

PART 4 ELECTROMAGNETISM

CHAPTER 23 ELECTRIC CHARGE, FORCE, AND FIELD

Problem

1. Suppose the electron and proton charges differed by one part in one billion. Estimate the net charge you would carry.

Solution

Nearly all of the mass of an atom is in its nucleus, and about one half of the nuclear mass of the light elements in living matter (H, O, N, and C) is protons. Thus, the number of protons in a 65 kg average-sized person is approximately $\frac{1}{2}(65 \text{ kg})/(1.67 \times 10^{-27} \text{ kg}) \approx 2 \times 10^{28}$, which is also the number of electrons, since an average person is electrically neutral. If there were a charge imbalance of $|q_{\text{proton}} - q_{\text{electron}}| = 10^{-9} e$, a person's net charge would be about $\pm 2 \times 10^{28} \times 10^{-9} \times 1.6 \times 10^{-19} \text{ C} = \pm 3.2 \text{ C}$, or several coulombs (huge by ordinary standards).

Problem

2. A typical lightning flash delivers about 25 C of negative charge from cloud to ground. How many electrons are involved?

Solution

The number is $Q/e = 25 \text{ C}/1.6 \times 10^{-19} \text{ C} = 1.56 \times 10^{20}$.

Problem

5. If the charge imbalance of Problem 1 existed, what would be the approximate force between you and another person 10 m away? Treat the people as point charges, and compare the answer with your weight.

Solution

The magnitude of the Coulomb force between two point charges of 3.2 C (see solution to Problem 1), at a distance of 10 m, is $kq^2/r^2 = (9 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2)(3.2 \text{ C}/10 \text{ m})^2 = 9.22 \times 10^8 \text{ N}$. This is approximately 1.45 million times the weight of an average-sized 65 kg person.

Problem

8. How far apart should an electron and proton be so the force of Earth's gravity on the electron is equal to the electric force arising from the proton? Your answer shows why gravity is unimportant on the molecular scale!

Solution

The electric force between a proton and an electron has magnitude ke^2/r^2 , while the weight of an electron is $m_e g$. These are equal when

$$r = \sqrt{ke^2/m_e g} = \sqrt{\frac{(9 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2)(1.6 \times 10^{-19} \text{ C})^2}{(9.11 \times 10^{-31} \text{ kg})(9.8 \text{ m/s}^2)}} = 5.08 \text{ m}$$

(almost fifty billion atomic diameters).

Problem

9. Two charges, one twice as large as the other, are located 15 cm apart and experience a repulsive force of 95 N. What is the magnitude of the larger charge?

Solution

The product of the charges is $q_1q_2 = r^2 F_{\text{Coulomb}}/k = (0.15 \text{ m})^2(95 \text{ N})/(9 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2) = 2.38 \times 10^{-10} \text{ C}^2$. If one charge is twice the other, $q_1 = 2q_2$, then $\frac{1}{2}q_1^2 = 2.38 \times 10^{-10} \text{ C}^2$ and $q_1 = \pm 21.8 \mu\text{C}$.

Problem

11. A proton is on the x -axis at $x = 1.6 \text{ nm}$. An electron is on the y -axis at $y = 0.85 \text{ nm}$. Find the net force the two exert on a helium nucleus (charge $+2e$) at the origin.

Solution

A unit vector from the proton's position to the origin is $-\hat{i}$, so the Coulomb force of the proton on the helium nucleus is $\mathbf{F}_{\text{p,He}} = k(e)(2e)(-\hat{i})/(1.6 \text{ nm})^2 = -0.180\hat{i} \text{ nN}$. (Use Equation 23-1, with q_1 for the proton, q_2 for the helium nucleus, and the approximate values of k and e given.) A unit vector from the electron's position to the origin is $-\hat{j}$, so its force on the helium nucleus is $\mathbf{F}_{\text{e,He}} = k(-e)(2e)(-\hat{j})/(0.85 \text{ nm})^2 = 0.638\hat{j} \text{ nN}$. The net Coulomb force on the helium nucleus is the sum of these. (The vector form of Coulomb's law and superposition, as explained in the solution to Problems 15 and 19, provides a more general approach.)

