Math 201 Lecture 18: Convolution

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- Many examples here are taken from the textbook. The first number in () refers to the problem number in the UA Custom edition, the second number in () refers to the problem number in the 8th edition.

0. Review

- Laplace transform of functions with jumps:

  1. Represent the function using unit jump functions:

$$g(t) = g_1(t) + [g_2(t) - g_1(t)]\, u(t - t_1) + \cdots + [g_k(t) - g_{k-1}(t)]\, u(t - t_{k-1}). \tag{1}$$

  2. Transforming each $g(t)\, u(t-a)$ as follows:

     a. Obtain $g(t+a)$.

     b. Obtain $\mathcal{L}\{g(t+a)\}$.

     c. $\mathcal{L}\{g(t)\, u(t-a)\} = e^{-as}\, \mathcal{L}\{g(t+a)\}(s)$.

- Inverse Laplace transform of functions involving e^{-as}:

  1. Write the function to be inverse transformed as $e^{-as}\, F(s)$.
  2. Obtain $f(t) = \mathcal{L}^{-1}\{F(s)\}$.
  3. Obtain $f(t-a)$.
  4. The inverse transform is $f(t-a)\, u(t-a)$.

- Laplace transform of periodic functions: If f has period T and is piecewise continuous on $[0, T]$, then

$$\mathcal{L}\{f\}(s) = \frac{\int_0^T e^{-st} f(t)\, \mathrm{d}t}{1 - e^{-sT}}. \tag{2}$$

- Quiz:

$$\mathcal{L}\{f(t)\},\, f(t) = \begin{cases} 1 & 0 < t < 2 \\ t & t > 2 \end{cases}; \qquad \mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s^3}\right\}. \tag{3}$$

1. Basic Information

- Equation with general right hand sides and general initial conditions.

  In practice, we often face the following situation: We need to compute the output of a system for many different inputs, sometimes we also need to find out a general relation between the inputs and the outputs.

  After mathematical modeling, this situation is translated to the following. Given a differential operator (corresponding to the system), we need to compute the solutions for many different right hand sides/initial values. For example, we may need to consider problems like

$$y'' + y = g(t); \qquad y(0) = a, \quad y'(0) = b \tag{4}$$

  and need to solve it for many different g's.

  We follow our three steps.

  1. Transform the equation. Using the initial values and denoting $Y(s) = \mathcal{L}\{y\},\ G(s) = \mathcal{L}\{g\}$, we have

$$(s^2 + 1)\, Y = G + s\, a + b. \tag{5}$$

2. Solve the transformed equation. We have

$$Y(s) = \frac{1}{s^2+1}\left[G(s) + s\,a + b\right]. \tag{6}$$

3. Transform back. We need to compute

$$y = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{H(s)\,\tilde{G}(s)\right\} \tag{7}$$

where $H(s) := \frac{1}{s^2+1}$, and $\tilde{G}(s) := G(s) + s\,a + b$.

- From the above we can make the following observations.

  A) The Laplace transform of the solution is the product of two functions. One of them, $H(s)$, is determined by the differential operator (the system); The other, $\tilde{G}(s)$, is only dependent on the right hand side and the initial values (the inputs).

     Thus it is possible to simplify the process of solving equations with many different inputs, as $H(s)$ remains the same.

  B) To be able to take advantage of the above observation, we need to be able to compute the inverse Laplace transforms of product of two functions. Or more specifically, if we know that $\mathcal{L}^{-1}\{H\} = h$, $\mathcal{L}^{-1}\left\{\tilde{G}\right\} = \tilde{g}$, can we obtain $\mathcal{L}^{-1}\left\{H\,\tilde{G}\right\}$ using h and $\tilde{g}$?

     The answer is yes:

$$\mathcal{L}^{-1}\left\{H\,\tilde{G}\right\} = h * \tilde{g}, \tag{8}$$

  the **convolution** of h and $\tilde{g}$.

- Transfer function.

Definition 1. (Transfer function/Impulse response function) *Consider the linear system*

$$a\,y'' + b\,y' + c\,y = g, \qquad y(0) = y'(0) = 0, \tag{9}$$

Let $Y = \mathcal{L}\{y\}$, $G = \mathcal{L}\{g\}$. The function $H(s) = Y/G$ is called the ***transfer function*** *of the linear system. Its inverse Laplace transform, $h(t) = \mathcal{L}^{-1}\{H(s)\}(t)$, is called the* ***impulse response function*** *for the system.*

Remark 2. We will see later that $h(t)$ solve the equation

$$a\,y'' + b\,y' + c\,y = \delta(t) \tag{10}$$

where $\delta(t)$ is a special function such that

$$\mathcal{L}\{\delta\} = 1. \tag{11}$$

It turns out that this $\delta(t)$ cannot be a usual function. It belongs to the so-called "generalized functions".

