Section 12.2: Quadric Surfaces

Goals:
1. To recognize and write equations of quadric surfaces
2. To graph quadric surfaces by hand

Definitions:
1. A **quadric surface** is the three-dimensional graph of an equation that can (through appropriate transformations, if necessary), be written in either of the following forms:

$$Ax^2 + By^2 + Cz^2 + J = 0 \text{ or } Ax^2 + By^2 + Iz = 0.$$

2. The intersection of a surface with a plane is called a **trace of the surface** in the plane.

Notes:
1. There are 6 kinds of quadric surfaces. Scroll down to get an idea of what they look like. Keep in mind that each graph shown illustrates just one of many possible orientations of the surface.
2. The traces of quadric surfaces are conic sections (i.e. a parabola, ellipse, or hyperbola).
3. The key to graphing quadric surfaces is making use of traces in planes parallel to the *xy*, *xz*, and *yz* planes.
4. The following pages are from the lecture notes of Professor Eitan Angel, University of Colorado. Keep scrolling down (or press the Page Down key) to advance the slide show.

Calculus III – Fall 2008

Lecture – Quadric Surfaces

Eitan Angel

University of Colorado

Monday, September 8, 2008

Introduction

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- Now we will discuss second-degree equations (called **quadric surfaces**). These are the three dimensional analogues of conic sections.
- To sketch the graph of a quadric surface (or any surface), it is useful to determine curves of intersection of the surface with planes parallel to the coordinate planes. These types of curves are called **traces**.

Definition

In Calculus II, we discuss second degree equations in x and y of the form

$$Ax^2 + By^2 + Cxy + Dx + Ey + F = 0,$$

which represents a conic section. If we are allowed to rotate and translate a conic section, it can be written in the standard form

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$$Ax^2 + By^2 + Cz^2 + J = 0 \qquad \text{or} \qquad Ax^2 + By^2 + Iz = 0$$

Ellipsoids

The quadric surface with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

is called an **ellipsoid** because its traces are ellipses. For instance, the horizontal plane with $z = k$ $(-c < k < c)$ intersects the surface in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{k^2}{c^2}$. Let's graph $\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{9} = 1$.

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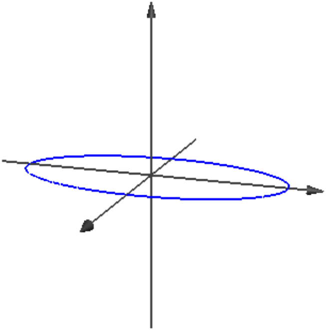

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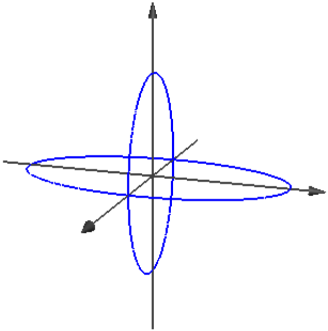

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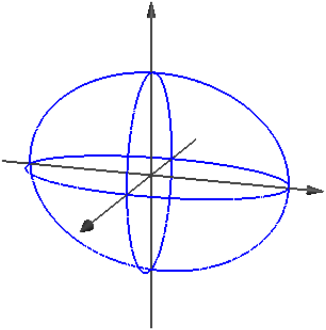

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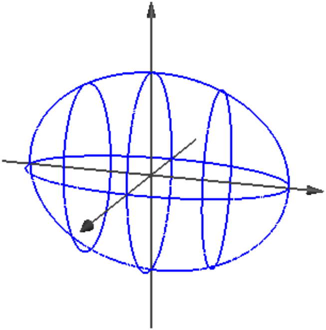

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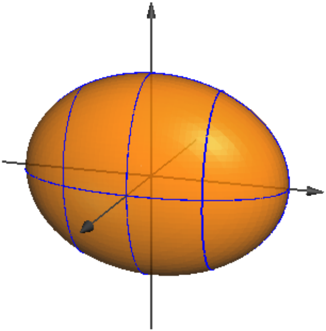
- The six intercepts are $(\pm a, 0, 0)$, $(0, \pm b, 0)$, and $(0, 0, \pm c)$.