Problem

16. A charge $3q$ is at the origin, and a charge $-2q$ is on the positive x -axis at $x = a$. Where would you place a third charge so it would experience no net electric force?

Solution

The reasoning of Example 23-3 implies that for the force on a third charge Q to be zero, it must be placed on the x -axis to the right of the (smaller) negative charge, i.e., at $x > a$. The net Coulomb force on a third charge so placed is $F_x = kQ[3qx^{-2} - 2q(x-a)^{-2}]$, so $F_x = 0$ implies that $3(x-a)^2 = 2x^2$, or $x^2 - 6xa + 3a^2 = 0$. Thus, $x = 3a \pm \sqrt{9a^2 - 3a^2} = (3 \pm \sqrt{6})a$. Only the solution $(3 + \sqrt{6})a = 5.45a$ is to the right of $x = a$.

Problem

19. In Fig. 23-39 take $q_1 = 68 \mu\text{C}$, $q_2 = -34 \mu\text{C}$, and $q_3 = 15 \mu\text{C}$. Find the electric force on q_3 .

Solution

Denote the positions of the charges by $\mathbf{r}_1 = \hat{j}$, $\mathbf{r}_2 = 2\hat{i}$, and $\mathbf{r}_3 = 2\hat{i} + 2\hat{j}$ (distances in meters). The vector form of Coulomb's law (in the solution to Problem 15) and the superposition principle give the net electric force on q_3 as:

$$\begin{aligned} \mathbf{F}_3 &= \mathbf{F}_{13} + \mathbf{F}_{23} = \frac{kq_1q_3(\mathbf{r}_3 - \mathbf{r}_1)}{|\mathbf{r}_3 - \mathbf{r}_1|^3} + \frac{kq_2q_3(\mathbf{r}_3 - \mathbf{r}_2)}{|\mathbf{r}_3 - \mathbf{r}_2|^3} = (9 \times 10^9 \text{ N})(15 \times 10^{-6})[(68 \times 10^{-6})(2\hat{i} + \hat{j})/5\sqrt{5} + (-34 \times 10^{-6})2\hat{j}/8] \\ &= (1.64\hat{i} - 0.326\hat{j}) \text{ N}, \end{aligned}$$

or $F_3 = \sqrt{F_{3x}^2 + F_{3y}^2} = 1.67 \text{ N}$ at an angle of $\theta = \tan^{-1}(F_{3y}/F_{3x}) = -11.2^\circ$ to the x -axis.

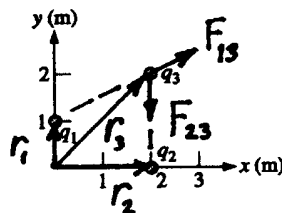


FIGURE 23-39 Problem 19 Solution.

Problem

22. Three identical charges $+q$ and a fourth charge $-q$ form a square of side a . (a) Find the magnitude of the electric force on a charge Q placed at the center of the square. (b) Describe the direction of this force.

Solution

The magnitudes of the forces on Q from each of the four charges are equal to $kqQ/(\sqrt{2}a/2)^2 = 2kqQ/a^2$. But the forces from the two positive charges on the same diagonal are in opposite directions, and cancel, while the forces from the positive and negative charges on the other diagonal are in the same direction (depending on the sign of Q) and add. Thus, the net force on Q has magnitude $2(2kqQ/a^2)$ and is directed toward (or away from) the negative charge for $Q > 0$ (or $Q < 0$).

Problem

24. Two identical small metal spheres initially carry charges q_1 and q_2 , respectively. When they're 1.0 m apart they experience a 2.5-N attractive force. Then they're brought together so charge moves from one to the other until they have the same net charge. They're again placed 1.0 m apart, and now they repel with a 2.5-N force. What were the original values of q_1 and q_2 ?