Example 3. Find the transfer function and the impulse response function for

$$y'' + 9\,y = g(t). \tag{12}$$

Solution. Taking Laplace transform we have (recall that when talking about transfer function, we always assume $y(0) = y'(0) = 0$.)

$$s^2\,Y + 9\,Y = G \Longrightarrow H = \frac{Y}{G} = \frac{1}{s^2+9}. \tag{13}$$

So the transfer function is

$$H(s) = \frac{1}{s^2+9}. \tag{14}$$

The impulse response function is then given by

$$h(t) = \mathcal{L}^{-1}\{H\} = \frac{1}{3}\sin 3\,t. \tag{15}$$

- Convolution.

Definition 4. (Convolution) *Let $f(t)$ and $g(t)$ be piecewise continuous on $[0,\infty)$. The* ***convolution*** *of $f(t)$ and $g(t)$, denoted $f*g$, is defined by*

$$(f*g)(t) := \int_0^t f(t-v)\, g(v)\, \mathrm{d}v. \tag{16}$$

Example 5. For example, the convolution of $f = t$ and $g = t^2$ is

$$t*t^2 = \int_0^t (t-v)\, v^2\, \mathrm{d}v = \int_0^t t\, v^2 - v^3\, \mathrm{d}v = \frac{1}{3}\, t^4 - \frac{1}{4}\, t^4 = \frac{t^4}{12}. \tag{17}$$

The key property of convolution is the following

Theorem 6. (Convolution Theorem) *Let $f(t)$ and $g(t)$ be piecewise continuous on $[0,\infty)$ and of exponential order α and set $F(s) = \mathcal{L}\{f\}(s)$ and $G(s) = \mathcal{L}\{g\}(s)$. Then*

$$\mathcal{L}\{f*g\}(s) = F(s)\, G(s), \tag{18}$$

or equivalently

$$\mathcal{L}^{-1}\{F(s)\, G(s)\} = (f*g)(t). \tag{19}$$

Example 7. (7.7 13; 7.7 13) Find the Laplace transform of

$$f(t) = \int_0^t (t-v)\, e^{3v}\, \mathrm{d}v. \tag{20}$$

Solution. We recognize that in fact

$$f(t) = t*e^{3t}. \tag{21}$$

Thus

$$\mathcal{L}\{f\} = \mathcal{L}\{t\}\, \mathcal{L}\{e^{3t}\} = \frac{1}{s^2}\, \frac{1}{s-3} = \frac{1}{s^2\,(s-3)}.$$

- What convolution can do:
  1. An alternative method of computing inverse Laplace transforms;
  2. Enable us to solve special integral-differential equations;
  3. Obtain formula for solution when the right hand side or initial values are not given.

2. Things to be Careful/Tricky Issues

See "Common Mistakes" for examples.

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3. Examples

- Convolution as alternative method fo inverse Laplace transforms:

Example 8. (7.7 5; 7.7 5) Compute

$$\mathcal{L}^{-1}\left\{\frac{1}{s\,(s^2+1)}\right\} \tag{22}$$

Solution. Using Convolution Theorem, we have

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s\,(s^2+1)}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}*\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}\\ &= 1*\sin t\\ &= \int_0^t \sin v\,\mathrm{d}v\\ &= 1-\cos t. \end{aligned}\tag{23}$$

