

Introduction to Dynamics and Vibrations

School of Engineering
Brown University

Home

5.4 Forced vibration of damped, single degree of freedom, linear spring mass systems.

Finally, we solve the most important vibration problems of all. In engineering practice, we are almost invariably interested in predicting the response of a structure or mechanical system to external forcing. For example, we may need to predict the response of a bridge or tall building to wind loading, earthquakes, or ground vibrations due to traffic. Another typical problem you are likely to encounter is to isolate a sensitive system from vibrations. For example, the suspension of your car is designed to isolate a sensitive system (you) from bumps in the road. Electron microscopes are another example of sensitive instruments that must be isolated from vibrations. Electron microscopes are designed to resolve features a few nanometers in size. If the specimen vibrates with amplitude of only a few nanometers, it will be impossible to see! Great care is taken to isolate this kind of instrument from vibrations. That is one reason they are almost always in the basement of a building: the basement vibrates much less than the floors above.

We will again use a spring-mass system as a model of a real engineering system. As before, the spring-mass system can be thought of as representing a single mode of vibration in a real system, whose natural frequency and damping coefficient coincide with that of our spring-mass system.

We will consider three types of forcing applied to the spring-mass system, as shown below:

External Forcing models the behavior of a system which has a time varying force acting on it. An example might be an offshore structure subjected to wave loading.

Base Excitation models the behavior of a vibration isolation system. The base of the spring is given a prescribed motion, causing the mass to vibrate. This system can be used to model a vehicle suspension system, or the earthquake response of a structure.

Rotor Excitation models the effect of a rotating machine mounted on a flexible floor. The crank with small mass m_0 rotates at constant angular velocity, causing the mass m to vibrate.

Of course, vibrating systems can be excited in other ways as well, but the equations of motion will always reduce to one of the three cases we consider here.

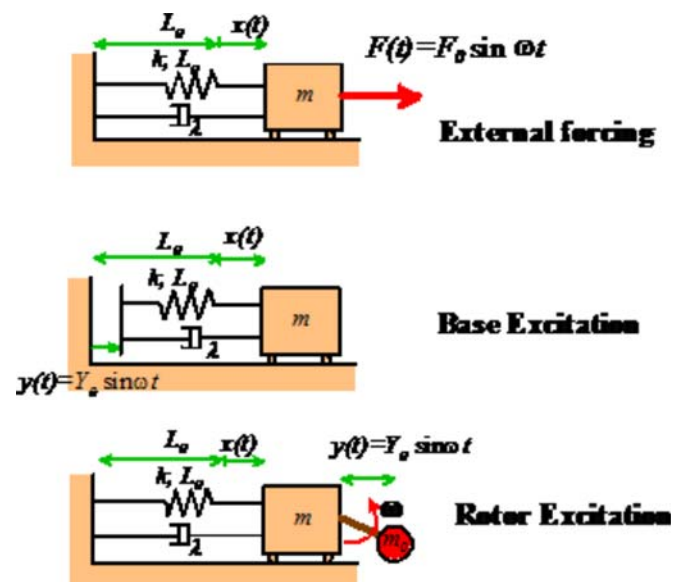
Notice that in each case, we will restrict our analysis to **harmonic excitation**. For example, the external force applied to the first system is given by

$$F(t) = F_0 \sin \omega t$$

The force varies harmonically, with amplitude F_0 and frequency ω . Similarly, the base motion for the second system is

$$y(t) = Y_0 \sin \omega t$$

and the distance between the small mass m_0 and the large mass m for the third system has the same form.



We assume that at time $t=0$, the initial position and velocity of each system is

$$x = x_0 \quad \frac{dx}{dt} = v_0$$

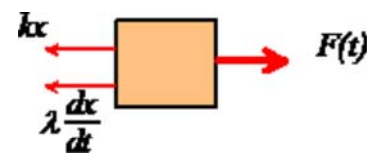
In each case, we wish to calculate the displacement of the mass x from its static equilibrium configuration, as a function of time t . It is of particular interest to determine the influence of forcing amplitude and frequency on the motion of the mass.

We follow the same approach to analyze each system: we set up, and solve the equation of motion.

5.4.1 Equations of Motion for Forced Spring Mass Systems

Equation of Motion for External Forcing

We have no problem setting up and solving equations of motion by now. First draw a free body diagram for the system, as show on the right



Newton's law of motion gives

$$m \frac{d^2 x}{dt^2} = F(t) - kx - \lambda \frac{dx}{dt}$$

Rearrange and substitute for $F(t)$

$$\frac{m}{k} \frac{d^2 x}{dt^2} + \frac{\lambda}{k} \frac{dx}{dt} + x = \frac{1}{k} F_0 \sin \omega t$$

Check out our list of solutions to standard ODEs. We find that if we set

$$\omega_n = \sqrt{\frac{k}{m}}, \quad \zeta = \frac{\lambda}{2\sqrt{km}}, \quad K = \frac{1}{k},$$

our equation can be reduced to the form

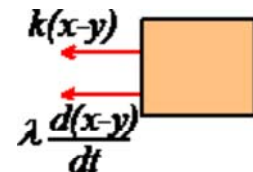
$$\frac{1}{\omega_n^2} \frac{d^2 x}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dx}{dt} + x = KF_0 \sin \omega t$$

which is on the list.

