EEE3112 - ELECTRICAL ENGINEERING PRACTICE

MECHANICAL COMPONENT

Static Mechanics:

- Deflection of beams with concentrated and distributed loading

Introduction

Before we can consider deflection we need to understand:

A. Bending Stresses and

B. Shearing Forces and Bending Moments

Bending Stress

3.1 : Pure bending

- A Length of beam is acted upon by a constant bending moment (zero shearing force), stresses set up on any cross-section must constitute a pure couple equal in magnitude to the bending moment.

- Hence it can be deduced that a part of the cross-section is in compression and the other in tension.

- Surfaces at which stress is zero is NN (neutral surface)

Fig. 3.1

Pure bending continued …

Let σ be the longitudinal stress in a filament *ab* at a distance (y) from NN, the

strain $\sigma/E = (ab - NN)/NN$

$= [(R+y)\theta - R\theta]/R\theta$

$= y/R$ or

Therefore $\sigma/y = E/R$ ---------------------(1)

Pure bending continued..

- It is apparent at this stage that, since *E/R* is constant, the stress is proportional to the distance from the neutral axis *XX* (***Fig 3.1 (c))***

- and that for purposes of economy and weight reduction the material should be concentrated as much as possible at the greatest distance from the neutral axis.
- Hence the universal adoption of ***I*** – beams for steel beams.

- **Pure bending continued …**
- Three equilibrium equations can be obtained for the system of parallel stresses on any cross-section
- If δA is an element of cross-sectional area at a distance y from the neutral axis (Fig 3.1 (b)) then for pure bending the net normal force on the cross-section must be zero, i.e.

$$\int \sigma . dA = 0 \text{ or } (E/R) \int y . dA = 0 \text{ from } (1)$$

- This is the condition that XX passes through the centroid of the section.

Pure bending continued..

The bending moment is balanced by the moment of the normal forces about XX, i.e.

$$M = \int \sigma y.dA$$

$$= (E/R)\int y^2.dA \qquad \text{from (1)}$$

$$= EI/R,$$

where I ($\int y^2.dA$) is the property of the cross-section known as the moment of inertia or second moment of area.

or

$$M/I = E/R \qquad \text{------------------(2)}$$

From (1) and (2)

$$\boxed{\sigma/y = M/I = E/R} \qquad \text{--------------(3)}$$

- **Pure bending continued …**
- In order to satisfy the conversion of signs, y should be taken as positive when measured outwards from the centre of curvature, and negative when inwards.
- The ratio $I/\hat{y}$ is called the section modulus Z, so that $\sigma_{max} = M/Z$.
- The bending moment which can be carried by a given section for a limiting maximum stress is called the **moment of resistance**

Pure bending continued …

- For pure bending about the neutral axis moments about axis YY must be zero

$$\int \sigma x.dA = 0 \text{ or } \int xy.dA = 0 \text{ from (1)}$$

- This integral is referred to as the product of inertia

- It is important to use consistent units in the bending formula, e.g

σ N/mm2,

y mm,

M Nmm,

I mm4,

E N/mm2,

R mm

3.2 Moments of Inertia

- You maybe familiar with moments of inertia of a rigid body, which is a property obtained by summing the products of particle mass and square of its distance from a given axis, for all the particles in the body. This function is involved in all problems of angular motion.

- By analogy with mass moment of inertia the summation of areas times distance squared from a fixed axis, which arose in the proof of the previous paragraph, is called the moment of inertia (I) of the cross-section about that axis.

- An alternative name is the second moment of area, the first moment being the sum of the areas times their distance from a given axis.

- **Moment of Inertia continued ...**
- By definition Ix $= \int y^2.dA$ about XX' axis (Fig 3.2) and Iy = $\int x^2.dA$
- The moment of inertia about an axis through O perpendicular to the figure is called the Polar Moment of Inertia

$$J = \int r^2.dA$$

$$= \int (x^2 + y^2).dA$$

$$= I_x + I_y$$

- This relation is referred to as the perpendicular axis theorem, and maybe stated as follows:
- "The sum of moments of inertia about any two axes in the plane is equal to the moment of inertia about the axis perpendicular to the plane, the three axes being concurrent".

Moment of Inertia continued..

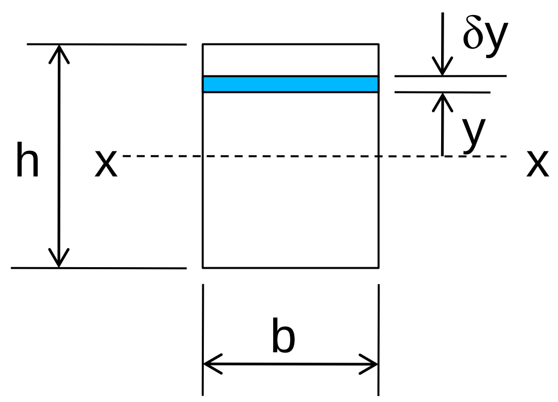
- **Circular Section**
- To calculate the polar moment of inertia about O (Fig 3.3), $\delta A = 2\pi r.\delta r$
- Therefore $J = \int r2.2\pi r.dr$

$= 2\pi[r4/4]$

$= \pi d4/32$

- **Circular Section continued …**
- But $J = Ix + Iy$, by perpendicular axis theorem, and since Ix and Iy are both equal being moments of inertia about a diameter
- $I\text{-dia} = 1/2J = \pi d4/64$
- For a hollow circular section of diameters D, d
- $J = (\pi/32)(D4\text{-}d4)$ and $I = (\pi/64)(D4\text{-}d4)$