Solution

The charges initially attract, so q_1 and q_2 have opposite signs, and $2.5 \text{ N} = -kq_1q_2/1 \text{ m}^2$. When the spheres are brought together, they share the total charge equally, each acquiring $\frac{1}{2}(q_1 + q_2)$. The magnitude of their repulsion is $2.5 \text{ N} = k\frac{1}{4}(q_1 + q_2)^2/1 \text{ m}^2$. Equating these two forces, we find a quadratic equation $\frac{1}{4}(q_1 + q_2)^2 = -q_1q_2$, or $q_1^2 + 6q_1q_2 + q_2^2 = 0$, with solutions $q_1 = (-3 \pm \sqrt{8})q_2$. Both solutions are possible, but since $3 + \sqrt{8} = (3 - \sqrt{8})^{-1}$, they merely represent a relabeling of the charges. Since $-q_1q_2 = 2.5 \text{ N} \cdot \text{m}^2/(9 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2) = (16.7 \mu\text{C})^2$, the solutions are $q_1 = \pm\sqrt{3 + \sqrt{8}}(16.7 \mu\text{C}) = \pm 40.2 \mu\text{C}$ and $q_2 = \mp 40.2 \mu\text{C} = (3 + \sqrt{8})^{-1}(\pm 40.2 \mu\text{C}) = \mp 6.90 \mu\text{C}$, or the same values with q_1 and q_2 interchanged.

Problem

30. A $65\text{-}\mu\text{C}$ point charge is at the origin. Find the electric field at the points (a) $x = 50 \text{ cm}$, $y = 0$; (b) $x = 50 \text{ cm}$, $y = 50 \text{ cm}$; (c) $x = -25 \text{ cm}$, $y = 75 \text{ cm}$.

Solution

The electric field from a point charge at the origin is $\mathbf{E}(\mathbf{r}) = kq\hat{\mathbf{r}}/r^2 = kq\mathbf{r}/r^3$, since $\hat{\mathbf{r}} = \mathbf{r}/r$. (a) For $\mathbf{r} = 0.5\hat{\mathbf{i}} \text{ m}$ and $q = 65 \mu\text{C}$, $\mathbf{E} = (9 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2)(65 \mu\text{C})\hat{\mathbf{i}}/(0.5 \text{ m})^2 = 2.34\hat{\mathbf{i}} \text{ MN/C}$. (b) At $\mathbf{r} = 0.5 \text{ m}(\hat{\mathbf{i}} + \hat{\mathbf{j}})$, $\mathbf{E} = (9 \times 65 \times 10^3 \text{ N} \cdot \text{m}^2/\text{C})(0.5 \text{ m})(\hat{\mathbf{i}} + \hat{\mathbf{j}})/(0.5\sqrt{2} \text{ m})^3 = (827 \text{ kN/C})(\hat{\mathbf{i}} + \hat{\mathbf{j}})$. (The field strength is 1.17 MN/C at 45° to the x axis.) (c) When $\mathbf{r} = (-0.25\hat{\mathbf{i}} + 0.75\hat{\mathbf{j}}) \text{ m}$, $\mathbf{E} = (5.85 \times 10^5 \text{ N} \cdot \text{m}^2/\text{C})(-0.25\hat{\mathbf{i}} + 0.75\hat{\mathbf{j}}) \text{ m}/[(-0.25)^2 + (0.75)^2]^{3/2} \text{ m}^3 = (-296\hat{\mathbf{i}} + 888\hat{\mathbf{j}}) \text{ kN/C}$ ($|\mathbf{E}| = 936 \text{ kN/C}$, $\theta_x = 108^\circ$).

Problem

31. In Fig. 23-40, point P is midway between the two charges. Find the electric field in the plane of the page (a) 5.0 cm directly above P , (b) 5.0 cm directly to the right of P , and (c) at P .