The (horrible) solution to this equation is given in the list of solutions. We will discuss the solution later, after we have analyzed the other two systems.

Equation of Motion for Base Excitation

Exactly the same approach works for this system. The free body diagram is shown in the figure. Note that the force in the spring is now $k(x-y)$ because the length of the spring is $L_0 + x - y$. Similarly, the rate of change of length of the dashpot is $d(x-y)/dt$.



Newton's second law then tells us that

$$\begin{aligned} m \frac{d^2 x}{dt^2} &= -k(x-y) - \lambda \left(\frac{dx}{dt} - \frac{dy}{dt} \right) \\ \Rightarrow \frac{m}{k} \frac{d^2 x}{dt^2} + \frac{\lambda}{k} \frac{dx}{dt} + x &= y + \frac{\lambda}{k} \frac{dy}{dt} \end{aligned}$$

Make the following substitutions

$$\omega_n = \sqrt{\frac{k}{m}}, \quad \zeta = \frac{\lambda}{2\sqrt{km}}, \quad K = 1$$

and the equation reduces to the standard form

$$\frac{1}{\omega_n^2} \frac{d^2 x}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dx}{dt} + x = K \left(y + \frac{2\zeta}{\omega_n} \frac{dy}{dt} \right)$$

Given the initial conditions

$$x = x_0 \quad \frac{dx}{dt} = v_0$$

and the base motion

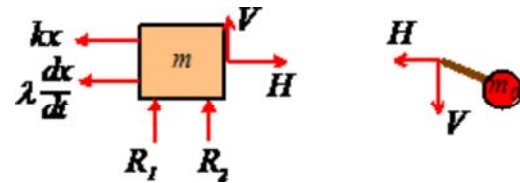
$$y(t) = Y_0 \sin \omega t$$

we can look up the solution in our handy list of solutions to ODEs.

Equation of motion for Rotor Excitation

Finally, we will derive the equation of motion for the third case.

Free body diagrams are shown below for both the rotor and the mass



Note that the horizontal acceleration of the mass m_0 is

$$a = \frac{d^2}{dt^2}(L_0 + x + y) = \frac{d^2 x}{dt^2} + \frac{d^2 y}{dt^2}$$

Hence, applying Newton's second law in the horizontal direction for both masses:

$$m \frac{d^2 x}{dt^2} = H - kx - \lambda \frac{dx}{dt}$$

$$m_0 \left(\frac{d^2 x}{dt^2} + \frac{d^2 y}{dt^2} \right) = -H$$

Add these two equations to eliminate H and rearrange

$$\frac{m + m_0}{k} \frac{d^2 x}{dt^2} + \frac{\lambda}{k} \frac{dx}{dt} + x = -\frac{m_0}{k} \frac{d^2 y}{dt^2}$$

To arrange this into standard form, make the following substitutions

$$\omega_n = \sqrt{\frac{k}{m + m_0}} \quad \zeta = \frac{\lambda}{2\sqrt{k(m + m_0)}} \quad K = \frac{m_0}{m + m_0}$$

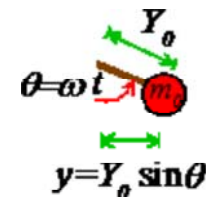
whereupon the equation of motion reduces to

$$\frac{1}{\omega_n^2} \frac{d^2 x}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dx}{dt} + x = -\frac{K}{\omega_n^2} \frac{d^2 y}{dt^2}$$

Finally, look at the picture to convince yourself that if the crank rotates with angular velocity ω , then

$$y(t) = Y_0 \sin \omega t$$

where Y_0 is the length of the crank.



The solution can once again be found in the list of solutions to ODEs.

5.4.2 Definition of Transient and Steady State Response.

If you have looked at the list of solutions to the equations of motion we derived in the preceding section, you will have discovered that they look horrible. Unless you have a great deal of experience with visualizing equations, it is extremely difficult to work out what the equations are telling

A Java applet posted at <http://www.brown.edu/Departments/Engineering/Courses/En4/java/forced.html> should help to visualize the motion. The applet will open in a new window so you can see it and read the text at the same time. The applet simply calculates the solution to the equations of motion using the formulae given in the list of solutions, and plots graphs showing features of the motion. You can use the sliders to set various parameters in the system, including the type of forcing, its amplitude and frequency; spring constant, damping coefficient and mass; as well as the position and velocity of the mass at time $t=0$. Note that you can control the properties of the spring-mass system in two ways: you can either set values for k , m and λ using the sliders, or you can set ω_n , K and ζ instead.

We will use the applet to demonstrate a number of important features of forced vibrations, including the following:

The steady state response of a forced, damped, spring mass system is independent of the initial conditions.

To convince yourself of this, run the applet (click on 'start' and let the system run for a while). Now, press 'stop'; change the initial position of the mass, and press 'start' again.

You will see that, after a while, the solution with the new initial conditions is exactly the same as it was before. Change the type of forcing, and repeat this test. You can change the initial velocity too, if you wish.

We call the behavior of the system as time gets very large the '**steady state**' response; and as you see, it is independent of the initial position and velocity of the mass.

The behavior of the system while it is approaching the steady state is called the '**transient**' response. The transient response depends on everything...

Now, reduce the damping coefficient and repeat the test. You will find that the system takes longer to reach steady state. Thus, the length of time to reach steady state depends on the properties of the system (and also the initial conditions).

