EE 3112

STATICS
&
DYNAMICS

COURSE NOTES

COMPILED BY M.O. GOMA

TABLE OF CONTENTS

LIST OF FIGURES

INTRODUCTION

This part of the course is meant to provide the student with additional knowledge in statics and to introduce concepts of dynamics which are relevant to electrical engineering.

A STATICS

A.1 Deflections of Beams with Concentrated and Distributed Loads

A.1.1 Introduction

In mechanics, the loading at any section of a simply-supported beam or cantilever can be resolved into a bending moment and a shearing force. The stresses due to these bending moments and shearing forces can be evaluated in a number of ways. Another aspect of the problem of bending is the calculation of the stiffness of a beam.

For practical use, a beam should not only be strong (enough) for its purpose, it must also have the requisite strength i.e. it should not deflect from its original position by more than a certain amount.

This section considers methods of finding the deflected form of a beam under a given system of external loads and having known conditions of support.

There are certain types of beams, such as those carried by more than two supports and beams with ends held in such a way that they must keep their original directions, for which bending moments and shear forces cannot be determined without knowing the deformations of the axis of the beam. Such problems are known as 'statically indeterminate'.

A.1.2 Strain energy due to bending

Consider a short length of beam δx, under the action of a bending moment M. if σ is the bending stress on an element of the x-section of area δA at a distance y from the neutral axis, the strain energy (work done by load in straining it) of the length δx is given by

$$\delta U = \int \frac{\sigma^2}{2E} \times volume \qquad \text{i.e. } \delta U = \frac{1}{2} Px, but, P = \sigma A, x = \frac{\sigma L}{E}$$

$$= \delta x \int \frac{\sigma^2}{2E} \cdot dA$$

$$= \frac{\delta x}{2E} \int \frac{M^2 y^2}{I^2} \cdot dA$$

But, $\int y^2 \cdot dA = I$

Hence $\delta U = \left(\frac{M^2}{2EI} \right) \cdot \delta x$

From the whole beam:

$$\delta U = \int \frac{M^2 \cdot dx}{2EI} \qquad \text{……………….(1.1)}$$

where EI = flexural rigidity (stiffness) of beam.

The strain energy due to bending can be equated to the work done by the load from which the deflection under the load can be obtained.

Example 1.1

A simply supported beam of length L carries a concentrated load W at distances a and b from the two ends. Find expressions for the total strain energy and the deflection under the load.

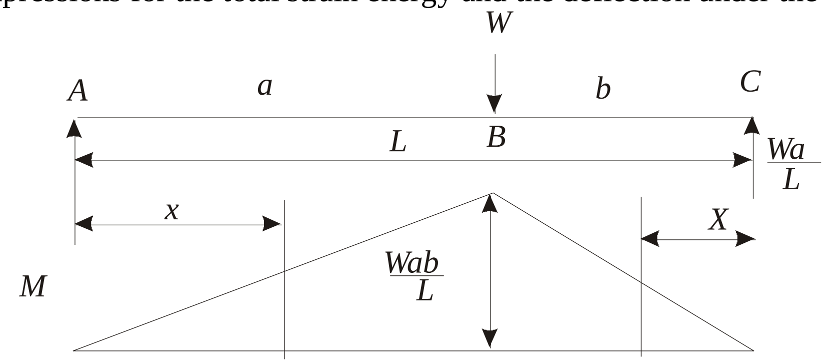


Figure A.1.1 Simply supported beam

Integrate for strain energy only over a length of beam for which a continuous expression for M can be obtained. In this case use separate integration for each section on either side of the concentrated load.

Thus for section A – B

$$M = \left(\frac{Wb}{L}\right)x$$

$$U_a = \int_0^a \frac{W^2 b^2 x^2}{2L^2 EI} \cdot dx = \frac{W^2 b^2}{2L^2 EI}\left[\frac{x^3}{3}\right]_0^a = \frac{W^2 a^3 b^2}{6L^2 EI}$$

Similarly for section B –C

$$U_b = \int_0^b \frac{W^2 a^2 x^2}{2L^2 EI} \cdot dx = \frac{W^2 a^2 b^3}{6EIL^2}$$

Total

$$U = U_a + U_b = \frac{W^2 a^2 b^2}{6EIL^2}(a+b)$$

$$U = \frac{W^2 a^2 b^2}{6EIL}$$

But, if δ is the deflection under the load, the strain energy must equal the work done by the load: i.e.

$$\frac{1}{2}W\delta = \frac{W^2 a^2 b^2}{6EIL} \Rightarrow \delta = \frac{Wa^2 b^2}{6EIL}$$

For a central load, $a = b = L/2$, and $\delta = \frac{W}{3EIL}\left(\frac{L^2}{4}\right)\cdot\left(\frac{L^2}{4}\right) \Rightarrow \delta = \frac{WL^3}{48EI}$

Note: This method of finding deflection is limited to cases where only one concentrated load is applied i.e. doing work, and then only gives the deflection under the load.

A.1.3 Elastic Bending of straight beams

A.1.3.1 Calculus Method

For bending about a principal axis of a straight beam of uniform x-section,

$$\frac{M}{EI} = \frac{1}{R} \qquad \text{………………….(1.2)}$$

Where R = radius of the circular arc which the beam assumes.
M = applied end couples M about a principal axis.
I = second moment of area about relevant principal axis.

Where a beam is subjected to shearing forces as well as bending moments, the axis of the beam is no longer bent into a circular arc but instead varies from section to section (as the bending varies) in accordance with equation (1.2).

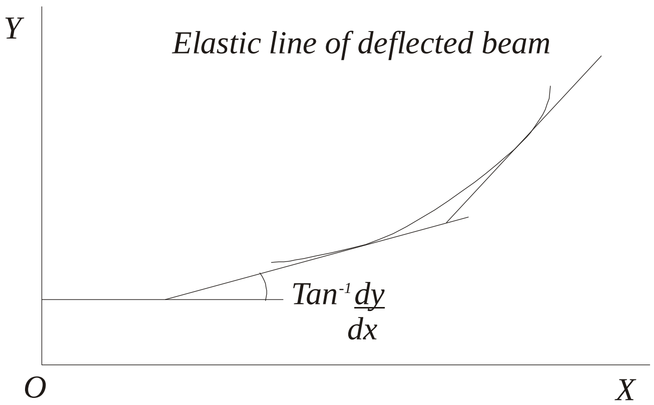


Figure A.1.2 Elastic line of a deflected beam

Hence, $$\frac{M}{EI} = \frac{1}{R} = \frac{\pm\frac{d^2y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}};$$ the sign depends on the convention for axes.

Now for beams met in engineering practice, the deflection y is small, thus the slope $\frac{dy}{dx}$ is small, and $\left(\frac{dy}{dx}\right)^2$ is negligible compared to unity.

Taking y +ve upwards, $\frac{d^2y}{dx^2}$ is +ve and

$$\frac{1}{R} = \frac{d^2y}{dx^2}$$

hence $\frac{M}{EI} = \frac{d^2y}{dx^2}$

or $EI \cdot \frac{d^2y}{dx^2} = M$ ……………………….(1.3)

Provided M can be expressed as a function of x, equation (1.3) can be integrated to give the slope $\frac{dy}{dx}$, and the deflection y, of the beam for any value of x.

- 2 constants of integration, evaluated from known values of slope or deflection at particular points.

Resulting expression gives a mathematical form for the deflected beam, or 'elastic line'.
Differentiating (1.3),

$EI \cdot \frac{d^3y}{dx^3} = \frac{dM}{dx} = F$ ……………………….(1.4), Where F is the shearing force

and $EI \cdot \frac{d^4y}{dx^4} = \frac{dF}{dx} = -w$ ……………………….(1.5), Where w is the intensity of vertical loading (i.e. the mean rate of loading) at any section.

Notes on Application

1. Take the X axis through the level of the supports.
2. Take the origin at one end, or at a point of zero slope.
3. For a built-in or fixed end, or where the deflection is a maximum, slope $\frac{dy}{dx} = 0$
4. For points on the X-axis, usually supports, the deflection y=0.

Example 1.2

Obtain expressions for the maximum slope and deflection of a cantilever of length L carrying

(a) a concentrated load W at its free end
(b) a uniformly distributed load w along its whole length.

(a)

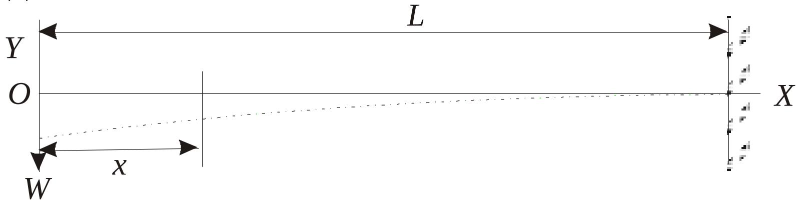


Figure A.1.3:Cantilever carrying a concentrated end load

At a distance x form the origin,

M = -Wx

And $EI \cdot \frac{d^2 y}{dx^2} = M = -Wx$ from equation (1.3)

Integrating $EI \cdot \frac{dy}{dx} = -\frac{Wx^2}{2} + A$

Apply Boundary conditions (BC's), $\frac{dy}{dx} = 0$ at $x = L$, hence $A = \frac{WL^2}{2}$

Integrating again $EIy = -\frac{Wx^3}{6} + \frac{WL^2 x}{2} + B$

Apply BC's: $y = 0$ at $y = l$

$$\therefore B = \frac{WL^3}{6} - \frac{WL^3}{2} = -\frac{WL^3}{3}$$

Hence Slope (Max) $= \frac{dy}{dx} = \frac{WL^2}{2EI}$ (i.e. at x=0)

$$y = -\frac{WL^3}{3EI}$$ (also at x=0)

If W was at a distance 'a' from O, find the slope and deflection of 'a' and deflection at L

$$y_a = \frac{Wa^3}{3EI}, \quad y_a = \frac{Wa^2}{2EI}, \quad y_L = \frac{Wa^2}{6EI}(3L - a)$$

(b)

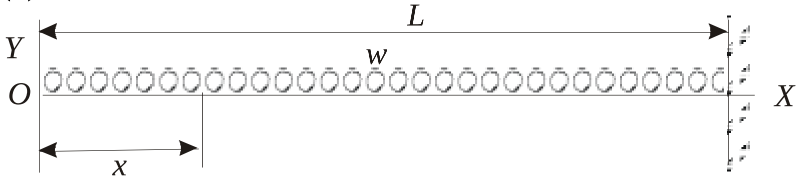


Figure A.1.4: Cantilever carrying a distributed load over its entire length

$$EI \cdot \frac{d^2 y}{dx^2} = M = -\frac{wx^2}{2}$$

Integrating $EI \cdot \frac{dy}{dx} = -\frac{wx^3}{6} + A$

Apply Boundary conditions (BC's), $\frac{dy}{dx} = 0$ at $x = L$, hence $A = \frac{wL^3}{6}$

Integrating again $EIy = -\frac{wx^4}{24} + \frac{wL^3 x}{6} + B$

Apply BC's: $y = 0$ at $y = l$

$$\therefore B = \frac{wL^4}{24} - \frac{wL^4}{6} = -\frac{wL^4}{8}$$

Hence Slope (Max) $= \frac{dy}{dx} = \frac{wL^3}{6EI}$ (i.e. at x=0)

$$y = -\frac{wL^4}{8EI}$$ (also at x=0)

A.1.3.2 MACAULAY'S METHOD

The Calculus method normally requires obtaining a separate expression for bending moment for each section of the beam between adjacent concentrated loads or reactions, each producing a different equation with its own constants of integrations. But for the simplest cases, the work will be laborious.

The method devised by Macaulay enables one continuous expression for bending moment to be obtained, and provided certain rules are followed, the constants of integration will be the same for all sections of the beam.

