12

Fourier Series

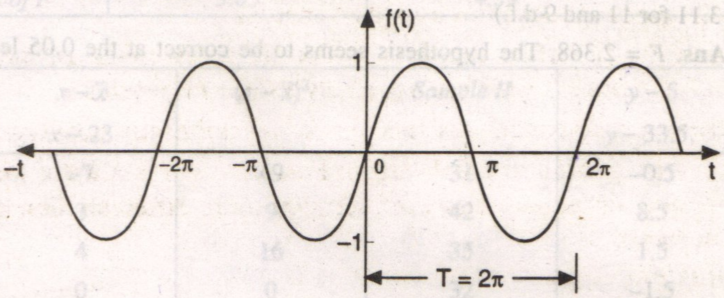
12.1 PERIODIC FUNCTIONS

If the value of each ordinate $f(t)$ repeats itself at equal intervals in the abscissa, then $f(t)$ is said to be a periodic function.

If $f(t) = f(t+T) = f(t+2T) = \ldots$ then T is called the period of the function $f(t)$.

For example :

$\sin x = \sin(x + 2\pi) = \sin(x + 4\pi) = \ldots$ so $\sin x$ is a periodic function with the period 2π. This is also called sinusoidal periodic function.



12.2 FOURIER SERIES

Here we will express a non-sinusoidal periodic function into a fundamental and its harmonics. A series of sines and cosines of an angle and its multiples of the form.

$$\frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \ldots + a_n \cos nx + \ldots$$
$$+ b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \ldots + b_n \sin nx + \ldots$$
$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

is called the *Fourier series, where* $a_0, a_1, a_2, \ldots a_n, \ldots b_1, b_2, b_3 \ldots b_n \ldots$ are constants.

A periodic function $f(x)$ can be expanded in a Fourier Series. The series consists of the following:

(*i*) A constant term a_0 (called d.c. component in electrical work).

(*ii*) A component at the fundamental frequency determined by the values of a_1, b_1.

(*iii*) Components of the harmonics (multiples of the fundamental frequency) determined by $a_2, a_3 \ldots b_2, b_3 \ldots$. And $a_0, a_1, a_2 \ldots, b_1, b_2 \ldots$ are known as *Fourier coefficients* or Fourier constants.

12.3. DIRICHLET'S CONDITIONS FOR A FOURIER SERIES

If the function $f(x)$ for the interval $(-\pi, \pi)$

(1) is single-valued (2) is bounded

(3) has at most a finite number of maxima and minima.

(4) has only a finite number of discontinuous

(5) is $f(x+2\pi) = f(x)$ for values of x outside $[-\pi, \pi]$, then

$$S_p(x) = \frac{a_0}{2} + \sum_{n=1}^{P} a_n \cos nx + \sum_{n=1}^{P} b_n \sin nx$$

converges to $f(x)$ as $P \to \infty$ at values of x for which $f(x)$ is continuous and to $\frac{1}{2}[f(x+0)+f(x-0)]$ at points of discontinuity.

12.4. ADVANTAGES OF FOURIER SERIES

1. Discontinuous function can be represented by Fourier series. Although derivatives of the discontinuous functions do not exist. (This is not true for Taylor's series).

2. The Fourier series is useful in expanding the periodic functions since outside the closed interval, there exists a periodic extension of the function.

3. Expansion of an oscillating function by Fourier series gives all modes of oscillation (fundamental and all overtones) which is extremely useful in physics.

4. Fourier series of a discontinuous function is not uniformly convergent at all points.

5. Term by term integration of a convergent Fourier series is always valid, and it may be valid if the series is not convergent. However, term by term, differentiation of a Fourier series is not valid in most cases.

12.5 USEFUL INTEGRALS

The following integrals are useful in Fourier Series.

(i) $\int_0^{2\pi} \sin nx\, dx = 0$ (ii) $\int_0^{2\pi} \cos nx\, dx = 0$

(iii) $\int_0^{2\pi} \sin^2 nx\, dx = \pi$ (iv) $\int_0^{2\pi} \cos^2 nx\, dx = \pi$

(v) $\int_0^{2\pi} \sin nx \cdot \sin mx\, dx = 0$ (vi) $\int_0^{2\pi} \cos nx \cos mx\, dx = 0$

(vii) $\int_0^{2\pi} \sin nx \cdot \cos mx\, dx = 0$ (viii) $\int_0^{2\pi} \sin nx \cdot \cos nx\, dx = 0$

(ix) $\int uv = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots$

where $u' = \frac{du}{dx}$, $u'' = \frac{d^2u}{dx^2}$ and so on and $v_1 = \int v\, dx$, $v_2 = \int v_1\, dx$ and so on.

(x) $\sin n\pi = 0$, $\cos n\pi = (-1)^n$ where $n \varepsilon I$

12.6 DETERMINATION OF FOURIER COEFFICIENTS (EULER'S FORMULAE)

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \ldots + a_n \cos nx + \ldots$$
$$+ b_1 \sin x + b_2 \sin 2x + \ldots + b_n \sin nx + \ldots \quad \ldots(1)$$

(*i*) **To find a_0:** Integrate both sides of (1) from $x = 0$ to $x = 2\pi$.

$$\int_0^{2\pi} f(x)\,dx = \frac{a_0}{2}\int_0^{2\pi} dx + a_1\int_0^{2\pi}\cos x\,dx + a_2\int_0^{2\pi}\cos 2x\,dx + \ldots + a_n\int_0^{2\pi}\cos nx\,dx + \ldots$$

$$+\, b_1\int_0^{2\pi}\sin x\,dx + b_2\int_0^{2\pi}\sin 2x\,dx + \ldots + b_n\int_0^{2\pi}\sin nx\,dx + \ldots$$

$$= \frac{a_0}{2}\int_0^{2\pi} dx, \text{ other integrals 0 by formulae (i) and (ii)}$$

$$\int_0^{2\pi} f(x)\,dx = \frac{a_0}{2}\,2\pi,\quad a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx \qquad \ldots(2)$$

(*ii*) **To find a_n:** Multiply each side of (1) by $\cos nx$ and integrate from $x = 0$ to $x = 2\pi$.

$$\int_0^{2\pi} f(x)\cos nx\,dx = \frac{a_0}{2}\int_0^{2\pi}\cos nx\,dx + a_1\int_0^{2\pi}\cos x\cos nx\,dx + \ldots + a_n\int_0^{2\pi}\cos^2 nx\,dx \ldots$$

$$+\, b_1\int_0^{2\pi}\sin x\cos nx\,dx + b_2\int_0^{2\pi}\sin 2x\cos nx\,dx + \ldots$$

$$= a_n\int_0^{2\pi}\cos^2 nx\,dx = a_n\pi \text{ (Other integrals = 0, by formulae on Page ii)}$$

$$\therefore \quad a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \qquad \ldots(3)$$

By taking $n = 1, 2 \ldots$ we can find the values of $a_1, a_2 \ldots$.

(*iii*) **To find b_n:** Multiply each side of (1) by $\sin nx$ and integrate from $x = 0$ to $x = 2\pi$.

$$\int_0^{2\pi} f(x)\sin nx\,dx = \frac{a_0}{2}\int_0^{2\pi}\sin nx\,dx + a_1\int_0^{2\pi}\cos x\sin nx\,dx + \ldots + a_n\int_0^{2\pi}\cos nx\sin nx\,dx + \ldots$$

$$+\, b_1\int_0^{2\pi}\sin x\sin nx\,dx + \ldots + b_n\int_0^{2\pi}\sin^2 nx\,dx + \ldots$$

$$= b_n\int_0^{2\pi}\sin^2 nx\,dx$$

(All other integrals = 0, Article No. 12.5)

$$= b_n\pi$$

$$\therefore \quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx \qquad \ldots(4)$$

Note : To get similar formula of a_0, $\frac{1}{2}$ has been written with a_0 in Fourier series.

Example 1. *Find the Fourier series representing*

$$f(x) = x,\quad 0 < x < 2\pi$$

and sketch its graph from $x = -\pi$ to $x = 4\pi$.

Solution. Let $f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + b_1\sin x + b_2\sin 2x + \ldots \qquad \ldots(1)$

Hence $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x\,dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{2\pi} = 2\pi$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\cos nx\,dx$$

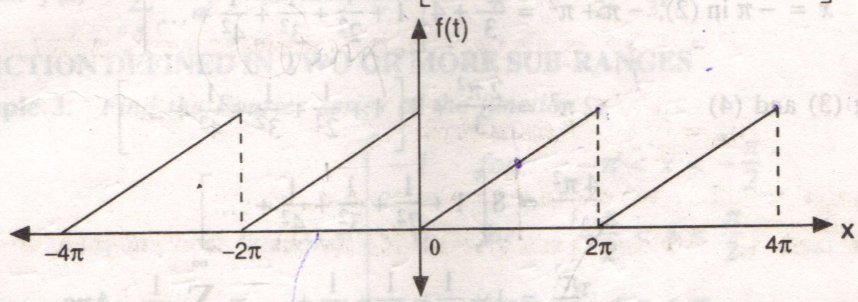
$$= \frac{1}{\pi}\left[x\frac{\sin nx}{n} - 1.\left(-\frac{\cos nx}{n^2}\right)\right]_0^{2\pi} = \frac{1}{\pi}\left[\frac{\cos 2n\pi}{n^2} - \frac{1}{n^2}\right] = \frac{1}{n^2\pi}(1-1) = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin nx\,dx$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - 1.\left(\frac{-\sin nx}{n^2}\right)\right]_0^{2\pi} = \frac{1}{\pi}\left[\frac{-2\pi\cos 2n\pi}{n}\right] = -\frac{2}{n}$$

Substituting the values of $a_0, a_1 a_2..., b_1, b_2...$ in (1)

$$x = \pi - 2\left[\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \ldots\right] \quad \textbf{Ans.}$$



Example 2. *Given that $f(x) = x + x^2$ for $-\pi < x < \pi$, find the Fourier expression of $f(x)$.*

Deduce that $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$ *(Madras 1995 S)*

Solution. Let $x + x^2 = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + \ldots + b_1\sin x + b_2\sin 2x + \ldots$ ...(1)

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2} + \frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left[\frac{\pi^2}{2} + \frac{\pi^3}{3} - \frac{\pi^2}{2} + \frac{\pi^3}{3}\right] = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \int_{-\pi}^{\pi}(x+x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[(x+x^2)\frac{\sin nx}{n} - (2x+1)\frac{(-\cos nx)}{n^2} + (2)\left(-\frac{\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(2\pi+1)\frac{\cos n\pi}{n^2} - (-2\pi+1)\frac{\cos(-n\pi)}{n^2}\right] = \frac{1}{\pi}\left[4\pi\frac{\cos n\pi}{n^2}\right] = \frac{4(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(x+x^2)\left(-\frac{\cos nx}{n}\right)-(2x+1)\left(\frac{-\sin nx}{n^2}\right)+2\,\frac{\cos nx}{n^3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-(\pi+\pi^2)\frac{\cos n\pi}{n}+2\,\frac{\cos n\pi}{n^3}-0+(-\pi+\pi^2)\frac{\cos n\pi}{n}-2\,\frac{\cos n\pi}{n^3}\right]$$

$$= \frac{1}{\pi}\left[-\frac{2\pi}{n}\cos n\pi\right] = -\frac{2}{n}(-1)^n$$

Substituting the values of a_0, a_n, b_n in (1) we get

$$x+x^2 = \frac{\pi^2}{3}+4\left[-\cos x+\frac{1}{2^2}\cos 2x-\frac{1}{3^2}\cos 3x+\ldots\right]$$
$$-2\left[-\sin x+\frac{1}{2}\sin 2x-\frac{1}{3}\sin 3x+\ldots\right] \quad \ldots(2)$$

Put $x = \pi$ in (2), $\pi+\pi^2 = \frac{\pi^2}{3}+4\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\right]$ ...(3)

Put $x = -\pi$ in (2), $-\pi+\pi^2 = \frac{\pi^2}{3}+4\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\right]$ ...(4)

Adding (3) and (4) $\quad 2\pi^2 = \frac{2\pi^2}{3}+8\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\right]$

$$\frac{4\pi^2}{3} = 8\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\right]$$

$$\frac{\pi^2}{6} = 1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots = \sum_{n=1}^{\infty}\frac{1}{n^2} \quad \textbf{Ans.}$$

Exercise 12.1

1. Find a Fourier series to represent, $f(x) = \pi - x$ for $0 < x < 2\pi$.

   Ans. $2\left[\sin x+\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x+\ldots+\frac{1}{n}\sin nx+\ldots\right]$

2. Find a Fourier series to represent $x - x^2$ from $x = -\pi$ to π and show that

   $$\frac{\pi^2}{12} = \frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots$$

   (Mysore 1997, Osmania 1995)

   Ans. $-\frac{\pi^2}{3}+4\left[\frac{\cos x}{1^2}-\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}-\frac{\cos 4x}{4^2}+\ldots\right]+2\left[\frac{\sin x}{1}-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-\frac{\sin 4x}{4}+\ldots\right]$

3. Find a Fourier series to represent: $f(x) = x\sin x$, for $0 < x < 2\pi$.

   (A.M.I.E.T.E., Summer 1997, Madras 1997, Mysore 1995)

   Ans. $-1+\pi\sin x-\frac{1}{2}\cos x+2\left[\frac{\cos 2x}{2^2-1}+\frac{\cos 3x}{3^2-1}+\frac{\cos 4x}{4^2-1}+\ldots\right]$

4. Find a Fourier series to represent the function $f(x) = e^x$, for $-\pi < x < \pi$ and hence derive a series for $\frac{\pi}{\sinh \pi}$.

