C$ where C is a constant.

You can see from the above that rule for integrating a simple function of x such as ax^b is:

i. To increase the power of x by 1 {thus b goes to b + 1}, and then

ii. To divide the whole function by b + 1

In particular:

i. $\int u^p du = \frac{u^{p+1}}{p+1} + C$ if $p \neq -1$

ii. $\int u^{-1} du = \ln|u| + C$ if $p = -1$

iii. $\int e^u du = e^u + C$

Example

Find the integral of $4x^2$

Solution

$$\int 4x^2 dx = \frac{4}{2+1}x^{2+1} + C = \frac{4}{3}x^3 + C$$

Types of Integrals

There are basically two types of integrals i.e the Definite and Indefinite Integrals.

1. Definite Integrals

   The Definite integral inputs a function and gives out a number which represents the area between the graph of the input and the x-axis. This is represented by the following:

   $$\int_b^a f(x)dx$$

   The Definite integral takes a specific value and is free from the variable x and the arbitrary constant C. In the formula above is called the upper limit of integration and b the lower limit.

   Example

   Find $\int_{-4}^{2}\left(x - \frac{1}{x^2}\right)dx$

Solution

$$\int_{-4}^{-2}\left(x-\frac{1}{x^2}\right)dx = \left[\frac{x^2}{2}+\frac{1}{x}\right]_{-4}^{-2} = \left(\frac{-2^2}{2}+\frac{1}{-2}\right)-\left(\frac{-4^2}{2}+\frac{1}{4}\right)=\left(2-\frac{1}{2}\right)-\left(8-\frac{1}{4}\right)=-6\frac{1}{4}$$

2. Indefinite Integrals

   This is the inverse operation to the derivative. It is represented by the following:

$$\int f(x)dx$$

The Integral of *kf*(*x*) where *k* is a constant

A constant factor in an integral can be moved outside the integral sign as follows:

$$\int kf(x)dx = k\int f(x)dx$$

Example: Find the integral of $11x^2$

Solution

$$\int 11x^2dx = 11\int x^2dx = 11\left(\frac{x^{2+1}}{3}+C\right) = 11\left(\frac{x^3}{3}+C\right) = \frac{11x^3}{3}+K$$

where K is a constant.

The Integral of *f*(*x*) + *g*(*x*) and *f*(x) – *g*(x)

When we wish to integrate the sum or difference of two functions, we integrate each term separately as follows:

i. $\int[f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx$

ii. $\int[f(x)-g(x)]dx = \int f(x)dx - \int g(x)dx$

Example

Find: $\int\left(3t^4+\sqrt{t}\right)dt$

Solution

$$\int\left(3t^4+\sqrt{t}\right)dt = \int 3t^4dt + \int\sqrt{t}dt = \frac{3}{5}t^5+\frac{2}{3}t^{\frac{3}{2}}+C$$

Integration and Differentiation in a business situation

Example

The total revenue obtained (in K000) from selling x hundred items in a particular day is given by R, which is a function of variable x.

Given that $\frac{dR}{dx} = 20-4x$:

a. Determine the total revenue function R

b. Find the number of items sold in one day that will maximise the total revenue and evaluate this total total revenue.

Solution

a. We are given $\frac{dR}{dx} = 20 - 4x$. Intergrating this must therefore give R. That is, $\int \frac{dR}{dx} dx = \int (20 - 4x)dx = 20x - 2x^2 + C$.

But when no items are sold (i.e. x = 0), there will be no revenue (i.e. = 0). Thus substituting = 0 into R above gives C = 0. So that, R = total revenue = $20x - 2x^2$

b. The value of x that maximises R is found by solving the equation $\frac{dR}{dx} = 0$, That is where $20 - 4x = 0$ which gives x = 5.

In other words, the total revenue is maximised if 500 items are sold in a day. The value of this total revenue is found by sudstituting x = 5 into R.

This gives: 20(5) – 2(5)2 = 100 – 50 = 50.

Thus, the maximum total revenue (obtained by selling 500 items) is K50,000.

Unit summary

Summary

In this unit you learned that Integration is simply the reverse process of differentiation and that there are two types of Integrals, namely the Definite and the Indefinite Integrals. You also learned about the basic notations for these integrals as well as the basic rules of integration. Finally you learnt how to apply integration in a business situation.

Assignment

Assignment

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Evaluate the following:

i. $\int (s^2 - 2s + 3)\, ds$

ii. $\int (p - 1)(2 - p)dp$

iii. $\int_{-2}^{0} (1 - t - t^2)dt$