# Math 2280 Section 002 [SPRING 2013][1]

Today well learn about a method for solving systems of differential equations, the *method of elimination*, that is very similar to the elimination methods we learned about in linear algebra. Well extend this analogy further by learning about polynomial differential operators, and how we can apply analogues of Cramers rule using these differential operators to solving systems of differential equations.

## 1 The Method of Elimination

For systems of differential equations, particularly linear systems, we can combine equations like we do in linear algebra to eliminate dependent variables. This simplifies the equation, and eventually can be used to re- duce the equation to a single linear differential equations (usually not of first-order) that we can then solve using the methods from chapter 3.

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**Example.** Use the method of elimination to find the solution to initial value problem

$$x' = -3x + 2y, y' = -3x + 4y; y(0) = 1, x(0) = 1$$

Solve for $x$ in the second equation to get

$$x = \frac{1}{3}(4y - y').$$

Differentiating both sides tells us that

$$x' = \frac{1}{3}(4y' - y'').$$

Now replace $x$ and $x'$ in the top equation by these expressions involving $y$

$$\begin{aligned}\frac{1}{3}(4y' - y'') &= -3\frac{1}{3}(4y - y') + 2y\\ 0 &= y'' - y' - 6y\end{aligned}$$

We have successfully "eliminated" $x$. We can solve this DE by finding the roots of the characteristic equation and then writing down the corresponding general solution. In this case, we get

$$y(t) = c_1e^{3t} + c_2e^{-2t}.$$

Using a similar process, we get

$$x(t) = d_1e^{3t} + d_2e^{-2t}.$$

The reason we're getting the same general solution for $x$ and $y$ is that this is a system of homogeneous linear DE's. For a nonhomogeneous system, we shouldn't expect the same general solution. We'll uncover the reason why in the section below. Anyway, let's solve for $c_1, c_2, d_1, d_2$. It looks like we don't have enough initial conditions to solve for these constants. However, we notice that

$$\begin{aligned}x'(0) &= -3x(0) + 2y(0) = -1\\ y'(0) &= -3x(0) + 4y(0) = 1\end{aligned}$$

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Using the initial conditions $x(0) = 1$ and $x'(0) = -1$, we get that

$$\begin{aligned} 1 &= x(0) = d_1 + d_2 \\ -1 &= x'(0) = 3d_1 - 2d_2, \end{aligned}$$

so $d_1 = \frac{1}{5}, d_2 = \frac{4}{5}$. Using the initial conditions $y(0) = 1$ and $y'(0) = 1$, we get

$$\begin{aligned} 1 &= y(0) = c_1 + c_2 \\ 1 &= y'(0) = 3c_1 - 2c_2, \end{aligned}$$

so $c_1 = \frac{3}{5}, c_2 = \frac{2}{5}$. The solution to the initial value problem is

$$y(t) = \frac{3}{5}e^{3t} + \frac{2}{5}e^{-2t}; x(t) = \frac{1}{5}e^{3t} + \frac{4}{5}e^{-2t}.$$

## 2 Polynomial Differential Operators

A *polynomial differential operator* is a map from functions to functions of the form

$$L = a_n D_n + a_{n-1} D_{n-1} + ... + a_1 D + a_0,$$

where $D$ represents the derivative operator.

**Example.** The differential operator

$$L = D^2 + 2D - 3$$

when applied to the function

$$f(x) = x^2 + 4x - 5$$

yields

$$\begin{aligned} L(f) &= D^2(x^2 + 4x - 5) + 2D(x^2 + 4x - 5) - 3(x^2 + 4x - 5) \\ &= (2) + (4x + 8) - (3x2 + 12x - 15) = -3x^2 - 8x + 25. \end{aligned}$$

If $n = 0$, the polynomial differential operator $L = a_0$ is just multiplying by a constant. We'll see that means the methods will learn today are actually generalizations of the methods you learned in linear algebra. In other words, solving systems of linear equations is a special case of solving systems of linear DE's.