   Ans. $\frac{2\sinh\pi}{\pi}\left[\left(\frac{1}{2}-\frac{1}{1^2+1}\cos x+\frac{1}{2^2+1}\cos 2x-\frac{1}{3^2+1}\cos 3x+\ldots\right)\right.$
   $\left.+\frac{1}{1^2+1}\sin x-\frac{2}{2^2+1}\sin 2x+\frac{3}{3^2+1}\sin 3x\ldots\right]$ $\quad \frac{\pi}{\sinh\pi} = 1+2\left[-\frac{1}{2}+\frac{1}{5}-\frac{1}{10}+\ldots\right]$

5. Obtain the Fourier series for $f(x) = e^{-x}$ in the interval $0 \le x < 2\pi$. *(Nagpur 1997)*

$$\textbf{Ans. } \frac{1-e^{-2\pi}}{\pi}\left[\frac{1}{2}+\frac{1}{2}\cos x+\frac{1}{5}\cos 2x+\frac{1}{10}\cos 3x+\frac{1}{2}\sin x+\frac{2}{5}\sin 2x+\frac{3}{10}\sin 3x+\ldots\right]$$

6. If $f(x) = \left(\dfrac{\pi - x}{2}\right)^2$, $0 < x < 2\pi$, show that $f(x) = \dfrac{\pi^2}{12} + \sum\limits_{n=1}^{\infty} \dfrac{\cos nx}{n^2}$ *(Madras 1998)*

7. Prove that $x^2 = \dfrac{\pi^2}{3} + 4\sum\limits_{n=1}^{\infty} (-1)^n \dfrac{\cos nx}{n^2}$, $-\pi < x < \pi$.

(Madras 1997, Mangalore 1997, Warangal 1996)

Hence show that (*i*) $\Sigma \dfrac{1}{n^2} = \dfrac{\pi^2}{6}$

(*ii*) $\Sigma \dfrac{1}{(2n-1)^2} = \dfrac{\pi^2}{8}$ *(Mangalore 1997)* (*iii*) $\Sigma \dfrac{1}{n^4} = \dfrac{\pi^4}{90}$ *(Madras 1997)*

8. If $f(x)$ is a periodic function defined over a period $(0, 2\pi)$ by $f(x) = \dfrac{(3x^2 - 6x\pi + 2\pi^2)}{12}$.

Prove that $f(x) = \sum\limits_{n=1}^{\infty} \dfrac{\cos nx}{n^2}$ and hence show that $\dfrac{\pi^2}{6} = 1 + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \ldots$

12.7 FUNCTION DEFINED IN TWO OR MORE SUB-RANGES

Example 3. *Find the Fourier series of the function*

$$f(x) = \begin{cases} -1 & \text{for } -\pi < x < -\dfrac{\pi}{2} \\ 0 & \text{for } -\dfrac{\pi}{2} < x < \dfrac{\pi}{2} \\ +1 & \text{for } \dfrac{\pi}{2} < x < \pi. \end{cases}$$

Solution. Let $f(x) = \dfrac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \ldots + b_1 \sin x + b_2 \sin 2x + \ldots$ ...(1)

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{-\pi/2}(-1)\,dx + \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 0\,dx + \frac{1}{\pi}\int_{\pi/2}^{\pi} 1\,dx$$

$$= \frac{1}{\pi}\Big[-x\Big]_{-\pi}^{-\pi/2} + \frac{1}{\pi}\Big[x\Big]_{\pi/2}^{\pi} = \frac{1}{\pi}\left[\frac{\pi}{2} - \pi + \pi - \frac{\pi}{2}\right] = 0$$

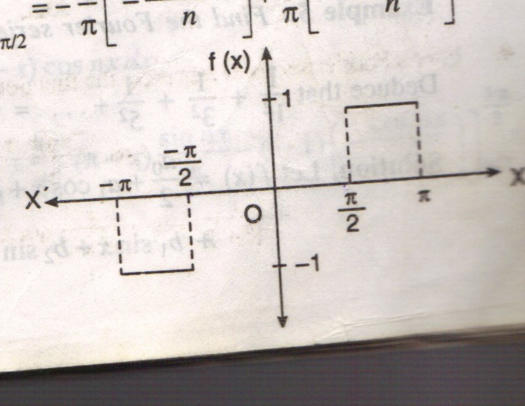
$$a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{-\pi/2}(-1)\cos nx\,dx + \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}(0)\cos nx\,dx + \frac{1}{\pi}\int_{\pi/2}^{\pi}(1)\cos nx\,dx$$

$$= -\frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_{-\pi}^{-\pi/2} + \frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_{\pi/2}^{\pi} = -\frac{1}{\pi}\left[-\frac{\sin\frac{n\pi}{2}}{n}\right] + \frac{1}{\pi}\left[-\frac{\sin\frac{n\pi}{2}}{n}\right] = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{-\pi/2}(-1)\sin nx\,dx + \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}(0)\sin nx\,dx$$



$$+\frac{1}{\pi}\int_{\pi/2}^{\pi}(1)\sin nx\,dx = \frac{1}{\pi}\left[\frac{\cos nx}{n}\right]_{-\pi}^{-\pi/2} - \frac{1}{\pi}\left[\frac{\cos nx}{n}\right]_{\pi/2}^{\pi}$$

$$= \frac{1}{n\pi}\left[\cos\frac{n\pi}{2} - \cos n\pi\right] - \frac{1}{n\pi}\left(\cos n\pi - \cos\frac{n\pi}{2}\right) = \frac{2}{n\pi}\left[\cos\frac{n\pi}{2} - \cos n\pi\right]$$

$$b_1 = \frac{2}{\pi},\; b_2 = -\frac{2}{\pi},\; b_3 = \frac{2}{3\pi}$$

Putting the values of a_0, a_n, b_n in (1) we get

$$f(x) = \frac{1}{\pi}\left[2\sin x - 2\sin 2x + \frac{2}{3}\sin 3x + \ldots\right] \quad \textbf{Ans.}$$

Example 4. *Find the Fourier series for the periodic function*

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$

$$f(x + 2\pi) = f(x)$$ *(Warangal 1996, Bangalore 1994)*

Solution. Let $f(x) = \frac{a_0}{2} + a_1\cos x + a_0\cos 2x + \ldots + b_1\sin x + b_2\sin 2x + \ldots$ ...(1)

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0} 0\,.\,dx + \frac{1}{\pi}\int_0^{\pi} x\,dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{1}{\pi}\left(\frac{\pi^2}{2}\right) = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} x\cos nx\,dx = \frac{1}{\pi}\left[x\cdot\frac{\sin nx}{n} - (1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] = -\frac{2}{n^2\pi} \text{ when } n \text{ is odd} \quad = 0 \text{ when } n \text{ is even.}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} x\sin nx\,dx = \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi} = \frac{1}{\pi}\left[-\pi\frac{(-1)^n}{n}\right] = -\frac{(-1)^n}{n}$$

Substituting the values of $a_0, a_1, a_2 \ldots b_1, b_2, \ldots$ in (i), we get

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right] + \left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \ldots\right] \quad \textbf{Ans.}$$

DISCONTINUOUS FUNCTIONS

At a point of discontinuity, Fourier series gives the value of $f(x)$ as the arithmetic mean of left and right limits.

At the point of discontinuity, $x = c$

At $x = c$, $f(x) = \frac{1}{2}\,[f(c-0) + f(c+0)]$

Example 5. *Find the Fourier series for* $f(x)$, *if* $f(x) = \begin{cases} -\pi & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$.

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$ *(Warangal, 1996)*

Solution. Let $f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + \ldots + a_n\cos nx + \ldots$

$$+\, b_1\sin x + b_2\sin 2x + \ldots + b_n\sin nx + \ldots \quad \ldots (1)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ dx$$

Then $a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\,dx + \int_0^{\pi} x\,dx\right] = \frac{1}{\pi}\left[-\pi\,(x)_{-\pi}^{0} + (x^2/2)_0^{\pi}\right] = \frac{1}{\pi}(-\pi^2 + \pi^2/2) = -\frac{\pi}{2}$;

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\cos nx\,dx + \int_0^{\pi} x\cos nx\,dx\right] = \frac{1}{\pi}\left[-\pi\left(\frac{\sin nx}{n}\right)_{-\pi}^{0} + \left(\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right)_0^{\pi}\right]$$

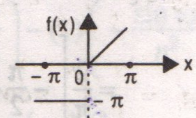
$$= \frac{1}{\pi}\left[0 + \frac{1}{n^2}\cos n\pi - \frac{1}{n^2}\right] = \frac{1}{\pi n^2}(\cos n\pi - 1), \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ \sin nx\,dx$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\sin nx\,dx + \int_0^{\pi} x\sin nx\,dx\right] = \frac{1}{\pi}\left[\left(\frac{\pi\cos nx}{n}\right)_{-\pi}^{0} + \left(-x\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right)_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}(1 - \cos n\pi) - \frac{\pi}{n}\cos n\pi\right] = \frac{1}{n}(1 - 2\cos n\pi)$$

$$f(x) = -\frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right) + 3\sin x - \frac{\sin 2x}{2} + \frac{3\sin 3x}{3} - \frac{\sin 4x}{4} \quad \ldots(2)$$

Putting $x = 0$ in (2), we get $f(0) = -\frac{\pi}{4} - \frac{2}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\infty\right)$ (3)



Now $f(x)$ is discontinuous at $x = 0$.

But $f(0-0) = -\pi$ and $f(0+0) = 0$ $\therefore$ $f(0) = \frac{1}{2}[f(0-0) + f(0+0)] = -\pi/2$

From (3), $-\frac{\pi}{2} = -\frac{\pi}{4} - \frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right]$ or $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$ **Proved**

Example 6. *Find the Fourier series expansion of the periodic function of period 2π, defined by*

$$f(x) = x, \quad \text{if } -\frac{\pi}{2} < x < \frac{\pi}{2}, \qquad f(x) = \pi - x, \quad \text{if } \frac{\pi}{2} < x < \frac{3\pi}{2}.$$

(A.M.I.E.T.E., Summer 1995)

Solution. Let $f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + \ldots + b_1\sin x + b_2\sin 2x + \ldots$

$$a_0 = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} x\,dx + \frac{1}{\pi}\int_{\pi/2}^{\frac{3\pi}{2}}(\pi - x)\,dx = \frac{1}{\pi}\left(\frac{x^2}{2}\right)_{-\pi/2}^{\pi/2} + \frac{1}{\pi}\left(\pi x - \frac{x^2}{2}\right)_{\pi/2}^{\frac{3\pi}{2}}$$

$$= \frac{1}{\pi}\left(\frac{\pi^2}{8} - \frac{\pi^2}{8}\right) + \frac{1}{\pi}\left(\frac{3\pi^2}{2} - \frac{9\pi^2}{8} - \frac{\pi^2}{2} + \frac{\pi^2}{8}\right) = \pi\left(\frac{3}{2} - \frac{9}{8} - \frac{1}{2} + \frac{1}{8}\right) = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} x\cos nx\,dx + \frac{1}{\pi}\int_{\pi/2}^{\frac{3\pi}{2}}(\pi - x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[x\frac{\sin nx}{n} - (1)\left(-\frac{\cos nx}{n^2}\right)\right]_{-\pi/2}^{\pi/2} + \frac{1}{\pi}\left[(\pi - x)\frac{\sin nx}{n} - (-1)\left(-\frac{\cos nx}{n^2}\right)\right]_{\pi/2}^{\frac{3\pi}{2}}$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2}\frac{\sin\frac{n\pi}{2}}{n} + \frac{\cos\frac{n\pi}{2}}{n^2} - \frac{\pi}{2}\frac{\sin\frac{n\pi}{2}}{n} - \frac{\cos\frac{n\pi}{2}}{n^2}\right]$$

$$+\frac{1}{\pi}\left[-\frac{\pi}{2}\frac{\sin\frac{3n\pi}{2}}{n} - \frac{\cos\frac{3n\pi}{2}}{n^2} - \frac{\pi}{2}\frac{\sin\frac{n\pi}{2}}{n} + \cos\frac{\frac{n\pi}{2}}{n^2}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{2n}\left(\sin\frac{3n\pi}{2} + \sin\frac{n\pi}{2}\right) - \frac{1}{n^2}\left(\cos\frac{3n\pi}{2} - \cos\frac{n\pi}{2}\right)\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n}\sin n\pi\cos\frac{n\pi}{2} + \frac{2}{n^2}\sin\frac{n\pi}{2}\sin n\pi\right] = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} x\sin nx\,dx + \frac{1}{\pi}\int_{\pi/2}^{3\pi/2}(\pi - x)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi/2} x\sin nx\,dx + \frac{1}{\pi}\int_{\pi/2}^{\frac{3\pi}{2}}(\pi - x)\sin nx\,dx$$

$$= \frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi/2} + \frac{1}{\pi}\left[(\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right]_{\pi/2}^{\frac{3\pi}{2}}$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{2}\frac{\cos\frac{n\pi}{2}}{n} + \frac{\sin\frac{n\pi}{2}}{n^2}\right] + \frac{1}{\pi}\left[\frac{\pi}{2}\frac{\cos\frac{3n\pi}{2}}{n} - \frac{\sin\frac{3n\pi}{2}}{n^2} + \frac{\pi}{2}\frac{\cos\frac{n\pi}{2}}{n} + \frac{\sin\frac{n\pi}{2}}{n^2}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{2}\frac{\cos\frac{n\pi}{2}}{n} + \frac{3\sin\frac{n\pi}{2}}{n^2} + \frac{\pi}{2}\frac{\cos 3\frac{n\pi}{2}}{n} - \frac{\sin\frac{3n\pi}{2}}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2n}\left(\cos\frac{3n\pi}{2} - \cos\frac{n\pi}{2}\right) + \frac{3}{n^2}\sin\frac{n\pi}{2} - \frac{1}{n^2}\sin\frac{3n\pi}{2}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n}\sin\frac{n\pi}{2}\sin n\pi + \frac{3}{n^2}\sin\frac{n\pi}{2} - \frac{1}{n^2}\sin\frac{3n\pi}{2}\right] = \frac{1}{n^2\pi}\left[3\sin\frac{n\pi}{2} - \sin\frac{3n\pi}{2}\right]$$