Concentrated loads

Step 1: Measuring x form one end, write down an expression for the bending moment in the last section of the beam, enclosing all distances less than x in square brackets

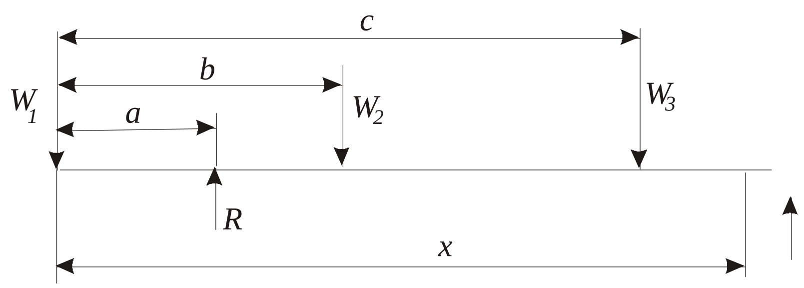


Figure A.1.5 Macaulay's method, Concentrated loading of beam

i.e. $$EI \cdot \frac{d^2 y}{dx^2} = M = -W_1 x + R[x-a] - W_2[x-b] - W_3[x-c]$$

Step2: Omit non-relevant. The expression in terms of the bending moment for all values of x subject to the condition that <u>all terms for which the quantity inside the square brackets is negative are omitted</u>.

i.e. If x is less than c the last term is omitted, if x is less than b then both the last two terms are omitted, and so on.

Step3: Integrate the brackets as a whole.

i.e. $$EI \cdot \frac{dy}{dx} = -W_1 \frac{x^2}{2} + \left(\frac{R}{2}\right)[x-a]^2 - \left(\frac{W_2}{2}\right)[x-b]^2 - \left(\frac{W_3}{2}\right)[x-c]^2 + A$$

and $$EIy = -W_1 \frac{x^3}{6} + \left(\frac{R}{6}\right)[x-a]^3 - \left(\frac{W_2}{6}\right)[x-b]^3 - \left(\frac{W_3}{6}\right)[x-c]^3 + Ax + B$$

Note: It can be shown that the constants of integration are common to all sections of the beam.

i.e. if $x = b - \Delta$

then $$EI \cdot \frac{dy}{dx} = -\frac{W_1}{2}(b-\Delta)^2 + \left(\frac{R}{2}\right)[b-\Delta-a]^2 + A$$

and $$EIy = -\frac{W_1}{6}(b-\Delta)^3 + \left(\frac{R}{6}\right)[b-\Delta-a]^3 + A(b-\Delta) + B$$

if $x = b + \Delta$

then $$EI \cdot \frac{dy}{dx} = -\frac{W_1}{2}(b+\Delta)^2 + \left(\frac{R}{2}\right)[b+\Delta-a]^2 - \frac{W_2}{2}\Delta^2 + A'$$

and $$EIy = -\frac{W_1}{6}(b+\Delta)^3 + \left(\frac{R}{6}\right)[b+\Delta-a]^3 - \frac{W_2}{6}\Delta^3 + A'(b+\Delta) + B'$$

As $\Delta \to 0$ these slope and deflection values must correspond (i.e. at $x = b$), from which it is seen that *A=A'* and *B=B'*. The values of A & B are found using BC's.

Uniformly Distributed loads

Consider the following

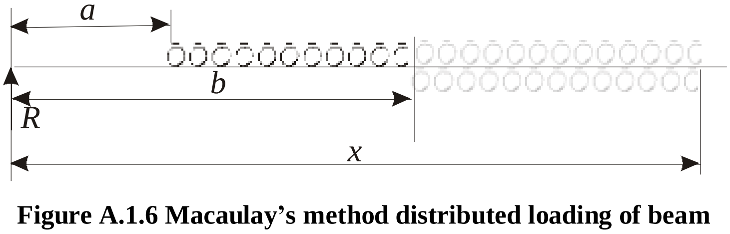


Figure A.1.6 Macaulay's method distributed loading of beam

In order to obtain an expression for the bending moment at a distance *x* from the end, which will apply for all values of *x*, it is necessary to continue the loading up to the section *x*, compensating with an equal –ve load from *b* to *x*, i.e.

$$M = Rx - \left(\frac{w}{2}\right)[x-a]^2 - \left(\frac{w_2}{2}\right)[x-b]^2$$

for x > a but < b, omit [x – b], and $M = Rx - \left(\frac{w}{2}\right)[x-a]^2$

Repeat the remaining steps of integration and constant enumeration as before.

Concentrated Bending Moment

Consider the following:

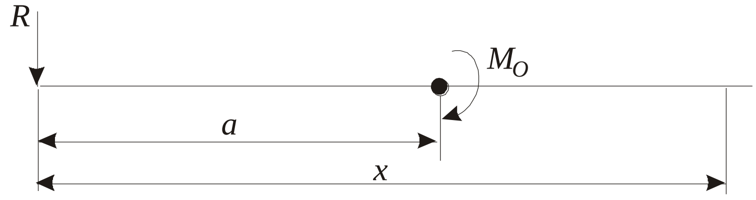


Figure A.1.7 Concentrated bending moment loading

$$EI \cdot \frac{d^2y}{dx^2} = M = -Rx + M_0[x-a]^0, \quad \text{then} \quad EI \cdot \frac{dy}{dx} = -R\frac{x^2}{2} + M_0[x-a] + A, \text{ etc}$$

Example 1.3

A simply supported beam of length L carries a load W at a distance a from one end, b from the other (a > b). Find the position and magnitude of the maximum deflection and show that its position is always with L/13, approximately of the centre.

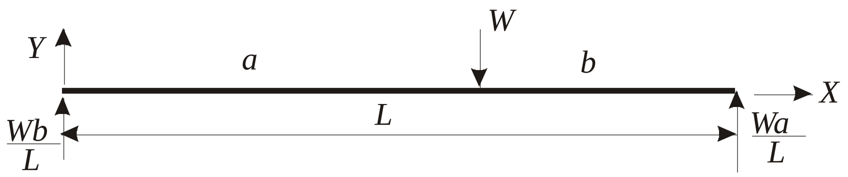


Figure A.1.8 Simply supported beam with concentrated load

The maximum deflection will occur on the length a since a > b.

$$EI \cdot \frac{d^2y}{dx^2} = M = -\left(\frac{Wb}{L}\right)x - W[x-a]$$

$$EI \cdot \frac{dy}{dx} = -\left(\frac{Wb}{L}\right)\frac{x^2}{2} - \left(\frac{W}{2}\right)[x-a]^2 + A \qquad \text{……….(i)}$$

$$EIy = -\left(\frac{Wb}{L}\right)\frac{x^3}{6} - \left(\frac{W}{6}\right)[x-a]^3 + Ax + B \qquad \text{……….(ii)}$$

Boundary conditions: At x = 0, y = 0 $\quad \therefore B = 0$

At x = L, y = 0 $\qquad \therefore AL = \left(\frac{Wb}{L}\right)\cdot\left(\frac{L^3}{6}\right) + \left(\frac{W}{6}\right)\cdot b^3$

$$\text{Giving } \therefore A = -\left(\frac{Wb}{6L}\right)\cdot\left(L^2 - b^2\right)$$

$\frac{dy}{dx} = 0$ at a value of x given by $\left(\frac{Wb}{6L}\right)\cdot\left(\frac{x^2}{2}\right) - \left(\frac{Wb}{6L}\right)\cdot\left(L^2 - b^2\right) = 0$, from (i), omitting [x – a] since x < a for zero slope when a > b.

This gives $x = \sqrt{\left[\frac{\left(L^2 - b^2\right)}{3}\right]}$ at the point of maximum deflection. Substituting in (ii) to find the value of the maximum deflection:

$$EIy = \left(\frac{Wb}{L}\right)\cdot\frac{\left(L^2 - b^2\right)^{\frac{3}{2}}}{6\times 3\sqrt{3}} - \left(\frac{Wb}{6L}\right)\cdot\frac{\left(L^2 - b^2\right)^{\frac{3}{2}}}{\sqrt{3}}, \text{ giving } y = \frac{Wb\left(L^2 - b^2\right)^{\frac{3}{2}}}{9\sqrt{3}EIL}$$

Distance of point of maximum deflection from centre = $\sqrt{\frac{\left(L^2 - b^2\right)}{3}} - \frac{L}{2}$

Which has a maximum value of $\frac{L}{\sqrt{3}} - \frac{L}{2}$, or approximately $\frac{L}{13}$

A.1.3.3 Moment – Area Method

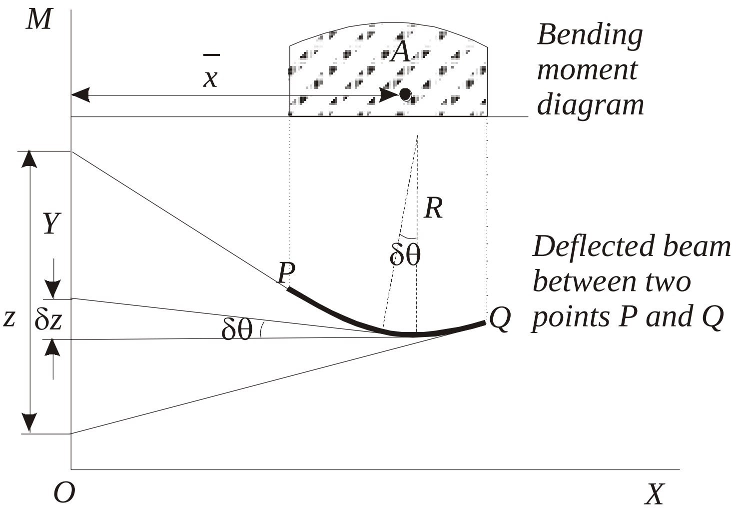


Figure A.1.9 Moment – area method

A – Area of BMD

$\bar{X}$ - Centroid of BMD from a chosen line OY Tangents at P and Q to the elastic line cut off an intercept z on OY.

$$\frac{d^2y}{dx^2} = \frac{M}{EI}$$

Integrating between P and Q

$$\left[\frac{dy}{dx}\right]_P^Q = \int \frac{Mdx}{EI}$$

If EI is constant

$$\left(\frac{dy}{dx}\right)_Q - \left(\frac{dy}{dx}\right)_P = \frac{A}{EI} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.6)$$

i.e. The increase of slope between any two points on a beam is equal to the net area of the BMD between those points divided by EI

If R is the radius of curvature of the beam at some point between P and Q, then the angle between the tangents at the ends of a short length δx is $\delta\theta$, where $\delta x = R \cdot \delta\theta$. The intercept of these tangents on OY is δz, and since the slope is small everywhere,

$$\delta z \approx x\delta\theta = \frac{x\delta x}{R} = \frac{Mx\delta x}{EI}$$

integrating, $z = \int \frac{Mx}{EI} \cdot \delta x = \frac{A\bar{x}}{EI}$ ……………………(1.7), if E is constant

i.e. The intercept on a given line between the tangents on the beam at any two points P and Q is equal to the net moment about that line of the BMD between P & Q divided by EI.

NOTE:

1. Account must be taken of +ve and –ve areas
2. It is frequently convenient to break down the BMD into a number of simple figures so that the moment is obtained from $\sum A\bar{x}$.
3. The intercept z is +ve when the tangent at Q strikes OY below the tangent at P
4. this method is particularly applicable where it produces a quicker solution than the mathematical solution i.e. generally those for which a point of zero slope is known. If this point is chosen as Q and OY is taken through P, then (1.6) reduces to slope at $P = -\frac{A}{EI}$ and (1.7) to deflection of P relative to $Q = \frac{A\bar{x}}{EI}$

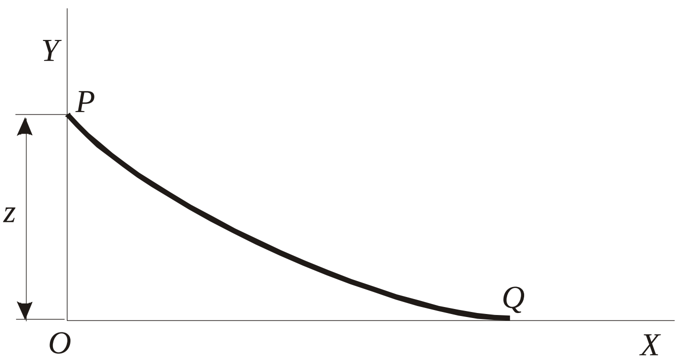


Figure A.1.10 Slope variation

i.e. the deflection at any point can be found by working between there and a poin t of zero slope and taking moments about the point where the deflection is required.

