Unit Step Function

The **unit step function** is just a piecewise function with a **jump discontinuity** at t = a. Recall from calculus that a jump discontinuity is an x-value for which the limit doesn't exist. Then general form of the unit step function is given below.

$$Mu(t-a)=\begin{cases} 0, & t<a \\ M, & t>a \end{cases}$$

The M in the function represents the height of the jump and a is the number of units shifted to the right.

First we learn how to turn a piecewise function that is continuous into a unit step function.

Examples:

$$g(t)=\begin{cases} 0, & 0<t<1 \\ 2, & 1<t<2 \\ 1, & 2<t<3 \\ 3, & 3<t \end{cases}$$

To write the unit step function, we analyze the behavior of the graph in terms of the jumps. The trick to writing it is to compute the height of the jump M by taking the new value and subtracting the old value at the new t-value.

→ First, g(t) = 3 until it reaches t = 1 and then it jumps up by 2 units.

$$g(t)=0+(2-0)u(t-1)=2u(t-1)$$

→ Next, g(t) = 2 until it reaches t = 2 and then it jumps down by 1 unit.

$$g(t)=2u(t-1)+(1-2)u(t-2)=2u(t-1)-u(t-2)$$

→ Third, g(t) = 1 until it reaches t = 3 and then it jumps up by 2 units.

$$g(t)=2u(t-1)-u(t-2)+(3-2)u(t-3)=2u(t-1)-u(t-2)+u(t-3)$$

So g(t) in terms of unit step functions is given by
$g(t)=2u(t-1)-u(t-2)+2u(t-3)$.

$$g(t)=\begin{cases} 0, & 0<t<2 \\ t+1, & t>2 \end{cases}$$

Again to write the unit step function, we analyze the behavior of the graph in terms of the jumps.

→ g(t) = 0 until it reaches t = 2 and then it jumps up by t + 1 units.

$$g(t)=0+(t-1-0)u(t-2)=(t-1)u(t-1)$$

So g(t) in terms of unit step functions is given by $g(t)=(t-1)u(t-1)$.

Now we know the unit step function, we can find its Laplace transform. It's given as

$$\mathcal{L}\{g(t)u(t-a)\} = e^{-as}\mathcal{L}\{g(t+a)\}$$

In simplest terms, the Laplace transform of any function times the unit step function shifted over to the right a units is equal to the exponential times the Laplace transform of the function g shift over to the left.

Example:

→ $f(t) = t^2 u(t-2)$

In this example, $g(t) = t^2$ and $a = 2$. So applying the Laplace transform gives us

$$\begin{aligned} L\{t^2 u(t-2)\} &= e^{-2s} L\{(t+2)^2\} \\ &= e^{-2s} L\{t^2 + 2t + 1\} \\ &= e^{-2s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right) \\ &= \frac{2e^{-2s}}{s^3} + \frac{2e^{-2s}}{s^2} + \frac{e^{-2s}}{s} \end{aligned}$$

Now we discuss the inverse Laplace transform. It's given as

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$$

So if you see an exponential term in the problem, then you know that the unit step function is involved. Don't forget the techniques discussed in finding the inverse Laplace transform. Recognition comes with practice.

Examples:

→ $\dfrac{e^{-3s}}{s^2}$

According to the the table of Laplace transforms, this takes the form of $\dfrac{n!}{s^{n+1}}$ where n = 1. But because of the exponential term, we know that the unit step function is involved where a = 3. So the inverse Laplace transform is given by

$$L^{-1}\left\{\frac{e^{-3s}}{s^2}\right\} = L^{-1}\left\{e^{-3s} \cdot \frac{1}{s^2}\right\} = (t-3)u(t-3)$$

→ $\dfrac{se^{-s}}{s^2+25}$

According to the the table of Laplace transforms, this takes the form of $\dfrac{s}{s^2+b^2}$ where b = 5. But because of the exponential term, we know that the unit step function is involved where a = 1. So the inverse Laplace transform is given by

$$L^{-1}\left\{\frac{se^{-s}}{s^2+25}\right\} = L^{-1}\left\{e^{-3s}\cdot\frac{s}{s^2+5^2}\right\} = \cos[5(t-1)]u(t-5) = \cos(5t-5)u(t-5)$$

→ $\dfrac{se^{-8s}}{s^2-3s+4}$

This doesn't seem to fit any of the forms we know. But because we can factor the denominator, we can use the method of partial fractions. Using partial fractions on the expression $\dfrac{s}{s^2-3s+4}$ gives the following:

→ Factor the denominator.

$$\frac{s}{s^2-3s+4} = \frac{s}{(s-1)(s-3)}$$

→ Find the decomposition.

$$\frac{s}{(s-1)(s-3)} = \frac{A}{s-1} + \frac{B}{s-3}$$

→ Multiply both sides by the common denominator and simplify the RHS.

$$s = A(s-3) + B(s-1)$$

$$s = As - 3A + Bs - B$$

→Equate coefficients.

$$1 = A + B$$

$$0 = -3A - B$$

→ Solve for coefficients.

A = -1/2 B = -3/2

→ Plug back into partial fraction decomposition.

$$\frac{s}{(s-1)(s-3)} = \frac{-1/2}{s-1} + \frac{-3/2}{s-3}$$

Now we have the composition and we compute the inverse Laplace transform where from the exponential term, a = 8. It's given as

$$\begin{aligned} L^{-1}\left\{\frac{se^{-8s}}{s^2-3s+4}\right\} &= L^{-1}\left\{e^{-8s}\left(\frac{-1/2}{s-1}+\frac{-3/2}{s-3}\right)\right\} \\ &= L^{-1}\left\{e^{-8s}\left(\frac{-1/2}{s-1}\right)\right\}+L^{-1}\left\{e^{-8s}\left(\frac{-3/2}{s-3}\right)\right\} \\ &= -\frac{1}{2}e^{(t-8)}u(t-8)-\frac{3}{2}e^{3(t-8)}u(t-8) \end{aligned}$$

So the inverse Laplace transform is given by

$$\underline{-\frac{1}{2}e^{(t-8)}u(t-8)-\frac{3}{2}e^{3(t-8)}u(t-8)}.$$