The observation that the system always settles to a steady state has two important consequences. Firstly, we rarely know the initial conditions for a real engineering system (who knows what the position and velocity of a bridge is at time $t=0$?) . Now we know this doesn't matter – the response is not sensitive to the initial conditions. Secondly, if we aren't interested in the transient response, it turns out we can greatly simplify the horrible solutions to our equations of motion.

When analyzing forced vibrations, we (almost) always neglect the transient response of the system, and calculate only the steady state behavior.

If you look at the solutions to the equations of motion we calculated in the preceding sections, you will see that each solution has the form

$$x(t) = x_h(t) + x_p(t)$$

The term $x_h(t)$ accounts for the transient response, and is always zero for large time. The second term gives the steady state response of the system.

Following standard convention, we will list only the steady state solutions below. You should bear in mind, however, that the steady state is only part of the solution, and is only valid if the time is large enough that the transient term can be neglected.

5.4.3 Summary of Steady-State Response of Forced Spring Mass Systems.

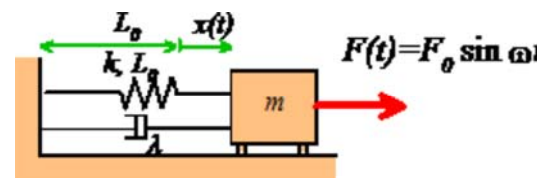
This section summarizes all the formulas you will need to solve problems involving forced vibrations.

Solution for External Forcing

Equation of Motion

$$\frac{1}{\omega_n^2} \frac{d^2x}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dx}{dt} + x = KF(t)$$

with



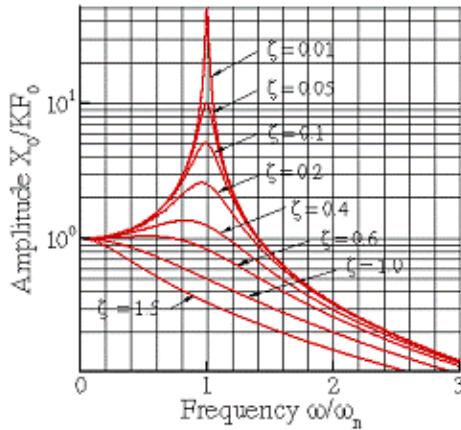
$$\omega_n = \sqrt{\frac{k}{m}}, \quad \zeta = \frac{\lambda}{2\sqrt{km}}, \quad K = \frac{1}{k}$$

Steady State Solution:

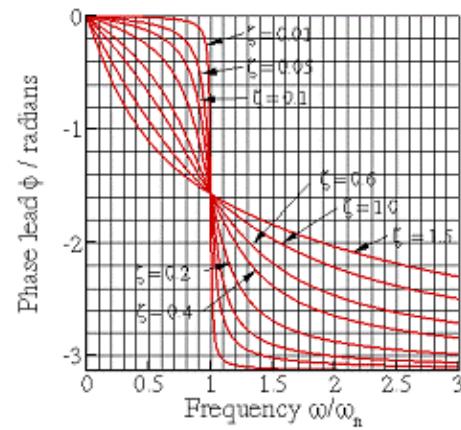
$$x(t) = X_0 \sin(\omega t + \phi)$$

$$X_0 = \frac{KF_0}{\left\{ \left(1 - \omega^2 / \omega_n^2\right)^2 + \left(2\zeta\omega / \omega_n\right)^2 \right\}^{1/2}} \quad \phi = \tan^{-1} \frac{-2\zeta\omega / \omega_n}{1 - \omega^2 / \omega_n^2}$$

The expressions for X_0 and ϕ are graphed below, as a function of ω / ω_n



(a)



(b)

Steady state vibration of a force spring-mass system (a) amplitude (b) phase.

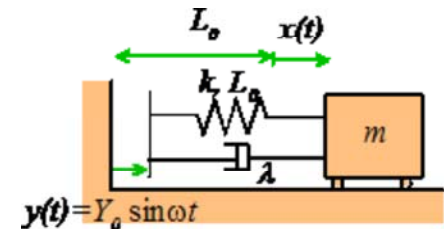
Solution for Base Excitation

Equation of Motion

$$\frac{1}{\omega_n^2} \frac{d^2x}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dx}{dt} + x = K \left(y + \frac{2\zeta}{\omega_n} \frac{dy}{dt} \right)$$

with

$$\omega_n = \sqrt{\frac{k}{m}}, \quad \zeta = \frac{\lambda}{2\sqrt{km}}, \quad K = 1$$