- **Moment of Inertia continued..**
- **Rectangular Section**

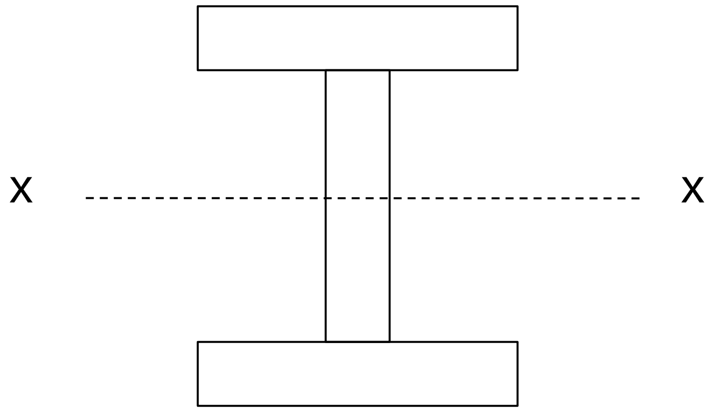


$\delta A = b\delta y$

$I_x = \int y^2.b\,dy$, where y varies from = $-h/2$ to $+h/2$

$I_x = bh^3/12$

- **I section**
- Derive the area moment of inertia (Ix) of an I-section beam



References

- G.H Ryder, Strength of Materials,

Next we look at

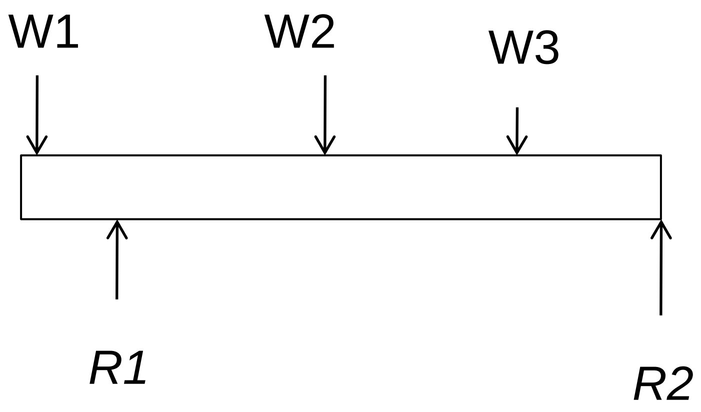
Shearing Force and Bending Moments in Beams

Shearing Force in beams

4.1 : Shearing Force

- The shearing force at any section of a beam represents the tendency for the portion of the beam to one side of the section to slide or shear laterally relative to the other portion.

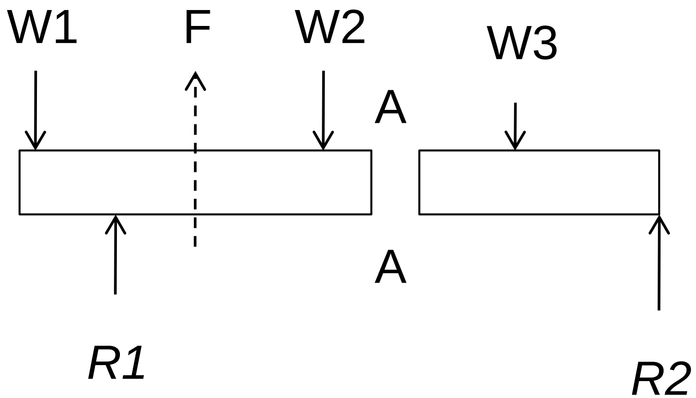
- A beam carrying loads W1, W2, and W3 is simply supported at two points.



The reactions at the supports being R1 and R2.

Shearing force continued…

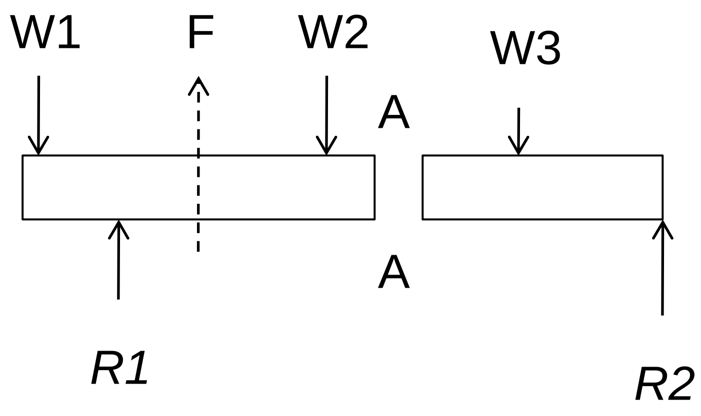
- Now imagine the beam to be divided into two portions by a section at AA.



- The resultant of the loads and reactions to the left of AA is F vertically upwards, and since the whole beam is in equilibrium, the resultant of the forces to the right of AA must also be F, acting downwards.

Shearing force continued..

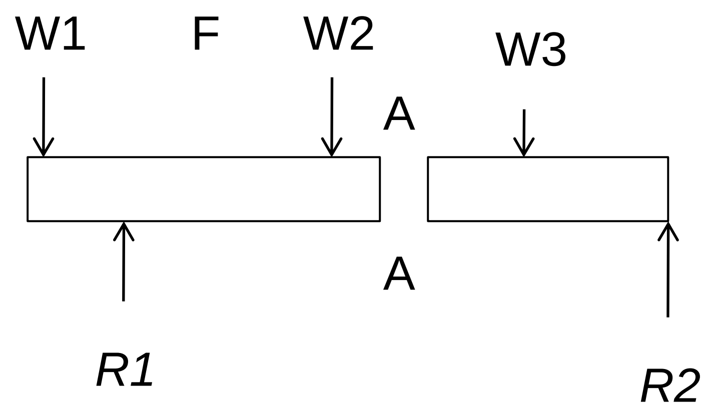
- F is called the Shearing Force (abbrev. S.F.) at the section AA and maybe defined as follows: **the shearing force at any section of a beam is the algebraic sum of the lateral components of the forces acting on either side of the section.**



- **Shearing force continued …**
- Where a force is in neither the axial nor the lateral direction it must be resolved in the usual way, the lateral component being taken into account in the shearing.
- Shearing force will be considered positive when the resultant of the forces to the left is upwards, or to the right is downwards.
- **Shearing force diagram** is one which shows the variation of shearing force along the length of the beam.