5. In applying this method, it is helpful to sketch the approximate shape of the deflected beam and by drawing the tangents at chosen points, it should be clear which intercept gives the relative deflection.
6. This method is advantageous for:
   (i) most cantilever problems
   (ii) symmetrically loaded simply supported beams (zero slope at the centre)
   (iii)built-in beams (zero slope at each end)
7. For uniformly distributed loads, the BMD is a parabola and the following properties of area and centroids hold.

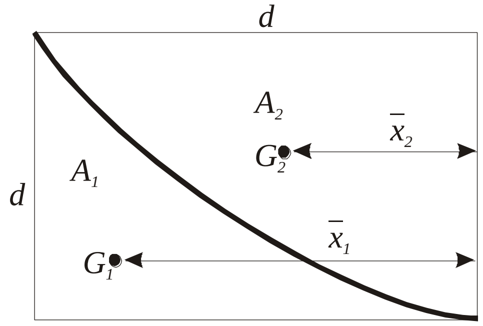


Figure A.1.11 Area under Bending moment diagram

- Bd is the surrounding rectangle
- Parabola is tangential to base

Then

$$A_1 = -\frac{1}{3}bd, \overline{x_1} = \frac{3}{4}b$$

$$A_2 = -\frac{2}{3}bd, \overline{x_2} = \frac{3}{8}b$$

Example 1.4

Obtain expression for the maximum slope and deflection of a simply supported beam of span L:
(a) with a concentrated load W at mid-span.
(b) With a uniformly distributed load w over the whole span.

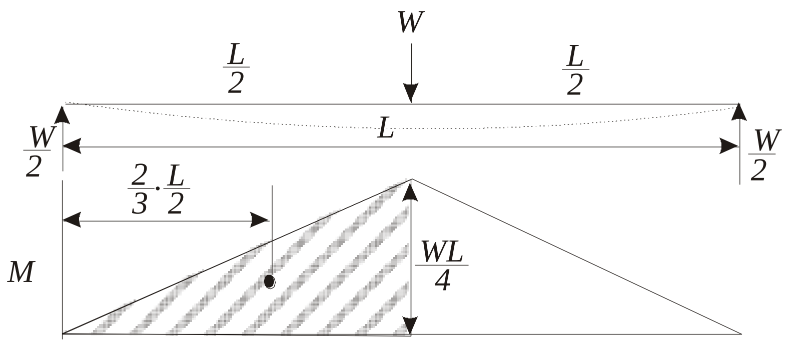


Figure A.1.12 concentrated load

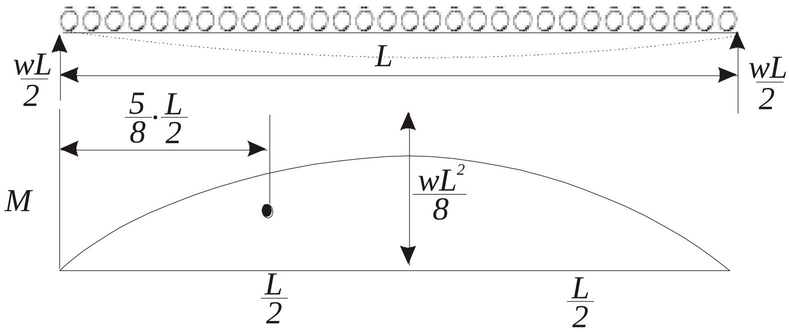


Figure A.1.13 Uniformly distributed load

Note: In both cases, by symmetry, the slope is zero at the centre, and the maximum slope and deflection can be found from the area of the BMD over half the beam i.e. 'P' at support 'Q' at centre.

(a) If A is the area of the BMD for half the beam.

$$A = \frac{1}{2} \cdot \left(\frac{WL}{4} \right) \cdot \left(\frac{L}{2} \right) = \frac{WL^2}{16}$$

then from (1.6), slope at support = $-\frac{A}{EI} = -\frac{WL^2}{16EI}$

from (1.7), deflection of support relative to centre = $\frac{A\bar{x}}{EI} = \frac{WL^2}{16} \cdot \frac{L/3}{EI} = \frac{WL^3}{48EI}$

(b) Shaded area $A = \frac{2}{3} \cdot \left(\frac{wL^2}{8} \right) \cdot \left(\frac{L}{2} \right) = \frac{wL^3}{24}$

Slope at support $= -\frac{A}{EI} = \frac{wL^3}{24EI}$

Deflection of support relative to centre $= \frac{A\bar{x}}{EI} = \left(\frac{wL^3}{24} \right) \cdot \left(\frac{5}{16} L \right) \cdot \frac{1}{EI} = \frac{5wL^4}{384EI}$

A.2 TORSIONAL STRESSES AND TWISTING IN CIRCULAR SHAFTS

- Important for shafts transmitting heavy torques, etc

If a shaft is acted upon by a pure torque T about its polar axis, shear stresses will be set up in directions perpendicular to the radius on all transverse sections.

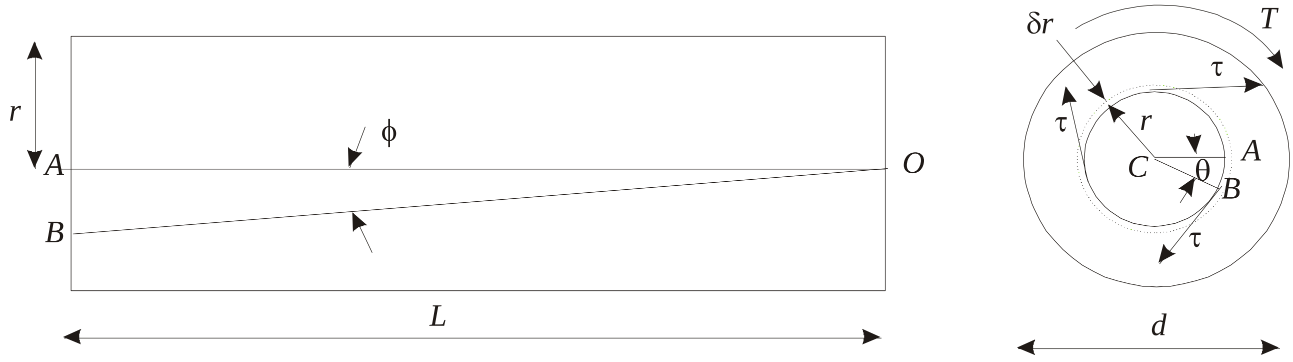


Figure A.2.1Torsion effect on circular shaft

θ is the angle of twist over a length L of the shaft. ϕ is the shear strain of elements at a distance r from the axis (ϕ is constant for constant T), so that a line originally OA twists to OB, and angle ACB = θ, the relative angle of twist of cross-sections a distance L apart.

Now, Arc AB = rθ =Lϕ approx, but ϕ = τ / G, where G is the modulus of rigidity.

By substituting and rearranging

$$\frac{\tau}{r} = \frac{G\theta}{L} \qquad \text{..........................(2.1)}$$

The torque can be equated to the sum of the moments of the tangential stresses on the elements 2πrδr, i.e.

$$T = \int \tau(2\pi r dr) r = \left(\frac{G\theta}{L}\right) \cdot \int (2\pi r dr) r^2 \text{ from (2.1)}$$

$$T = \left(\frac{G\theta}{L}\right) \cdot J \qquad \text{..........................(2.2),}$$

where J = Polar moment of inertia or Polar 2nd moment of area or torsion constant.

Combining (2.1) & (2.2)

$$\frac{T}{J} = \frac{\tau}{r} = \frac{G\theta}{L} \qquad \text{..........................(2.3)}$$

Showing that, for a given torque, the shear stress is proportional to the radius.

For a solid shaft $J = \frac{\pi D^4}{32}$

And the maximum stress $\hat{\tau} = \frac{16T}{\pi D^3}$, at $r = \frac{D}{2}$

For a hollow shaft $J = \frac{\pi}{32}\left(D^4 - d^4\right)$

And the maximum stress $\hat{\tau} = \frac{16 D \cdot T}{\pi\left(D^4 - d^4\right)}$, at $r = \frac{D}{2}$

Torsional stiffness is defined as torque per radian twist; i.e. $k = \frac{T}{\theta} = \frac{GJ}{L}$

Example 2.1

A ships propeller shaft has external and internal ϕ_s of 25cm and 15cm. What power can be transmitted at 110 rpm with a maximum shearing stress of 75MN/m^2, and what will then be the twist in degrees of a 10m length of the shaft? Take G = 80GN/m^2.

Solution

$r_1 = 0.125$m, $r_2 = 0.075$m, L = 10m

$J = \frac{\pi}{2}\left((0.125)^4 - (0.075)^4\right) = 0.334 m^4$, and τ = 75MN/m^2.

Then $T = \frac{J\tau}{r_1} = \frac{(0.334\times10^{-3})\cdot(75\times10^6)}{0.125} = 200 KNm$

At 110 rpm, the power generated is $(200\times10^3)\cdot\left(2\pi\cdot\frac{110}{60}\right) = 2.31\times10^6\ Nms^{-1}$

The angle of twist is $\theta = \frac{TL}{GJ} = \frac{(200\times10^3)\cdot(10)}{(80\times10^9)\cdot(0.334\times10^{-3})} = 0.075\, radians, 4.3°$

A.3 SPRINGS

A.3.1 Introduction

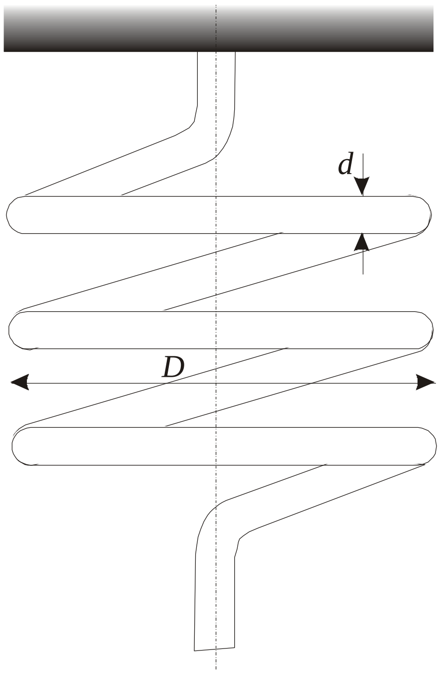
The common purpose of springs is to absorb energy and restore it slowly or rapidly according to the function of the particular spring under consideration, e.g. clock spring.
The other common use of springs is for absorbing shock, e.g. in motor vehicles.
The properties of springs which are of interest to the engineer are

(i) Its capacity for absorbing shock
(ii) The deformation given by a given load or vice versa.
(iii) Its natural frequency of vibrations (sometimes)

Springs can be of the helical, torsional (spiral) or leaf type.

A.3.2 Helical springs

These may be of the close-coiled or open coiled type. Helical springs are defined when the axis of the wire describes a helix. Only the close-coiled spring is dealt with in this course.



D = mean coiled diameter
d = wire diameter
n = number of coils

A.3.2.1 Close-coiled springs

When then coils of a helical spring are so close together that they may be regarded as lying in planes at right angles to axis of the helix, the angle of the helix is very small.

