The University of Zambia
Department of Mathematics and Statistics
Mat 3110-Engineering Mathematics II

Assignment 1- Submit by Friday, 19th March, 2021.

Find a power series solution to the following ordinary differential equations in powers of x.

1. $(1-x^2)y'' - 2xy' + 2y = 0$.
2. $y'' + (1+x^2)y = 0$.
3. $y'' - 4xy' + (4x^2-2)y = 0$.

Solutions

1. Let $y = \sum_{n=0}^{\infty} a_n x^n$, then $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} (n-1)n a_n x^{n-2}$ Plugging these into the equation, we get

$$(1-x^2)\sum_{n=2}^{\infty}(n-1)na_nx^{n-2} - 2x\sum_{n=1}^{\infty}na_nx^{n-1} + 2\sum_{n=0}^{\infty}a_nx^n = 0$$

$$\sum_{n=2}^{\infty}(n-1)na_nx^{n-2} - \sum_{n=2}^{\infty}(n-1)na_nx^n - \sum_{n=1}^{\infty}2na_nx^n + \sum_{n=0}^{\infty}2a_nx^n = 0$$

$$\sum_{n=2}^{\infty}(n+1)(n+2)a_{n+2}x^n - \sum_{n=2}^{\infty}(n-1)na_nx^n - \sum_{n=1}^{\infty}2na_nx^n + \sum_{n=0}^{\infty}2a_nx^n = 0$$

$$2a_2 + 6a_3x + \sum_{n=2}^{\infty}(n+1)(n+2)a_{n+2}x^n - \sum_{n=2}^{\infty}(n-1)na_nx^n - 2_ax - \sum_{n=2}^{\infty}2na_nx^n$$

$$+\,2a_0 + 2a_1x + \sum_{n=2}^{\infty}2a_nx^n = 0$$

$$2(a_2-a_0) + 6a_3 + \sum_{n=2}^{\infty}\left[(n+1)(n+2)a_{n+2} - (n(n-1)+2n+2)\,a_n\right]x^n = 0$$

$$2(a_2-a_0) + 6a_3 + \sum_{n=2}^{\infty}\left[(n+1)(n+2)a_{n+2} - \left(n^2-n+2\right)a_n\right]x^n = 0$$

$$2(a_2-a_0) + 6a_3 + \sum_{n=2}^{\infty}\left[(n+1)(n+2)a_{n+2} - (n-1)(n+2)a_n\right]x^n = 0$$

From the above equation, we get the following.

$a_2 = a_0$, $a_3 = 0$ and

$$a_{n+2} = \frac{n-1}{n+1}a_n.$$

Thus,

$$n=2,\quad a_4 = \frac{a_2}{3} = \frac{a_0}{3}$$

$$n=3,\quad a_5 = \frac{2a_3}{4} = 0$$

$$n=4,\quad a_6 = \frac{3a_4}{5} = \frac{a_0}{5}$$

$$a_7 = 0$$

$$n=6,\quad a_8 = \frac{5a_6}{7} = \frac{a_0}{7}$$

Hence,

$$y = a_0 + a_1x + a_0x^2 + \frac{a_0}{3}x^4 + \frac{a_0}{5}x^6 + \cdots$$

2. Let $y = \sum_{n=0}^{\infty} a_n x^n$, then $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} (n-1) n a_n x^{n-2}$ Plugging these into the equation, we get

$$\sum_{n=2}^{\infty}(n-1)na_nx^{n-2} + (1+x^2)\sum_{n=0}^{\infty} a_nx^n = 0$$
$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n + \sum_{n=0}^{\infty} a_nx^n + \sum_{n=0}^{\infty} a_nx^{n+2} = 0$$
$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n + \sum_{n=0}^{\infty} a_nx^n + \sum_{n=2}^{\infty} a_{n-2}x^n = 0$$
$$2a_2 + 6a_3x + \sum_{n=2}^{\infty}(n+1)(n+2)a_{n+2}x^n + a_0 + a_1x + \sum_{n=2}^{\infty} a_nx^n + \sum_{n=2}^{\infty} a_{n-2}x^n = 0$$
$$(2a_2 + a_0) + (6a_3 + a_1)x \sum_{n=2}^{\infty} \left[(n+1)(n+2)a_{n+2} + a_n + a_{n-2}\right] x^n = 0$$

We get the following.

$a_2 = \frac{-a_0}{2}$, $a_3 = \frac{-a_1}{3!}$ and

$$a_{n+2} = \frac{-a_n - a_{n-2}}{(n+2)(n+1)}.$$

Thus,

$$n=2, \quad a_4 = \frac{-a_2 - a_0}{4 \times 3} = \frac{\frac{a_0}{2} - a_0}{4 \times 3} = -\frac{a_0}{4!}$$
$$n=3, \quad a_5 = \frac{-a_3 - a_1}{5 \times 4} = \frac{\frac{a_1}{3!} - a_1}{5 \times 4} = -\frac{5a_1}{5!}$$
$$n=4, \quad a_6 = \frac{-a_4 - a_2}{6 \times 5} = \frac{\frac{a_0}{4!} + \frac{a_0}{2}}{6 \times 5} = \frac{13a_0}{4!}$$
$$n=5, \quad a_7 = \frac{-a_5 - a_3}{7 \times 6} = \frac{\frac{5a_1}{5!} + \frac{a_1}{3!}}{7 \times 6} = \frac{25a_0}{7!}$$

Hence,

$$y = a_0 + a_1x + -\frac{a_0}{2}x^2 - \frac{a_1}{3!}x^3 - \frac{a_0}{4!}x^4 - \frac{5a_1}{5!}x^5 + \frac{13a_0}{6!}x^6 + \cdots$$

3. Let $y = \sum_{n=0}^{\infty} a_n x^n$, then $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} (n-1) n a_n x^{n-2}$ Plugging these into the equation, we get

$$\sum_{n=2}^{\infty}(n-1)na_nx^{n-2} - 4x\sum_{n=1}^{\infty}na_nx^{n-1} + (4x^2-2)\sum_{n=0}^{\infty}a_nx^n = 0$$

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n - \sum_{n=1}^{\infty}4na_nx^n + \sum_{n=0}^{\infty}4a_nx^{n+2} - \sum_{n=0}^{\infty}2a_nx^n = 0$$

$$2a_2 + 6a_3x + \sum_{n=2}^{\infty}(n+1)(n+2)a_{n+2}x^n - 4a_1x - \sum_{n=2}^{\infty}4na_nx^n + \sum_{n=2}^{\infty}4a_{n-2}x^{n-2} - 2a_0 - 2a_1x + \sum_{n=2}^{\infty}2a_n$$

$$(2a_2+a_0) + (6a_3-4a_1-2a_1)x\sum_{n=2}^{\infty}\left[(n+1)(n+2)a_{n+2} - 4na_n + 4a_{n-2} - 2a_n\right]x^n = 0$$

We get the following.

$a_2 = a_0$, $a_3 = a_1$ and

$$a_{n+2} = \frac{(4n+2)a_n - 4a_{n-2}}{(n+2)(n+1)}.$$

Thus,