4.2 Bending Moment

- In a similar manner it can be argued that the moment about the section AA of the forces to the left is ***M*** clockwise, then the moment of the forces to the right of AA must be ***M*** anticlockwise.



- ***M*** is called the **Bending Moment** (abbr. B.M.) at AA, and is defined as: the algebraic sum of the moments about the section of all the forces acting on either side of the section.

- **Bending Moment continued …**
- Bending moment will be considered positive when the moment on the left portion is clockwise, and on the right portion anticlockwise.
- This is referred to as sagging bending moment since it tends to make the beam concave upwards at AA.
- Negative bending moment is termed hogging.
- A **bending moment diagram** is one which shows the variation of bending moment along the length of the beam.

4.3 Types of Load

- A **beam** is normally horizontal, the loads being vertical, other cases which occur being looked upon as exceptions

- A **concentrated load** is one which is considered to act at a point, although in practice it must really be distributed over a small area.

- A **distributed load** is one which is spread in some manner over the length of the beam.

- The **rate of loading w** maybe uniform, or may vary from point to point along the beam.

4.4 Types of Support

- A **simple or free support** is one on which the beam is rested, and which exerts a reaction on the beam.

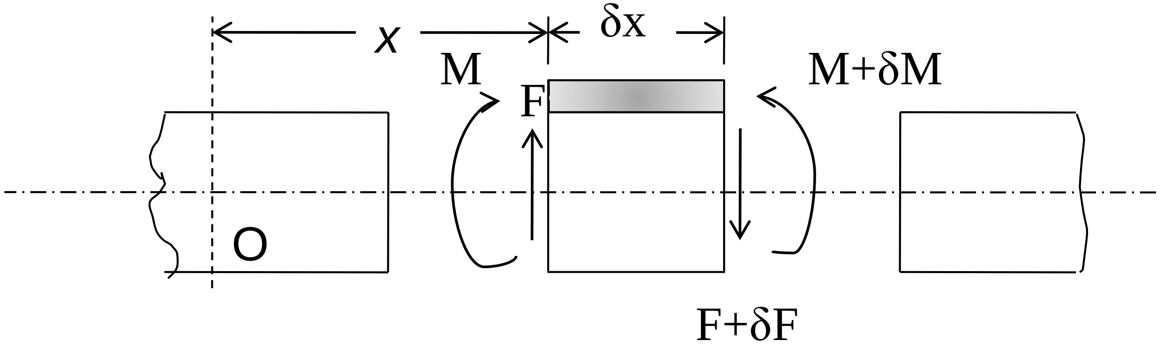
- A **built-in or encastre support** is frequently met with, **the effect being to fix the direction of the beam at the support**.

- In order to do this the support must exert a ‘fixing” moment M and a reaction R on the beam.

- A beam thus fixed at one end is called a cantilever

4.5 Relations between w, F, and M

- Consider a short length δx imagined to be a slice cut out from a loaded beam at a distance x from a fixed origin O
- Let the shearing force at the section x be F, and at $x + \delta x$ be $F + \delta F$.
- Similarly the bending moment is M at x, and $M + \delta M$ at $x + \delta x$.
- If w is the mean rate of loading on the length δx, the total load is $w\delta x$, acting approximately (exactly, if uniformly distributed) through the center C.



Relations between w, F, and M Continued …

- The elements must be in equilibrium under the action of these forces and couples, and the following equations are obtained.
- Taking moments about C:
- $M + F.\delta x/2 + (F + \delta F)\delta x/2 = M + \delta M$
- Neglecting the product $\delta F.\delta x$, and taking the limit, gives
- $F = dM/dx$ ---------------------(1)

- **Relations between w, F, and M Continued …**

- Resolving vertically

$$w\delta x + F + \delta F = F$$

$$w = -dF/dx \text{ --------------------} \quad (2)$$

$$= -d^2M/dx^2 \text{ from (1) -----------} \quad (3)$$

- **Relations between w, F, and M Continued …**
- From equation (1) it can be seen that, if M is varying continuously, zero shearing force corresponds to maximum or minimum bending moment.

➢ $F = dM/dx$ --------------------(1)

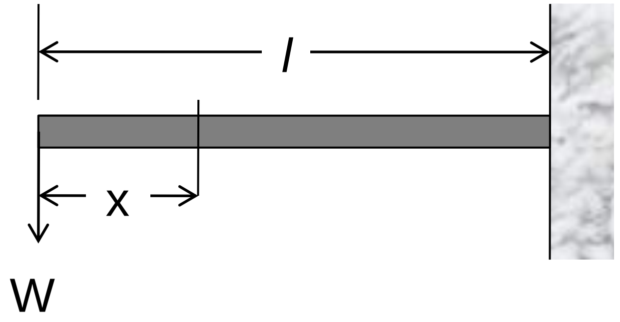
- It will be seen later however that "peaks" in the bending moment diagram frequently occur at concentrated loads or reactions, and are not then given by $F = dM/dx = 0$, although they may represent the greatest bending moment on the beam.
- Consequently it is not always sufficient to instigate the points of zero shearing force when determining the maximum bending moment.
- At a point on the beam where the type of bending is changing from sagging to hogging, the bending moment must be zero, and this is called a point of **inflection or contraflecture**.

- **Relations between w, F, and M Continued …**
- By integrating equation (1) between two values of x = a and x = b, then $M_b - M_a = \int F\,dx$
- Showing that the increase in bending moment between two sections is given by the area under the shearing force diagram

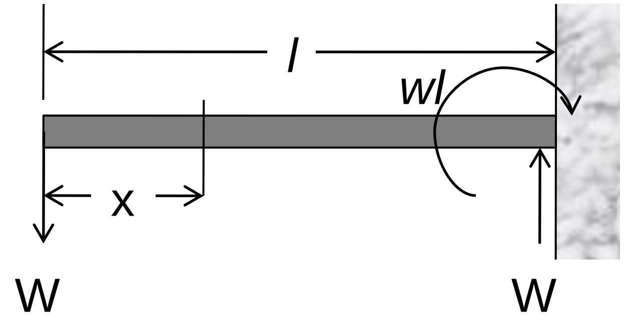
- **Relations between w, F, and M Continued …**
- Similarly, integrating equation (2)
- $Fa - Fb = \int w dx$
- = the area under the load distribution diagram.
- Integrating equation (3) gives
- $Ma - Mb = \iint w dx . dx$
- These relations prove very valuable when the rate of loading cannot be expressed in algebraic form, and provide a means of graphical solution.