(a) Under axial load

Since the angle of the helix is small the action on any cross-section is approximately a pure torque $= \frac{WD}{2}$, and the bending moment and shear effect maybe neglected. The wire is therefore being twisted like a shaft, and if θ is the total angle of W along the axis of the coils, $x = -\frac{D}{2}\theta$ approximately. Applying the formula for torsion of shafts, making substitutions for torque and x and also noting that the length of the wire $L \approx \pi Dn$

then $\frac{T}{J} \Rightarrow \frac{W \cdot D/2}{\frac{\pi d^4}{32}} = \frac{2\hat{\tau}}{d} = \frac{G \cdot 2x/D}{\pi Dn}$

or $\frac{8WD}{\pi d^4} = \frac{\hat{\tau}}{d} = \frac{Gx}{\pi D^2 n}$ ……………………………….(3.1)

Given that the spring stiffness $k = \frac{W}{x} = \frac{Gd^4}{8D^3 n}$, and the strain energy $U = \frac{1}{2}Wx$, then substituting in terms of $\hat{\tau}$ from (3.1),

$U = \left(\frac{\hat{\tau}}{G}\right) \times volume\ of\ metal$ ……………………………………….………(3.2)

(b) Under axial torque T

This action produces approximately a pure bending moment of magnitude T at all cross sections. The total strain energy is therefore

$U = \frac{T^2 L}{2EI} = \frac{T^2 \cdot \pi Dn}{2E\frac{\pi d^4}{64}} = \frac{32T^2 Dn}{Ed^4}$ ……………………………………….(3.3)

But if T causes a rotation of one end of the spring through an angle θ, about the axis, relative to the other end,

$U = \frac{1}{2}T\theta$

Equation (3.3) gives $\theta = \frac{TL}{EI} = \frac{64TDn}{Ed^4}$ ....……………………………….(3.4)

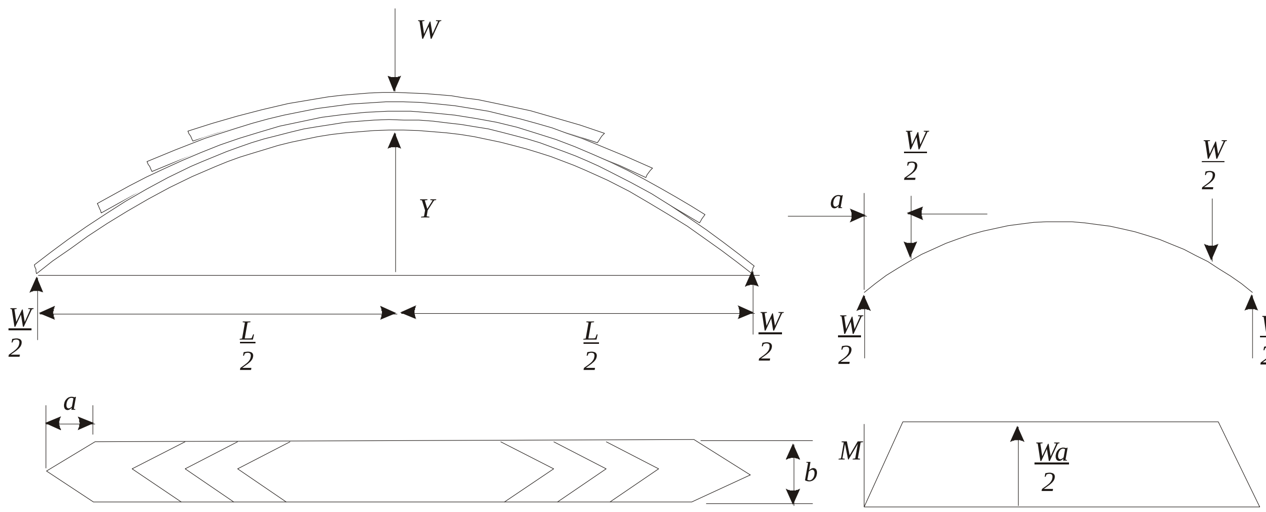
Maximum bending stress, $\sigma = \frac{T\,d/2}{\frac{\pi d^4}{64}} = \frac{32T}{\pi d^3}$ ……………………………….(3.5)

A.3.3 Leaf Springs

This type of spring is commonly used in motor vehicles and railway wagons. It is made up of a number of leaves of equal width and thickness, but varying length, placed in laminations and loaded as a beam.

Consider the following:

L = the span (assumed constant)
b = width of leaves
t = thickness of leaves
W = central vertical load
y = rise of crown above level of ends



(a) (b)

If the leaves are initially curved to circular arcs of the same radius R_o, contact between the leaves will only take place at their ends, and consequently any one leaf will be loaded as shown in fig(b). Over the central portion both M and I are constant. Over the end portions both for the whole leaf M/I is constant. But $\frac{M}{EI} = \frac{1}{R} - \frac{1}{R_0}$ and since R_o is assumed constant, the radius of curvature R in the strained state must be the same for all leaves and contact continues through the ends only. It can be shown that the spring can be approximated to a beam of constant depth and varying width since all plates maintain the same radius of curvature the bending moment of the equivalent section is directly proportional to the distance from either end, and I also varies uniformly, it can be shown that the spring is equivalent to a beam of uniform strength (i.e. same maximum stress at all sections).

Consider the central section:

$M = -\frac{WL}{4}$ tending to decrease the curvature

$I = \frac{nbt^3}{12}$, n = number of leaves

By the geometry of circle,

$y(2R - y) = \left(\frac{L}{2}\right)\cdot\left(\frac{L}{2}\right)$ and treating y as small compared with R, this gives $\frac{1}{R} = \frac{8y}{L^2}$

substituting in $\frac{M}{EI} = \frac{1}{R} - \frac{1}{R_0} \Rightarrow \frac{-WL/4}{E \cdot \frac{nbt^3}{12}} = \frac{8}{L^2}\left(y - y_0\right)$

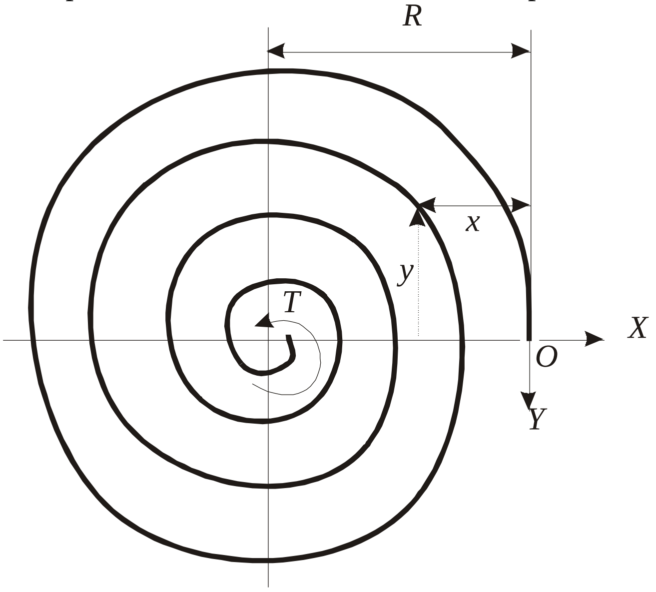
and the deflection $\delta = y_0 - y = \frac{3WL^3}{8nbt^3E}$ ……………………………...……….(3.6)

The load required to straighten the spring is called the **'Proof Load'** and is given by $\frac{8nbt^3Ey_0}{3L^3}$

The maximum bending stress $\sigma = \left(\frac{M}{I}\right)\cdot\left(\frac{t}{2}\right) = \frac{\left(WL/4\right)\cdot\left(t/2\right)}{\frac{nbt^3}{12}} = \frac{3WL}{2nbt^2}$ ……..(3.7)

A.3.4 Flat Spiral Springs

This is the type of spring used in clockwork mechanisms, and consists of a uniform thin strip wound into a spiral in one plane, and pinned at its outer end. The spring is 'wound up' by applying a torque to a spindle attached to the centre of the spiral.



If T is the torque tending to wind up the spring, and X and Y the components of reaction at the one end of the spring, O, taking moments about the spindle axis gives

$T = YR$ ……………………………………………(3.8) where R = maximum radius of the spiral

At any point in the spring, defined by co-ordinates x and y, the bending moment = $Y_x - X_y$ tending to increase the curvature.

The strain energy $U = \int \frac{(Y_x - X_y)^2 ds}{2EI} = \int \frac{((T/R)_x - X_y)^2 ds}{2EI}$ from (3.8)

Since O is a fixed point $\frac{\partial U}{\partial x} = 0$ giving $X = \left(\frac{T}{R}\right)\frac{\int xyds}{\int y^2 ds} = 0$ by symmetry

Then $\theta = \frac{\partial U}{\partial T} = \left(\frac{2T}{R^2}\right)\int \frac{x^2 ds}{2EI}$

But $\int \frac{x^2 ds}{2EI} \cong \left(\frac{R^2}{4} + R^2\right) \cdot L$, treating a spiral as a uniform disc, where L is the total length of the spiral.

Therefore $\theta = 1.25\frac{TL}{EI}$ ……………………….…………..(3.9)

Strain energy $= \frac{1}{2}T\theta = 1.25\frac{T^2 L}{2EI}$ from (3.9)

Maximum bending moment = 2YR at the left-hand edge = 2T from (3.8)

Maximum stress $\hat{\sigma} = \frac{2T}{z} = \frac{12T}{bt^2}$ ….………………..(3.10) Where b = width,

t = thickness of spring material.

B DYNAMICS

B.1 Moments of Inertia of Plane Figures and 3-D Symmetrical Objects

If the mass of every particle of a body is multiplied by the square of its distance from an axis, the summation of these quantities for the whole body is termed the moment of Inertia of the body about the axis and is denoted by I

Consider the following:

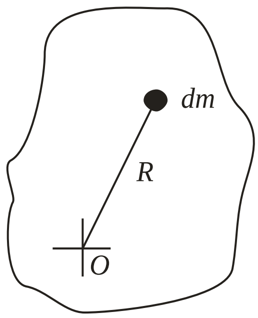


Figure B.1.1 Moment of Inertia

The moment of inertia of a particle of mass dm, at a distance r from an axis through O perpendicular to the plane of the paper is dmr^2. Hence the moment of inertia of the whole body about O, is:

$$I = \int r^2 dm \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.1)$$

This integral represents an important property of a body particularly in the force analysis of any body that has rotational acceleration about a given axis. It is a measure of resistance to rotational acceleration of the body.

The moment of inertia may be expressed alternatively as:

$$I = \sum r_i^2 m_i \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.2)$$

Where r_i is the radial distance from the inertia axis to the representative particle of mass m_i.

If the density ρ, is constant throughout the body, the moment of inertia becomes

$$I = \rho \int r^2 dV \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.3)$$

B.1.1 Radius of Gyration

If the total mass of the body is m, the moment of inertia may be written as

$$I = mk^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.4)$$

Where k is termed the of gyration and is radius at which the mass would have to be concentrated to give the same value of I.

Thus $$k = \sqrt{\frac{I}{M}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.5)$$

B.1.2 Theorem of Parallel Axes

The moment of inertia of a body about any axis is equal to the moment of inertia of the body about a parallel axis through the centre of mass together with the product of the mass and the square of the distance between the axes.

Let I_G be the moment of inertia of a body about an axis through the centre of mass, G (Fig B.1.2)

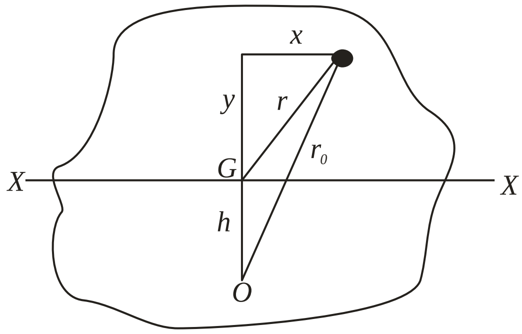


Figure B.1.2 Theorem of Parallel axes

It is required to find the moment of inertia about a parallel axis through O, which is a distance h from G.

Moment of inertia of particle $O = r_0^2 dm$

$$= dm\left[x^2 + (h+y)^2\right]$$
$$= dm\left[x^2 + h^2 + 2hy + y^2\right]$$
$$= dm\left[r^2 + h^2 + 2hy\right]$$

Therefore the moment of inertia of the body about O,

$$I_0 = \int r^2 dm + \int h^2 dm + 2h\int y dm$$
$$= I_G + mh^2 + 2h \times \text{(total moment of the mass about XX)}$$

Since XX passes through the centre of mass G, the total moment of the mass about XX is zero. Hence

$$I_0 = I_G + mh^2 \quad \text{……………………..(4.6)}$$

B.1.3 Moments of Inertia for Common Cases

(a) Uniform disc or cylinder, radius r. I about the central axis is $\frac{mr^2}{2}$ and $k = \frac{r}{\sqrt{2}}$

(b) Hollow disc or cylinder, external and internal radii R and r respectively. I about the central axis is $\frac{m(R^2 + r^2)}{2}$.