Solution

Take the origin of x - y coordinates at the midpoint, as indicated, and use Equation 23-5. Let $\mathbf{r}_\pm = \pm(2.5 \text{ cm})\hat{\mathbf{j}}$ denote the positions of the charges, and \mathbf{r} that of the field point. A unit vector from one charge to the field point is $(\mathbf{r} - \mathbf{r}_\pm)/|\mathbf{r} - \mathbf{r}_\pm|$,

so the spacial factors in Coulomb's law are $\hat{\mathbf{r}}_i = \frac{\mathbf{r}_i}{r_i} = \frac{\mathbf{r} - \mathbf{r}_\pm}{|\mathbf{r} - \mathbf{r}_\pm|}$. (a) For $\mathbf{r} = (5.0 \text{ cm})\hat{\mathbf{j}}$, $\mathbf{r}_1 = \mathbf{r} - \mathbf{r}_+ = (5.0 \text{ cm})\hat{\mathbf{j}} - (2.5 \text{ cm})\hat{\mathbf{j}} = (2.5 \text{ cm})\hat{\mathbf{j}}$, and $\mathbf{r}_2 = \mathbf{r}_- = \mathbf{r} - \mathbf{r}_- = (7.5 \text{ cm})\hat{\mathbf{j}}$. Then

$$\mathbf{E} = k \left(\frac{q_1 \mathbf{r}_1}{r_1^3} + \frac{q_2 \mathbf{r}_2}{r_2^3} \right) = \left(9 \times 10^9 \frac{\text{N} \cdot \text{m}^2}{\text{C}^2} \right) (2 \mu\text{C}) \left[\frac{\hat{\mathbf{j}}}{(2.5 \text{ cm})^2} - \frac{\hat{\mathbf{j}}}{(7.5 \text{ cm})^2} \right] = (25.6 \text{ MN/C})\hat{\mathbf{j}}.$$

(b) For $\mathbf{r} = (5.0 \text{ cm})\hat{\mathbf{i}}$,

$$\mathbf{E} = \left(9 \times 10^9 \frac{\text{N} \cdot \text{m}^2}{\text{C}^2} \right) \left(\frac{2 \mu\text{C}}{\text{cm}^2} \right) \left[\frac{(5.0\hat{\mathbf{i}} - 2.5\hat{\mathbf{j}})}{(5.0^2 + (-2.5)^2)^{3/2}} - \frac{(5.0\hat{\mathbf{i}} + 2.5\hat{\mathbf{j}})}{(5.0^2 + 2.5^2)^{3/2}} \right] = -(5.15 \text{ MN/C})\hat{\mathbf{j}}.$$

(c) For $\mathbf{r} = 0$,

$$\mathbf{E} = \left(9 \times 10^9 \frac{\text{N} \cdot \text{m}^2}{\text{C}^2} \right) \left(\frac{2 \mu\text{C}}{\text{cm}^2} \right) \left[\frac{-\hat{\mathbf{j}}}{(2.5)^2} - \frac{\hat{\mathbf{j}}}{(2.5)^2} \right] = -(57.6 \text{ MN/C})\hat{\mathbf{j}}.$$

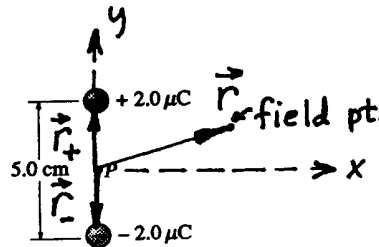


FIGURE 23-40 Problem 31 Solution.

Problem

32. A $1.0\text{-}\mu\text{C}$ charge and a $2.0\text{-}\mu\text{C}$ charge are 10 cm apart, as shown in Fig. 23-41. Find a point where the electric field is zero.

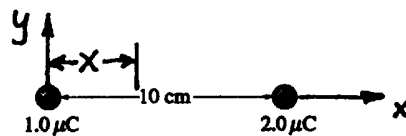


FIGURE 23-41 Problem 32 Solution.