Examples

- **4.6 Concentrated Loads**
- **Example 1.** A cantilever of length l carries a concentrated load W at its free end. Draw the S.F. and B.M. diagrams



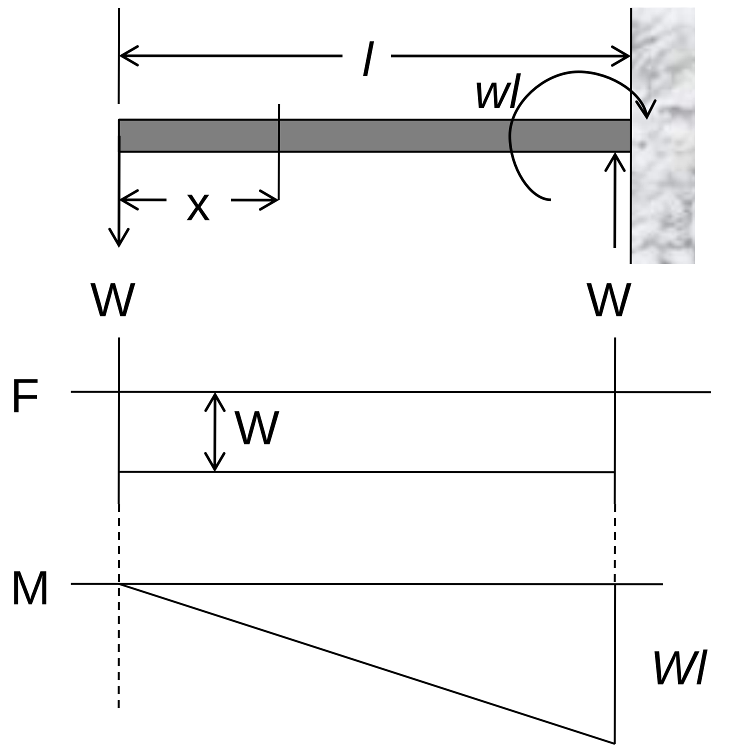
- **Example 1 continued …**
- **Solution:**
- At a section a distance x from the free end, consider the forces to the left.
- Then F = -W and is constant along the whole beam (i.e for all x).



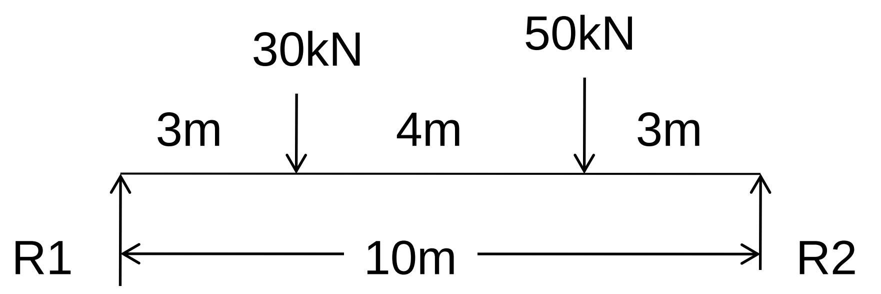
- Taking moments about the section gives M = -Wx, so that the maximum bending moment occurs at the fixed end., i.e. Mmax = Wl (hogging)
- From equilibrium considerations, the fixing moment applied at the built in end is Wl, and the reaction is W.

Concentrated load continued

- The S.F. and B.M. diagrams are therefore as shown:



- **Example 2.** A beam 10m long is simply supported at its ends and carries concentrated loads of 30kN and 50kN at distances of 3 m from each end. Draw the S.F. and B.M. diagrams.



- **SOLUTION**
- First calculate the reactions R1 and R2 at the supports

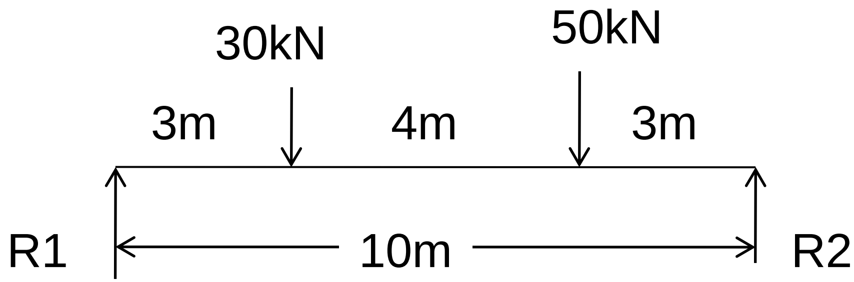
- By moments about R2:
- R1 x 10 = 30x7+50x3
- Therefore R1 = 36kN
- and R2 = 30+50-R1=44kN
- Let x be the distance of the section from the left – hand end
- **Shearing force:**

  0 < x < 3, **F** = **R1** = 36kN

  3 < x < 7, **F** = **R1**-30=6kN

  7 < x < 10, **F** = **R1** – 30 – 50 = -44kN

- Note that the last value = -R2, which provides a check on the working



- **Bending Moment:**

  $0 < x < 3$, $M = R1x = 36x$ kNm

  $3 < x < 7$, $M = R1x - 30(x-3) = 6x + 90$ kNm

  $7 < x < 10$, $M = R1x - 30(x-3) - 50(x - 7)$

  $= -44x + 440$ kNm

Having developed the background Now we are ready for:

Deflection of Beams

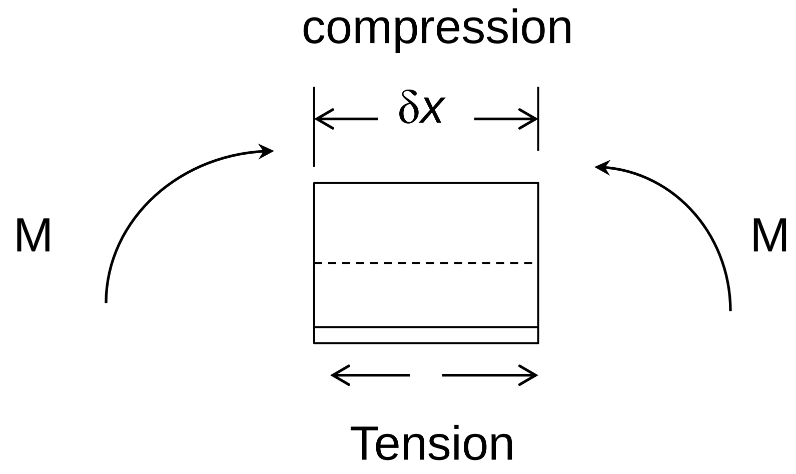
5.1 : Strain Energy due to Bending

From the theory of pure bending the bending moment is balanced by the moment of the normal forces, i.e. $\sigma/y = M/I = E/R$

$$\delta U = \int (\frac{\sigma^2}{2E}) x volume$$

$$= \delta x \int \frac{\sigma^2}{2E} . dA$$

$$= (\frac{\delta x}{2E}) \int \frac{M^2 y^2}{I^2} dA$$



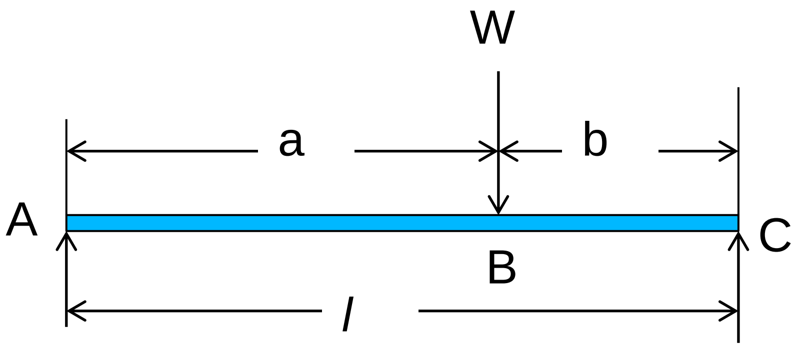
But $\int y^2 . dA = I$ Hence $\delta U = (\frac{M^2}{2EI}) \delta x$

For the whole beam $U = \int \frac{M^2 . dx}{2EI}$

The product EI is called the flexural rigidity of the beam

Example 1.

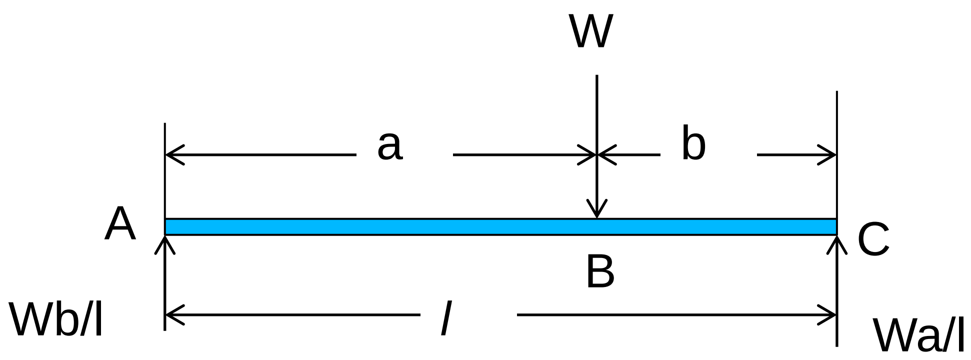
- A simply supported beam of length l carries a concentrated load ***W*** at distances of ***a*** and ***b*** from the two ends. Find expressions for the total strain energy of the beam and the deflection under the load.
- **SOLUTION**
- Determine the reactions at ***A*** and ***B***



Example continued..

- The integration for strain energy can only be applied over a length of beam for which a continuous expression for M can be obtained.

- This usually implies a separate integration for each section between two concentrated loads or reactions.



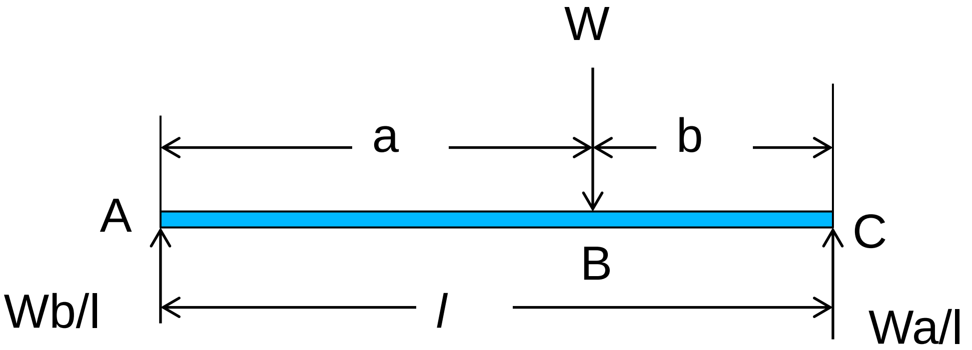
- **Example continued …**
- For the section AB