(c) Thin disc, radius r. I about a diameter is $\frac{mr^2}{4}$ and $k = \frac{r}{2}$

(d) Thin uniform rod, length L. I about an axis through the centre perpendicular to the length is $\frac{mL^2}{12}$. I about parallel axis through one end is $\frac{mL^2}{3}$

(e) Solid cylinder, radius r, and length L. I about an axis through the centre perpendicular to the length is $m \cdot \left(\frac{r^2}{4} + \frac{L^2}{12} \right)$

(f) Solid sphere, radius r. I about a diametrical axis is $\frac{2}{5} mr^2$

B.2 Energy, Momentum and Impulse

B.2.1 Energy

Definition: Energy is the capacity to do work, mechanical energy being equal to the work done on a body in altering either its position, velocity or shape.

Potential energy (PE) of a body is the energy it possesses due to its position and is equal to the work done in raising it from some datum level.

$PE = mgh$ ……………………………..(4.7)

Kinetic energy (KE) of a body is the energy it possesses due to its velocity.

$KE = \frac{1}{2} mv^2$ …………………………(4.8) for rectilinear and curvilinear translation

and $KE = \frac{1}{2} I_0 \omega^2$ …………………………(4.9) for rotation

For general plane motion of a body having both translation and rotation.

$KE = \frac{1}{2} m\bar{v}^2 + \frac{1}{2} \bar{I}_0 \omega^2$ ………………...(4.10)

where $\bar{I}$ = moment of inertia of the body about its mass centre.

The strain energy (SE) of an elastic body is the energy stored when the body is deformed.

$SE = \frac{1}{2} Sx^2$ …………………………(4.11) where S = stiffness.

B.2.1.1 Principle of Conservation of Energy

Energy can be neither created nor destroyed. It may exist in various forms e.g. Mechanical, Electrical or heat energy. A loss of energy in any one form is always accompanied by an equivalent increase in another form.

i.e. Total energy = KE + PE + SE …………(4.12)

B.2.2 Momentum and Impulse

Definition: The linear momentum of a general mass system is the vector sum of the linear momenta of all particles of th system where the product of the mass and velocity is the linear momentum of each particle i.e. $G = \sum m_i v_i$

For a system of constant total mass, the linear momentum is given by

$G = m\bar{v}$ ……………………….(4.13)

The effect of the resultant force ΣF on the linear momentum of a body over a finite period of time may be obtained by integrating $\sum F = m\dot{v} = \frac{d}{dt}(mv)$ from time t_1 to t_2.

i.e. $\int_{t_1}^{t_2} \sum F dt = G_2 - G_1$ …………………..(4.14)

since $\sum F = \dot{G}$

Definition: The product of force and time is defined as **Linear Impulse** and equation (4.14) states that the total linear impulse on m equals the corresponding change in linear momentum.

Note:

1. The impulse integral is in the general case a vector which may change both in magnitude and direction during the time interval. Thus it will be necessary to express ΣF and G in component form and then combine the integrated components. Thus the x-component of equation (4.14) becomes the scalar equation $\int_{t_1}^{t_2} \sum F_x dt = (mv_x)_2 - (mv_x)_1$ and similarly for the y and z components.
2. In evaluating the impulse, it is necessary to include the effect of all forces acting on m except those whose magnitudes are negligible. The only reliable method of accounting for the effects of all forces is to isolate the particles or body in question by drawing its free-body diagram.
3. In addition to the equations of linear impulse and linear momentum, there exists a parallel set of equations of angular impulse and angular momentum (moment of momentum). The angular momentum of a mass system about a fixed point is defined as the vector sum of the moments of linear momenta about the fixed point of all particles of the system and is

   $H_0 = \sum (r_i \bullet m_i v_i)$ ………………………..(4.15)

   if the fixed point is the mass centre,

   $\overline{H} = \sum (\rho_i \bullet m_i v_i)$ …………………………(4.16)

   Where ρ_i is the position vector of m_i wrt G.

   It can be shown that equation (4.16) can be written as

   $\overline{H} = \overline{I}\omega$ ……………………………..(4.17) [kgm²/s]

   Where $\overline{I} = \sum m_i \rho_i^2$ is the moment of inertia of the body about its centre of mass.

   Angular Impulse is defined as the product of moment and time and is equal to the change of angular momentum.

   Thus,

   $\int_{t_1}^{t_2} \sum \overline{M} dt = \overline{H}_2 - \overline{H}_1$ ……………………….…(4.18) [kgm²/s]

   where $\overline{M}$ is the summation of the moments of the linear momenta of all particles accounted for by $\overline{H} = \overline{I}\omega$.

   Equation (4.18) is the total angular impulse acting on the body about its mass centre during the internal t_1 and t_2.
4. When a\body rotates about about a fixed point O (not the centre of mass) on the body or body extension, angular momentum may be written as

   $H_0 = (\overline{I}\omega + m\overline{r}^2\omega)$……………………………(4.19)

   where $\overline{I}\omega$ = angular momentum about a parallel axis through G

$m\bar{r}^2\omega$ = moment of linear momentum about O

$\bar{r}$ = the distance from the centre of mass to O.

but $I_0 = \bar{I} + m\bar{r}^2\omega$

so that $H_0 = I_0\omega$ ……………………………..(4.20)

Principle of Conservation of Momentum

The total linear or angular momentum of a system of masses in one direction (or about any one axis) remains constant unless acted upon by an external force in that direction (or torque about that axis).

Example 4.1

A rigid uniform beam AB, 6m long, is supported vertically with the end B resting on the ground. End A is released and the beam is allowed to fall. It turns about the end B which remains in its original position. A point C on the beam , 3.6m from B, strikes the edge of horizontal step. After impact, the beam rotates about the edge of the step without slipping. Determine the height of the step if the beam comes momentarily to rest in the horizontal position.

Solution

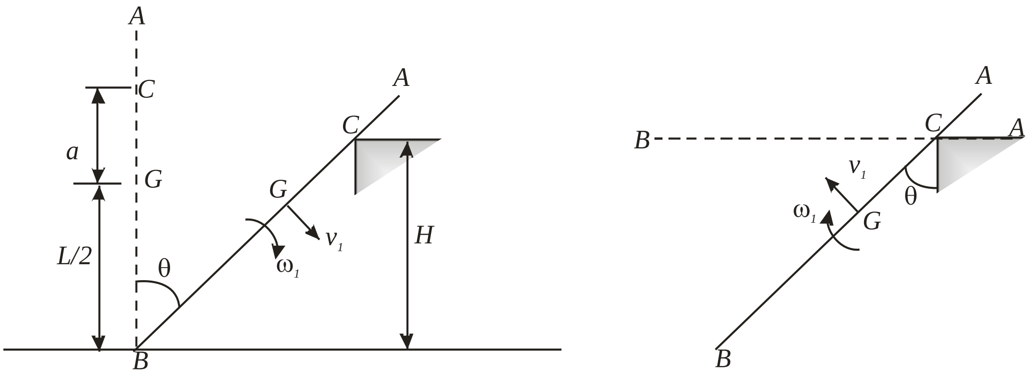


Figure B.2.1 Schematic of Example 4.1

Let ω_1 and ω_2 be the angular velocities of the beam immediately before and after impact, and v_1 and v_2 be the linear velocities of G immediately before and after impact.

In falling from the vertical position

Loss of PE = gain of KE

$$mg \cdot \frac{1}{2}(1-\cos\theta) = \frac{1}{2} I_B \omega_1^2 = \frac{1}{2}\frac{mL^2}{3}\omega_1^2$$

i.e.

$$\therefore \omega_1 = \sqrt{\left[\frac{3g}{L}(1-\cos\theta)\right]} ----(1)$$

in moving to the horizontal position after impact

gain of PE = loss of KE

$$mga\cos\theta = \frac{1}{2} I_c \omega_2^2 = \frac{1}{2} m\left(\frac{L^2}{12} + a^2\right)\omega_2^2$$

i.e.

$$\therefore \omega_2 = \sqrt{\left[\frac{24ga\cos\theta}{L^2 + 12a^2}\right]} ----(2)$$

During impact, moment of momentum of the beam about C remains constant since the impulsive force at C has no moment about that point,
Taking clockwise momentum as +ve

$$I_G\omega_1 - mv_1 a = I_G\omega_2 + mv_2 a$$

But $v_1 = \omega_1 \frac{L}{2}$ and $v_2 = \omega_2 a$

$$\therefore m\frac{L^2}{12}\omega_1 - m\omega_1\frac{L}{2}a = m\frac{L^2}{12}\omega_2 - m\omega_2 a^2$$

Substituting for w_1 and w_2 from equations (1) & (2),

$$\left(\frac{L^2}{12} - \frac{La}{2}\right)\sqrt{\left[\frac{3g}{L}(1-\cos\theta)\right]} = \left(\frac{L^2}{12} + a^2\right)\sqrt{\left[\frac{24ga\cos\theta}{L^2+12a^2}\right]}$$

Thus for L = 6m and a = 0.6m, cosθ = 0.1515

$\therefore$ Height of step h = 3.6cosθ
= 0.546m

B.3 Balancing of Machines

Definition: Balancing is the technique of correcting or eliminating unwanted inertia forces and moments in rotating machinery. *If these forces or moments left unchecked, may lead to failure of the machine since they induce vibrations during operations.*

B.3.1 Static and Dynamic Balance

When a shaft carrying several eccentric masses in **static** balance, the **centre of mass of the system lies in the axis of the shaft** so that the shaft and the attached masses remain in any position in which it is placed. *For static balance resultant force must be zero*. When the shaft rotates, however, centrifugal forces act upon the masses, and if **these are not rotating in the same plane**, couples also act upon the shaft.

For complete dynamic balance, therefore

(i) the resultant force acting upon the shaft must be zero and
(ii) the resultant couple acting upon the shaft must be zero

B.3.2 Balancing of Masses Rotating in the Same Plane

Consider the following:

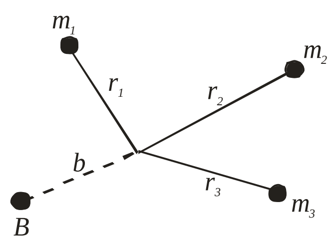


Figure B.3.1 Balancing of masses rotating in the same plane

If m_1, m_2 and m_3 are the out-of-balance masses and r_1, r_2 and r_3 are the respective radii of rotation, then for dynamic balance, the vector sum of the centrifugal forces must be zero,

i.e. Centrifugal force = $\sum m\omega^2 r = 0$ ……………..(5.1) where ω is the angular speed of the shaft

$\Rightarrow \sum mr = 0$ ……………..(5.2) since ω^2 is the same for each mass.

If a force polygon with sides representing the magnitudes and directions of the mass-arm products m_1r_1, m_2r_2 and m_3r_3 is drawn, the closing side represents the product of the balance mass B, and its radii of rotation b.

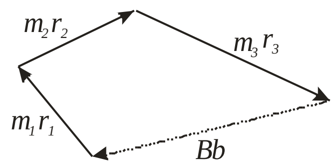


Figure B.3.2 Force Polygon

Note:

1. The condition $\sum mr = 0$ is also the condition for static balance.
2. The sides of the force polygon are drawn in the respective directions of the respective radii of rotation to scale.
3. The force polygon will not close if the system has any static unbalance.

B.3.3 Balancing of Masses rotating in Different Planes – Dalby's Method

Consider the following:

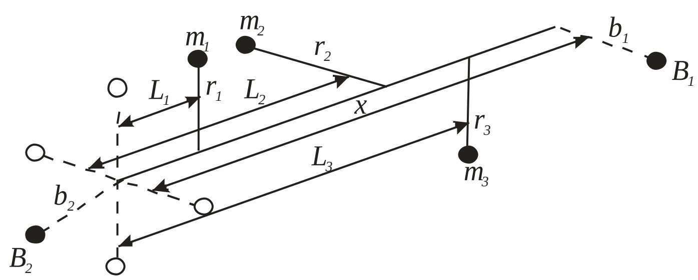


Figure B.3.3 Balancing masses rotating in different planes

If the out-of-balance masses, m_1, m_2 and m_3 are situated at distances L_1, L_2, and L_3 from a reference plane, the forces m_1r_1, m_2r_2 and m_3r_3 may be transferred to the reference plane by the addition of couples of magnitude $m_1r_1L_1$, $m_2r_2L_2$ and $m_3r_3L_3$, acting in the planes containing the respective forces and the shaft axis.