Example 9. (7.7 12; 7.7 12) Compute

$$\mathcal{L}^{-1}\left\{\frac{s+1}{(s^2+1)^2}\right\}.\tag{24}$$

Solution. We have

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s+1}{(s^2+1)^2}\right\} &= \mathcal{L}^{-1}\left\{\frac{s+1}{s^2+1}\cdot\frac{1}{s^2+1}\right\}\\ &= \mathcal{L}^{-1}\left\{\frac{s+1}{s^2+1}\right\}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}\\ &= [\cos t+\sin t]*\sin t\\ &= \int_0^t \cos(t-v)\sin v\,\mathrm{d}v+\int_0^t \sin(t-v)\sin v\,\mathrm{d}v\\ &= \int_0^t \frac{1}{2}[\sin t-\sin(t-2\,v)]\,\mathrm{d}v+\int_0^t \frac{1}{2}[\cos(t-2\,v)-\cos t]\,\mathrm{d}v\\ &= \frac{1}{2}t\,(\sin t-\cos t)-\frac{1}{2}\left[\int_0^t \sin(t-2\,v)\,\mathrm{d}v+\int_0^t \cos(t-2\,v)\,\mathrm{d}v\right]\\ &= \frac{1}{2}t\,(\sin t-\cos t)-\frac{1}{4}[\cos(t-2\,v)|_{v=0}^{v=t}-\sin(t-2\,v)|_0^t]\\ &= \frac{1}{2}t\,(\sin t-\cos t)-\frac{1}{2}\sin t. \end{aligned}\tag{25}$$

- Solving integral-differential equations:

Example 10. (7.7 15; 7.7 15) Solve

$$y(t)+3\int_0^t y(v)\sin(t-v)\,\mathrm{d}v=t.\tag{26}$$

Solution. We try following the same three steps.

1. Transform the equation. The key is to notice that the integral term is in fact a convolution:

$$\int_0^t y(v)\sin(t-v)\,\mathrm{d}v=y(t)*\sin t.\tag{27}$$

   Thus taking Laplace transform of both sides, we have

$$Y(s)+3\,Y(s)\,\frac{1}{1+s^2}=\frac{1}{s^2}.\tag{28}$$

2. Solve the transformed equation. We have

$$Y(s)=\frac{1+s^2}{(4+s^2)\,s^2}=\frac{1}{s^2}-3\,\frac{1}{(4+s^2)\,s^2}=\frac{1}{4}\frac{1}{s^2}-\frac{3}{4}\frac{1}{s^2+4}.\tag{29}$$

3. Transform back. Compute

$$y(t)=\mathcal{L}^{-1}\left\{\frac{1}{4}\frac{1}{s^2}-\frac{3}{4}\frac{1}{s^2+4}\right\}=\frac{t}{4}-\frac{3}{8}\sin 2\,t.\tag{30}$$

Example 11. (7.7 21; 7.7 21) Solve

$$y'(t) + y(t) - \int_0^t y(v)\sin(t-v)\,\mathrm{d}v = -\sin t, \qquad y(0) = 1. \tag{31}$$

Solution. Again,

1. Transform the equation. We have

$$\mathcal{L}\{y'\} = s\,Y - y(0) = s\,Y - 1. \tag{32}$$

and

$$\mathcal{L}\left\{\int_0^t y(v)\sin(t-v)\,\mathrm{d}v\right\} = Y(s)\,\mathcal{L}\{\sin t\} = \frac{Y(s)}{1+s^2}. \tag{33}$$

So the equation is transformed into

$$s\,Y + Y - 1 - \frac{Y(s)}{1+s^2} = -\frac{1}{1+s^2}. \tag{34}$$

2. Solve the transformed equation. We have

$$Y = \frac{s}{s^2+s+1}. \tag{35}$$

3. Transform back. Compute

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{Y\} \\ &= \mathcal{L}^{-1}\left\{\frac{s+1/2}{(s+1/2)^2+\left(\sqrt{3}/2\right)^2} - \frac{1/2}{(s+1/2)^2+\left(\sqrt{3}/2\right)^2}\right\} \\ &= e^{-\frac{1}{2}t}\left[\cos\frac{\sqrt{3}\,t}{2} - \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}\,t}{2}\right]. \end{aligned} \tag{36}$$

- Obtain formula of solution for general right hand side:

Example 12. (7.7 1; 7.7 1) Use the convolution theorem to obtain a formula for the solution to the given initial value problem.

$$y'' - 2\,y' + y = g(t); \qquad y(0) = -1, \quad y'(0) = 1. \tag{37}$$

Solution. Using

$$\mathcal{L}\{y''\} = s^2\,Y - s\,y(0) - y'(0) = s^2\,Y + s - 1, \tag{38}$$

$$\mathcal{L}\{y'\} = s\,Y - y(0) = s\,Y + 1, \tag{39}$$

we have the transformed equation

$$(s^2 - 2\,s + 1)\,Y = G(s) - s + 3 \Longrightarrow Y(s) = \frac{1}{(s-1)^2}\,G(s) - \frac{1}{(s-1)} + \frac{2}{(s-1)^2}. \tag{40}$$