Polynomial differential operators commute. So, if we have two differential operators, $L_1$ and $L_2$, then $L_2 L_1(f) = L_1 L_2(f)$. Note that this is not generally the case for all differential operators. Its not even necessarily the case for polynomial operators with variable coefficients. Any system of two linear differential equations with constant coefficients can be written in the form

$$\begin{aligned} L_1 x + L_2 y &= f_1(t) \\ L_3 x + L_4 y &= f_2(t) \end{aligned}$$

If we act on the top row with the operator $L_3$, and on the bottom row with the operator $L_1$ we get

$$\begin{aligned} L_3(L_1x + L_2)y = L_3L_1x + L_3L_2y = L_3f_1(t) \\ L_1(L_3x + L_4)y = L_1L_3x + L_1L_4y = L_1f_2(t). \end{aligned}$$

Because $L_1$ and $L_3$ are linear operators, they behave well with sums of functions. That's why the first equality holds in the top and bottom equation. If we then subtract the first equation from the second, using the fact that the operators commute, we get

$$(L_1L_4 - L_2L_3)y = L_1f_2 - L_3f_1,$$

in the single variable $y$. Alternatively, we could have eliminated $y$ in a like manner from the original system and obtained the equation

$$(L_1L_4 - L_2L_3)x = L_4f_1 - L_2f_2.$$

Note that the same operator, $(L_1L_4 - L_2L_3)$, appears on the left hand side of both equations. This operator is called the *operational determinant*:

$$\begin{vmatrix} L_1 & L_2 \\ L_3 & L_4 \end{vmatrix} = L_1L_4 - L_2L_3,$$

and the above two equalities are the operational versions of Cramers rule:

$$\begin{aligned} \begin{vmatrix} L_1 & L_2 \\ L_3 & L_4 \end{vmatrix} x = \begin{vmatrix} f_1 & L_2 \\ f_2 & L_4 \end{vmatrix}, \\ \begin{vmatrix} L_1 & L_2 \\ L_3 & L_4 \end{vmatrix} y = \begin{vmatrix} L_1 & f_1 \\ L_3 & f_2 \end{vmatrix}. \end{aligned}$$

We're using the notation for determinants, but don't forget that $L_1$, $L_2$, $L_3$, $L_4$ are differential operators. This is just notation that we've adopted to emphasize the parallels between solving systems of linear equations and solving systems of differential equations.

When $f_1$ and $f_2$ are both zero (in other words, are DE's are homogeneous), notice that we get the same DE for $x$ and $y$.

If the operational determinant is identically zero, there can be either no solutions or infinitely many solutions to the ODE. This is called the *degenerate* case. Let me clarify what I mean by "infinitely many solutions." What we're used to seeing is a DE with a general solution that represents infinitely many solutions, but there is a unique solution once we specify some initial conditions. But in this degenerate case, I can choose *any* differentiable function I want for $x_1$, and there will be an $x_2$ such that the pair $x_1, x_2$ solve the given system of differential equations.

One final note–this approach that we've applied here for systems with two variables generalizes to systems with an arbitrary numbers of variables, although the computations involved for Cramers rule become more and more nasty.

Here's a couple of examples to illustrate this technique:

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**Example.** Show that the following system is degenerate, and then determine whether it has infinitely many or no solutions.

$$\begin{aligned}(D+2)x + (D+2)y &= e^{-3t}\\ (D+3)x + (D+3)y &= e^{-2t}\end{aligned}$$

In this case, $L_1 = L_2 = D+2$, $L_3 = L_4 = D+3$. If there are infinitely many solutions, then the operational determinant will be 0 and the equations below will still hold:

$$\begin{vmatrix} L_1 & L_2 \\ L_3 & L_4 \end{vmatrix} x = \begin{vmatrix} f_1 & L_2 \\ f_2 & L_4 \end{vmatrix},$$
$$\begin{vmatrix} L_1 & L_2 \\ L_3 & L_4 \end{vmatrix} y = \begin{vmatrix} L_1 & f_1 \\ L_3 & f_2 \end{vmatrix}.$$