Then, for balance, the resultant force in the reference plane must be zero, i.e. $\sum mr = 0$, and the resultant couple in the reference plane must be zero

i.e. $\sum mrL = 0$ ……………………………(5.3)

These two conditions determine the necessary mass-arm products B_1b_1 and B_2b_2 for balance. The couple polygon.

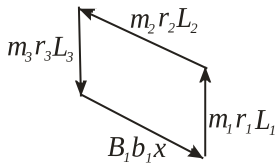


Figure B.3.4 Couple Polygon

The closing side of the mrL polygon in fig B.3.4 represents the magnitude and direction of the couple required for equilibrium, B_1b_1x.

The closing side of the force (mr) polygon, represents the magnitude and direction of the force, B_2b_2, required in the reference plane for equilibrium.

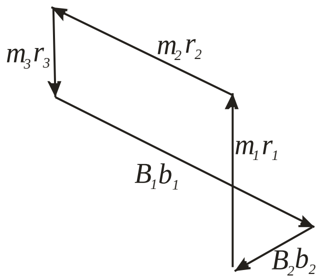


Figure B.3.5 Force Polygon

Note:

1. The reference plane is normally chosen to coincide with the plane of revolution of one of the unknown masses, thus eliminating the couple produced by this mass.
2. In constructing the couple polygon, it is usual to draw the couple vectors in the directions of the respective forces instead of 90˚ anticlockwise to them, as is conventionally done for couple vectors (Polygon shape is unaffected).
3. Where the reference plane divides the planes of revolution of the masses, the vectors for the couples to one side of the plane are drawn radially outwards and those to the other side, being regarded as negative, are drawn radially inwards

Example 5.1

A rotating shaft carries four masses A, B, C and D, rigidly attached to it, the centres of mass are at 30mm, 36mm, 39mm and 33mm respectively from the axis of rotation; A, C and D are 75kg, 5kg and 4kg; the axial distance between A and B is 400mm and that between B and C is 500mm; the eccentricities of A and C are at 90˚ to one another.

Find for complete balance:

(a) the angles between A, B, and D.
(b) the axial distance between the planes of revolution of C and D, and
(c) the mass B.

Solution

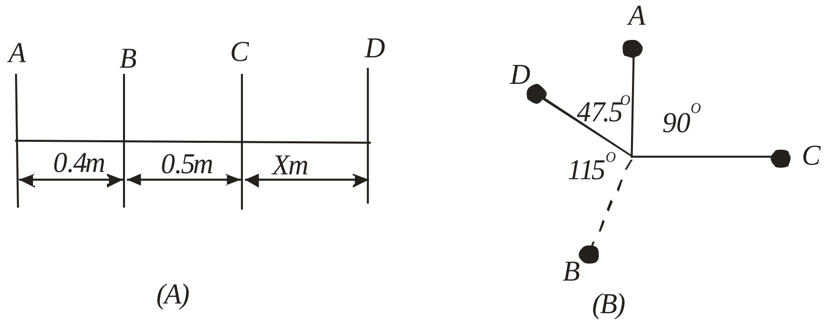


Figure B.3.6 Example 5.1 schematic

Choose the plane of B as the reference plane since it contains one of the unknown masses. Distances measured to the right of it are regarded as +ve.

Using the given data, compile the following table:

Plane	m (kg)	r (mm)	mr (kgmm)	L (m)	mrL (kgmm.m)
A	7.5	30	225	-0.4	-90
B	m	36	36m	0	0
C	5	39	195	0.5	97.5
D	4	33	132	0.5 + x	66 + 132x

Construct the couple polygon, from the data in the mrL column, the direction of the couple due to A being downwards since it is –ve.

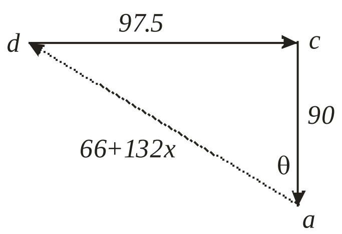


Figure B.3.7 Couple polygon for example 5.1

By measurement, the closing side, ad = 132.6 = 66 + 132x kgmm.m
From which x = 0.505m and θ = 47.5°

Construct the force polygon, using the data in the mr column and the known directions of the forces A, C & D.

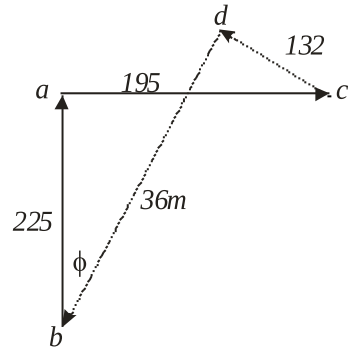


Figure B.3.8 Force polygon for example 5.1

By measurement, db = 330 = 36m kgmm

m = 9.16kg
also ϕ = 17.5° from which the angle between B & D = 115°

The relative positions of masses are shown in fig B.3.6(b)

B.4 Introduction to Mechanical Vibrations

Vibrations produce unwanted noise, high stresses, wear, poor reliability and frequently, premature failure of one or more of the parts.

Definitions: Any motion that exactly repeats itself after a certain interval of time is a periodic motion and is called a **vibration**.

Vibrations may be either free or forced. A mechanical element is said to have a **free vibration** if the periodic motion continues after the cause of the original disturbance is removed, but if a vibrating motion persists because of the continuing existence of a disturbing force, then it is called a **forced vibration**.

Any free vibration of a mechanical system will eventually cease because of loss of energy. These energy losses are accounted for by using a single factor called the **damping factor**. Thus a heavily damped system is one in which the vibration decays rapidly.

- The **period** of a vibration is the time of a single event or cycle.
- The **frequency** is the number of cycles or periods occurring in unit time.
- The **natural frequency** is the frequency of a free vibration.

Note: If the forcing frequency becomes equal to the natural frequency of a system, then **resonance** is said to occur.

Consider a concentrated mass mounted on an elastic spring and subjected to a retarding force and a disturbing force. The mass is acted upon by an applied force F = f(t). Also the mass is retarded by a force with magnitude proportional to the velocity.

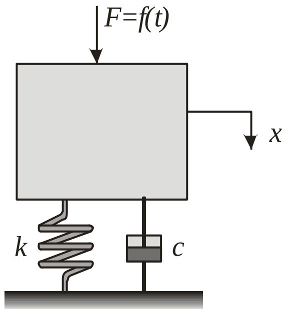


Figure B.4.1 Spring mass system with dashpot

Note: Frictional retardation whose magnitude is proportional to the velocity is termed viscous damping. Other types of damping forces may be encountered such as dry friction or Coloumb damping which is essentially independent of velocity, internal damping due to material hysteresis losses, turbulent flow damping where the retarding force is more nearly proportional to the square of the velocity, and magnetic damping.

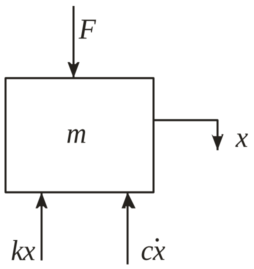


Figure B.4.2 Free Body Diagram of mass

Applying Newton's 2nd law of motion i.e. $\sum F = m\ddot{x}$

$$F - kx - c\dot{x} = m\ddot{x}$$

$$\Rightarrow m\ddot{x} + c\dot{x} + kx = F \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6.1)$$

Note: The vibrating system of fig B.4.1 has one degree of freedom because the position of the mass can be completely defined by a single coordinate. Hence the system is classified as a forced, single degree of freedom system with damping.

B.4.1 Free Response

When the disturbing force F is zero, equation (6.1) becomes a homogeneous second order equation. Its solution describes the motion and response of the mass m.

B.4.1.1 Undamped response

The mass vibrates freely without energy loss and the equation of motion becomes

$$m\ddot{x} + kx = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6.2)$$

Assuming that the solution is of the form

$$X = A\sin\omega t\ldots\ldots\ldots\ldots\ldots()$$

Then $\dot{X} = A\omega\cos\omega t$

$$\ddot{X} = -A\omega^2\sin\omega t$$

Substituting into (6.2) gives

$$-mA\omega^2\sin\omega t + kA\sin\omega t = 0$$

from which $\omega^2 = \frac{k}{m}$

$$\Rightarrow \omega = \sqrt{\frac{k}{m}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6.3)$$

Note: It can be easily shown that $X = B\cos\omega t$ ---(ii), is also a solution of equation (6.2). A general solution is obtained by adding equation (i) and (ii) to give:

$$X = A\sin\omega t + B\cos\omega t \quad \ldots\ldots\ldots\ldots\ldots\ldots(6.4)$$

where A & B are integration constants, which depend on the manner in which the motion was begun i.e. the initial conditions (at t = 0)

Define $\omega_n = \sqrt{\frac{k}{m}}$ ……………………………..(6.5) [rad/sec] where ω_n is the natural frequency

The solution can now be written in the form

$$X = A\sin\omega_n t + B\cos\omega_n t \quad \ldots\ldots\ldots\ldots\ldots\ldots(6.6)$$

Note: In order to evaluate the solution, the initial conditions must be known.

i.e. at $t = 0, x = x_0, \dot{x} = 0$

Substituting into (6.6)

$$\Rightarrow x_0 = A(0) + B(1)$$

$$\Rightarrow B = x_0$$

Differentiating (6.6); $\dot{X} = A\omega_n \cos\omega_n t - B\omega_n \sin\omega_n t$

Substituting $\dot{X} = 0$ at t = 0;

$$0 = A\omega_n(1) - B\omega_n(0)$$

$$\Rightarrow A = 0$$

Substituting for A & B into equation (6.6) gives

$$X = X_0 \cos\omega_n t \quad \text{……….………………….(6.7)}$$

Note: For most systems, ω_n is a constant because the mass and the spring constant do not change (they are system properties).

Period of vibration $T = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{m}{k}}$ ……………(6.8)

And the frequency of vibration

$$f = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \quad \text{………………………..(6.9) [Hz]}$$

Note:

1. Equation (6.4) can alternatively be expressed as

   $X = X_0 \cos(\omega_n t - \phi)$ ……………………..(6.10)

   where X_0 is the amplitude of vibrations

   and $X = \sqrt{A^2 + B^2}$

   and ϕ is the phase angle given by $\tan\phi = \frac{B}{A}$ and denotes the angular lag of the motion wrt the cosine function.

2. In the absence of a disturbing force F and a damping force $X = X_0 \cos\omega_n t$, the energy of the system is conserved so that

   (KE + PE) = Const (Energy Method)

   and $\frac{d}{dt}(KE + PE) = 0$

   Substitution of the expressions for KE and PE gives

   $\frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2\right) = m\dot{x}\ddot{x} + kx\dot{x} = 0$ i.e. equation (6.2)

   $\Rightarrow m\ddot{x} + kx = 0$

   Derivation of the equation of motion by differentiating the energy equation is useful for **systems with interacting members** and **reactive forces which do no work**.

3. Energy considerations may be used to determine the natural frequency for a **linear conservative system** without having to derive the equations of motion. Since energy is

conserved, the maximum KE occurs at the position x = 0 and must equal the maximum PE at $x = x_0$ or

$KE_{max} = PE_{max}$ (Rayleigh's Method)

Where PE = 0 when $KE = KE_{max}$.

e.g. from the solution for **simple harmonic motion** (shm), the maximum velocity is $\dot{x}_{max} = X_0\omega$, so that $\frac{1}{2}m(x_0\omega)^2 = \frac{1}{2}kx_0^2$ from which $\omega = \sqrt{\frac{k}{m}}$

B.4.1.2 Damped Response

When the damping force is not negligible,

$$m\ddot{x} + c\dot{x} + kx = 0 \quad \text{……………………………..(6.11)}$$

Assume a solution of the form

$$x = Ae^{\omega t} \quad \text{……………………………………..(6.12)}$$

giving $\dot{x} = A\omega e^{\omega t}$ and $\ddot{x} = A\omega^2 e^{\omega t}$ substituting into equation(6.11) gives the characteristic equation

$$m\omega^2 + c\omega + k = 0$$

The roots are

$$\omega = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \quad \text{…………………..(6.13)}$$

The equation (6.12) can be written as

$$x = B_1 e^{\omega_1 t} + B_2 e^{\omega_2 t} \quad \text{…………………..………(6.14)}$$

where ω_1 & ω_2 are the two roots of equation (6.13) and B_1 & B_2 the constants of integration evaluated by initial conditions.