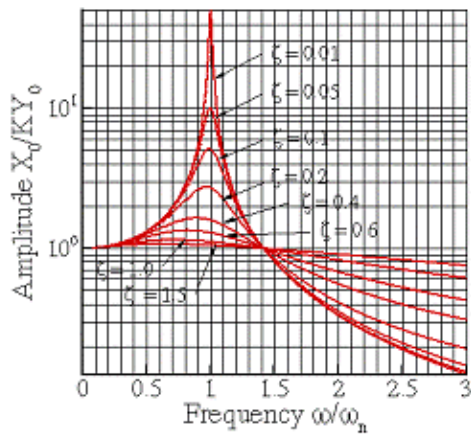


Steady State solution

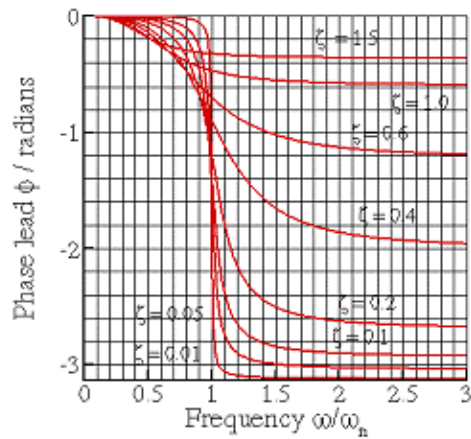
$$x(t) = X_0 \sin(\omega t + \phi)$$

$$X_0 = \frac{KY_0 \left\{ 1 + \left(2\zeta\omega / \omega_n \right)^2 \right\}^{1/2}}{\left\{ \left(1 - \omega^2 / \omega_n^2 \right)^2 + \left(2\zeta\omega / \omega_n \right)^2 \right\}^{1/2}} \quad \phi = \tan^{-1} \frac{-2\zeta\omega^3 / \omega_n^3}{1 - (1 - 4\zeta^2)\omega^2 / \omega_n^2}$$

The expressions for X_0 and ϕ are graphed below, as a function of ω / ω_n



(a)



(b)

Steady state vibration of a base excited spring—mass system (a) Amplitude and (b) phase

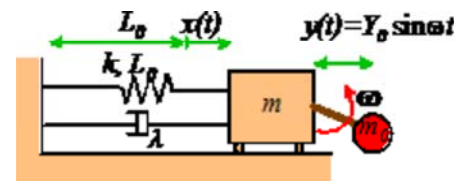
Solution for Rotor Excitation

Equation of Motion

$$\frac{1}{\omega_n^2} \frac{d^2x}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dx}{dt} + x = -\frac{K}{\omega_n^2} \frac{d^2y}{dt^2}$$

with

$$\omega_n = \sqrt{\frac{k}{m+m_0}} \quad \zeta = \frac{\lambda}{2\sqrt{k(m+m_0)}} \quad K = \frac{m_0}{m+m_0}$$

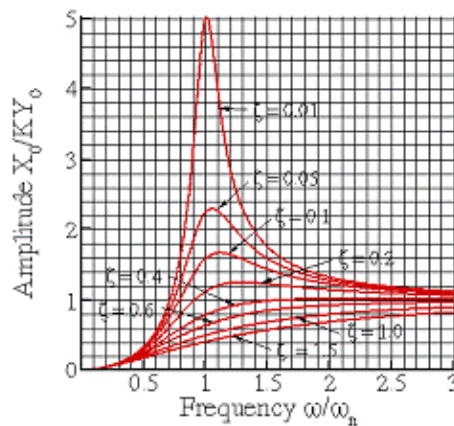


Steady state solution

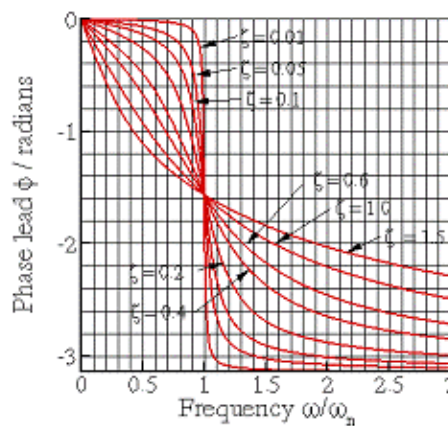
$$x(t) = X_0 \sin(\omega t + \phi)$$

$$X_0 = \frac{KY_0 \omega^2 / \omega_n^2}{\left\{ \left(1 - \omega^2 / \omega_n^2\right)^2 + \left(2\zeta \omega / \omega_n\right)^2 \right\}^{1/2}} \quad \phi = \tan^{-1} \frac{-2\zeta \omega / \omega_n}{1 - \omega^2 / \omega_n^2}$$

The expressions for X_0 and ϕ are graphed below, as a function of ω / ω_n



(a)



(b)

Steady state vibration of a rotor excited spring—mass system (a) Amplitude (b) Phase

5.4.4 Features of the Steady State Response of Spring Mass Systems to Forced Vibrations.

Now, we will discuss the implications of the results in the preceding section.

- The steady state response is always harmonic, and has the same frequency as that of the forcing.

To see this mathematically, note that in each case the solution has the form $x(t) = X_0 \sin(\omega t + \phi)$. Recall that ω defines the frequency of the force, the frequency of base excitation, or the rotor angular velocity. Thus, the frequency of vibration is determined by the forcing, not by the properties of the spring-mass system. This is unlike the free vibration response.

You can also check this out using our applet. To switch off the transient solution, click on the checkbox labeled 'show transient'. Then, try running the applet with different values for k , m and λ , as well as different forcing frequencies, to see what happens. As long as you have switched off the transient solution, the response will always be harmonic.

- The amplitude of vibration is strongly dependent on the frequency of excitation, and on the properties of the spring—mass system.

To see this mathematically, note that the solution has the form $x(t) = X_0 \sin(\omega t + \phi)$. Observe that X_0 is the amplitude of vibration, and look at the preceding section to find out how the amplitude of vibration varies with frequency, the natural frequency of the system, the damping factor, and the amplitude of the forcing. The formulae for X_0 are quite complicated, but you will learn a great deal if you are able to sketch graphs of X_0 as a function of ω / ω_n for various values of ζ .