$$M = (Wb/l)x$$

$$U_a = \int_0^a \frac{W^2b^2x^2}{2l^2EI}.dx$$

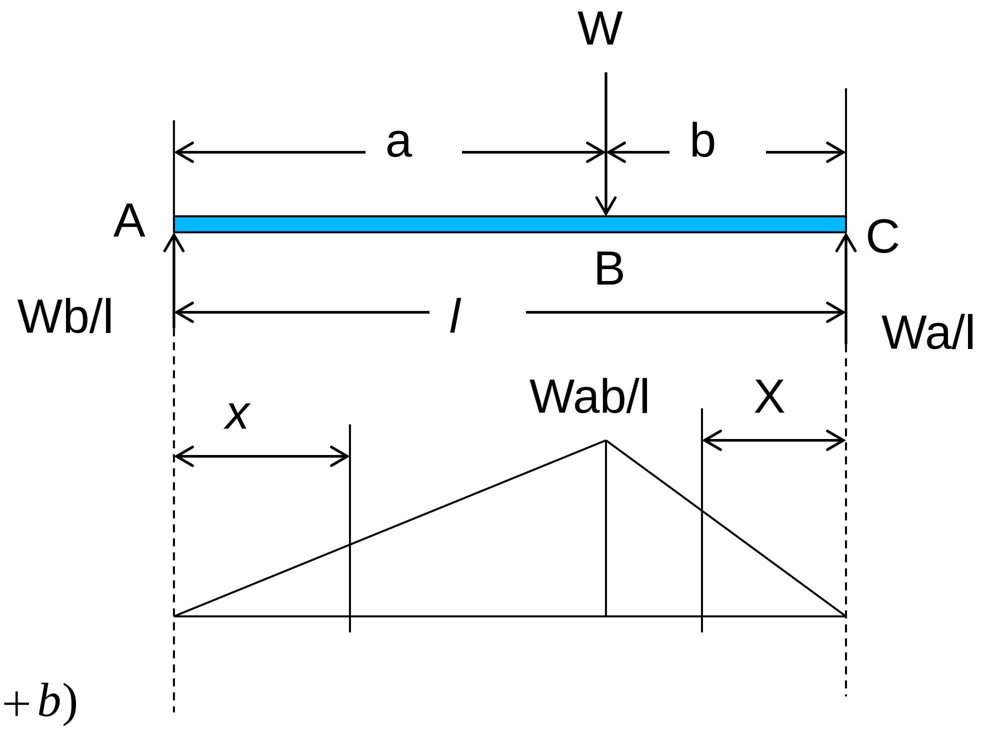
$$= \frac{W^2b^2}{2l^2EI}\left[\frac{x^3}{3}\right]_0^a$$

$$= \frac{W^2a^3b^2}{6EIl^2}$$



Example 1 continued …

By taking a variable *X* measured from C



$$M = (Wa/l)X$$

$$U_b = \int_0^b \frac{W^2 a^2 X^2}{2l^2 EI} . dX$$

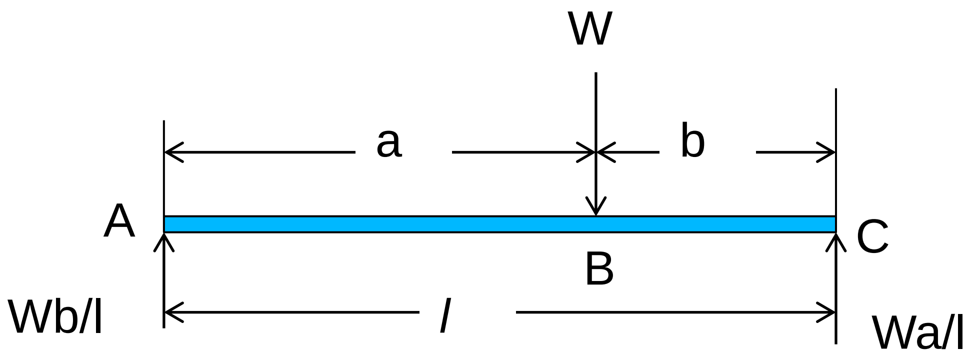
$$= \frac{W^2 a^2 b^3}{6EIl^2}$$

$$\text{Total } U = U_a + U_b = \left(\frac{W^2 a^2 b^2}{6EIl^2}\right)(a+b)$$

$$= \frac{W^2 a^2 b^2}{6EIl}$$

- Example 1 continued…

$$U = \frac{W^2 a^2 b^2}{6EIl}$$



But if δ is the deflection under the load, the strain energy must equal the work done by the load (gradually applied) i.e

$$\frac{1}{2} W\delta = \frac{W^2 a^2 b^2}{6EIl}$$

$$\therefore \ \delta = \frac{W a^2 b^2}{3EIl}$$

Example 1 continued…

- This method of finding deflection is limited to cases where only one concentrated load is applied (and only gives the deflection under the load).
- A more general application of strain energy to deflection is found in **Castigliano's Theorem.**

Calculus Method

5.2 Deflection by Calculus

- It was proved that for bending about a principle axis

$$\frac{M}{EI} = \frac{1}{R}$$

From calculus (analytic geometry) $\frac{1}{R} = \frac{\pm \frac{d^2 y}{dx^2}}{\left[1 + (\frac{dy}{dx})^2\right]^{3/2}}$

For beams met in engineering practice the slope dy/dx is small and maybe neglected in comparison with 1

$$Hence \quad \frac{1}{R} = \frac{d^2 y}{dx^2}$$

- **Deflection by calculus continued…**

Hence $$\frac{M(x)}{EI} = \frac{d^2y}{dx^2}$$

or $$EI.\frac{d^2y}{dx^2} = M(x)$$

Can be integrated to give the **slope dy/dx** and the **deflection y**

•Differentiating

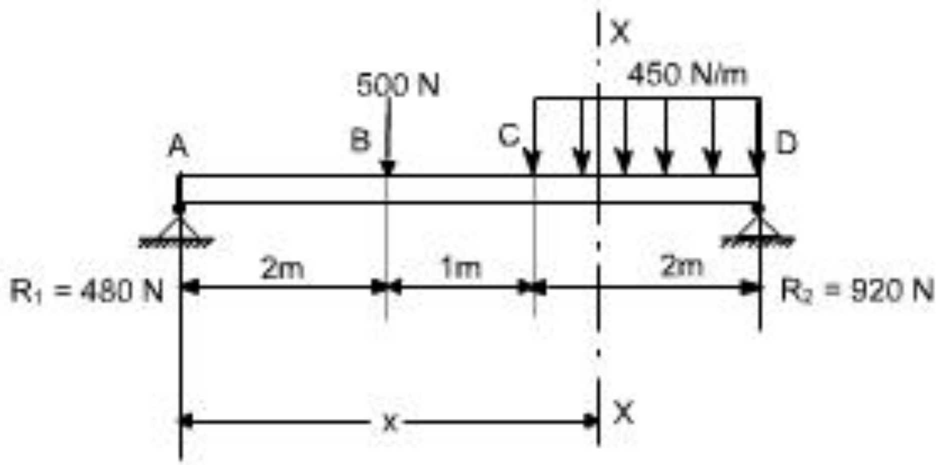
$$EI\frac{d^3y}{dx^3} = \frac{dM(x)}{dx} = F(x)$$

$$EI.\frac{d^4y}{dx^4} = \frac{dF}{dx} = -w$$

Macauley’s Method..