Substituting the values of $a_0, a_1, a_2 \ldots b_1, b_2, \ldots$ we get

$$f(x) = \frac{4}{\pi}\left[\frac{\sin x}{1^2} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \ldots\right]$$ **Ans.**

Example 7. *Find the Fourier series of the function defined as*

$$f(x) = \begin{cases} x + \pi & \text{for } 0 \le x \le \pi \\ -x - \pi & -\pi \le x < 0 \end{cases} \quad \text{and} \quad f(x + 2\pi) = f(x).$$

Solution. $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\,dx + \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(-x - \pi)\,dx + \frac{1}{\pi}\int_0^{\pi}(x + \pi)\,dx = \frac{1}{\pi}\left(-\frac{x^2}{2} - \pi x\right)_{-\pi}^{0} + \frac{1}{\pi}\left(\frac{x^2}{2} + \pi x\right)_0^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{\pi^2}{2} - \pi^2\right) + \frac{1}{\pi}\left(\frac{\pi^2}{2} + \pi^2\right) = \pi\left(\frac{1}{2} - 1\right) + \pi\left(\frac{1}{2} + 1\right) = \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(-x-\pi)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}(x+\pi)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[(-x-\pi)\frac{\sin nx}{n} - (-1)\left\{-\frac{\cos nx}{n^2}\right\}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[(x+\pi)\frac{\sin nx}{n} - (1)\left\{-\frac{\cos nx}{n^2}\right\}\right]_{0}^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{1}{n^2} + \frac{(-1)^n}{n^2}\right] + \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] = \frac{2}{n^2\pi}[(-1)^n - 1] = \frac{-4}{n^2\pi} \quad \text{if } n \text{ is odd.}$$

$= 0$ if n is even.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(-x-\pi)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}(x+\pi)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(-x-\pi)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right]_{-\pi}^{0} + \frac{1}{\pi}\left[(x+\pi)\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_{0}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}\right] + \frac{1}{\pi}\left[-\frac{2\pi}{n}(-1)^n + \frac{\pi}{n}\right] = \frac{1}{n}[(1) - 2(-1)^n + (1)] = \frac{2}{n}[1 - (-1)^n]$$

$= \frac{4}{n}$, if n is odd.

$= 0$, if n is even.

Fourier series is

$$f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + \dots + b_1\sin x + b_2\sin 2x + \dots$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \dots\right) + 4\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \dots\right)$$

Ans.

Exercise 12.2

1. Find the Fourier series of the function

$$f(x) = \begin{cases} -1 & \text{for } -\pi < x < 0 \\ 1 & \text{for } 0 < x < \pi \end{cases}$$

period = 2π

where $f(x + 2\pi) = f(x)$. **Ans.** $\frac{4}{\pi}\left[\frac{1}{1}\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \frac{1}{7}\sin 7x + \dots\right]$

2. Find the Fourier series for the function

$$f(x) = \begin{cases} -\frac{\pi}{4} & \text{for } -\pi < x < 0 \\ \frac{\pi}{4} & \text{for } 0 < x < \pi \end{cases}$$

and $f(-\pi) = f(0) = f(\pi) = 0$, $f(x) = f(x + 2\pi)$ for all x.

Deduce that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

Ans. $\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \ldots\ldots$

3. Find the Fourier series of the function

$$f(x) = \begin{cases} 0 & \text{for} \quad -\pi \le x \le 0 \\ 1 & \text{for} \quad 0 < x < \frac{\pi}{2} \\ 0 & \text{for} \quad \frac{\pi}{2} \le x \le \pi \end{cases}$$

4. Obtain a Fourier series to represent the following periodic function

$$f(x) = 0 \quad \text{when} \quad 0 < x < \pi$$
$$f(x) = 1 \quad \text{when} \quad \pi < x < 2\pi$$

Ans. $\frac{1}{2} - \frac{2}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots\right)$

5. Find the Fourier expansion of the function defined in a single period by the relations.

$$f(x) = \begin{cases} 1 & \text{for} \quad 0 < x < \pi \\ 2 & \text{for} \quad \pi < x < 2\pi \end{cases}$$

and from it deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$

Ans. $\frac{3}{2} - \frac{2}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots\right)$

6. Find a Fourier series to represent the function

$$f(x) = \begin{cases} 0 & \text{for} \quad -\pi < x \le 0 \\ \frac{1}{4}\pi x & \text{for} \quad 0 < x < \pi \end{cases}$$

and hence deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$

Ans. $\frac{\pi^2}{16} + \sum_{n=1}^{\infty}\left(\frac{[(-1)^n - 1]}{4n^2}\cos nx - \frac{(-1)^n \pi}{4n}\sin nx + \ldots\right)$

7. Find the Fourier series for $f(x)$, if

$$f(x) = -\pi \quad \text{for} \quad -\pi < x \le 0$$
$$= x \quad \text{for} \quad 0 < x < \pi$$
$$= \frac{-\pi}{2} \quad \text{for} \quad x = 0$$

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$

(Warangal 1996)

Ans. $-\frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right) + 3\sin x - \frac{1}{2}\sin 2x + \frac{3}{3}\sin 3x - \frac{1}{4}\sin 4x + \ldots$

8. Obtain a Fourier series to represent the function

$$f(x) = |x| \quad \text{for} \quad -\pi < x < \pi$$

and hence deduce $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$

(Madras 1997, Mangalore 1997, A.M.I.E.T.E., Summer 1996)

Ans. $\frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots\right]$

9. Expand as a Fourier series, the function $f(x)$ defined as

$$f(x) = \pi + x \quad \text{for} \quad -\pi < x < -\frac{\pi}{2}$$
$$= \frac{\pi}{2} \quad \text{for} \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$
$$= \pi - x \quad \text{for} \quad \frac{\pi}{2} < x < \pi$$

Ans. $\frac{3\pi}{8} + \frac{2}{\pi}\left[\frac{1}{1^2}\cos x - \frac{2}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x + \ldots\right]$

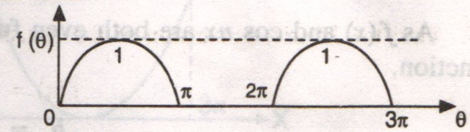
10. Obtain a Fourier series to represent the function

$$f(x) = |\sin x| \quad \text{for} \quad -\pi < x < \pi \qquad \left\{\textbf{Hint}\ \begin{array}{ll} f(x) = -\sin x & \text{for} \quad -\pi < x < 0 \\ = \sin x & \text{for} \quad 0 < x < \pi \end{array}\right\}$$

Ans. $\frac{2}{\pi} - \frac{4}{\pi}\left[\frac{1}{3}\cos 2x + \frac{1}{15}\cos 4x + \frac{1}{35}\cos 6x + \ldots\right]$

11. An alternating current after passing through a rectifier has the form

$$i = I\sin\theta \quad \text{for} \quad 0 < \theta < \pi$$
$$= 0 \quad \text{for} \quad \pi < \theta < 2\pi$$



Find the Fourier series of the function.

(*Delhi 1997*) **Ans.** $\frac{I}{\pi} - \frac{2I}{\pi}\left(\frac{\cos 2\theta}{3} + \frac{\cos 4\theta}{15} + \ldots\right) + \frac{I}{2}\sin\theta$

12. If $f(x) = 0$ for $-\pi < x < 0$

$= \sin x$ for $0 < x < \pi$

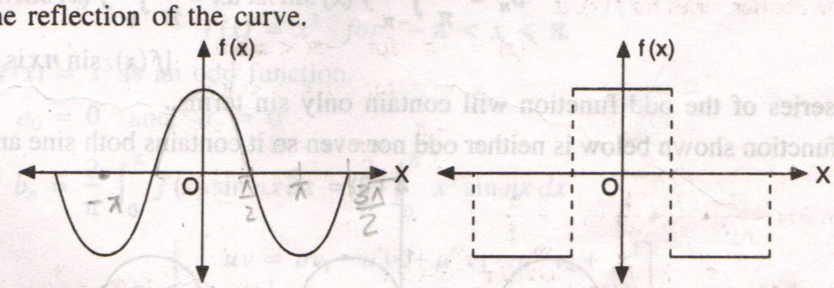
Prove that $f(x) = \frac{1}{\pi} + \frac{\sin x}{2} - \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{\cos 2mx}{4m^2 - 1}$. Hence show that $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} \ldots \infty = \frac{1}{4}(\pi - 2)$

12.8 EVEN FUNCTION

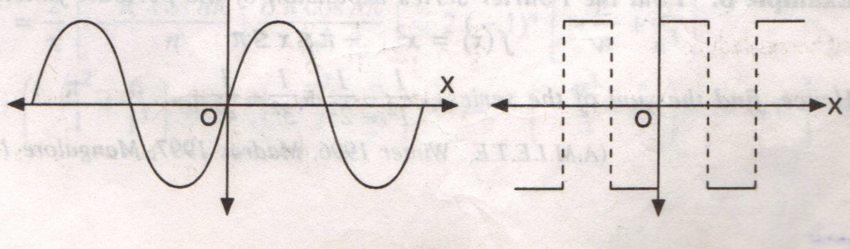
A function $f(x)$ is said to be even (or symmetric) function if,

$$f(-x) = f(x)$$

The graph of such a function is symmetrical with respect to y-axis [$f(x)$ axis]. Here y-axis is a mirror for the reflection of the curve.



(*b*) **Odd Function**



The area under such a curve from $-\pi$ to π is double the area from 0 to π.

$$\therefore \qquad \int_{-\pi}^{\pi} f(x)\, dx = 2\int_{0}^{\pi} f(x)\, dx$$

A function $f(x)$ is called odd (or skew symmetric) function if

$$f(-x) = -f(x)$$

Here the area under the curve from $-\pi$ to π is zero.

$$\int_{-\pi}^{\pi} f(x)\, dx = 0$$

Expansion of an even function:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{2}{\pi}\int_{0}^{\pi} (f(x)\, dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\, dx$$

As $f(x)$ and $\cos nx$ are both even functions $\therefore$ The product of $f(x).\cos nx$ is also an even function.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = 0$$

As $\sin nx$ is an odd function so $f(x).\sin nx$ is also an odd function. We need not to calculate b_n. It saves our labour a lot.

The series of the even function will contain only cos terms.

Expansion of an odd function :

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = 0$$

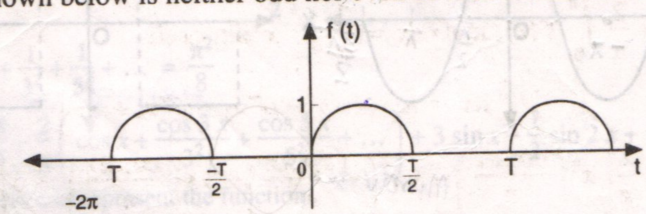
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = 0 \quad [f(x).\cos nx \text{ is odd function.}]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\, dx$$

$[f(x).\sin nx$ is even function.$]$

The series of the odd function will contain only sin terms.