Hyperboloids of One Sheet

The quadric surface with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

is called a **hyperboloid of one sheet**. The z-axis is called the **axis** of this hyperboloid. Let's graph $x^2 + y^2 - \frac{z^2}{4} = 1$.

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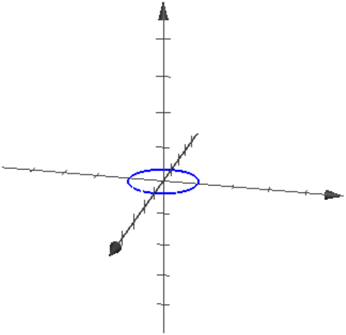

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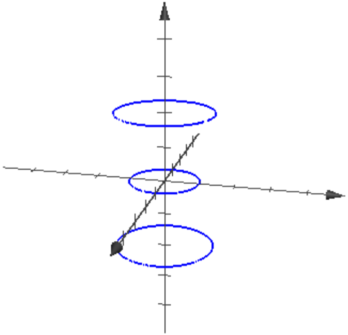

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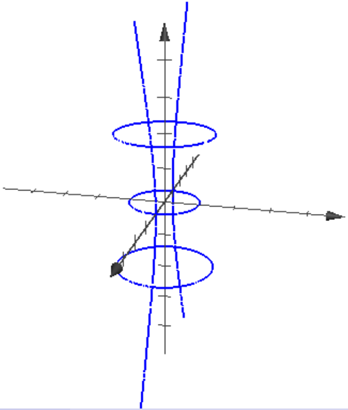

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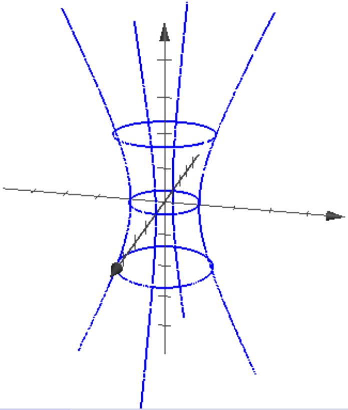

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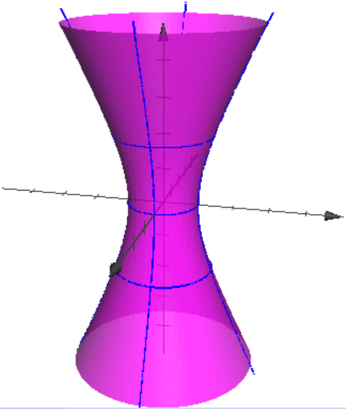
- So we have a decent idea of what a hyperboloid of one sheet looks like.



Hyperboloids of Two Sheets

The quadric surface with equation

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

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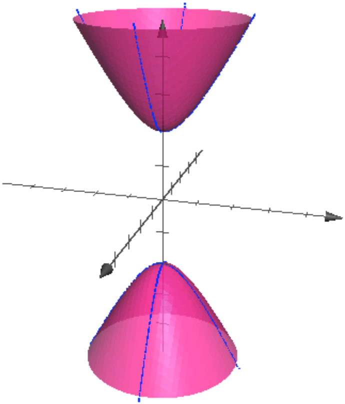
Traces in the xz- and yz-planes are the hyperbolas

$$-x^2 + \frac{z^2}{4} = 1 \qquad \text{and} \qquad -y^2 + \frac{z^2}{4} = 1$$

If $|k| > c = 2$, the horizontal plane $z = k$ intersects the surface in the ellipse

$$x^2 + y^2 = k^2 - 1$$

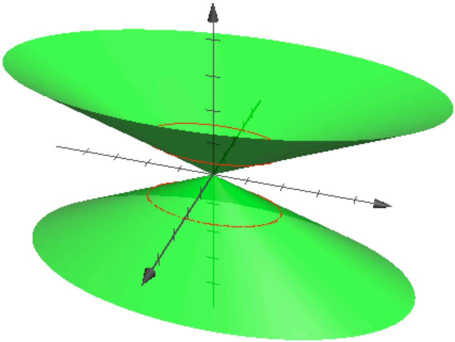


Cones

The quadric surface with equation

$$z^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

is called a **cone**. To graph the cone $z^2 = x^2 + \frac{y^2}{4}$, find the traces in the planes $z = \pm 1$: the ellipses $x^2 + \frac{y^2}{4} = 1$.