Note:

1. If $c < 2\sqrt{km}$, the roots of (6.13) are complex.
   With the substitutions

   $$b = \frac{c}{2m} \text{ and } q = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}$$

   the roots are $\omega_1 = -b + iq$ and $\omega_2 = -b - iq$

   The solution then becomes

   $$x(t) = e^{-bt}\left(C_1 \sin qt + C_2 \cos qt\right) \quad \text{…...(6.15a)}$$

   or $$x(t) = e^{-bt}\sin(qt - \phi) \quad \text{………………(6.15b)}$$

   where $x_0 = \sqrt{C_1^2 + C_2^2}$ and $\tan\phi = \frac{C_2}{C_1}$

   The motion is oscillatory with a decreasing amplitude bounded by the limiting curves $x = \pm x_0 e^{-bt}$

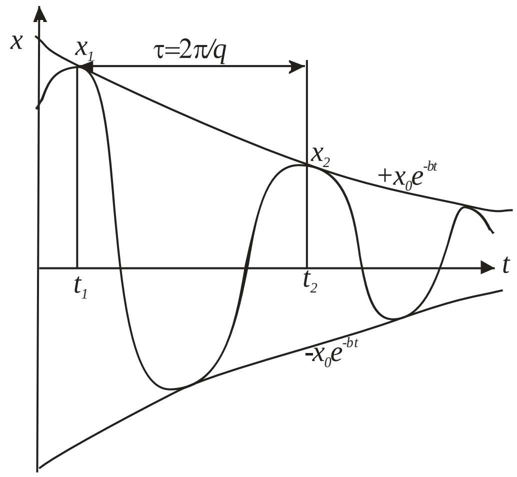


Figure B.4.3 Damped Response

The period $\tau = \dfrac{2\pi}{q}$ and is greater than case with no damping

If two successive amplitudes are measured as x_1 and x_2, with $t_2 = t_1 + T$, then

$$\frac{x_1}{x_2} = \frac{x_0 e^{-bt_1}}{x_0 e^{-b(t_1+\tau)}} = e^{b\tau} \quad \text{……..………..(6.16)}$$

The quantity $b\tau$ is known as the logarithmic decrement and is a direct measure of the damping coefficient.

If the critical value of $C = C_c = 2\sqrt{km}$ (the critical damping coefficient) then log;

$$\delta = \ln\frac{x_1}{x_2} = b\tau = \frac{2\pi\xi}{\sqrt{1-\xi^2}}$$

Where $\xi = \dfrac{c}{c_c}$ ………………………….(6.17)

Thus $q = \omega_n\sqrt{1-\xi^2}$ ……………………(6.18)

The natural frequency of the damped vibration.

With $x_0 e^{-bt_1}$, vibration is said to be **underdamped**.

2. If $c > 2\sqrt{km}$, $\xi < 1$, both roots are real and there is no vibration. The mass slowly returns to its neutral position without oscillation. The motion is said to be **overdamped**. The solution is then

$$x = e^{-bt}\left(C_1 e^{q't} + C_2 e^{-q't}\right) \quad \text{……………(6.19)}$$

where $q' = \sqrt{\left(\dfrac{c}{2m}\right)^2 - \dfrac{k}{m}}$

3. If $C = C_c = 2\sqrt{km}, \xi = 1$, the motion is said be **critically damped** and the condition represents the transition between a damped vibration and an overdamped motion. The roots of the characteristic equation are equal and

$$x = e^{-bt}\left(C_1 + C_2 t\right) \quad \text{…………………..(6.20)} \quad \text{there is no oscillation.}$$

Fig B.4.4 shows the graphical representation of the free vibration.

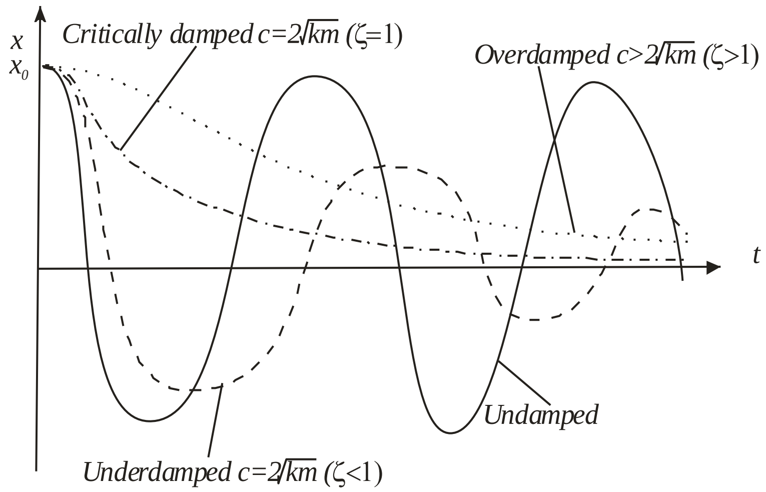


Figure B.4.4 Free Vibration

4. x(t) is oscillatory only if the system is underdamped.
5. The natural frequency of damped vibration, q (or ω_d) is less than ω_n
6. In all cases of ξ, x(t) eventually dies out regardless of the initial conditions
7. Damping reduces the amplitude of oscillation but increases the time for the motion to return to the neutral position.

B.4.2 Forced Response

Given equation (6.1)

$$m\ddot{x} + c\dot{x} + kx = F$$

From theory the linear differential equations, the completely solution of the homogeneous linear equation has 2 components, given by

$$x = x_c + x_p \quad \text{……………………………….…(6.21)}$$

x_c = Complimentary (transient) solution for F = 0

x_p = Particular (steady state) solution

The response may be steady state or transient depending on the forcing function. Only the steady state is dealt with in this course.

B.4.2.1 Steady-State Response

Forcing function has a steady oscillation if the forcing function is harmonic, such as rotating unbalance the forcing function is described by

$F = F_0 \sin\omega t$ ……………………………..….(6.22) where F_0 = amplitude

ω = forcing frequency

then equation (6.21) becomes

$$m\ddot{x} + c\dot{x} + kx = F_0 \sin\omega t \quad \text{……………………..(6.23)}$$

The complementary solution is equation (6.15a)

$x_c = e^{-bt}\left(C_1 \sin qt + C_2 \cos qt\right)$ this dies out with time.

The particular solution, is also expected to be harmonic and is of the form:

$$x_p = x_0 \cos(\omega t - \phi) \quad \text{………………………….(6.24)}$$

or $(= x_0 \sin(\omega t - \phi))$

$\left(= A\sin\omega t + B\cos\omega t\right)$

where x_0 (amplitude) and ϕ (phase angle) are constants to be determined between F_0 & x_0.

Substituting into (6.23) gives:

$$x_0\left[\left(k-m\omega^2\right)\cos(\omega t-\phi)-c\omega\sin(\omega t-\phi)\right]=F_0\cos\omega t \quad \text{……..(6.25)}$$

using trigonometry relations:

$$\cos(\omega t-\phi)=\cos\omega t\cos\phi+\sin\omega t\sin\phi$$

$$\sin(\omega t-\phi)=\sin\omega t\cos\phi-\cos\omega t\sin\phi$$

in equation (6.25) and equating the coefficients of cosωt and sinωt on both sides of the resulting equation gives:

$$\begin{aligned} x_0\left[\left(k-m\omega^2\right)\cos\phi-c\omega\sin\phi\right]&=F_0 \\ x_0\left[\left(k-m\omega^2\right)\sin\phi-c\omega\cos\phi\right]&=0 \end{aligned} \quad \text{…………………...........(6.26)}$$

Solving (6.26) simultaneously gives:

$$x_0=\frac{F_0}{\sqrt{\left[\left(k-m\omega^2\right)^2+(c\omega)^2\right]}} \quad \text{………………………………...(6.27)}$$

and $\phi=\tan^{-1}\left(\dfrac{c\omega}{k-m\omega^2}\right)$ ………………………………………...(6.28)

The complete solution is now written as

$$x(t)=e^{-bt}\left(C_1\sin qt+C_2\cos qt\right)+\frac{F_0\sin(\omega t-\phi)}{\sqrt{\left[\left(k-m\omega^2\right)^2+(c\omega)^2\right]}} \quad \text{…..(6.29)}$$

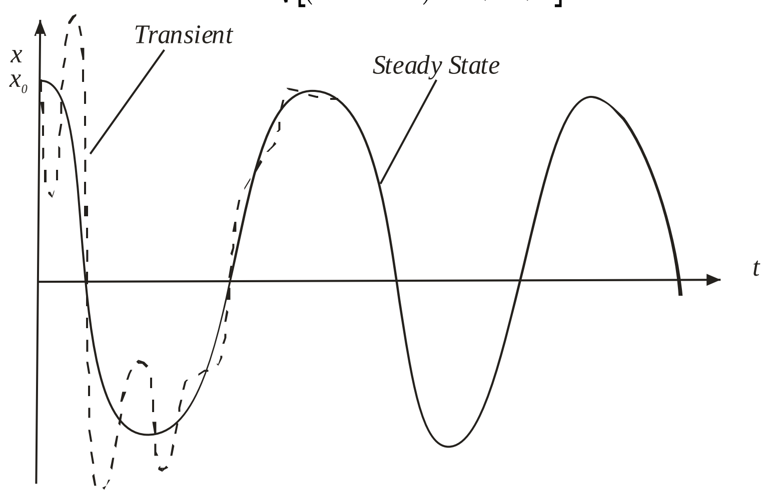


Figure B.4.5 Transient and Steady state behaviour

Note: If the amplitude x_0 is divided by the static displacement $\delta_0=\dfrac{F_0}{k}$ which m would have under the action of a steady force F_0 only, and making the following substitutions:

$$\omega_n=\sqrt{\frac{k}{m}}\,; \qquad \xi=\frac{c}{c_c}=\frac{c}{2m\omega_n}=\frac{c}{2\sqrt{mk}}\,;\frac{c}{m}=2\xi\omega_n$$

$$\delta_0=\frac{F_0}{k}\,; \qquad r=\frac{\omega}{\omega_n} \quad \text{…. frequency ratio}$$

then the ratio $\dfrac{x_0}{\delta_0}$ known as the magnification ratio becomes

$$\frac{x_0}{\delta_0} = \frac{1}{\sqrt{\left(-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}}$$

$$\frac{x_0}{\delta_0} = \frac{1}{\sqrt{\left(1-r^2\right)^2 + \left(2\xi r\right)^2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6.30)$$

with $\phi = \tan^{-1}\left(\frac{2\xi r}{1-r^2}\right)$ ……………………………………………(6.31)

Equation (6.30) shows that x_0 becomes extremely large as ω approaches ω_n if c is small. In the limit c=0, $x_0 \to \infty$ and $\omega \to \omega_n$ i.e. **resonance**

The variations of $\frac{x_0}{\delta_0}$ and ϕ with r and the damping ratio ξ are shown below