The function shown below is neither odd nor even so it contains both sine and cosine terms



Example 8. *Find the Fourier series expansion of the periodic function of period* 2π

$$f(x) = x^2, \quad -\pi \le x \le \pi$$

Hence, find the sum of the series $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$

(*A.M.I.E.T.E., Winter 1996, Madras 1997, Mangalore 1997, Warangal 1996*)

Solution. $f(x) = x^2, \quad -\pi \le x \le \pi$

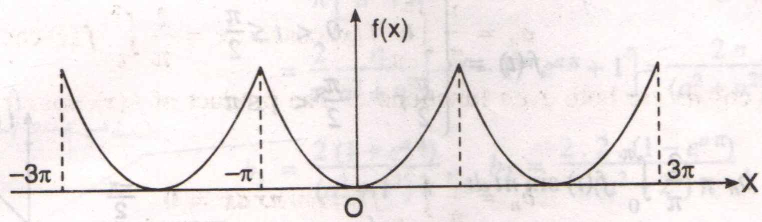
This is an even function. $\therefore \; b_n = 0$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,dx$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - (2x)\left(-\frac{\cos nx}{n^2}\right) + (2)\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi^2 \sin n\pi}{n} + \frac{2\pi\cos n\pi}{n^2} - \frac{2\sin n\pi}{n^3}\right] = \frac{4(-1)^n}{n^2}$$



Fourier series is $\quad f(x) = \dfrac{a_0}{2} + a_1\cos x + a_2\cos 2x + a_3\cos 3x + \ldots + a_n\cos nx + \ldots$

$$x^2 = \frac{\pi^2}{3} - 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \ldots\right]$$

On putting $x = 0$, we have

$$0 = \frac{\pi^2}{3} - 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2}\ldots\right]$$

or $\quad \dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2}\ldots = \dfrac{\pi^2}{12}$ **Proved.**

Example 9. *Obtain a Fourier expression for*

$$f(x) = x^3 \quad \textit{for} \quad -\pi < x < \pi.$$

Solution. $f(x) = x^3$ is an odd function.

$\therefore \quad a_0 = 0$ and $a_n = 0$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^3\sin nx\,dx$$

$$\left[\int uv = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots\right]$$

$$= \frac{2}{\pi}\left[x^3\left(\frac{\cos nx}{n}\right) - 3x^2\left(-\frac{\sin nx}{n^2}\right) + 6x\left(\frac{\cos nx}{n^3}\right) - 6\left(\frac{\sin nx}{n^4}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\pi^3\cos n\pi}{n} + \frac{6\pi\cos n\pi}{n^3}\right] = 2.(-1)^n\left[-\frac{\pi^2}{n} + \frac{6}{n^3}\right]$$

$$\therefore \quad x^3 = 2\left[-\left(-\frac{\pi^2}{1} + \frac{6}{1^3}\right)\sin x + \left(-\frac{\pi^2}{2} + \frac{6}{2^3}\right)\sin 2x - \left(-\frac{\pi^2}{3} + \frac{6}{3^3}\right)\sin 3x\ldots\right]$$ **Ans.**

12.9 HALF-RANGE SERIES, PERIOD 0 TO π

The given function is defined in the interval (0, π) and it is immaterial whatever the function may be outside the interval (0, π). To get the series of cosines only we assume that $f(x)$ is an even function in the interval $(-\pi, \pi)$.

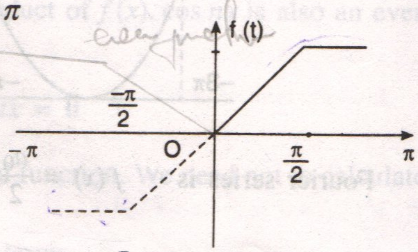
$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx \quad \text{and} \quad b_n = 0$$

To expand $f(x)$ as a sine series we extend the function in the interval $(-\pi, \pi)$ as an odd function.

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx \quad \text{and} \quad a_n = 0$$

Example 10. *Represent the following function by a Fourier sin series :*

$$f(t) = \begin{cases} t, & 0 < t \le \dfrac{\pi}{2} \\ \dfrac{\pi}{2}, & \dfrac{\pi}{2} < t \le \pi \end{cases}$$



Solution. $b_n = \frac{2}{\pi}\int_0^{\pi} f(t)\sin nt\,dt$

$$= \frac{2}{\pi}\int_0^{\pi/2} t\sin nt\,dt + \frac{2}{\pi}\int_{\pi/2}^{\pi}\frac{\pi}{2}\sin nt\,dt$$

$$= \frac{2}{\pi}\left[t\left(-\frac{\cos nt}{n}\right) - (1)\left(-\frac{\sin nt}{n^2}\right)\right]_0^{\pi/2} + \frac{2}{\pi}\frac{\pi}{2}\left[-\frac{\cos nt}{n}\right]_{\pi/2}^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{2}\frac{\cos\frac{n\pi}{2}}{n} + \frac{\sin\frac{n\pi}{2}}{n^2}\right] + \left[-\frac{\cos n\pi}{n} + \frac{\cos\frac{n\pi}{2}}{n}\right]$$

$$b_1 = \frac{2}{\pi}\left[-\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right] + \left[-\cos\pi + \cos\frac{\pi}{2}\right] = \frac{2}{\pi}[0+1] + [1] = \frac{2}{\pi} + 1$$

$$b_2 = \frac{2}{\pi}\left[-\frac{\pi}{2}\frac{\cos\pi}{2} + \frac{\sin\pi}{2^2}\right] + \left[-\frac{\cos 2\pi}{2} + \frac{\cos\pi}{2}\right] = \frac{2}{\pi}\left[-\frac{\pi}{2}\frac{(-1)}{2} + 0\right] + \left[-\frac{1}{2} - \frac{1}{2}\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi}{4}\right] - 1 = \frac{1}{2} - 1 = -\frac{1}{2}$$

$$b_3 = \frac{2}{\pi}\left[-\frac{\pi}{2}\frac{\cos\frac{3\pi}{2}}{3} + \frac{\sin\frac{3\pi}{2}}{3^2}\right] + \left[-\frac{\cos 3\pi}{3} + \frac{\cos\frac{3\pi}{2}}{3}\right]$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{2}(0) - \frac{1}{9}\right] + \left[\frac{1}{3} + 0\right] = -\frac{2}{9\pi} + \frac{1}{3}$$

$$f(t) = \left(\frac{2}{\pi} + 1\right)\sin t - \frac{1}{2}\sin 2t + \left(-\frac{2}{9\pi} + \frac{1}{3}\right)\sin 3t + \ldots \quad \textbf{Ans.}$$

Example 11. *Find the Fourier sine series for the function*

$$f(x) = e^{ax} \quad for \quad 0 < x < \pi$$

where a is constant.

Solution. $$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2} [a \sin bx - b \cos bx]$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} e^{ax} \sin nx \, dx$$

$$= \frac{2}{\pi}\left[\frac{e^{ax}}{a^2+n^2}(a \sin nx - n \cos nx)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{e^{a\pi}}{a^2+n^2}(a \sin n\pi - n \cos n\pi) + \frac{n}{a^2+n^2}\right]$$

$$= \frac{2}{\pi}\frac{n}{a^2+n^2}\left[-(-1)^n e^{a\pi} + 1\right] = \frac{2n}{(a^2+n^2)\pi}\left[1 - (-1)^n e^{a\pi}\right]$$

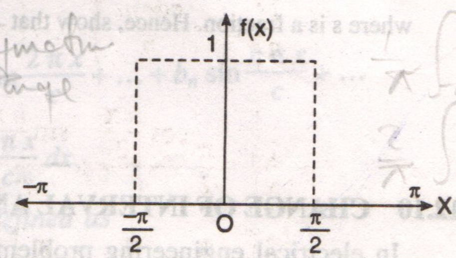
$$b_1 = \frac{2(1+e^{a\pi})}{(a^2+1^2)\pi}, \quad b_2 = \frac{2.2.(1-e^{a\pi})}{.(a^2+2^2)\pi}$$

$$e^{ax} = \frac{2}{\pi}\left[\frac{1+e^{a\pi}}{a^2+1^2}\sin x + \frac{2(2-e^{a\pi})}{a^2+2^2}\sin 2x + \dots\right]$$ **Ans.**

Exercise 12.3

1. Find the Fourier cosine series for the function

$$f(x) = \begin{cases} 1 & \text{for} \quad 0 < x < \frac{\pi}{2} \\ 0 & \text{for} \quad \frac{\pi}{2} < x < \pi. \end{cases}$$



Ans. $\frac{1}{2} + \frac{2}{\pi}\left[\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - \dots\right]$

2. Find a series of cosine of multiples of x which will represent $f(x)$ in $(0, \pi)$ where

$$f(x) = 0 \quad \text{for} \quad 0 < x < \frac{\pi}{2}$$

$$f(x) = \frac{\pi}{2} \quad \text{for} \quad \frac{\pi}{2} < x < \pi$$

Deduce that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \infty = \frac{\pi}{4}$ **Ans.** $\frac{\pi}{4} - \cos x + \frac{1}{3}\cos 3x - \frac{1}{5}\cos 5x + \dots$

3. Express $f(x) = x$ as a sin series in $0 < x < \pi$. **Ans.** $2\left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \dots\right]$

4. Find the cosine series for $f(x) = \pi - x$ in the interval $0 < x < \pi$.

Ans. $\frac{\pi}{2} + \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\right)$

5. If $f(x) = x, \quad$ for $\quad 0 < x < \frac{\pi}{2}$

$= \pi - x, \quad$ for $\quad \frac{\pi}{2} < x < \pi$

Show that:

(i) $f(x) = \dfrac{4}{\pi}\left(\sin x - \dfrac{1}{3^2}\sin 3x + \dfrac{1}{5^2}\sin 5x - \ldots\right)$ (*Madras 1998, Mysore 1997, Rewa 1994*)

(ii) $f(x) = \dfrac{\pi}{4} - \dfrac{2}{\pi}\left(\dfrac{1}{1^2}\cos 2x + \dfrac{1}{3^2}\cos 6x + \dfrac{1}{5^2}\cos 10x + \ldots\right)$ (*Delhi 1997, Patel 1997*)

6. Obtain the half-range cosine series for $f(x) = x^2$ in $0 < x < \pi$.

$$\textbf{Ans. } \frac{\pi^2}{3} - \frac{4}{\pi}\left(\cos x - \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x - \ldots\right)$$

7. Find (*i*) sine series and (*ii*) cosine series for the function

$$f(x) = e^x \quad \text{for } 0 < x < \pi.$$

$$\textbf{Ans. } (i)\ \frac{2}{\pi}\sum_{1}^{\infty} n\left[\frac{1-(-1)^n e^{\pi}}{n^2+1}\right]\sin nx \quad (ii)\ \frac{e^{\pi}-1}{\pi} - \frac{2}{\pi}\sum_{1}^{\infty}\frac{1-(-1)^n e^{\pi}}{n^2+1}\cos nx$$

8. If $f(x) = x + 1$, for $0 < x < \pi$, find its Fourier (*i*) sine series (*ii*) cosine series. Hence deduce that

(i) $1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \ldots = \dfrac{\pi}{4}$ (ii) $1 + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \dfrac{1}{7^2} + \ldots = \dfrac{\pi^2}{8}$

$$\textbf{Ans. } (i)\ \frac{2}{\pi}\left[(\pi+2)\sin x - \frac{\pi}{2}\sin 2x + \frac{1}{3}(\pi+2)\sin 3x - \frac{\pi}{4}\sin 4x + \ldots\right]$$

$$(ii)\ \frac{\pi}{2} + 1 - 4\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right)$$

9. Find the Fourier series expansion of the function

$$f(x) = \cos(sx), \quad -\pi \le x \le \pi$$

where s is a fraction. Hence, show that $\cot\theta = \dfrac{1}{\theta} + \dfrac{2\theta}{\theta^2-\pi^2} + \dfrac{2\theta}{\theta^2-4\pi^2} + \ldots$

(*A.M.I.E.T.E., Summer 1997*)

$$\textbf{Ans. } \frac{\sin \pi x}{\pi s} + \frac{1}{\pi}\Sigma\left[\frac{\sin(s\pi + n\pi)}{s+n} + \frac{\sin(s\pi - n\pi)}{s-n}\right]\cos nx$$

12.10 CHANGE OF INTERVAL AND FUNCTIONS HAVING ARBITRARY PERIOD

In electrical engineering problems, the period of the function is not always 2π but T or $2c$. This period must be converted to the length 2π. The independent variable x is also to be changed proportionally.

Let the function $f(x)$ be defined in the interval $(-c, c)$. Now we want to change the function to the period of 2π so that we can use the formulae of a_n, b_n as discussed in article 12.6.

$\because$ $2c$ is the interval for the variable x.

$\therefore$ 1 is the interval for the variable $= \dfrac{x}{2c}$

$\therefore$ 2π is the invertal for the variable $= \dfrac{x\,2\pi}{2c} = \dfrac{\pi x}{c}$

so put $\qquad z = \dfrac{\pi x}{c}$ or $x = \dfrac{zc}{\pi}$

Thus the function $f(x)$ of period $2c$ is transformed to the function

$f\left(\dfrac{cz}{\pi}\right)$ or $F(z)$ of period 2π.