Transforming back, we have

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\,G(s)\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} * g(t) = (e^t\,t) * g(t); \tag{41}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} = e^t; \tag{42}$$

$$\mathcal{L}^{-1}\left\{\frac{2}{(s-1)^2}\right\} = 2\,e^t\,t. \tag{43}$$

So the formula for the solution reads

$$y(t) = 2\,t\,e^t - e^t + (e^t\,t) * g(t) = 2\,t\,e^t - e^t + \int_0^t e^{t-v}\,(t-v)\,g(v)\,\mathrm{d}v. \tag{44}$$

Finally let's do an example that combined both 7.6 and 7.7.

Example 13. (7.6 39; 7.6 39) Solve

$$y'' + 5\,y' + 6\,y = g(t), \qquad y(0) = 0, \quad y'(0) = 2, \tag{45}$$

where

$$g(t) = \begin{cases} 0 & 0 \leqslant t < 1 \\ t & 1 < t < 5 \\ 1 & 5 < t \end{cases}. \tag{46}$$

Solution.

1. Transform the equation.

   a. Transform the LHS. Compute

   $$\mathcal{L}\{y''\} = s^2\,Y - s\,y(0) - y'(0) = s^2\,Y - 2; \tag{47}$$

   $$\mathcal{L}\{y'\} = s\,Y - y(0) = s\,Y. \tag{48}$$

   So

   $$\mathcal{L}\{y'' + 5\,y' + 6\,y\} = (s^2 + 5\,s + 6)\,Y - 2. \tag{49}$$

   b. Transform the RHS. We first need to represent g using unit step functions. There are two discontinuities at $t = 1, 5$. We write

   $$g(t) = 0 + A(t)\,u(t-1) + B(t)\,u(t-5) \tag{50}$$

   and determine

   $$A(t) = t, \qquad B(t) = 1 - t. \tag{51}$$

   So

   $$g(t) = t\,u(t-1) + (1-t)\,u(t-5). \tag{52}$$

   Recalling

   $$\mathcal{L}\{g(t)\,u(t-a)\} = e^{-a\,s}\,\mathcal{L}\{g(t+a)\}(s) \tag{53}$$

   We compute

   $$\begin{aligned} \mathcal{L}\{t\,u(t-1) + (1-t)\,u(t-5)\} &= e^{-s}\,\mathcal{L}\{t+1\} + e^{-5s}\,\mathcal{L}\{-t-4\} \\ &= e^{-s}\left(\frac{1}{s} + \frac{1}{s^2}\right) - e^{-5s}\left(\frac{1}{s^2} + \frac{4}{s}\right). \end{aligned} \tag{54}$$

   Thus the transformed equation reads

   $$(s^2 + 5\,s + 6)\,Y = e^{-s}\left(\frac{1}{s} + \frac{1}{s^2}\right) - e^{-5s}\left(\frac{1}{s^2} + \frac{4}{s}\right) + 2. \tag{55}$$

2. Solve the transformed equation. We have

   $$Y(s) = e^{-s}\,\frac{s+1}{s^2\,(s^2 + 5\,s + 6)} - e^{-5s}\,\frac{4\,s+1}{s^2\,(s^2 + 5\,s + 6)} + \frac{2}{s^2 + 5\,s + 6}. \tag{56}$$

3. Transform back. We need to use the formula

   $$\mathcal{L}^{-1}\{e^{-a\,s}\,F(s)\} = f(t-a)\,u(t-a). \tag{57}$$

   Thus we have to compute the inverse transforms of $\frac{s+1}{s^2\,(s^2+5\,s+6)}$ and $\frac{4\,s+1}{s^2\,(s^2+5\,s+6)}$. Due to linearity of the inverse transform, we first compute

   $$\mathcal{L}^{-1}\left\{\frac{s}{s^2\,(s^2 + 5\,s + 6)}\right\} \text{ and } \mathcal{L}^{-1}\left\{\frac{1}{s^2\,(s^2 + 5\,s + 6)}\right\}. \tag{58}$$