You can also use our [applet](#) to study the influence of forcing frequency, the natural frequency of the system, and the damping coefficient. If you plot position-v-time curves, make sure you switch off the transient solution to show clearly the steady state behavior. Note also that if you click on the 'amplitude -v- frequency' radio button just below the graphs, you will see a graph showing the steady state amplitude of vibration as a function of forcing frequency. The current frequency of excitation is marked as a square dot on the curve (if you don't see the square dot, it means the frequency of excitation is too high to fit on the scale – if you lower the excitation frequency and press 'start' again you should see the dot appear). You can change the properties of the spring mass system (or the natural frequency and damping coefficient) and draw new amplitude-v-frequency curves to see how the response of the system has changed.

Try the following tests

- Keeping the natural frequency fixed (or k and m fixed), plot amplitude-v-frequency graphs for various values of damping coefficient (or the dashpot coefficient). What happens to the maximum amplitude of vibration as damping is reduced?
- Keep the damping coefficient fixed at around 0.1. Plot graphs of amplitude-v-frequency for various values of the natural frequency of the system. How does the maximum vibration amplitude change as natural frequency is varied? What about the frequency at which the maximum occurs?
- Keep the dashpot coefficient fixed at a lowish value. Plot graphs of amplitude-v-frequency for various values of spring stiffness and mass. Can you reconcile the behavior you observe with the results of test (ii)?
- Try changing the type of forcing to base excitation and rotor excitation. Can you see any differences in the amplitude-v-frequency curves for different types of forcing?
- Set the damping coefficient to a low value (below 0.1). Keep the natural frequency fixed. Run the program for different excitation frequencies. Watch what the system is doing. Observe the behavior when the excitation frequency coincides with the natural frequency of the system. Try this test for each type of excitation.

- If the forcing frequency is close to the natural frequency of the system, and the system is lightly damped, huge vibration amplitudes may occur. This phenomenon is known as **resonance**.

If you ran the tests in the preceding section, you will have seen the system resonate. Note that the system resonates at a very similar frequency for each type of forcing.

As a general rule, engineers try to avoid resonance like the plague. Resonance is bad vibrations, man. Large amplitude vibrations imply large forces; and large forces cause material failure. There are exceptions to this rule, of course. Musical instruments, for example, are supposed to resonate, so as to amplify sound. Musicians who play string, wind and brass instruments spend years training their lips or bowing arm to excite just the right vibration modes in their instruments to make them sound perfect.

- There is a phase lag between the forcing and the system response, which depends on the frequency of excitation and the properties of the spring-mass system.

The response of the system is $x(t) = X_0 \sin(\omega t + \phi)$. Expressions for ϕ are given in the preceding section. Note that the phase lag is always negative.

You can use the applet to examine the physical significance of the phase lag. Note that you can have the program plot a graph of phase-v-frequency for you, if you wish.

It is rather unusual to be particularly interested in the phase of the vibration, so we will not discuss it in detail here.

5.4.5 Engineering implications of vibration behavior

The solutions listed in the preceding sections give us general guidelines for engineering a system to avoid (or create!) vibrations.

Preventing a system from vibrating: Suppose that we need to stop a structure or component from vibrating – e.g. to stop a tall building from swaying. Structures are always deformable to some extent – this is represented qualitatively by the spring in a spring-mass system. They always have mass – this is represented by the mass of the block. Finally, the damper represents energy dissipation. Forces acting on a system generally fluctuate with time. They probably aren't perfectly harmonic, but they usually do have a fairly well defined frequency (visualize waves on the ocean, for example, or wind gusts. Many vibrations are man-made, in which case their frequency is known – for example vehicles traveling on a road tend to induce vibrations with a frequency of about 2Hz, corresponding to the bounce of the car on its suspension).

So how do we stop the system from vibrating? We know that its motion is given by

$$x(t) = X_0 \sin(\omega t + \phi)$$

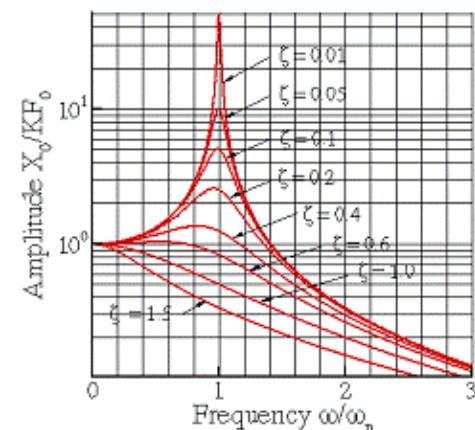
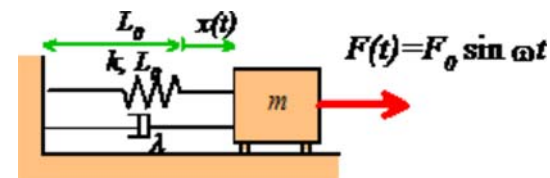
$$X_0 = \frac{KF_0}{\left\{ \left(1 - \omega^2 / \omega_n^2\right)^2 + (2\zeta\omega / \omega_n)^2 \right\}^{1/2}}$$

$$\phi = \tan^{-1} \frac{-2\zeta\omega / \omega_n}{1 - \omega^2 / \omega_n^2}$$

$$\omega_n = \sqrt{\frac{k}{m}}, \quad \zeta = \frac{\lambda}{2\sqrt{km}}, \quad K = \frac{1}{k}$$