Solution

The field can be zero only along the line joining the charges (the x -axis). To the left or right of both charges, the fields due to each are in the same direction, and cannot add to zero. Between the two, a distance $x > 0$ from the $1 \mu\text{C}$ charge, the electric field is $\mathbf{E} = k[q_1 \hat{\mathbf{i}}x^{-2} + q_2(-\hat{\mathbf{i}})(10 \text{ cm} - x)^{-2}]$, which vanishes when $1 \mu\text{C}x^{-2} = 2 \mu\text{C}(10 \text{ cm} - x)^{-2}$, or $x = 10 \text{ cm}(\sqrt{2} + 1) = 4.14 \text{ cm}$.

Problem

39. The dipole moment of the water molecule is $6.2 \times 10^{-30} \text{ C} \cdot \text{m}$. What would be the separation distance if the molecule consisted of charges $\pm e$? (The effective charge is actually less because electrons are shared by the oxygen and hydrogen atoms.)

Solution

The distance separating the charges of a dipole is $d = p/q = 6.2 \times 10^{-30} \text{ C} \cdot \text{m} / 1.6 \times 10^{-19} \text{ C} = 38.8 \text{ pm}$.

Problem

46. Two identical rods of length ℓ lie on the x -axis and carry uniform charges $\pm Q$, as shown in Fig. 23-43. (a) Find an expression for the electric field strength as a function of position x for points to the right of the right-hand rod. (b) Show that your result has the $1/x^3$ dependence of a dipole field for $x \gg \ell$. (c) What is the dipole moment of this configuration? *Hint:* See Equation 23-7b.

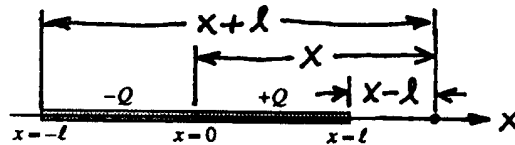


FIGURE 23-43 Problem 46 Solution.

Solution

- (a) The field due to each rod, for a point on their common axis, can be obtained from Example 23-7: $E_+ = kQ/x(x - \ell)$, to the right, and $E_- = kQ/x(x + \ell)$, to the left. The resultant field (positive right) is

$$E = E_+ - E_- = \frac{kQ}{x} \left(\frac{1}{x - \ell} - \frac{1}{x + \ell} \right) = \frac{2kQ\ell}{x(x^2 - \ell^2)}.$$

- (b) For $x \gg \ell$, $E \approx 2kQ\ell/x^3$. (c) Comparison with Equation 23-7b shows that the rods appear like a dipole with moment $p = Q\ell$.

Problem

48. Figure 23-44 shows a thin, uniformly charged disk of radius R . Imagine the disk divided into rings of varying radii r , as suggested in the figure. (a) Show that the area of such a ring is very nearly $2\pi r dr$. (b) If the surface charge density on the disk is $\sigma \text{ C/m}^2$, use the result of (a) to write an expression for the charge dq on an infinitesimal ring. (c) Use the result of (b) along with the result of Example 23-8 to write the infinitesimal electric field dE of this ring at a point on the disk axis, taken to be the positive x axis. (d) Integrate over all such rings (that is, from $r = 0$ to $r = R$), to show that the net electric field on the disk axis is

$$E = 2\pi k\sigma \left(1 - \frac{x}{\sqrt{x^2 + R^2}} \right).$$

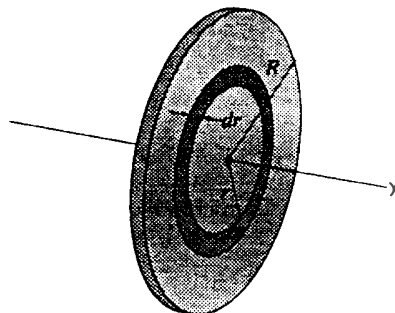


FIGURE 23-44 Problem 48.