- If the loading conditions change along the span of beam, there is corresponding change in moment equation.
- This requires that a separate moment equation be written between each change of load point and that two integration be made for each such moment equation. Evaluation of the constants introduced by each integration can become very involved.
- Fortunately, these complications can be avoided by writing single moment equation in such a way that it becomes continuous for entire length of the beam in spite of the discontinuity of loading.
- **Note :** In Macaulay's method some author's take the help of unit function approximation (i.e. Laplace transform) in order to illustrate this method, however both are essentially the same.

- **Macauley's continued …**
- For example consider the beam shown in fig below:
- Let us write the general moment equation using the definition $M = (\sum M)_L$, Which means that we consider the effects of loads lying on the left of an exploratory section. The moment equations for the portions AB,BC and CD are written as follows



$$M_{AB} = 480\ x\ \text{N.m}$$

$$M_{BC} = \left[480\ x - 500\,(x-2)\right] \text{N.m}$$

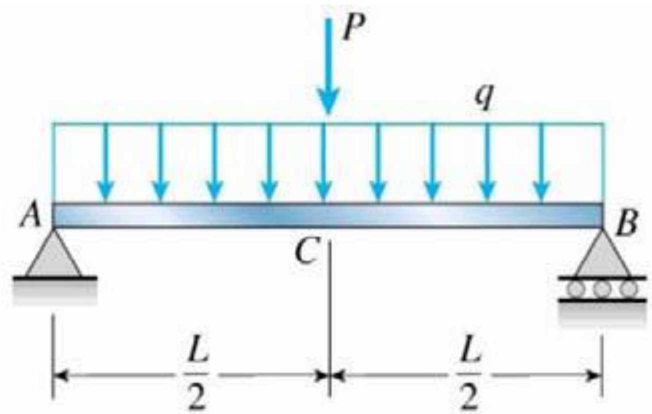
$$M_{CD} = \left[480\ x - 500\,(x-2) - \frac{450}{2}(x-3)^2\right] \text{N.m}$$

- It may be observed that the equation for M_{CD} will also be valid for both M_{AB} and M_{BC} provided that the terms (x - 2) and (x - 3)^{2}are neglected for values of x less than 2 m and 3 m, respectively. In other words, the terms (x - 2) and (x - 3)^2 are nonexistent for values of x for which the terms in parentheses are negative.

- As an clear indication of these restrictions,one may use a nomenclature in which the usual form of parentheses is replaced by pointed brackets, namely, ‹ ›. With this change in nomenclature, we obtain a single moment equation

$$M = \left(480x - 500\left(x - 2\right) - \frac{450}{2}\left(x - 3\right)^2\right) N.m$$

- **Method of Superposition**
- the slope and deflection of beam caused by several different loads acting
- simultaneously can be found by superimposing the slopes and deflections
- caused by the loads acting separately



(a)

References

- G.H Ryder, Strength of Materials

Deflection by Castigliano's Theorem

To use this method…
You should have some background with:

- Deflection of beam/cylinder due to:
  - Axial loading
  - Bending
  - Torsion

- Calculating normal and polar moments of inertia

- Deriving equations for linear changes in quantities

- Using singularity functions (for more advanced applications; no examples here explicitly show it, but it is often used in conjunction with castigliano's Theorem).

- **Definition**

✓ Determining the deflection of beams typically requires repeated integration of singularity functions **(done in previous slides of this lecture).**

- Castigliano’s Theorem lets us use strain energies at the locations of forces to determine the deflections

- The Theorem also allows for the determining of deflections for objects with changing cross sectional areas.

Definition.

Castigliano's Theorem is given as:

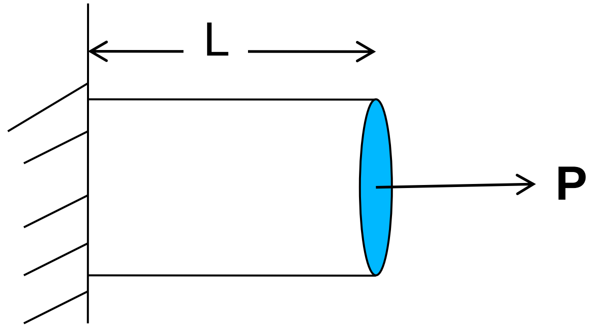
$$\delta = \frac{\partial U}{\partial P}$$

Where δ is the deflection, U is the strain energy and P is the force (or torque) at a certain point

Variations

Different loading conditions require different strain energies.

For axial loading:

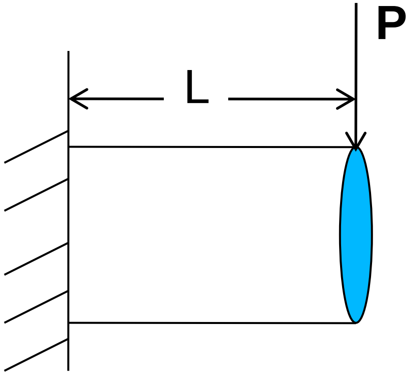


$$U = \int_0^L \frac{P^2 dx}{2EA}$$

Where P is the load, E is the material's Young's Modulus (usually in Gpa), A is the cross sectional area, and L is the length

Variations

For a material in bending:

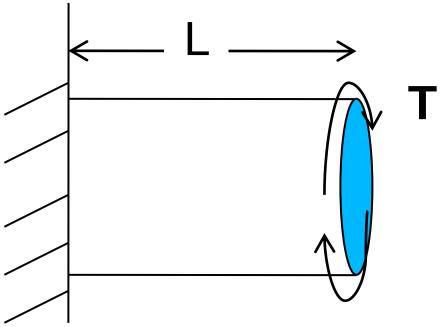


$$U = \int_0^L \frac{M^2 dx}{2EI}$$

Where **M** is the moment applied, and **I** is the area moment of inertia.