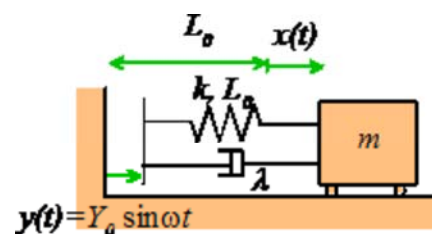
To minimize vibrations, we must design the system to make the vibration amplitude X_0 as small as possible. The formula for X_0 is a bit scary,

which is why we plot graphs of the solution. The graphs show that we will observe vibrations with large amplitudes if (i) The frequency ω / ω_n is close to 1; and (ii) the damping ζ is small. At first sight, it looks like we could minimize

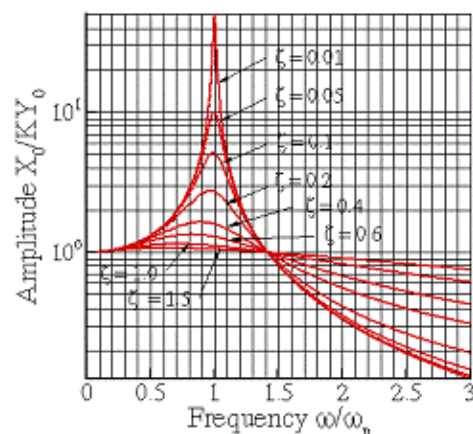


vibrations by making ω / ω_n very large. This is true in principle, and can be done in some designs, e.g. if the force acts on a very localized area of the structure, and will only excite a single vibration mode. For most systems, this approach will not work, however. This is because real components generally have a very large number of natural frequencies of vibration, corresponding to different vibration modes. We could design the system so that ω / ω_n is large for the mode with the lowest frequency – and perhaps some others – but there will always be other modes with higher frequencies, which will have smaller values of ω / ω_n . There is a risk that one of these will be close to resonance. Consequently, we generally design the system so that $\omega / \omega_n \ll 1$ for the mode with the lowest natural frequency. In fact, design codes usually specify the minimum allowable value of ω / ω_n for vibration critical components. This will guarantee that $\omega / \omega_n \ll 1$ for all modes, and hence the vibration amplitude $X_0 \rightarrow KF_0 = F_0 / k$. This tells us that the best approach to avoid vibrations is to make the structure as stiff as possible. This will make the natural frequency large, and will also make F_0 / k small.

Designing a suspension or vibration isolation system. Suspensions, and vibration isolation systems, are examples of base excited systems. In this case, the system really consists of a mass (the vehicle, or the isolation table) on a spring (the shock absorber or vibration isolation pad). We expect that the base will vibrate with some characteristic frequency ω . Our goal is to design the system to minimize the vibration of the mass.



Our vibration solution predicts that the mass vibrates with displacement



$$x(t) = X_0 \sin(\omega t + \phi)$$

$$X_0 = \frac{KY_0 \{1 + (2\zeta\omega / \omega_n)^2\}^{1/2}}{\left\{ (1 - \omega^2 / \omega_n^2)^2 + (2\zeta\omega / \omega_n)^2 \right\}^{1/2}}$$

$$\phi = \tan^{-1} \frac{-2\zeta\omega^3 / \omega_n^3}{1 - (1 - 4\zeta^2)\omega^2 / \omega_n^2}$$

$$\omega_n = \sqrt{\frac{k}{m}}, \quad \zeta = \frac{\lambda}{2\sqrt{km}}, \quad K = 1$$

Again, the graph is helpful to understand how the vibration amplitude X_0 varies with system parameters.

Clearly, we can minimize the vibration amplitude of the mass by making $\omega / \omega_n \gg 1$. We can do this by making the spring stiffness as small as possible (use a soft spring), and making the mass large. It also helps to make the damping ζ small. This is counter-intuitive – people often think that the energy dissipated by the shock absorbers in their suspensions that makes them work. There are some disadvantages to making the damping too small, however. For one thing, if the system is lightly damped, and is disturbed somehow, the subsequent transient vibrations will take a very long time to die out. In addition, there is always a risk that the frequency of base excitation is lower than we expect – if the system is lightly damped, a potentially damaging resonance may occur.

Suspension design involves a bit more than simply minimizing the vibration of the mass, of course – the car will handle poorly if the wheels begin to leave the ground. A very soft suspension generally has poor handling, so the engineers must trade off handling against vibration isolation.

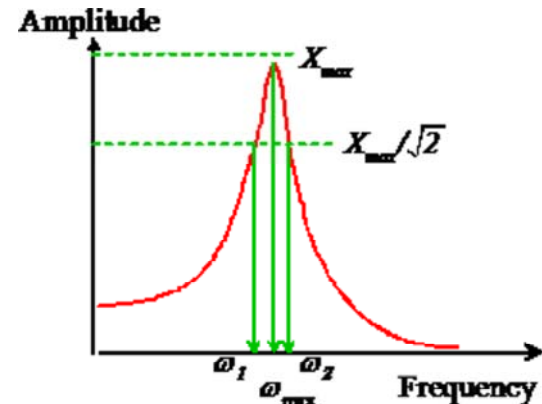
5.4.6 Using Forced Vibration Response to Measure Properties of a System.