$F(z)$ can be expanded in the Fourier series.

$$F(z) = f\left(\frac{cz}{\pi}\right) = \frac{a_0}{2} + a_1 \cos z + a_2 \cos 2z + b_1 \sin z + b_2 \sin 2z + \ldots$$

where $$a_0 = \frac{1}{\pi}\int_0^{2\pi} F(z)\,dz = \frac{1}{\pi}\int_0^{2\pi} f\left(\frac{cz}{\pi}\right) dz$$

$$= \frac{1}{c}\int_0^{2c} f(x)\, d\left(\frac{\pi x}{c}\right) = \frac{1}{c}\int_0^{2c} f(x)dx \quad \textbf{put } z = \frac{\pi x}{c}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} F(z)\cos nz\,dz = \frac{1}{\pi}\int_0^{2\pi} f\left(\frac{cz}{\pi}\right)\cos nx\,dz \qquad \text{Put } z = \frac{\pi x}{c}$$

$$= \frac{1}{\pi}\int_0^{2c} f(x)\cos\frac{n\pi x}{c}\, d\left(\frac{\pi x}{c}\right) = \frac{1}{c}\int_0^{2c} f(x)\cos\frac{n\pi x}{c}\,dx.$$

Similarly $$b_n = \frac{1}{c}\int_0^{2c} f(x)\sin\frac{n\pi x}{c}\,dx.$$

Cor. Half range series [Interval (o, c)]

Cosine series:

$$f(x) = \frac{a_0}{2} + a_1\cos\frac{\pi x}{c} + a_2\cos\frac{2\pi x}{c} + \ldots + a_n\cos\frac{n\pi x}{c} + \ldots$$

where $$a_0 = \frac{2}{c}\int_0^c f(x)\,dx, \quad a_n = \frac{2}{c}\int_0^c f(x)\cos\frac{n\pi x}{c}\,dx$$

Sine series.

$$f(x) = b_1\sin\frac{\pi x}{c} + b_2\sin\frac{2\pi x}{c} + \ldots + b_n\sin\frac{n\pi x}{c} + \ldots$$

where $$b_n = \frac{2}{c}\int_0^c f(x)\sin\frac{n\pi x}{c}\,dx.$$

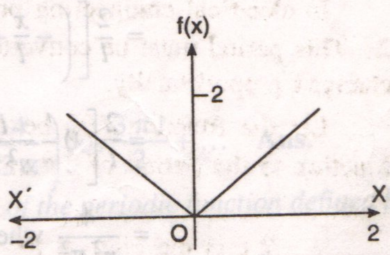
Example 12. *A periodic function of period 4 is defined as*

$$f(x) = |x|, \; -2 < x < 2.$$

Find its Fourier series expansion.

Solution.

$$f(x) = |x| \qquad -2 < x < 2$$
$$f(x) = x \qquad 0 < x < 2$$
$$= -x \qquad -2 < x < 0$$



$$a_0 = \frac{1}{c}\int_{-c}^{c} f(x)\,dx = \frac{1}{2}\int_0^2 x\,dx + \frac{1}{2}\int_{-2}^{0}(-x)\,dx$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\right]_0^2 + \frac{1}{2}\left[\frac{-x^2}{2}\right]_{-2}^{0} = \frac{1}{4}(4-0) + \frac{1}{4}(0+4) = 2$$

$$a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}\,dx = \frac{1}{2}\int_0^2 x\cos\frac{n\pi x}{2}\,dx + \frac{1}{2}\int_{-2}^{0}(-x)\cos\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\left[x\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right) - (1)\left(-\frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)\right]_0^2$$

$$+\frac{1}{2}\left[(-x)\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right)-(-1)\left(-\frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)\right]_{-2}^{0}$$

$$=\frac{1}{2}\left[0+\frac{4}{n^2\pi^2}(-1)^n-\frac{4}{n^2\pi^2}\right]+\frac{1}{2}\left[0-\frac{4}{n^2\pi^2}+\frac{4}{n^2\pi^2}(-1)^n\right]$$

$$=\frac{1}{2}\frac{4}{n^2\pi^2}[(-1)^n-1-1+(-1)^n]=\frac{4}{n^2\pi^2}[(-1)^n-1]$$

$$=-\frac{8}{n^2\pi^2} \quad \text{if } n \text{ is odd.}$$

$$=0 \quad \text{if } n \text{ is even}$$

$b_n = 0$ as $f(x)$ is even function.

Fourier series is

$$f(x)=\frac{a_0}{2}+a_1\cos\frac{\pi x}{c}+a_2\cos\frac{2\pi x}{c}+\ldots+b_1\sin\frac{\pi x}{c}+b_2\sin\frac{2\pi x}{c}+\ldots$$

$$f(x)=1-\frac{8}{\pi^2}\left[\frac{\cos\frac{\pi x}{2}}{1^2}+\frac{\cos\frac{3\pi x}{2}}{3^2}+\frac{\cos\frac{5\pi x}{2}}{5^2}+\ldots\right] \quad \textbf{Ans.}$$

Example 13. *Find Fourier half-range even expansion of the function,*

$$f(x)=(-x/l)+1,\ 0\le x\le l.$$

Solution. $a_0=\frac{2}{l}\int_0^l f(x)\,dx=\frac{2}{l}\int_0^l\left(-\frac{x}{l}+1\right)dx$

$$=\frac{2}{l}\left[-\frac{x^2}{2l}+x\right]_0^l=\frac{2}{l}\left[-\frac{l^2}{2l}+l\right]=\frac{2l}{l}\left[-\frac{1}{2}+1\right]=1$$

$$a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx=\frac{2}{l}\int_0^l\left(-\frac{x}{l}+1\right)\cos\frac{n\pi x}{l}\,dx$$

$$=\frac{2}{l}\left[\left(-\frac{x}{l}+1\right)\left(\frac{l}{n\pi}\sin\frac{n\pi x}{l}\right)-\left(-\frac{1}{l}\right)\left(-\frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right)\right]_0^l$$

$$=\frac{2}{l}\left[0-\frac{l}{n^2\pi^2}\cos n\pi+\frac{l}{n^2\pi^2}\right]=\frac{2}{l}\frac{l}{n^2\pi^2}[-(-1)^n+1]=\frac{2}{n^2\pi^2}[1-(-1)^n]$$

$$=\frac{4}{n^2\pi^2} \quad \text{when } n \text{ is odd.}$$

$$=0 \quad \text{when } n \text{ is even.}$$

$$f(x)=\frac{1}{2}+\frac{4}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{l}+\frac{1}{3^2}\cos\frac{3\pi x}{l}+\frac{1}{5^2}\cos\frac{5\pi x}{l}\ldots\right] \quad \textbf{Ans.}$$

Example 14. *Find the Fourier half-range cosine series of the function*

$$f(t)=2t, \quad 0<t<1$$
$$=2(2-t), \quad 1<t<2$$

(Kuvempu 1996, A.M.I.E.T.E., Summer 1997 1996)

Solution. $f(t)=2t, \quad 0<t<1$

$$=2(2-t), \quad 1<t<2$$

Let $$f(t) = \frac{a_0}{2} + a_1 \cos\frac{\pi t}{c} + a_2 \cos\frac{2\pi t}{c} + a_3 \cos\frac{3\pi t}{c} + \dots$$
$$+ b_1 \sin\frac{\pi t}{c} + b_2 \sin\frac{2\pi t}{c} + b_3 \sin\frac{3\pi t}{c} + \dots \qquad \dots(1)$$

Here $c = 2$, because it is half range series.

Hence $$a_0 = \frac{2}{c}\int_0^c f(t)\,dt = \frac{2}{2}\int_0^1 2t\,dt + \frac{2}{2}\int_1^2 2(2-t)\,dt$$

$$= \left[t^2\right]_0^1 + \left[2\left(2t - \frac{t^2}{2}\right)\right]_1^2 = 1 + \left[(4t - t^2)\right]_1^2 = 1 + (8 - 4 - 4 + 1) = 2$$

$$a_n = \frac{2}{c}\int_0^c f(t)\cos\frac{n\pi t}{c}\,dt = \frac{2}{2}\int_0^1 2t\cos\frac{n\pi t}{2}\,dt + \frac{2}{2}\int_1^2 2(2-t)\cos\frac{n\pi t}{2}\,dt$$

$$= \left[2t\left(\frac{2}{n\pi}\sin\frac{n\pi t}{2}\right) - (2)\left(-\frac{4}{n^2\pi^2}\cos\frac{n\pi t}{2}\right)\right]_0^1$$
$$+ \left[(4-2t)\left(\frac{2}{n\pi}\sin\frac{n\pi t}{2}\right) - (-2)\left(-\frac{4}{n^2\pi^2}\cos\frac{n\pi t}{2}\right)\right]_1^2$$

$$= \left[\frac{4}{n\pi}\sin\frac{n\pi}{2} + \frac{8}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{8}{n^2+\pi^2}\right] + \left[0 - \frac{8}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{4}{n\pi}\sin\frac{n\pi}{2} + \frac{8}{n^2\pi^2}\cos\frac{n\pi}{2}\right]$$

$$= \frac{8}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{8}{n^2\pi^2} - \frac{4}{n\pi}\sin\frac{n\pi}{2} = \frac{8}{n^2\pi^2}\left[\cos\frac{n\pi}{2} - 1 - \frac{n\pi}{2}\sin\frac{n\pi}{2}\right]$$

If $n = 1$, $$a_1 = \frac{8}{\pi^2}\left[0 - 1 - \frac{\pi}{2}\right] = -\frac{8}{\pi^2} - \frac{4}{\pi}.$$

If $n = 2$, $$a_2 = \frac{8}{4\pi^2}[-1-1] = -\frac{16}{4\pi^2} = -\frac{4}{\pi^2}$$

If $n = 3$, $$a_3 = \frac{8}{9\pi^2}\left[0 - 1 + \frac{3\pi}{2}\right] = -\frac{8}{9\pi^2} + \frac{4}{3\pi}$$

Putting the values of $a_0, a_1, a_2, a_3 \dots$ in (1) we get

$$f(t) = 1 - \left(\frac{8}{\pi^2} + \frac{4}{\pi}\right)\cos\frac{\pi t}{2} - \frac{4}{\pi^2}\cos\frac{2\pi t}{2} + \left(-\frac{8}{9\pi^2} + \frac{4}{3\pi}\right)\cos\frac{3\pi t}{2} + \dots \quad \textbf{Ans.}$$

Example 15. *Obtain the Fourier cosine series expansion of the periodic function defined by*

$$f(t) = \sin\left(\frac{\pi t}{r}\right), \quad 0 < t < l$$

Solution. $$f(t) = \sin\left(\frac{\pi t}{l}\right), \quad 0 < t < l$$

$$a_0 = \frac{2}{l}\int_0^l \sin\left(\frac{\pi t}{l}\right)dt = \frac{2}{l}\left(-\frac{l}{\pi}\cos\frac{\pi t}{l}\right)_0^l = -\frac{2}{\pi}(\cos\pi - \cos 0) = -\frac{2}{\pi}(-1-1) = \frac{4}{\pi}$$

$$a_n = \frac{2}{l}\int_0^l \sin\left(\frac{\pi t}{l}\right)\cos\frac{n\pi t}{l}\,dt = \frac{1}{l}\int_0^l \left[\sin\left(\frac{\pi t}{l} + \frac{n\pi t}{l}\right) - \sin\left(\frac{n\pi t}{l} - \frac{\pi t}{l}\right)\right]dt$$

$$= \frac{1}{l}\int_0^l \sin(n+1)\frac{\pi t}{l}\,dt - \frac{1}{l}\int_0^l \sin(n-1)\frac{\pi t}{l}\,dt$$

$$= \frac{1}{l}\left[-\frac{l}{(n+1)\pi}\cos\frac{(n+1)\pi t}{l}\right]_0^l - \frac{1}{l}\left[\frac{l}{(n-1)\pi}\cos\frac{(n-1)\pi t}{l}\right]_0^l$$

$$= \frac{-1}{(n+1)\pi}[\cos(n+1)\pi - \cos 0] + \frac{1}{(n-1)\pi}[\cos(n-1)\pi - \cos 0]$$

$$= \frac{1}{(n+1)\pi}[(-1)^{n+1} - 1] + \frac{1}{(n-1)\pi}[(-1)^{n+1} - 1]$$

$$= (-1)^{n+1}\left[-\frac{1}{(n+1)\pi} + \frac{1}{(n-1)\pi}\right] + \frac{1}{(n+1)\pi} - \frac{1}{(n-1)\pi}$$

$$= (-1)^{n+1}\frac{2}{(n^2-1)\pi} - \frac{2}{(n^2-1)\pi} = \frac{2}{(n^2-1)\pi}[(-1)^{n+1} - 1]$$

$$= \frac{-4}{(n^2-1)\pi} \text{ when } n \text{ is even}$$

$$= 0 \quad \text{when } n \text{ is odd.}$$

The above formula for finding the value of a_1 is not applicable.