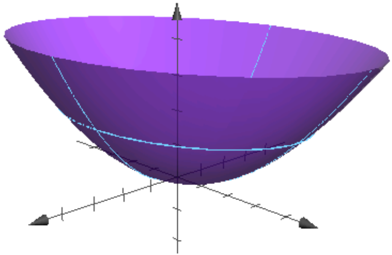
Elliptic Paraboloid

The quadric surface with equation

$$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

is called an **elliptic paraboloid** (with **axis** the z-axis) because its traces in horizontal planes $z = k$ are ellipses, whereas its traces in vertical planes $x = k$ or $y = k$ are parabolas, e.g., the trace in the yz-plane is the parabola $z = \frac{c}{b^2} y^2$.

- The case where $c > 0$ is illustrated (in fact $z = \frac{x^2}{4} + \frac{y^2}{9}$).



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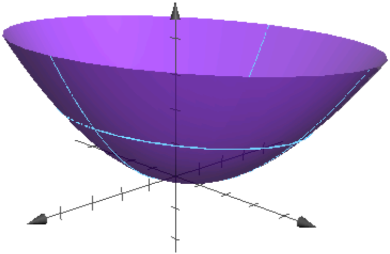

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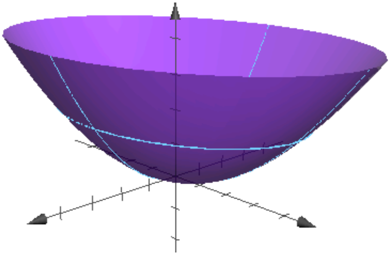
- When $x = 0$, $z = \frac{x^2}{4}$ and when $y = 0$, $z = \frac{y^2}{9}$.



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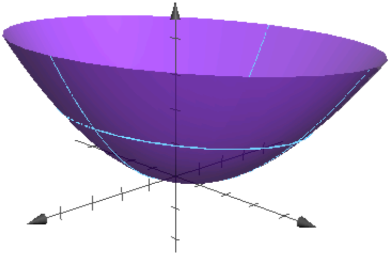
- When $x = 0$, $z = \frac{x^2}{4}$ and when $y = 0$, $z = \frac{y^2}{9}$.
- When $c < 0$, the paraboloid opens downwards.



Hyperbolic Paraboloid

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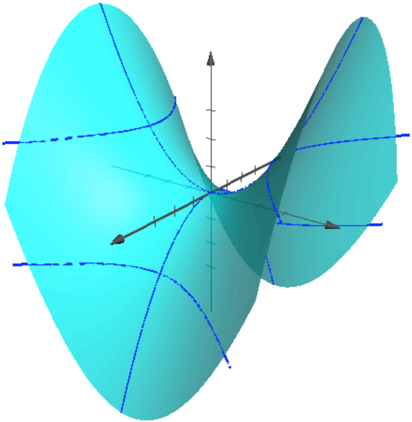
is called a **hyperbolic paraboloid** (with **axis** the z-axis) because its traces in horizontal planes $z = k$ are hyperbolas, whereas its traces in vertical planes $x = k$ or $y = k$ are parabolas (which open in opposite directions).



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Put the equation in standard form:

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This is a hyperboloid of two sheets, but now the axis is the y-axis.

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The traces in the xy- and yz-planes are hyperbolas

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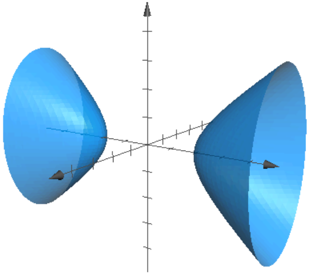

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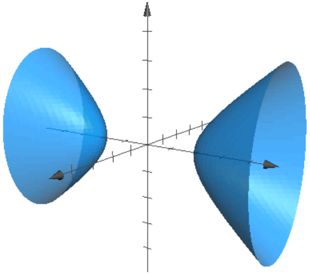

There is no trace in the xz-plane, but traces in the vertical planes $y = k$ for $|k| > 2$ are the ellipses $\quad x^2 + \frac{z^2}{2} = \frac{k^2}{4} - 1, \qquad y = k.$

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