We have

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s}{s^2\,(s^2+5\,s+6)}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s\,(s^2+5\,s+6)}\right\}\\ &= \mathcal{L}^{-1}\left\{\frac{1}{s}\,\frac{1}{(s+2)\,(s+3)}\right\}\\ &= \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}*\mathcal{L}^{-1}\left\{\frac{1}{s+2}-\frac{1}{s+3}\right\}\\ &= 1*(e^{-2t}-e^{-3t}\}\\ &= \int_0^t (e^{-2v}-e^{-3v})\,\mathrm{d}v\\ &= \frac{1}{2}-\frac{1}{2}\,e^{-2t}-\frac{1}{3}+\frac{1}{3}\,e^{-3t}\\ &= \frac{1}{6}-\frac{1}{2}\,e^{-2t}+\frac{1}{3}\,e^{-3t}. \end{aligned}\tag{59}$$

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s^2\,(s^2+5\,s+6)}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}*\mathcal{L}^{-1}\left\{\frac{1}{s+2}-\frac{1}{s+3}\right\}\\ &= t*(e^{-2t}-e^{-3t})\\ &= \int_0^t (t-v)\,(e^{-2v}-e^{-3v})\,\mathrm{d}v\\ &= \int_0^t (t-v)\,\mathrm{d}\left(-\frac{1}{2}\,e^{-2v}+\frac{1}{3}\,e^{-3v}\right)\\ &= (t-v)\left(-\frac{1}{2}\,e^{-2v}+\frac{1}{3}\,e^{-3v}\right)\Big|_{v=0}^{v=t}-\int_0^t -\frac{1}{2}\,e^{-2v}+\frac{1}{3}\,e^{-3v}\,\mathrm{d}(t-v)\\ &= \frac{1}{2}\,t-\frac{1}{3}\,t+\int_0^t \left(-\frac{1}{2}\,e^{-2v}+\frac{1}{3}\,e^{-3v}\right)\mathrm{d}v\\ &= \frac{1}{6}\,t+\frac{1}{4}\,e^{-2t}-\frac{1}{4}-\frac{1}{9}\,e^{-3t}+\frac{1}{9}\\ &= \frac{1}{6}\,t+\frac{1}{4}\,e^{-2t}-\frac{1}{9}\,e^{-3t}-\frac{5}{36}. \end{aligned}\tag{60}$$

Thus we have

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2\,(s^2+5\,s+6)}\right\}=\frac{1}{36}+\frac{t}{6}-\frac{1}{4}\,e^{-2t}+\frac{2}{9}\,e^{-3t}\tag{61}$$

which leads to

$$\mathcal{L}^{-1}\left\{e^{-s}\,\frac{s+1}{s^2\,(s^2+5\,s+6)}\right\}=\left(\frac{1}{36}+\frac{t-1}{6}-\frac{1}{4}\,e^{-2(t-1)}+\frac{2}{9}\,e^{-3(t-1)}\right)u(t-1);\tag{62}$$

For the other term we have

$$\mathcal{L}^{-1}\left\{\frac{4\,s+1}{s^2\,(s^2+5\,s+6)}\right\}=\frac{19}{36}+\frac{t}{6}-\frac{7}{4}\,e^{-2t}+\frac{11}{9}\,e^{-3t}\tag{63}$$

which leads to

$$\mathcal{L}^{-1}\left\{e^{-5s}\,\frac{4\,s+1}{s^2\,(s^2+5\,s+6)}\right\}=\left(\frac{19}{36}+\frac{t-5}{6}-\frac{7}{4}\,e^{-2(t-5)}+\frac{11}{9}\,e^{-3(t-5)}\right)u(t-5).\tag{64}$$

Finally we quickly compute

$$\mathcal{L}^{-1}\left\{\frac{2}{s^2+5\,s+6}\right\}=2\,\mathcal{L}^{-1}\left\{\frac{1}{s+2}-\frac{1}{s+3}\right\}=2\,e^{-2t}-2\,e^{-3t}.\tag{65}$$

Summarizing, we reach

$$\begin{aligned}y(t) &= \left(\frac{1}{36}+\frac{t-1}{6}-\frac{1}{4}\,e^{-2(t-1)}+\frac{2}{9}\,e^{-3(t-1)}\right)u(t-1)\\ &\quad -\left(\frac{19}{36}+\frac{t-5}{6}-\frac{7}{4}\,e^{-2(t-5)}+\frac{11}{9}\,e^{-3(t-5)}\right)u(t-5)+2\,e^{-2t}-2\,e^{-3t}. \end{aligned}\tag{66}$$

Remark 14. Note that for the above problem, it is actually more efficient to compute the inverse transforms using partial fractions. For example, to compute