$$a_1 = \frac{2}{l}\int_0^l \sin\frac{\pi t}{l}\cos\frac{\pi t}{l}\,dt = \frac{1}{l}\int_0^l \sin\frac{2\pi t}{l}\,dt$$

$$= \frac{1}{l}\left(-\frac{l}{2\pi}\cos\frac{2\pi t}{l}\right)_0^l = -\frac{1}{2\pi}(\cos 2\pi - \cos 0) = 0 = \frac{1}{2l}(1-1) = 0$$

$$f(t) = \frac{a_0}{2} + a_1\cos\frac{\pi t}{l} + a_2\cos\frac{2\pi t}{l} + a_3\cos\frac{3\pi t}{l} + a_4\cos\frac{4\pi t}{l} + \dots$$

$$= \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{1}{3}\cos\frac{2\pi t}{l} + \frac{1}{15}\cos\frac{4\pi t}{l} + \frac{1}{35}\cos\frac{6\pi t}{l} + \dots\right] \quad \textbf{Ans.}$$

Example 16. *Find the Fourier series expansion of the periodic function of period l*

$$f(x) = \frac{l}{2} + x, \quad -\frac{l}{2} < x \le 0$$

$$= \frac{l}{2} - x, \quad 0 < x < \frac{l}{2}$$

(A.M.I.E.T.E., Winter 1996)

Solution. Let
$$f(x) = \frac{a_0}{2} + a_1\cos\frac{\pi x}{c} + a_2\cos\frac{2\pi x}{c} + \dots$$
$$+ b_1\sin\frac{\pi x}{c} + b_2\sin\frac{2\pi x}{c} + \dots \quad \dots(1)$$

Here $2c = 1$ or $c = \frac{1}{2}$

$$a_0 = \frac{1}{c}\int_{-c}^{c} f(x)\,dx = \frac{1}{1/2}\int_{-1/2}^{0}\left(\frac{1}{2} + x\right)dx + \frac{1}{1/2}\int_0^{1/2}\left(\frac{1}{2} - x\right)dx$$

$$= 2\left[\frac{x}{2} + \frac{x^2}{2}\right]_{-1/2}^{0} + 2\left[\frac{x}{2} - \frac{x^2}{2}\right]_0^{1/2} = 2\left[\frac{1}{4} - \frac{1}{8}\right] + 2\left[\frac{1}{4} - \frac{1}{8}\right] = \frac{1}{2}$$

$$a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}\,dx$$

$$= \frac{1}{1/2}\int_{-1/2}^{0}\left(\frac{1}{2}+x\right)\cos\frac{n\pi x}{1/2}\,dx + \frac{1}{1/2}\int_{0}^{1/2}\left(\frac{1}{2}-x\right)\cos\frac{n\pi x}{1/2}\,dx$$

$$= 2\int_{-1/2}^{0}\left(\frac{1}{2}+x\right)\cos 2n\pi x\,dx + 2\int_{0}^{1/2}\left(\frac{1}{2}-x\right)\cos 2n\pi x\,dx$$

$$= 2\left[\left(\frac{1}{2}+x\right)\frac{\sin 2n\pi x}{2n\pi} - (1)\left(-\frac{\cos 2n\pi x}{4n^2\pi^2}\right)\right]_{-1/2}^{0}$$

$$+ 2\left[\left(\frac{1}{2}-x\right)\frac{\sin 2n\pi x}{2n\pi} - (-1)\left(\frac{-\cos 2n\pi x}{4n^2\pi^2}\right)\right]_{0}^{1/2}$$

$$= 2\left[0+\frac{1}{4n^2\pi^2}-\frac{(-1)^n}{4n^2\pi^2}\right] + 2\left[0-\frac{(-1)^n}{4n^2\pi^2}+\frac{1}{4n^2\pi^2}\right] = \frac{1}{\pi^2}\left[\frac{1}{n^2}-\frac{(-1)^n}{n^2}\right]$$

$$= \frac{2}{n^2\pi^2} \quad \text{if } n \text{ is odd}$$

$$= 0 \quad \text{if } n \text{ is even}$$

$$b_n = \frac{1}{c}\int_{-c}^{c} f(x)\sin\frac{n\pi x}{c}\,dx$$

$$= \frac{1}{1/2}\int_{-1/2}^{0}\left(\frac{1}{2}+x\right)\sin\frac{n\pi x}{1/2}\,dx + \frac{1}{1/2}\int_{0}^{1/2}\left(\frac{1}{2}-x\right)\sin\frac{n\pi x}{1/2}\,dx$$

$$= 2\int_{-1/2}^{0}\left(\frac{1}{2}+x\right)\sin 2n\pi x\,dx + 2\int_{0}^{1/2}\left(\frac{1}{2}-x\right)\sin 2n\pi x\,dx$$

$$= 2\left[\left(\frac{1}{2}+x\right)\left(\frac{-\cos 2n\pi x}{2n\pi}\right) - (1)\left(-\frac{\sin 2n\pi x}{4n^2\pi^2}\right)\right]_{-1/2}^{0}$$

$$+ 2\left[\left(\frac{1}{2}-x\right)\left(-\frac{\cos 2n\pi x}{2n\pi}\right) - (-1)\left(-\frac{\sin 2n\pi x}{4n^2\pi^2}\right)\right]_{0}^{1/2}$$

$$= 2\left[-\frac{1}{4n\pi}\right] + 2\left[\frac{1}{4n\pi}\right] = 0$$

Substituting the values of $a_0, a_1, a_2, a_3, \ldots b_1, b_2, b_3 \ldots$ in (1) we have

$$f(x) = \frac{1}{4} + \frac{2}{\pi^2}\left[\frac{\cos 2\pi x}{1^2} + \frac{\cos 6\pi x}{3^2} + \frac{\cos 10\pi x}{5^2} + \ldots\right] \quad \textbf{Ans.}$$

Example 17. *Prove that*

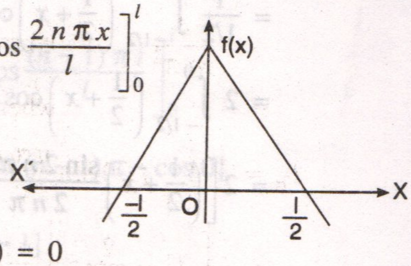
$$\frac{l}{2} - x = \frac{l}{\pi}\sum_{1}^{\infty}\frac{1}{n}\sin\frac{2n\pi x}{l}, \quad 0 < x < l.$$

Solution. $f(x) = \frac{l}{2} - x$

$$a_0 = \frac{1}{l/2}\int_0^l f(x)\,dx = \frac{2}{l}\int_0^l \left(\frac{l}{2}-x\right)dx = \frac{2}{l}\left[\frac{lx}{2}-\frac{x^2}{2}\right]_0^l = 0$$

$$a_n = \frac{1}{l/2}\int_0^l f(x)\cos\frac{n\pi x}{l/2}\,dx = \frac{2}{l}\int_0^l\left(\frac{1}{2}-x\right)\cos\frac{2n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[\left(\frac{l}{2}-x\right)\frac{1}{2n\pi}\sin\frac{2n\pi x}{l} + (-1)\frac{l^2}{4n^2\pi^2}\cos\frac{2n\pi x}{l}\right]_0^l$$



$$= \frac{2}{l}\left[0-\frac{l^2}{4n^2\pi^2}\cos 2n\pi+\frac{l^2}{4n^2\pi^2}\right]$$

$$= \frac{2}{l}\frac{l^2}{4n^2\pi^2}(-\cos 2n\pi+1) = \frac{l}{2n^2\pi^2}(-1+1) = 0$$

$$b_n = \frac{1}{l/2}\int_0^l f(x)\sin\frac{n\pi x}{l/2}\,dx = \frac{2}{l}\int_0^l\left(\frac{1}{2}-x\right)\sin\frac{2n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[\left(\frac{1}{2}-x\right)\left(-\frac{1}{2n\pi}\cos\frac{2n\pi x}{l}\right)-(-1)\left(-\frac{l^2}{4n^2\pi^2}\sin\frac{2n\pi x}{l}\right)\right]_0^l$$

$$= \frac{2}{l}\left[\frac{l}{2}\frac{l}{2n\pi}\cos 2n\pi - 0 + \frac{l}{2}\cdot\frac{l}{2n\pi}(1)\right] = \frac{2}{l}\left[\frac{l^2}{2n\pi}\right] = \frac{l}{n\pi}$$

Fourier series is

$$f(x) = \frac{a_0}{2}+a_1\cos\frac{n\pi x}{l/2}+a_2\cos\frac{2n\pi x}{l/2}+a_3\cos\frac{3n\pi x}{l/2}+\ldots$$

$$+\,b_1\sin\frac{n\pi x}{l/2}+b_2\sin\frac{2n\pi x}{l/2}+b_3\sin\frac{3n\pi x}{l/2}+\ldots$$

$$\frac{l}{2}-x = \frac{l}{\pi}\sin\frac{2\pi x}{l}+\frac{l}{2\pi}\sin\frac{4\pi x}{l}+\frac{l}{3\pi}\sin\frac{6\pi x}{l}+\ldots$$

$$= \frac{l}{\pi}\sum_1^\infty\frac{1}{n}\sin\frac{2n\pi x}{l}$$

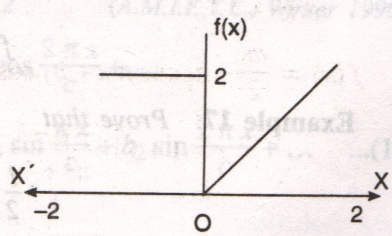
Proved.

Example 18. *Find the Fourier series corresponding to the function f (x) defined in (− 2, 2) as follows*

$$f(x) = 2 \quad \text{in} \quad -2 \le x \le 0$$
$$= x \qquad 0 < x < 2.$$

Solution. Here the interval is (− 2, 2) and $c = 2$

$$a_0 = \frac{1}{c}\int_{-c}^{c} f(x)\,dx = \frac{1}{2}\left[\int_{-2}^{0} 2.\,dx+\int_0^2 x.\,dx\right]$$



$$= \frac{1}{2}\left[[2x]_{-2}^{0}+\left(\frac{x^2}{2}\right)_0^2\right] = \frac{1}{2}[4+2] = 3$$

$$a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\left(\frac{n\pi x}{c}\right)dx = \frac{1}{2}\left[\int_{-2}^{0} 2\cos\frac{n\pi x}{2}\,dx+\int_0^2 x\cos\frac{n\pi x}{2}\,dx\right]$$

$$= \frac{1}{2}\left[\frac{4}{n\pi}\left(\sin\frac{n\pi x}{2}\right)_{-2}^{0}+\left(x\frac{2}{n\pi}\sin\frac{n\pi x}{2}+\frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)_0^2\right]$$

$$= \frac{1}{2}\left[\frac{4}{n^2\pi^2}\cos n\pi - \frac{4}{n^2\pi^2}\right] = \frac{2}{n^2\pi^2}[(-1)^n - 1]$$

$$= -\frac{4}{n^2\pi^2} \quad \text{when } n \text{ is odd}$$

$$= 0 \quad \text{when } n \text{ is even.}$$

$$b_n = \frac{1}{c}\int_{-c}^{c} f(x)\sin\frac{n\pi x}{c}\,dx = \frac{1}{2}\int_{-2}^{0} 2\sin\frac{n\pi x}{2}\,dx + \frac{1}{2}\int_{0}^{2} x\sin\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\left[2\left(-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right)\right]_{-2}^{0} + \frac{1}{2}\left[x\left(-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right) + (1)\frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right]_0^2$$

$$= \frac{1}{2}\left[-\frac{4}{n\pi} + \frac{4}{n\pi}\cos n\pi\right] + \frac{1}{2}\left[-\frac{4}{n\pi}\cos n\pi + \frac{4}{n^2\pi^2}\sin n\pi\right] = \frac{1}{2}\left[-\frac{4}{n\pi}\right] = -\frac{2}{n\pi}$$

$$f(x) = \frac{a_0}{2} + a_1\cos\frac{\pi x}{c} + a_2\cos\frac{2\pi x}{c} + a_3\cos\frac{3\pi x}{c} + \ldots$$

$$+ b_1\sin\frac{\pi x}{c} + b_2\sin\frac{2\pi x}{c} + b_3\sin\frac{3\pi x}{c} + \ldots$$

$$= \frac{3}{2} - \frac{4}{\pi^2}\left\{\frac{1}{1^2}\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \ldots\right\}$$

$$-\frac{2}{\pi}\left\{\frac{1}{1}\sin\frac{\pi x}{2} + \frac{1}{2}\sin\frac{2\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} + \ldots\right\} \quad \textbf{Ans.}$$

Example 19. *Expand $f(x) = e^x$ in a cosine series over (0, 1).*

Solution. $f(x) = e^x$ and $c = 1$

$$a_0 = \frac{2}{c}\int_0^c f(x)\,dx = \frac{2}{1}\int_0 e^x\,dx = 2(e-1)$$

$$a_n = \frac{2}{c}\int_0^c f(x)\cos\frac{n\pi x}{c}\,dx = \frac{2}{1}\int_0^1 e^x\cos\frac{n\pi x}{1}\,dx$$

$$= 2\left[\frac{e^x}{n^2\pi^2+1}(n\pi\sin n\pi x + \cos n\pi x)\right]_0^1 = 2\left[\frac{e}{n^2\pi^2+1}(n\pi\sin n\pi + \cos n\pi) - \frac{1}{n^2\pi^2+1}\right]$$

$$= \frac{2}{n^2\pi^2+1}[(-1)^n e - 1]$$

$$f(x) = \frac{a_0}{2} + a_1\cos\pi x + a_2\cos 2\pi x + a_3\cos 3\pi x + \ldots\ldots\ldots\ldots$$

$$e^x = e - 1 + 2\left[\frac{-e-1}{\pi^2+1}\cos\pi x + \frac{e-1}{4\pi^2+1}\cos 2\pi x + \frac{-e-1}{9\pi^2+1}\cos 3\pi x + \ldots\right] \quad \textbf{Ans.}$$