We often measure the natural frequency and damping coefficient for a mode of vibration in a structure or component, by measuring the forced vibration response of the system.

Here is how this is done. We find some way to apply a harmonic excitation to the system (base excitation might work; or you can apply a force using some kind of actuator, or you could deliberately mount an unbalanced rotor on the system).

Then, we mount accelerometers on our system, and use them to measure the displacement of the structure, at the point where it is being excited, as a function of frequency.

We then plot a graph, which usually looks something like the picture on the right. We read off the maximum response X_{\max} , and draw a horizontal line at amplitude $X_{\max} / \sqrt{2}$. Finally, we measure the frequencies ω_1 , ω_2 and ω_{\max} as shown in the picture.



We define the **bandwidth** of the response $\Delta\omega$ as

$$\Delta\omega = \omega_2 - \omega_1$$

Like the logarithmic decrement, the bandwidth of the forced harmonic response is a measure of the damping in a system.

It turns out that we can estimate the natural frequency of the system and its damping coefficient using the following formulae

$$\zeta \approx \frac{\Delta\omega}{2\omega_{\max}} \quad \omega_n \approx \omega_{\max}$$

The formulae are accurate for small ζ - say $\zeta < 0.2$.

To understand the origin of these formulae, recall that the amplitude of vibration due to external forcing is given by

$$X_0 = \frac{KF_0}{\sqrt{(1 - \omega^2 / \omega_n^2)^2 + (2\zeta\omega / \omega_n)^2}}$$

We can find the frequency at which the amplitude is a maximum by differentiating with respect to ω , setting the derivative equal to zero and solving the resulting equation for frequency. It turns out that the maximum amplitude occurs at a frequency

$$\omega_{\max} = \omega_n \sqrt{1 - 2\zeta^2}$$

For small ζ , we see that

$$\omega_{\max} \approx \omega_n$$

Next, to get an expression relating the bandwidth $\Delta\omega$ to ζ , we first calculate the frequencies ω_1 and ω_2 . Note that the maximum amplitude of vibration can be calculated by setting $\omega = \omega_{\max}$, which gives

$$X_{\max} = \frac{KF_0}{2\zeta\sqrt{1 - \zeta^2}}$$

Now, at the two frequencies of interest, we know $X_0 = X_{\max} / \sqrt{2}$, so that ω_1 and ω_2 must be solutions of the equation

$$\frac{KF_0}{2\zeta\sqrt{1-\zeta^2}} \frac{1}{\sqrt{2}} = \frac{KF_0}{\sqrt{\left(1-\omega^2/\omega_n^2\right)^2 + (2\zeta\omega\omega_n)^2}}$$

Rearrange this equation to see that

$$\omega^4 - 2\omega^2\omega_n^2(1-2\zeta^2) + \omega_n^4 - 8\zeta^2\omega_n^4(1-\zeta^2) = 0$$

This is a quadratic equation for ω^2 and has solutions

$$\omega_1 = \left\{ \omega_n^2(1-2\zeta^2) - 2\omega_n^2\zeta\sqrt{1-\zeta^2} \right\}^{1/2}$$

$$\omega_2 = \left\{ \omega_n^2(1-2\zeta^2) + 2\omega_n^2\zeta\sqrt{1-\zeta^2} \right\}^{1/2}$$

Expand both expressions in a Taylor series about $\zeta = 0$ to see that

$$\omega_1 \approx \omega_n(1-\zeta)$$

$$\omega_2 \approx \omega_n(1+\zeta)$$

so, finally, we confirm that

$$\Delta\omega = \omega_2 - \omega_1 = 2\zeta\omega_n$$

5.4.7 Example Problems in Forced Vibrations

Example 1: A structure is idealized as a damped spring—mass system with stiffness 10 kN/m; mass 2Mg; and dashpot coefficient 2 kNs/m. It is subjected to a harmonic force of amplitude 500N at frequency 0.5Hz. Calculate the steady state amplitude of vibration.

Start by calculating the properties of the system:

$$\omega_n = \sqrt{\frac{k}{m}} = 2.23 \text{ rad/s} \quad \zeta = \frac{\lambda}{2\sqrt{km}} = 0.224 \quad K = \frac{1}{k} = \frac{1}{10000} \text{ m/N}$$

Now, the list of solutions to forced vibration problems gives

$$x(t) = X_0 \sin(\omega t + \phi)$$

$$X_0 = \frac{KF_0}{\left\{ \left(1-\omega^2/\omega_n^2\right)^2 + (2\zeta\omega/\omega_n)^2 \right\}^{1/2}} \quad \phi = \tan^{-1} \frac{-2\zeta\omega/\omega_n}{1-\omega^2/\omega_n^2}$$

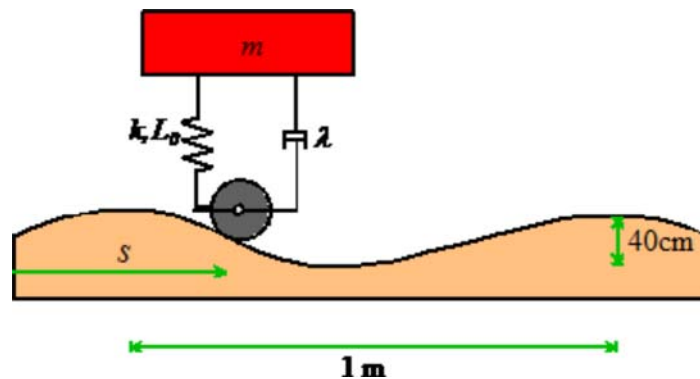
For the present problem:

$$\omega = 0.5 \times 2\pi \text{ rad/s} \Rightarrow \omega/\omega_n = \pi/2.23 = 1.41$$