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2\,(s^2+5\,s+6)}\right\} \tag{67}$$

we write

$$\frac{s+1}{s^2\,(s^2+5\,s+6)} = \frac{A\,s+B}{s^2} + \frac{C}{s+2} + \frac{D}{s+3} \tag{68}$$

which leads to

$$s+1 = (A\,s+B)\,(s+2)\,(s+3) + C\,s^2\,(s+3) + D\,s^2\,(s+2). \tag{69}$$

Setting s to be special values, we get

$$s=0 \implies 1=6\,B \implies B=\frac{1}{6}; \tag{70}$$

$$s=-2 \implies -1=4\,C \implies C=-\frac{1}{4}; \tag{71}$$

$$s=-3 \implies -2=-9\,D \implies D=\frac{2}{9}; \tag{72}$$

Finally comparing coefficients for the highest order term s^3 we get

$$0=A+C+D \implies A=\frac{1}{36}. \tag{73}$$

4. Notes and Comments

- Other properties of convolution.

Theorem 15. (Other properties of convolution) *Let $f(t), g(t)$, and $h(t)$ be piecewise continuous on $[0,\infty)$. Then*

a) $f*g = g*f$;

b) $f*(g+h) = (f*g)+(f*h)$;

c) $(f*g)*h = f*(g*h)$;

d) $f*0 = 0$.

Proof. One way to prove these properties is to use definition (p.425 in the textbook). Here we use the Convolution Theorem. We use F, G, H to denote the Laplace transforms of f, g, h.

a) We have

$$f*g = \mathcal{L}^{-1}\{\mathcal{L}\{f*g\}\} = \mathcal{L}^{-1}\{FG\} = \mathcal{L}^{-1}\{G\,F\} = g*f. \tag{74}$$

b) We have

$$\begin{aligned} f*(g+h) &= \mathcal{L}^{-1}\{\mathcal{L}\{f*(g+h)\}\} \\ &= \mathcal{L}^{-1}\{F\,(G+H) \\ &= \mathcal{L}^{-1}\{FG+FH\} \\ &= \mathcal{L}^{-1}\{FG\}+\mathcal{L}^{-1}\{FH\} \\ &= f*g+f*h. \end{aligned} \tag{75}$$

c) We have

$$\begin{aligned} (f*g)*h &= \mathcal{L}^{-1}\{\mathcal{L}\{(f*g)*h\}\} \\ &= \mathcal{L}^{-1}\{\mathcal{L}\{f*g\}\,H\} \\ &= \mathcal{L}^{-1}\{(FG)\,H\} \\ &= \mathcal{L}^{-1}\{F\,(G\,H)\} \\ &= f*\mathcal{L}^{-1}\{G\,H\} \\ &= f*(g*h). \end{aligned} \tag{76}$$

d) is obvious.

□

- Proof of convolution theorem.

Proof. We use definition.

$$\begin{aligned}\mathcal{L}\{f*g\}(s) &= \int_0^\infty e^{-st}\int_0^t f(t-v)\, g(v)\, \mathrm{d}v\, \mathrm{d}t \\ &= \iint_{0<v<t} e^{-st} f(t-v)\, g(v)\, \mathrm{d}v\, \mathrm{d}t \\ &= \int_0^\infty \left[\int_v^\infty e^{-st} f(t-v)\, g(v)\, \mathrm{d}t\right] \mathrm{d}v \\ &= \int_0^\infty g(v) \left[\int_v^\infty e^{-st} f(t-v)\, \mathrm{d}t\right] \mathrm{d}v. \end{aligned} \tag{77}$$

To evaluate the inner integral, we set $x = t - v$. Then $\mathrm{d}t = \mathrm{d}x$, $e^{-st} = e^{-sx} e^{-sv}$. Thus

$$\int_v^\infty e^{-st} f(t-v)\, \mathrm{d}t = e^{-sv}\int_0^\infty e^{-sx} f(x)\, \mathrm{d}x = e^{-sv} F(s). \tag{78}$$

Substituting into the original integral, we have

$$\mathcal{L}\{f*g\}(s) = \int_0^\infty g(v)\left[\int_v^\infty e^{-st} f(t-v)\, \mathrm{d}t\right] \mathrm{d}v = F(s)\int_0^\infty g(v)\, e^{-sv}\, \mathrm{d}v = F(s)\, G(s). \tag{79}$$

Thus ends the proof. □