Solution

(a) The area of an annulus of radii $R_1 < R_2$ is just $\pi(R_2^2 - R_1^2)$. For a thin ring, $R_1 = r$ and $R_2 = r + dr$, so the area is $\pi[(r + dr)^2 - r^2] = \pi(2r dr + dr^2)$. When dr is very small, the square term is negligible, and $dA = 2\pi r dr$. (This is equal to the circumference of the ring times its thickness.) (b) For surface charge density σ , $dq = \sigma dA = 2\pi\sigma r dr$. (c) From Example 23-8, $dE_x = k(dq)x(x^2 + r^2)^{-3/2} = 2\pi k\sigma xr(x^2 + r^2)^{-3/2} dr$, which holds for x positive away from the ring's center. (d) Integrating from $r = 0$ to R , one finds $E_x = \int_0^R dE_x$, or

$$E_x = 2\pi k\sigma x \int_0^R \frac{r dr}{(x^2 + r^2)^{3/2}} = 2\pi k\sigma x \left[\frac{-1}{\sqrt{x^2 + r^2}} \right]_0^R = 2\pi k\sigma \left[\frac{x}{|x|} - \frac{x}{(x^2 + R^2)^{1/2}} \right].$$

(Note: For $x > 0$, $|x| = x$ and the field is $E_x = 2\pi k\sigma[1 - x(x^2 + R^2)^{-1/2}]$. However, for $x < 0$, $|x| = -x$ and $E_x = 2\pi k\sigma[-1 + |x|(x^2 + R^2)^{-1/2}]$. This is consistent with symmetry on the axis, since $E_x(x) = -E_x(-x)$.)

Problem

50. A semicircular loop of radius a carries positive charge Q distributed uniformly over its length. Find the electric field at the center of the loop (point P in Fig. 23-45). *Hint:* Divide the loop into charge elements dq as shown in Fig. 23-45, and write dq in terms of the angle $d\theta$. Then integrate over θ to get the net field at P .

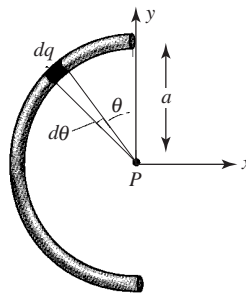


FIGURE 23-45 Problem 50 Solution.

Solution

This problem is the same as Problem 73, with $\theta_0 = 0$. Thus, $\mathbf{E}(P) = 2kQ\hat{\mathbf{i}}/\pi a^2$.

Problem

63. What is the line charge density on a long wire if a $6.8\text{-}\mu\text{g}$ particle carrying 2.1 nC describes a circular orbit about the wire with speed 280 m/s ?

Solution

The solution to Problem 61 reveals that $\lambda = -mv^2/2kq = -(6.8 \times 10^{-9}\text{ kg})(280\text{ m/s})^2/2(9 \times 10^9\text{ N}\cdot\text{m}^2/\text{C}^2)(2.1 \times 10^{-9}\text{ C}) = -14.1\text{ }\mu\text{C/m}$. (In this case, the force on a positively charged orbiting particle is attractive for a wire with negative linear charge density.)

Problem

65. A dipole with dipole moment $1.5\text{ nC}\cdot\text{m}$ is oriented at 30° to a 4.0-MN/C electric field. (a) What is the magnitude of the torque on the dipole? (b) How much work is required to rotate the dipole until it's antiparallel to the field?

Solution

(a) The torque on an electric dipole in an external electric field is given by Equation 23-11; $\tau = |\mathbf{p} \times \mathbf{E}| = pE \sin \theta = (1.5 \text{ nC} \cdot \text{m})(4.0 \text{ MN/C}) \sin 30^\circ = 3.0 \text{ mN} \cdot \text{m}$. (b) The work done against just the electric force is equal to the change in the dipole's potential energy (Equation 23-12); $W = \Delta U = (-\mathbf{p} \cdot \mathbf{E})_f - (-\mathbf{p} \cdot \mathbf{E})_i = pE(\cos 30^\circ - \cos 180^\circ) = (1.5 \text{ nC} \cdot \text{m}) \times (4.0 \text{ MN/C})(1.866) = 11.2 \text{ mJ}$.