Variations

For a material in torsion



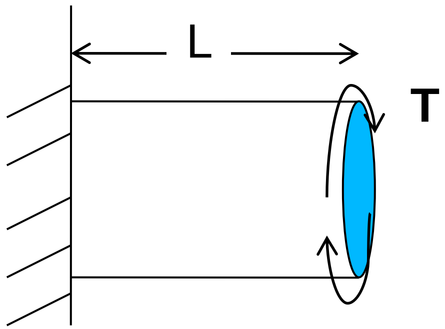
$$U = \int_0^L \frac{T^2 dx}{2GJ}$$

Where T is the torque applied, G is the Modulus of Rigidity, and J is the polar moment of inertia.

- **Variations**
- Note: except for the Young’s Modulus and Modulus of Rigidity (E and G), it is not guaranteed that the other variables are not functions of x.

- Sometimes dimensions of the material change as functions of x, and thus the moments of inertia change; and sometimes the forces applied may vary with x.

- Examples
- Imagine a cylinder attached to a fixed wall, with constant diameter d = 4 cm and length L = 2 m, and a torque of 8 Nm is applied. Assume G = 120 Gpa.



- To find the displacement of the cylinder, we use Castigliano's Theorem with the strain energy for torsion

- Continuing example:
- deflection

$$\delta = \frac{\partial}{\partial T}\left[\int_0^L \frac{T^2 dx}{2GJ}\right]$$

- With

$$J = \frac{\pi}{32} d^4$$

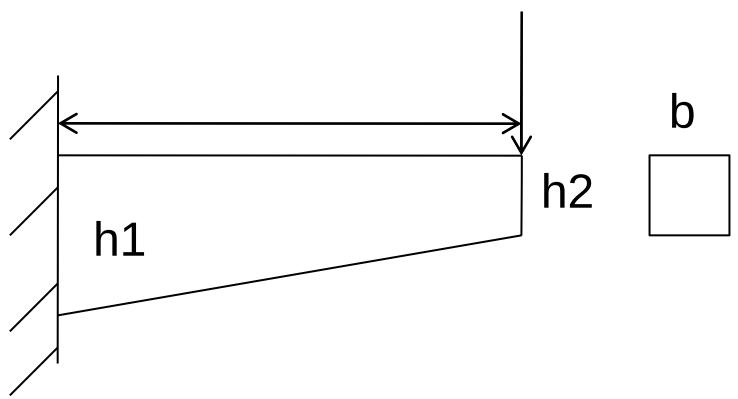
- The area and moment of inertia are not changing, so we can easily find the displacement.

- Continuing example:
- Because differentiating and integrating are linear operations, the partial derivative can be placed inside the integral:

$$\delta = \frac{\partial}{\partial T}\left[\int_0^L \frac{T^2 dx}{2GJ}\right] = \int_0^L \frac{\partial}{\partial T}\left[\frac{T^2}{2GJ}\right] dx = \int_0^L \frac{T dx}{GJ}$$

$$\delta = \frac{TL}{GJ} = \frac{8.2}{120x10^9 x \frac{\pi}{32} x0.4^4} = 0.5305mm$$

- Examples
- Imagine having a beam with a changing cross section shown below, with an initial height of 3 m and final height of 1 m, with a constant base length of 2 m. The beam has a length of 6 m, with a Young's Modulus of 120 GPa, and a force is applied with magnitude P = 10 kN.



- We will use Castigliano's Theorem applied for bending to solve for the deflection where M is applied.

$$\delta = \frac{\partial}{\partial P}\left[\int_0^L \frac{M^2 dx}{2EI}\right]$$

- To find M, we need to consider the circumstances. At the wall (x = 0) the moment felt is the maximum moment or PL, but at the end of the beam, the moment is zero because moments at the locations do not contribute to the overall moments.

- **Continuing example:**
- **And so;** $M(x) = PL - Px$

- The height is also a function of x, and the initial and final heights can be used to formulate an equation:

- $$h(x) = \frac{h_f - h_i}{L} x + h_i = -\frac{1}{3} x + 3$$

- Continuing example:
- And so the moment of inertia, as a function of x, is:

$$I(x) = \frac{1}{12}bh^3 = \frac{1}{12}(2)(-\frac{1}{3}x+3)^3$$

- Substituting the functions we have derived into the equation for displacement:

$$\delta = \int_o^l \frac{\partial}{\partial P}\left[\frac{(PL-Px)^2}{2E\left[\frac{1}{6}(-\frac{1}{3}x+3)^3\right]}\right]dx$$

- Continuing example:

$$\delta = \int_0^L \frac{(PL - Px)(L - x)dx}{E\left[\frac{1}{6}(-\frac{1}{3}x + 3)^3\right]}$$

$$\delta = \int_0^L \frac{(60000 - 10000x)(6 - x)dx}{E\left[\frac{1}{6}(-\frac{1}{3}x + 3)^3\right]}$$

$$\delta = 1.311\mu m$$

- **Summary**

- After this lecture, you should be confident in:

- Identifying a situation (whether in axial loading, bending, or torsion) where Castigliano's Theorem may be applied to solve for deflection in a beam or cylinder.

- Generate equations for the changes in height, base, or even force across the length of a beam/cylinder.

references

- Mechanics of Materials – Beer/Johnson 5th Edition
- Section 11.13 “Deflections by Castgliano’s Theorem”

- **SOURCE: A tutorial by “THE ARC”**

  *www.iit.edu/arc/workshops/pdfs/**Castigliano**_s_**Theorem**.pdf*