Exercise 12.4

1. Find the Fourier series to represent $f(x)$, where

$$f(x) = -a \qquad -c < x < 0$$
$$= a \qquad 0 < x < c$$

Ans. $\frac{4a}{\pi}\left[\sin\frac{\pi x}{c} + \frac{1}{3}\sin\frac{3\pi x}{c} + \frac{1}{5}\sin\frac{5\pi x}{c} + \ldots\right]$

2 Find the half-range sine series for the function

$$f(x) = 2x - 1 \qquad 0 < x < 1.$$

$$\textbf{Ans. } -\frac{2}{\pi}\left[\sin 2\pi x+\frac{1}{2}\sin 4\pi x+\frac{1}{3}\sin 6\pi x+\ldots\right]$$

3. Express $f(x) = x$ as a cosine, half range series in $0 < x < 2$.

$$\textbf{Ans. } 1-\frac{8}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{2}+\frac{1}{3^2}\cos\frac{3\pi x}{2}+\frac{1}{5^2}\cos\frac{5\pi x}{2}+\ldots\right]$$

4. Find the Fourier series of the function

$$f(x) = \begin{bmatrix} -2 & \text{for} & -4 < x < -2 \\ x & \text{for} & -2 < x < 2 \\ 2 & \text{for} & 2 < x < 4 \end{bmatrix}$$

$$\textbf{Ans. } \frac{4}{\pi}+\frac{8}{\pi^2}\sin\frac{\pi x}{4}-\frac{2}{\pi}\sin\frac{2\pi x}{4}+\left(\frac{4}{3\pi}-\frac{8}{3^2\pi}\right)\sin\frac{3\pi x}{4}-\frac{2}{2\pi}\sin\frac{4\pi x}{4}+\ldots$$

5. Find the Fourier series to represent

$$f(x) = x^2 - 2 \quad \text{from} \quad -2 < x < 2.$$

$$\textbf{Ans. } -\frac{2}{3}-\frac{16}{\pi^2}\left[\cos\frac{\pi x}{2^2}-\frac{1}{4}\cos\pi x+\frac{1}{9}\cos\frac{3\pi x}{2}\ldots\right]$$

6. If $f(x) = e^{-x}$ $\quad -c < x < c$, show that

$$f(x) = (e^c - e^{-c})\left\{\frac{1}{2c}-c\left(\frac{1}{c^2+\pi^2}\cos\frac{\pi x}{c}-\frac{1}{c^2+4\pi^2}\cos\frac{2\pi x}{c}+\ldots\right)\right.$$

$$\left.-\pi\left(\frac{1}{c^2+\pi^2}\sin\frac{\pi x}{c}-\frac{2}{c^2+4\pi^2}\sin\frac{2\pi x}{c}\ldots\right)\right\}$$ *(Hamirpur 1996, Mysore 1994)*

7. A sinusodial voltage E sin ωt is passed through a half wave rectifier which clips the negative portion of the wave. Develop the resulting portion of the function

$$u(t) = 0 \quad \text{when} \quad -\frac{T}{2} < t < 0$$

$$= E\sin\omega t \quad \text{when} \quad 0 < t < \frac{T}{2} \quad \left(T = \frac{2\pi}{\omega}\right)$$

(Mangalore 1997)

$$\textbf{Ans. } \frac{E}{\pi}+\frac{E}{2}\sin\omega t-\frac{2E}{\pi}\left[\frac{1}{1.3}\cos 2\omega t+\frac{1}{3.5}\cos 4\omega t+\frac{1}{5.7}\cos 6\omega t+\ldots\right]$$

8. A periodic square wave has a period 4. The function generating the square is

$$f(t) = 0 \quad \text{for} \quad -2 < t < -1$$

$$= k \quad \text{for} \quad -1 < t < 1$$

$$= 0 \quad \text{for} \quad 1 < t < 2$$

Find the Fourier series of the function.

$$\textbf{Ans. } f(t) = \frac{k}{2}+\frac{2k}{\pi}\left[\cos\frac{\pi t}{2}-\frac{1}{3}\cos\frac{3\pi t}{2}+\ldots\right]$$

9. Find a Fourier series to represent x^2 in the interval $(-l, l)$. *(Nagpur 1997)*

$$\textbf{Ans. } \frac{l^2}{3}-\frac{4l^2}{\pi^2}\left[\cos\pi x-\frac{\cos\pi x}{2^2}+\frac{\cos 3\pi x}{3^2}\right]$$

12.11. PARSEVAL'S FORMULA

$$\int_{-c}^{c}[f(x)]^2\,dx = c\left\{\frac{1}{2}a_0^2+\sum_{n=1}^{\infty}\left(a_n^2+b_n^2\right)\right\}$$

Solution. We know that $\quad f(x) = \dfrac{a_0}{2}+\displaystyle\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{c}+b_n\sin\frac{n\pi x}{c}\right) \quad \ldots (1)$

Multiplying (1) by $f(x)$, we get

$$[f(x)]^2 = \frac{a_0}{2} f(x) + \sum_{n=1}^{\infty} a_n f(x) \cos \frac{n\pi x}{c} + \sum_{n=1}^{\infty} b_n f(x) \sin \frac{n\pi x}{c} \quad ...(2)$$

Integrating term by term from $-C$ to C, we have

$$\int_{-c}^{c} [f(x)]^2 dx = \frac{a_0}{2} \int_{-c}^{c} f(x)\, dx + \sum_{n=1}^{\infty} a_n \int_{-c}^{c} f(x) \cos \frac{n\pi x}{c} dx$$
$$+ \sum_{n=1}^{\infty} b_n \int_{-c}^{c} f(x) \sin \frac{n\pi x}{c} dx \quad ...(3)$$

In article 12.10, we have the following results

$$\int_{-c}^{c} f(x)\, dx = c\, a_0$$

$$\int_{-c}^{c} f(x) \cos \frac{n\pi x}{c} dx = c\, a_n$$

$$\int_{-c}^{c} f(x) \sin \frac{n\pi x}{c} dx = c\, b_n$$

On putting these integrals in (3), we get

$$\int_{-c}^{c} [f(x)]^2 dx = c \frac{a_0^2}{2} + \sum_{n=1}^{\infty} c\, a_n^2 + \sum c\, b_n^2 = c\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)\right]$$

This is the Parseval's formula

Note. 1. If $0 < x < 2C$, then $\int_0^{2c} [f(x)]^2 dx = c\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)\right]$

2. If $0 < x < C$ (Half range cosine series), $\int_0^C [f(x)]^2 = \frac{c}{2}\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2\right]$

3. If $0 < x < C$ (Half range sin series), $\int_0^C [f(x)]^2 dx = \frac{c}{2}\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} b_n^2\right]$

4. R.M.S. $= \left\{\frac{\int_a^b [f(x)]^2 dx}{b-a}\right\}^{\frac{1}{2}}$

Example 20. *By using the sin series for $f(x) = 1$ in $0 < x < \pi$ show that*

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots\ldots$$

(*Hamirpur 1996*)

Solution. sin series is $f(x) = \sum b_n \sin nx$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} (1) \sin nx\, dx = \frac{2}{\pi}\left(\frac{-\cos nx}{n}\right)_0^{\pi} = \frac{-2}{n\pi}[\cos n\pi - 1] = \frac{-2}{n\pi}[(-1)^n - 1]$$

$$= \frac{4}{n\pi} \quad \text{if } n \text{ is odd}$$

$$= 0 \quad \text{if } n \text{ is even}$$

Then the sin series is

$$1 = \frac{4}{\pi}\sin x + \frac{4}{3\pi}\sin 3x + \frac{4}{5\pi}\sin 5x + \frac{4}{7\pi}\sin 7x + \ldots\ldots$$

$$\int_0^c [f(x)]^2\,dx = \frac{c}{2}\left[b_1^2 + b_2^2 + b_3^2 + b_4^2 + b_5^2 + \ldots\ldots\right]$$

$$\int_0^\pi (1)^2\,dx = \frac{\pi}{2}\left[\left(\frac{4}{\pi}\right)^2 + \left(\frac{4}{3\pi}\right)^2 + \left(\frac{4}{5\pi}\right)^2 + \left(\frac{4}{7\pi}\right)^2 + \ldots\ldots\right]$$

$$[x]_0^\pi = \left(\frac{\pi}{2}\right)\left(\frac{16}{\pi^2}\right)\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots\ldots\right]$$

$$\pi = \frac{\pi}{2}\left(\frac{16}{\pi^2}\right)\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots\ldots\right]$$

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots\ldots$$ **Proved.**

Example 21. *If* $f(x) = \begin{cases} \pi x, & 0 < x < 1 \\ \pi(2-x), & 1 < x < 2 \end{cases}$

sing half range cosine series, show that

$$\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\ldots = \frac{\pi^4}{96}$$

Solution. Half range cosine series is

$$f(x) = \frac{a_0}{2} + \Sigma\, a_n \cos\frac{n\pi x}{c}$$

where
$$a_0 = \frac{2}{c}\int_0^c f(x)\,dx = \frac{2}{2}\left[\int_0^1 \pi x\,dx + \int_1^2 \pi(2-x)\,dx\right]$$

$$= \pi\left(\frac{x^2}{2}\right)_0^1 + \pi\left(2x - \frac{x^2}{2}\right)_1^2 = \frac{\pi}{2} + \pi\left[(4-2) - \left(2 - \frac{1}{2}\right)\right]$$

$$= \pi$$

$$a_n = \frac{2}{c}\int_0^c f(x)\cos\frac{n\pi x}{c}\,dx$$

$$= \frac{2}{2}\left[\int_0^1 \pi x\cos\frac{n\pi x}{2}\,dx + \int_1^2 \pi(2-\pi)\cos\frac{n\pi x}{2}\,dx\right]$$

$$= \pi\left[x\,\frac{\sin\frac{\pi x}{2}}{\frac{n\pi}{2}} - \left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right]_0^1 + \pi\left[(2-x)\,\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}} - (-1)\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right]_1^2$$

$$= \pi\left[\frac{2}{n\pi}\sin\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{4}{n^2\pi^2}\right] + \pi\left[0 - \frac{4}{n^2\pi^2}\cos n\pi - \frac{2}{n\pi}\sin\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi}{2}\right]$$

$$= \pi\left[\frac{8}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{4}{n^2\pi^2} - \frac{4}{n^2\pi^2}\cos n\pi\right] = \frac{4}{n^2\pi}\left[2\cos\frac{n\pi}{2} - 1 - \cos n\pi\right]$$

$$a_1 = 0,\ a_2 = \frac{-4}{\pi},\ a_3 = 0,\ a_4 = 0,\ a_5 = 0,\ a_6 = \frac{-4}{9\pi} \ldots\ldots$$

$$\int_0^c [f(x)]^2\,dx = \frac{c}{2}\left[\frac{a_0^2}{2} + a_1^2 + a_2^2 + a_3^2 + \ldots\ldots\right]$$

$$\int_0^1 (\pi x)^2\,dx + \int_1^2 \pi^2 (2-x)^2\,dx = \frac{2}{2}\left[\frac{\pi^2}{2} + \frac{16}{\pi^2} + \frac{16}{81\pi^2} + \ldots\ldots\right]$$

$$\pi^2\left[\frac{x^3}{3}\right]_0^1 - \pi^2\left[\frac{(2-x)^3}{3}\right]_1^2 = \frac{\pi^2}{2} + \frac{16}{\pi^2} + \frac{16}{81\pi^2} + \ldots\ldots$$

$$\frac{\pi^2}{3} - \pi^2\left(0 - \frac{1}{3}\right) = \frac{\pi^2}{2} + \frac{16}{\pi^2}\left[1 + \frac{1}{81} + \ldots\ldots\right]$$

$$\frac{2\pi^2}{3} - \frac{\pi^2}{2} = \frac{16}{\pi^2}\left[1 + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\ldots\right]$$

$$\frac{\pi^2}{6} = \frac{16}{\pi^2}\left[1 + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\ldots\right]$$

$$\frac{\pi^4}{96} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\ldots$$ **Ans.**

Example 22. *Prove that for $0 < x < \pi$*

(a) $x(\pi - x) = \frac{\pi^2}{6} - \left[\frac{\cos 2x}{1^2} + \frac{\cos 4x}{2^2} + \frac{\cos 6x}{3^2} + \ldots\ldots\right]$

(b) $x(\pi - x) = \frac{8}{\pi}\left[\frac{\sin x}{1^2} + \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} + \ldots\ldots\right]$