If there are *no* solutions, then the operational determinant will be 0 but the above equations will not hold (we'll get that 0 is equal to something nonzero). Here, the operational determinant is

$$\begin{vmatrix} L_1 & L_2 \\ L_3 & L_4 \end{vmatrix} = L_1L_4 - L_2L_3 = (D+2)(D+3) - (D+2)(D+3) = 0.$$

Notice that

$$\begin{vmatrix} f_1 & L_2 \\ f_2 & L_4 \end{vmatrix} = (D+3)e^{-3t} - (D+2)e^{-2t} = 0$$

and that

$$\begin{vmatrix} L_1 & f_1 \\ L_3 & f_2 \end{vmatrix} = (D+2)e^{-2t} - (D+3)e^{-3t} = 0.$$

That means we have infinitely many solutions. In this instance, I can pick $x$ to be ANY differential function of $t$ I want and then the pair

$$x(t), y(t) = -x(t) - e^{-3t} + e^{-2t}$$

will be a solution to the system of differential equations. (Do you see why?)

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**Example.** Find the general solution to the system of differential equations:

$$\begin{aligned}(D^2+D)x + D^2y &= 2e^t\\ (D^2-D-1)x + (D^2-D)y &= 0.\end{aligned}$$

$L_1 = D^2 + D$, $L_2 = D^2$, $L_3 = D^2 - D - 1$, and $L_4 = D^2 - D$. That means the operational determinant is

$$(D^2+D)(D^2-D) - (D^2)(D^2-D-1) = D^4 - D^2 - D^4 + D^3 + D^2 = D^3.$$

We have that

$$\begin{vmatrix} 2e^t & L_2 \\ 0 & L_4 \end{vmatrix} = (D^2-D)2e^t = 2e^t - 2e^t = 0$$

and that

$$\begin{vmatrix} L_1 & 2e^t \\ L_3 & 0 \end{vmatrix} = -(D^2 - D - 1)2e^t = 2e^t.$$

Combining this calculations give us the system

$$\begin{aligned} D^3 x &= 0 \\ D^3 y &= 2e^t, \end{aligned}$$

which we can solve using methods we've learned before. Check that you get the general solutions

$$\begin{aligned} x(t) &= c_1 t^2 + c_2 t + c_3 \\ y(t) &= d_1 t^2 + d_2 t + d_3 + 2e^t. \end{aligned}$$

We're not done yet; there are restrictions on what $c_1, c_2, c_3, d_1, d_2, d_3$ can be. By plugging $x$ and $y$ into our original system of DE's we can get relations on the $c_i$ and $d_i$.

$$2e^t = (D^2 + D)x + D^2 y = [(2c_1) + (2c_1 t + c_2)] + (2d_1 + 2e^t),$$

so $2c_1 + 2d_1 + c_2 = 0$ and $2c_1 = 0$. Furthermore,

$$0 = (D^2 - D - 1)x + (D^2 - D)y = [(2c_1) - (2c_1 t + c_2) - (c_1 t^2 + c_2 t + c_3)] + [(2d_1) - (2d_1 t + d_2)],$$

which tells you that $2c_1 - c_2 - c_3 + 2d_1 - d_2 = 0$ (and also the same identities as the previous equation). In other words,

$$\begin{aligned} x(t) &= c_2 t + c_3 \\ y(t) &= -\frac{1}{2} c_2 t^2 + (-2c_2 + c_3)t + d_3 + 2e^t. \end{aligned}$$

I would need some initial conditions to solve for $c_2$, $c_3$, and $d_3$. Notice that the solution space is a vector space of dimension 3. (Why?)