Substituting numbers into the expression for the vibration amplitude shows that

$$X_0 = 43 \text{ mm}$$

Example 2: A car and its suspension system are idealized as a damped spring—mass system, with natural frequency 0.5Hz and damping coefficient 0.2. Suppose the car drives at speed V over a road with sinusoidal roughness. Assume the roughness wavelength is 10m, and its amplitude is 20cm. At what speed does the maximum amplitude of vibration occur, and what is the corresponding vibration amplitude?



Let s denote the distance traveled by the car, and let L denote the wavelength of the roughness and H the roughness amplitude. Then, the height of the wheel above the mean road height may be expressed as

$$y = H \sin\left(\frac{2\pi s}{L}\right)$$

Noting that $s = Vt$, we have that

$$y(t) = H \sin\left(\frac{2\pi V}{L}t\right)$$

i.e., the wheel oscillates vertically with harmonic motion, at frequency $\omega = 2\pi V / L$.

Now, the suspension has been idealized as a spring—mass system subjected to base excitation. The steady state vibration is

$$x(t) = X_0 \sin(\omega t + \phi)$$

$$X_0 = \frac{KY_0 \left\{1 + (2\zeta\omega / \omega_n)^2\right\}^{1/2}}{\left\{(1 - \omega^2 / \omega_n^2)^2 + (2\zeta\omega / \omega_n)^2\right\}^{1/2}} \quad \phi = \tan^{-1} \frac{-2\zeta\omega^3 / \omega_n^3}{1 - (1 - 4\zeta^2)\omega^2 / \omega_n^2}$$

For light damping, the maximum amplitude of vibration occurs at around the natural frequency. Therefore, the critical speed follows from

$$\omega = \frac{2\pi V}{L} = \omega_n \Rightarrow V = \omega_n L / 2\pi = 5\text{m/s} = 18\text{ km/hr}$$

Note that $K=I$ for base excitation, so that the amplitude of vibration at $\omega / \omega_n = 1$ is approximately

$$X_0 \approx \frac{Y_0}{2\zeta} = 20 / 0.4 = 50\text{cm}$$

Note that at this speed, the suspension system is making the vibration worse. The amplitude of the car's vibration is greater than the roughness of the road. Suspensions work best if they are excited at frequencies well above their resonant frequencies.

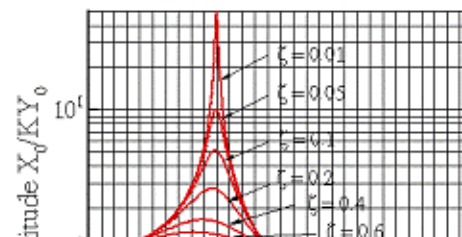
Example 3: The suspension system discussed in the preceding problem has the following specifications. For the roadway described in the preceding section, the amplitude of vibration may not exceed 35cm at any speed. At 55 miles per hour, the amplitude of vibration must be less than 10cm. The car weighs 3000lb. Select values for the spring stiffness and the dashpot coefficient.

We must first determine values for ζ and ω_n that will satisfy the design specifications. To this end:

- (i) The specification requires that

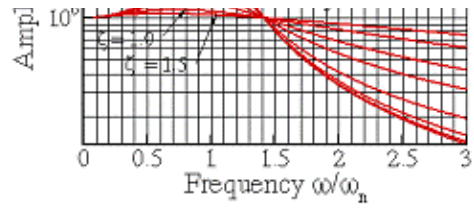
$$\frac{X_0}{Y_0} < \frac{35}{20} = 1.75$$

at resonance. Examine the graph of X_0 / KY_0 shown with the solutions to the equations of motion. Recall that $K=I$ for a base excited spring—mass system. Observe that, with $\zeta \approx 0.4$, the



amplification factor never exceeds 1.75.

- (ii) Now, the frequency of excitation at 55mph is
- $$\frac{2\pi \times 0.447 \times 55}{10} = 15.45 \text{ rad/s}$$



We must choose system parameters so that, at this excitation frequency, $X_0 / Y_0 < 10 / 20 = 1/2$. Examine the graph showing the response of a base excited spring—mass system again. We observe that, for $\zeta = 0.4$, $X_0 / Y_0 < 0.5$ for $\omega / \omega_n > 2$. Therefore, we pick $\omega_n = \omega / 2 = 7.7 \text{ rad/s}$.

Finally, we can compute properties of the system. We have that

$$\omega_n = \sqrt{\frac{k}{m}} \Rightarrow 7.7 = \sqrt{\frac{k}{0.44 \times 3000}} \Rightarrow k = 78 \text{ kN/m}$$

Similarly

$$\zeta = \frac{\lambda}{2\sqrt{mk}} \Rightarrow \lambda = 2 \times 0.4 \times \sqrt{78000 \times 0.44 \times 3000} = 8 \text{ kNs/m}$$