Deduce from (a) and (b) respectively that

(c) $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$ *(d)* $\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi}{945}$

Solution. Half range cosine series

$$a_0 = \frac{2}{\pi}\int_0^\pi x(\pi - x) = \frac{2}{\pi}\left[\frac{\pi x^2}{2} - \frac{x^3}{3}\right]_0^\pi = \frac{2}{\pi}\left[\frac{\pi^3}{2} - \frac{\pi^3}{3}\right] = \frac{\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^\pi x(\pi - x)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[(\pi x - x^2)\frac{\sin nx}{n} - (\pi - 2x)\left(\frac{-\cos nx}{n^2}\right) + (-2)\left(\frac{-\sin nx}{n^3}\right)\right]_0^\pi$$

$$= \frac{2}{\pi}\left[0 - \frac{\pi(-1)^n}{n^2} + 0 - \frac{\pi}{n^2}\right] = \frac{2}{\pi}\left(\frac{\pi}{n^2}\right)\left[-(-1)^n - 1\right]$$

$$= \frac{-4}{n^2}$$ when n is even

$$= 0$$ when n is odd

Hence, $$x(\pi - x) = \frac{\pi^2}{6} - 4\left[\frac{\cos 2x}{2^2} + \frac{\cos 4x}{4^2} + \ldots\ldots\right]$$

By Parseval's formula

$$\frac{2}{\pi}\int_0^{\pi} x^2(\pi-x)^2\,dx = \frac{a_0^2}{2}+\Sigma\, a_n^2$$

$$\frac{2}{\pi}\int_0^{\pi}(\pi^2x^2-2\pi x^3+x^4)\,dx = \frac{1}{2}\left(\frac{\pi^4}{9}\right)+16\left[\frac{1}{2^4}+\frac{1}{4^4}+\frac{1}{6^4}+\ldots\ldots\right]$$

$$\frac{2}{\pi}\left[\frac{\pi^2x^3}{3}-\frac{2\pi x^4}{4}+\frac{x^5}{5}\right]_0^{\pi} = \frac{\pi^4}{18}+\left[\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\ldots\ldots\right]$$

$$\frac{2}{\pi}\left[\frac{x^5}{3}-\frac{2\pi^5}{4}+\frac{\pi^5}{5}\right] = \frac{\pi^4}{18}+\left[\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\ldots\ldots\right]$$

$$\frac{\pi^4}{15} = \frac{\pi^4}{18}+\sum_{n=1}^{\infty}\frac{1}{n^4} \quad\text{or}\quad \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$$

(*b*) Half range sin series

$$b_n = \frac{2}{\pi}\int_0^{\pi} x(\pi-x)\sin nx\,dx$$

$$= \frac{2}{\pi}\left[(\pi x-x^2)\left(\frac{-\cos nx}{n}\right)-(\pi-2x)\left(\frac{-\sin nx}{n^2}\right)+(-2)\frac{\cos nx}{n^3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-2\frac{(-1)^n}{n^3}+\frac{2}{n^3}\right] = \frac{4}{\pi n^3}\left[-(-1)^n+1\right]$$

$$= \frac{8}{n^3\pi} \qquad \text{when } n \text{ is odd}$$

$$= 0 \qquad \text{when } n \text{ is even}$$

$$\therefore\quad x(\pi-x) = \frac{8}{\pi}\left[\frac{\sin x}{1^3}+\frac{\sin 3x}{3^3}+\frac{\sin 5x}{5^3}+\ldots\ldots\right]$$

By Parseval's formula

$$\frac{2}{\pi}\int_0^{\pi} x^2(\pi-x)^2\,dx = \Sigma\, b_n^2$$

$$\frac{\pi^2}{15} = \frac{64}{\pi^2}\left[\frac{1}{1^6}+\frac{1}{3^6}+\frac{1}{5^6}+\ldots\ldots\right]$$

$$\frac{\pi^4}{960} = \frac{1}{1^6}+\frac{1}{3^6}+\frac{1}{5^6}$$

$$\text{Let } S = \frac{1}{1^6}+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\ldots\ldots = \left(\frac{1}{1^6}+\frac{1}{3^6}+\frac{1}{5^6}+\ldots\ldots\right)+\left(\frac{1}{2^6}+\frac{1}{4^6}+\frac{1}{6^6}+\ldots\ldots\right)$$

$$S = \frac{\pi^4}{960}+\left(\frac{1}{2^6}+\frac{1}{4^6}+\frac{1}{6^6}+\ldots\right) = \frac{\pi^4}{960}+\frac{1}{2^6}\left[\frac{1}{1^6}+\frac{1}{2^6}+\frac{1}{3^6}+\ldots\right]$$

$$S = \frac{\pi^4}{960}+\frac{S}{64}$$

$$S-\frac{S}{64} = \frac{\pi^4}{960} \quad\text{or}\quad \frac{63s}{64} = \frac{\pi^4}{960}$$

$$S = \frac{\pi^4}{960} \times \frac{64}{63} = \frac{\pi^4}{945}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^4}{945}$$ **Proved.**

Exercise 12.5

1. Prove that in $0 < x < c$,

$$x = \frac{c}{2} - \frac{4c}{\pi^2}\left(\cos\frac{\pi x}{c} + \frac{1}{3^2}\cos\frac{3\pi x}{c} + \frac{1}{5^2}\cos\frac{5\pi x}{c} + \ldots\right)$$

and deduce that

(*i*) $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots = \frac{\pi^4}{96}$ (*ii*) $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \ldots = \frac{\pi^4}{90}$

12.12. FOURIER SERIES IN COMPLEX FORM

Fourier series of a function $f(x)$ of period $2l$ is

$$f(x) = \frac{a_0}{2} + a_1\cos\frac{\pi x}{l} + a_2\cos\frac{2\pi x}{l} + \ldots + a_n\cos\frac{n\pi x}{l} + \ldots$$

$$+ b_1\sin\frac{\pi x}{l} + b_2\sin\frac{2\pi x}{l} + \ldots + b_n\sin\frac{n\pi x}{l} + \ldots \quad \ldots (1)$$

We know that $\cos x = \frac{e^{ix} + e^{-ix}}{2}$ and $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$

On putting the value of $\cos x$ and $\sin x$ in (*i*), we get

$$f(x) = \frac{a_0}{2} + a_1\frac{e^{\frac{i\pi x}{l}} + e^{\frac{-i\pi x}{l}}}{2} + a_2\frac{e^{\frac{2i\pi x}{l}} + e^{\frac{-2i\pi x}{l}}}{2} + \ldots + b_1\frac{e^{\frac{i\pi x}{l}} - e^{\frac{i\pi x}{l}}}{2i} + b_2\frac{e^{\frac{2i\pi x}{l}} - e^{\frac{-2i\pi x}{l}}}{2i} + \ldots$$

$$= \frac{a_0}{2} + (a_1 - ib_1)e^{\frac{i\pi x}{l}} + (a_2 - ib_2)e^{\frac{2i\pi x}{l}} + \ldots + (a_1 + ib_1)e^{\frac{-i\pi x}{l}} + (a_2 + ib_2)e^{\frac{-2i\pi x}{l}} + \ldots$$

$$= c_0 + c_1 e^{\frac{i\pi x}{l}} + c_2 e^{\frac{2i\pi x}{l}} + \ldots + c_{-1}e^{\frac{-i\pi x}{l}} + c_{-2}e^{\frac{2i\pi x}{l}} + \ldots$$

$$= c_0 + \sum_{n=1}^{\infty} c_n e^{\frac{in\pi x}{l}} + \sum_{n=1}^{\infty} c_{-n} e^{\frac{-in\pi x}{l}}$$

$$c_n = \frac{1}{2}(a_n - ib_n), \quad c_{-n} = \frac{1}{2}(a_n + ib_n)$$

where $c_0 = \frac{a_0}{2}, = \frac{1}{2}\frac{1}{l}\int_0^{2l} f(x)\,dx$

$$c_n = \frac{1}{2}\left[\frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx - \frac{i}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx\right] c_n = \frac{1}{2}\frac{1}{l}\int_0^{2l} f(x)\left\{\cos\frac{n\pi x}{l} - i\sin\frac{n\pi x}{l}\right\}dx$$

$$c_n = \frac{1}{2l}\int_0^{2l} f(x)\,e^{\frac{-in\pi x}{l}}\,dx$$

$$c_{-n} = \frac{1}{2l}\int_0^{2l} f(x)\,e^{\frac{in\pi x}{l}}\,dx$$

Example 23. *Obtain the complex form of the Fourier series of the function*

$$f(x) = \begin{cases} 0 & -\pi \le x \le 0 \\ 1 & 0 \le x \le \pi \end{cases}$$

13

Laplace Transformation

13.1 INTRODUCTION

Laplace transforms help in solving the differential equations with boundary values without finding the general solution and the values of the arbitrary constants.

13.2 LAPLACE TRANSFORM

Definition. Let $f(t)$ be function defined for all positive values of t, then

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

provided the integral exists, is called the **Laplace Transform** of $f(t)$. It is denoted as

$$L[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

13.3 IMPORTANT FORMULAE

(1) $L(1) = \frac{1}{s}$ (2) $L(t^n) = \frac{n!}{s^{n+1}}$, when $n = 0, 1, 2, 3....$

(3) $L(e^{at}) = \frac{1}{s-a}$ $(s > a)$

(4) $L(\cosh at) = \frac{s}{s^2 - a^2}$ $(s^2 > a^2)$

(5) $L(\sinh at) = \frac{a}{s^2 - a^2}$ $(s^2 > a^2)$

(6) $L \sin(at) = \frac{a}{s^2 + a^2}$ $(s > 0)$

(7) $L(\cos at) = \frac{s}{s^2 + a^2}$ $(s > 0)$

1. $L(1) = \frac{1}{s}$

Proof. $L(1) = \int_0^\infty 1 . e^{-st}\, dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = -\frac{1}{s}\left[\frac{1}{e^{st}}\right]_0^\infty = -\frac{1}{s}[0 - 1] = \frac{1}{s}$

Hence $L(1) = \frac{1}{s}$ **Proved.**

2. $L(t^n) = \frac{n!}{s^{n+1}}$, where n and s are positive.

God, bless me.

Proof. $$L(t^n) = \int_0^\infty e^{-st}\, t^n\, dt$$

Putting $$st = x \quad \text{or} \quad t = \frac{x}{s} \quad \text{or} \quad dt = \frac{dx}{s}$$

Thus, we have, $L(t^n) = \int_0^\infty e^{-x}\left(\frac{x}{s}\right)^n \frac{dx}{s}$ or $L(t^n) = \frac{1}{s^{n+1}}\int_0^\infty e^{-x}\,.\,x^n\, dx$

or $$L(t^n) = \frac{n\,!}{s^{n+1}} \qquad \left[\begin{array}{l} \Gamma(n+1) = \int_0^\infty e^{-x}\,.\,x^n\,dx \\ \text{and} \quad \Gamma(n+1) = n\,! \end{array}\right]$$ **Proved**

3. $L(e^{at}) = \frac{1}{s-a}$ where $s > a$

Proof. $L(e^{at}) = \int_0^\infty e^{-st}\,.\,e^{at}\,dt = \int_0^\infty e^{-st+at}\,.\,dt$

$$= \int_0^\infty e^{(-s+a)t}\,.\,dt = \int_0^\infty e^{-(s-a)t}\,.\,dt = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty = -\frac{1}{s-a}\left[\frac{1}{e^{(s-a)t}}\right]_0^\infty$$

$$= \frac{-1}{(s-a)}(0-1) = \frac{1}{s-a}$$ **Proved**

4. $L(\cosh at) = \frac{s}{s^2-a^2}$

Proof. $$L(\cosh at) = L\left[\frac{e^{at}+e^{-at}}{2}\right] \qquad \left(\because \cosh at = \frac{e^{at}+e^{-at}}{2}\right)$$

$$= \frac{1}{2}L(e^{at}) + \frac{1}{2}L(e^{-at})$$

$$= \frac{1}{2}\left[\frac{1}{s-a}+\frac{1}{s+a}\right] \qquad \left[L(e^{at}) = \frac{1}{s-a}\right]$$

$$= \frac{1}{2}\left[\frac{s+a+s-a}{s^2-a^2}\right] = \frac{s}{s^2-a^2}$$ **Proved.**

5. $L(\sinh at) = \frac{a}{s^2-a^2}$

Proof. $L(\sinh at) = L\left[\frac{1}{2}(e^{at}-e^{-at})\right]$

$$= \frac{1}{2}[L(e^{at}) - L(e^{-at})] = \frac{1}{2}\left[\frac{1}{s-a}-\frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{s+a-s+a}{s^2-a^2}\right]$$

$$= \frac{a}{s^2-a^2}$$ **Proved.**

6. $L(\sin at) = \frac{a}{s^2+a^2}$

Proof. $$L(\sin at) = L\left[\frac{e^{iat}-e^{-iat}}{2i}\right] \qquad \left[\because \sin at = \frac{e^{iat}-e^{-iat}}{2i}\right]$$