Problem

67. Two identical dipoles, each of charge q and separation a , are a distance x apart as shown in Fig. 23-49. By considering forces between pairs of charges in the different dipoles, calculate the net force between the dipoles. (a) Show that, in the limit $a \ll x$, the force has magnitude $6kp^2/x^4$, where $p = qa$ is the dipole moment. (b) Is the force attractive or repulsive?

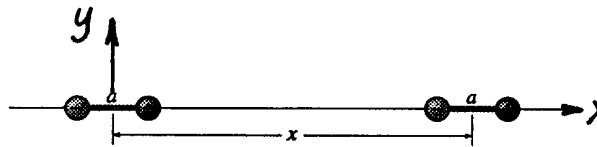


FIGURE 23-49 Problem 67 Solution.

Solution

All the forces are along the same line, so take the origin at the center of the left-hand dipole and the positive x axis in the direction of the right-hand dipole in Fig. 23-49. The right-hand dipole has charges $+q$ at $x + a/2$, $-q$ at $x - a/2$, each of which experiences a force from both charges of the left-hand dipole, which are $+q$ at $a/2$ and $-q$ at $-a/2$. (There are forces between four pairs of charges.) The Coulomb force on a charge in the right-hand dipole, due to one in the left-hand one, is $kq_r q_l / (x_r - x_l)^2 \hat{\mathbf{i}}$ (see solution to Problem 15), so the total force on the right-hand dipole is

$$F_x = kq^2 \hat{\mathbf{i}} \left[\frac{1}{x^2} - \frac{1}{(x+a)^2} - \frac{1}{(x-a)^2} + \frac{1}{x^2} \right] = -\frac{2kq^2 a^2 (3x^2 - a^2)}{x^2 (x^2 - a^2)^2} \hat{\mathbf{i}}.$$

(a) In the limit $a \ll x$, $F_x \rightarrow -2kq^2 a^2 (3x^2) / x^6 = -6kq^2 a^2 / x^4 = -6kp^2 / x^4$, where $p = qa$ is the dipole moment of both dipoles. (b) The force on the right-hand dipole is in the negative x direction, indicating an attractive force.

Problem

78. Two small spheres with the same mass m and charge q are suspended from massless strings of length ℓ , as shown in Fig. 23-52. Each string makes an angle θ with the vertical. Show that the charge on each sphere is $q = \pm 2\ell \sin \theta \sqrt{mg \tan \theta / k}$.

Solution

The magnitudes of the forces on either sphere (drawn acting on the righthand one in Fig. 23-52) are $F_{\text{grav}} = mg$, $F_{\text{elec}} = kq^2 / (2\ell \sin \theta)^2$, and T (the unknown string tension). In equilibrium, $\sum F_x = \sum F_y = 0$, or $0 = F_{\text{elec}} - T \sin \theta$ and $0 = T \cos \theta - mg$. Eliminating T , we obtain $\tan \theta = F_{\text{elec}} / mg = kq^2 / mg(2\ell \sin \theta)^2$, or $q = \pm 2\ell \sin \theta \sqrt{mg \tan \theta / k}$.

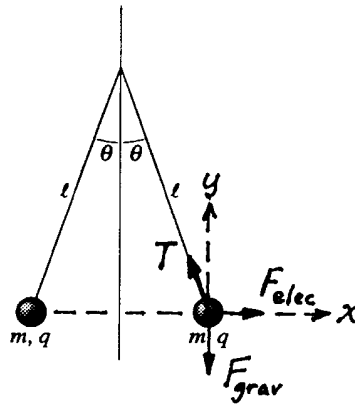


FIGURE 23-52 Problem 78 Solution.