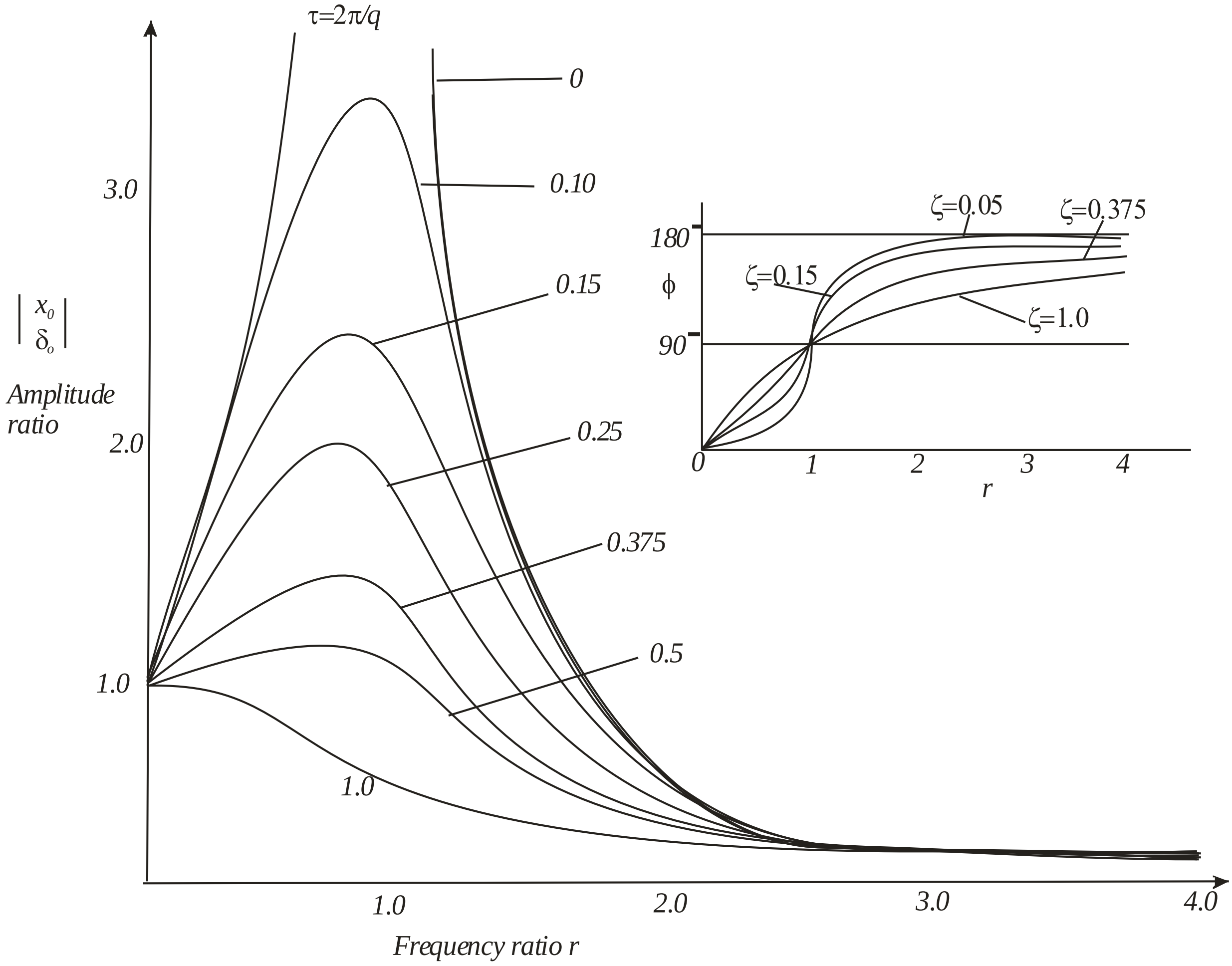


Figure B.4.6 Amplitude ratio vs Frequency ratio

There is an equivalent analysis for torsional vibration

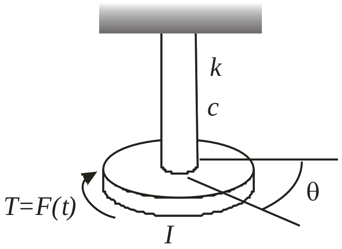


Figure B.4.7 Torsional system

From $\sum T = I\ddot{\theta} \Rightarrow -k\theta - c\dot{\theta} + f(t) + (-I\ddot{\theta}) = 0$

$\Rightarrow I\ddot{\theta} + c\dot{\theta} + k\theta = f(t)$ ………………….…(6.32) same as eqn (6.23) giving same solutions

Thus a torsional system can be simulated merely by substituting torsional notation for the usual linear notation.

B.4.3 Vibration Isolation

It is usually desirable to reduce as much as possible, the forced vibrations which are generated in generated in engineering structures and machines. Vibration reduction can be accomplished in any of the following ways:

1. Reduction or elimination of the exciting force by balancing or other removal
2. Introduction of sufficient damping to limit the amplitude
3. Operation at a forced frequency sufficiently different from the natural frequency so as to avoid resonance
4. Isolation of the body from the vibration source by providing elastic mountings of the proper stiffness

Only (4) is dealt with here

Definition: Vibration isolation involves the isolation of a resilient member (or isolator) between the vibrating mass and the source of vibration so that a reduction in the dynamic response of the system is achieved under specified conditions of excitation.

Vibration isolation ca be used in two types of situations:

(i) Where the foundation (base) is protaected against large unbalanced or impulsive forces (forging and stamping presses). The force transmitted to the base is given by:

$F_t(t) = kx(t) + c\dot{x}(t)$ …………………(6.33)

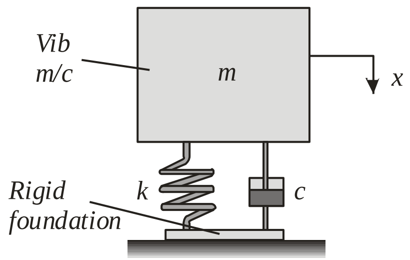


Figure B.4.8 Reciprocating and rotating machines

(ii) Where the system is protected against the motion of its foundation or base (protection of delicate equipment or instrument from the motion of its container). If the instrument is modeled as a single degree of freedom system;

The force transmitted to the instrument is given by

$F_t(t) = m\ddot{x}(t) = k[x(t) - y(t)] + c[\dot{x}(t) - \dot{y}(t)]$ ..(6.34) where x(t) – y(t) is the relative displacement of the spring.

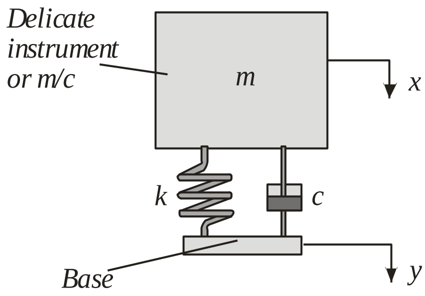


Figure B.4.9 Vibrating base

Consider the following

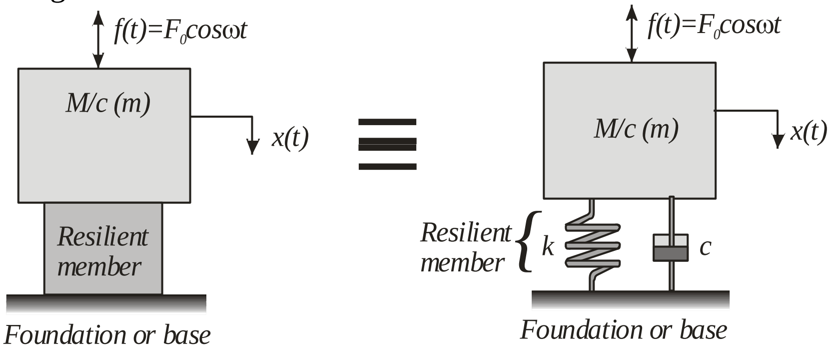


Figure B.4.10 M/C and Resilient member on rigid base

Equation of motion: $m\ddot{x} + c\dot{x} + kx = F_0 \cos\omega t$

Since transient solution dies out after some time, only the steady state solution is left. Hence

$$x = x_0 \cos(\omega t - \phi)$$

where $x_0 = \dfrac{F_0}{\sqrt{\left[\left(k - m\omega^2\right)^2 + (c\omega)^2\right]}}$ and $\phi = \tan^{-1}\left(\dfrac{c\omega}{k - m\omega^2}\right)$

The force transmitted to the foundation through the spring & the dashpot, $F_t(t)$ is given by

$$F_t(t) = kx(t)c\dot{x}(t) = kx_0 \cos(\omega t - \phi) - c\omega x_0 \sin(\omega t - \phi) \quad \text{……….(6.35)}$$

The magnitude of the total transmitted force (F_T) is given by

$$F_T = \sqrt{(kx)^2 + (c\dot{x})^2} = x_0\sqrt{k^2 + (\omega c)^2}$$

$$F_T = \frac{F_0\sqrt{k^2 + (\omega c)^2}}{\sqrt{\left[\left(k - m\omega^2\right)^2 + (c\omega)^2\right]}} \quad \text{…………………………………(6.36)}$$

Definition: The transmissibility or transmission ratio of the isolator (T_r) is defined as the ratio of the magnitude of the force transmitted to that of the exciting force.

$$T_r = \frac{F_T}{F_0} = \sqrt{\frac{k^2 + (c\omega)^2}{\left(k - m\omega^2\right)^2 + (c\omega)^2}}$$

$$\Rightarrow T_r = \sqrt{\frac{1+(2\xi\frac{\omega}{\omega_n})^2}{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2+(2\xi\frac{\omega}{\omega_n})^2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6.37)$$

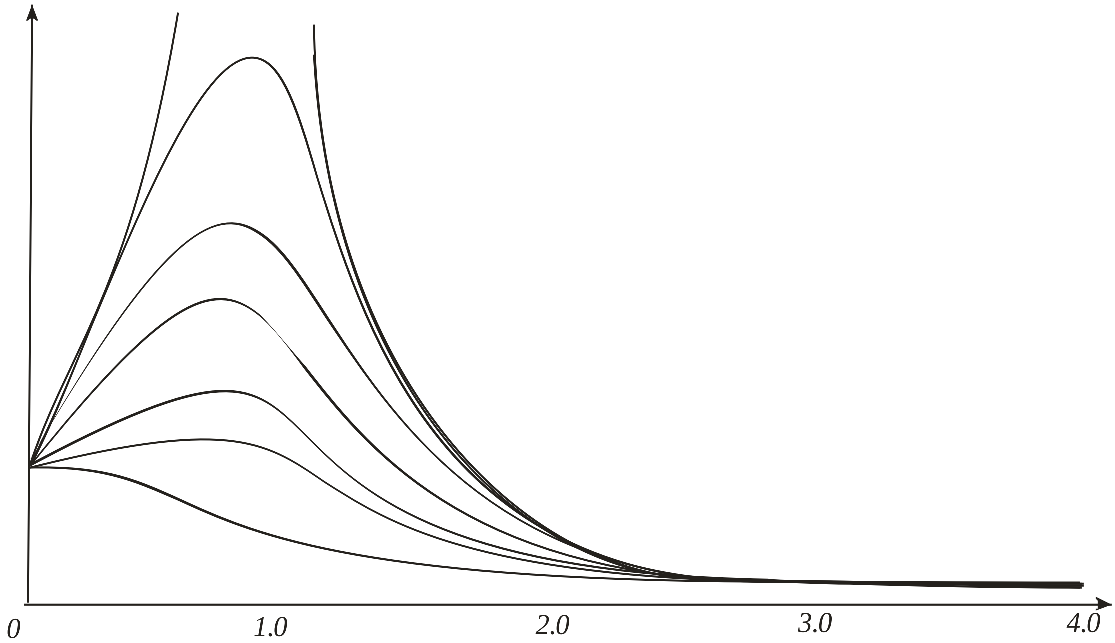


Figure B.4.11 Variation of Transmission ratio (T_r) with ω

Note:

1. In order to achieve isolation, the force transmitted to the foundation needs to be less than the exciting force.
2. Fig B.4.11 it is seen that the forcing frequency has to be greater than $\sqrt{2}$ times the natural frequency of the system in order to achieve isolation.
3. The magnitude of the force transmitted to the foundation can be reduced by decreasing the natural frequency of the system (ω_n)
4. The force transmitted to the foundation can also be reduced by decreasing the damping ratio.
5. Other common types of forcing functions are a moving base and rotating unbalance. These have different expressions for the transmissibility

(i) moving base

$m\ddot{z}+c\dot{z}+kz=-m\ddot{y}$ where z = x – y

displacement transmissibility $T_d = \frac{x}{y} = \sqrt{\frac{1+(2\xi r)^2}{(1-r^2)^2+(2\xi r)^2}}$

(ii) Rotating unbalance

$$m\ddot{x}+c\dot{x}+kx = F_0\sin\omega t = me\omega^2\sin\omega t$$

$$T_r = \frac{F_T}{F_0} = \frac{F_T}{me\omega^2} = \frac{F_T}{mer^2\omega_n^2}$$

$$\Rightarrow \frac{F_T}{me\omega_n^2} = r^2\sqrt{\frac{1+(2\xi r)^2}{(1-r^2)^2+(2\xi r)^2}}$$