

The University of Zambia
Department of Mathematics and Statistics
Mat 3110-Engineering Mathematics II

Assignment 1- Submit by Friday, 19th March, 2021.

Find a power series solution to the following ordinary differential equations in powers of x .

1. $(1 - x^2)y'' - 2xy' + 2y = 0$.
2. $y'' + (1 + x^2)y = 0$.
3. $y'' - 4xy' + (4x^2 - 2)y = 0$.

Solutions

1. Let $y = \sum_{n=0}^{\infty} a_n x^n$, then $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} (n-1) n a_n x^{n-2}$. Plugging these into the equation, we get

$$\begin{aligned}
 (1-x^2) \sum_{n=2}^{\infty} (n-1)na_nx^{n-2} - 2x \sum_{n=1}^{\infty} na_nx^{n-1} + 2 \sum_{n=0}^{\infty} a_nx^n &= 0 \\
 \sum_{n=2}^{\infty} (n-1)na_nx^{n-2} - \sum_{n=2}^{\infty} (n-1)na_nx^n - \sum_{n=1}^{\infty} 2na_nx^n + \sum_{n=0}^{\infty} 2a_nx^n &= 0 \\
 \sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2}x^n - \sum_{n=2}^{\infty} (n-1)na_nx^n - \sum_{n=1}^{\infty} 2na_nx^n + \sum_{n=0}^{\infty} 2a_nx^n &= 0 \\
 2a_2 + 6a_3x + \sum_{n=2}^{\infty} (n+1)(n+2)a_{n+2}x^n - \sum_{n=2}^{\infty} (n-1)na_nx^n - 2a_1x - \sum_{n=2}^{\infty} 2na_nx^n \\
 + 2a_0 + 2a_1x + \sum_{n=2}^{\infty} 2a_nx^n &= 0 \\
 2(a_2 + a_0) + 6a_3 + \sum_{n=2}^{\infty} [(n+1)(n+2)a_{n+2} - (n(n-1) + 2n + 2)a_n] x^n &= 0 \\
 2(a_2 + a_0) + 6a_3 + \sum_{n=2}^{\infty} [(n+1)(n+2)a_{n+2} - (n^2 - n + 2)a_n] x^n &= 0 \\
 2(a_2 + a_0) + 6a_3 + \sum_{n=2}^{\infty} [(n+1)(n+2)a_{n+2} - (n-1)(n+2)a_n] x^n &= 0
 \end{aligned}$$

From the above equation, we get the following.

$a_2 = -a_0$, $a_3 = 0$ and

$$a_{n+2} = \frac{n-1}{n+1}a_n.$$

Thus,

$$\begin{aligned}
 n = 2, \quad a_4 &= \frac{a_2}{3} = -\frac{a_0}{3} \\
 n = 3, \quad a_5 &= \frac{2a_3}{4} = 0 \\
 n = 4, \quad a_6 &= \frac{3a_4}{5} = -\frac{a_0}{5} \\
 n = 5, \quad a_7 &= 0 \\
 n = 6, \quad a_8 &= \frac{5a_6}{7} = -\frac{a_0}{7}
 \end{aligned}$$

Hence,

$$y = a_0 + a_1x - a_0x^2 - \frac{a_0}{3}x^4 - \frac{a_0}{5}x^6 + \dots$$

2. Let $y = \sum_{n=0}^{\infty} a_n x^n$, then $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} (n-1) n a_n x^{n-2}$ Plugging these into the equation, we get

$$\begin{aligned} \sum_{n=2}^{\infty} (n-1) n a_n x^{n-2} + (1+x^2) \sum_{n=0}^{\infty} a_n x^n &= 0 \\ \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} a_n x^{n+2} &= 0 \\ \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n &= 0 \\ 2a_2 + 6a_3 x + \sum_{n=2}^{\infty} (n+1)(n+2) a_{n+2} x^n + a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n &= 0 \\ (2a_2 + a_0) + (6a_3 + a_1) x + \sum_{n=2}^{\infty} [(n+1)(n+2) a_{n+2} + a_n + a_{n-2}] x^n &= 0 \end{aligned}$$

We get the following.

$$a_2 = \frac{-a_0}{2}, \quad a_3 = \frac{-a_1}{3!} \text{ and}$$

$$a_{n+2} = \frac{-a_n - a_{n-2}}{(n+2)(n+1)}.$$

Thus,

$$\begin{aligned} n=2, \quad a_4 &= \frac{-a_2 - a_0}{4 \times 3} = \frac{\frac{a_0}{2} - a_0}{4 \times 3} = -\frac{a_0}{4!} \\ n=3, \quad a_5 &= \frac{-a_3 - a_1}{5 \times 4} = \frac{\frac{a_1}{3!} - a_1}{5 \times 4} = -\frac{5a_1}{5!} \\ n=4, \quad a_6 &= \frac{-a_4 - a_2}{6 \times 5} = \frac{\frac{a_0}{4!} + \frac{a_0}{2}}{6 \times 5} = \frac{13a_0}{4!} \\ n=5, \quad a_7 &= \frac{-a_5 - a_3}{7 \times 6} = \frac{\frac{5a_1}{5!} + \frac{a_1}{3!}}{7 \times 6} = \frac{25a_0}{7!} \end{aligned}$$

Hence,

$$y = a_0 + a_1 x + \frac{a_0}{2} x^2 - \frac{a_1}{3!} x^3 - \frac{a_0}{4!} x^4 - \frac{5a_1}{5!} x^5 + \frac{13a_0}{6!} x^6 + \dots$$

3. Let $y = \sum_{n=0}^{\infty} a_n x^n$, then $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} (n-1) n a_n x^{n-2}$ Plugging these into the equation, we get

$$\begin{aligned} & \sum_{n=2}^{\infty} (n-1)na_nx^{n-2} - 4x \sum_{n=1}^{\infty} na_nx^{n-1} + (4x^2 - 2) \sum_{n=0}^{\infty} a_nx^n = 0 \\ & \sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2}x^n - \sum_{n=1}^{\infty} 4na_nx^n + \sum_{n=0}^{\infty} 4a_nx^{n+2} - \sum_{n=0}^{\infty} 2a_nx^n = 0 \\ & 2a_2 + 6a_3x + \sum_{n=2}^{\infty} (n+1)(n+2)a_{n+2}x^n - 4a_1x - \sum_{n=2}^{\infty} 4na_nx^n + \sum_{n=2}^{\infty} 4a_{n-2}x^{n-2} - 2a_0 - 2a_1x + \sum_{n=2}^{\infty} 2a_nx^2 \\ & (2a_2 + a_0) + (6a_3 - 4a_1 - 2a_1)x + \sum_{n=2}^{\infty} [(n+1)(n+2)a_{n+2} - 4na_n + 4a_{n-2} - 2a_n]x^n = 0 \end{aligned}$$

We get the following.

$a_2 = a_0$, $a_3 = a_1$ and

$$a_{n+2} = \frac{(4n+2)a_n - 4a_{n-2}}{(n+2)(n+1)}.$$

Thus,