Problem

82. A rod of length \$2\ell\$ lies on the \$x\$-axis, centered on the origin. It carries a line charge density given by \$\lambda = \lambda_0(x/\ell)\$, where \$\lambda_0\$ is a constant. (a) What is the net charge on the rod? (b) Find an expression for electric field strength at all points \$x > \ell\$. (c) Show that your result has the \$1/x^3\$ dependence of dipole field when \$x \gg \ell\$. *Hint:* For \$\ell \ll x\$, \$\ln(x - \ell)/(x + \ell)\$ becomes approximately \$-2\ell/x - 2\ell^3/3x^3\$. (d) By comparing with Equation 23-7b, determine the dipole moment of the rod.

Solution

(a) \$Q = \int_{-\ell}^{\ell} \lambda \, dx = \int_{-\ell}^{\ell} (\lambda_0 x/\ell) \, dx = (\lambda_0/\ell) \frac{1}{2} x^2 \Big|_{-\ell}^{\ell} = 0\$. (b) An element of charge, \$dq = \lambda \, dx'\$, produces an electric field \$d\mathbf{E} = k \, dq \hat{\mathbf{r}}/(x - x')^2\$ at position \$x > \ell\$ on the \$x\$-axis, where we are using \$x'\$ for the variable of integration. Thus,

$$\begin{aligned} \mathbf{E}(x) &= \int_{-\ell}^{\ell} \left(k \frac{\lambda_0}{\ell} \right) \hat{\mathbf{i}} \frac{x' \, dx'}{(x - x')^2} = \left(\frac{k\lambda_0}{\ell} \hat{\mathbf{i}} \right) \int_{-\ell}^{\ell} dx' \left[\frac{x}{(x - x')^2} - \frac{1}{(x - x')} \right] \\ &= \left(\frac{k\lambda_0}{\ell} \right) \hat{\mathbf{i}} \left[\frac{x}{x - \ell} - \frac{x}{x + \ell} + \ln \left(\frac{x - \ell}{x + \ell} \right) \right] = k\lambda_0 \hat{\mathbf{i}} \left[\frac{2x}{x^2 - \ell^2} + \frac{1}{\ell} \ln \left(\frac{x - \ell}{x + \ell} \right) \right]. \end{aligned}$$

(c) For \$\ell \ll x\$,

$$\frac{2x}{x^2 - \ell^2} = \frac{2}{x} \left(1 - \frac{\ell^2}{x^2} \right)^{-1} = \frac{2}{x} \left(1 + \frac{\ell^2}{x^2} + \dots \right),$$

while

$$\begin{aligned} \frac{1}{\ell} \ln \left(\frac{x - \ell}{x + \ell} \right) &= \frac{1}{\ell} \left[\ln \left(1 - \frac{\ell}{x} \right) - \ln \left(1 + \frac{\ell}{x} \right) \right] = \frac{1}{\ell} \left[-\frac{\ell}{x} - \frac{1}{2} \frac{\ell^2}{x^2} - \frac{1}{3} \frac{\ell^3}{x^3} + \dots - \left(\frac{\ell}{x} - \frac{1}{2} \frac{\ell^2}{x^2} + \frac{1}{3} \frac{\ell^3}{x^3} + \dots \right) \right] \\ &= -\frac{2}{x} - \frac{2\ell^2}{3x^3} + \dots \end{aligned}$$

Thus,

$$\mathbf{E}(x) = k\lambda_0 \hat{\mathbf{i}} \left[\frac{2}{x} + \frac{2\ell^2}{x^3} + \dots - \frac{2}{x} - \frac{2\ell^2}{3x^3} + \dots \right], \quad \frac{4k\lambda_0 \ell^2 \hat{\mathbf{i}}}{3x^3}$$

(d) On the axis of a point dipole, Equation 23-7b gives $\mathbf{E}(x) = 2kp\hat{\mathbf{i}}/x^3$, so comparison reveals that $p = 2\lambda_0 \ell^2/3$. (Note: Explicit use of the formula $p = \int_{-\ell}^{\ell} x dq = \int_{-\ell}^{\ell} \lambda_0 x^2 dx$ gives the same result.)