Chapter 8
DEFLECTION OF BEAMS BY INTEGRATION

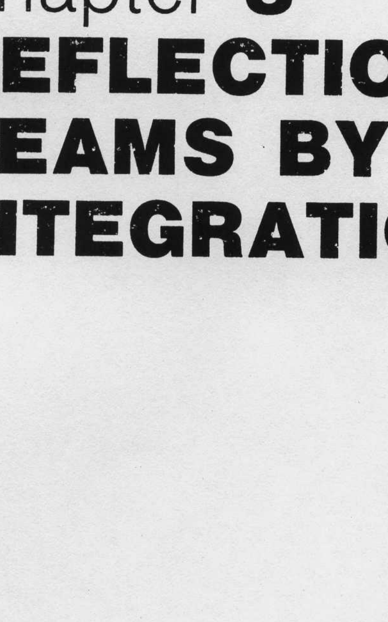


Fig. 8.1

8.1. INTRODUCTION

We saw in Sec. 4.4 that a prismatic beam subjected to pure bending is bent into an arc of circle and that, within the elastic range, the curvature of the neutral surface may be expressed as

$$\frac{1}{\rho} = \frac{M}{EI} \tag{4.21}$$

where M is the bending moment, E the modulus of elasticity, and I the moment of inertia of the cross section about its neutral axis.

When a beam is subjected to a transverse loading, Eq. (4.21) remains valid for any given transverse section, provided that Saint-Venant's principle applies. However, both the bending moment and the curvature of the neutral surface will vary from section to section. Denoting by x the distance of the section from the left end of the beam, we write

$$\frac{1}{\rho} = \frac{M(x)}{EI} \tag{8.1}$$

Consider, for example, a cantilever beam AB of length L subjected to a concentrated load $\mathbf{P}$ at its free end A (Fig. 8.1a). We have $M(x) = -Px$ and, substituting into (8.1),

$$\frac{1}{\rho} = -\frac{Px}{EI}$$

which shows that the curvature of the neutral surface varies linearly with x, from zero at A, where ρ_A itself is infinite, to $-PL/EI$ at B, where $|\rho_B| = EI/PL$ (Fig. 8.1b).

Consider now the overhanging beam AD of Fig. 8.2, which supports two concentrated loads as shown. From the free-body diagram of the beam (Fig. 8.3a), we find that the reactions at the supports are $R_A = 1$ kN and $R_C = 5$ kN, respectively, and draw the corresponding bending-moment diagram (Fig. 8.3*b*). We note from the diagram that M, and thus the curvature of the beam, are zero at both ends of the beam, and also at a point E located at $x = 4$ m. Between A and E the bending moment is positive and the beam is concave upward; between E and D the bending moment is negative and the beam is concave downward (Fig. 8.3*c*). We also note that the largest value of the curvature (i.e., the smallest value of the radius of curvature) occurs at the support C, where $|M|$ is maximum.

From the information obtained on its curvature, we may get a fairly good idea of the shape of the deformed beam. However, the analysis and design of a beam usually require more precise information on the *deflection* and the *slope* of the beam at various points. Of particular importance is the knowledge of the *maximum deflection* of the beam. In this chapter we shall use Eq. (8.1) to obtain a relation between the deflection y measured at a given point Q on the axis of the beam and the distance x of that point from some fixed origin (Fig. 8.4). The relation obtained is the equation of the *elastic curve*, i.e., the equation of the curve into which the axis of the beam is transformed under the given loading (Fig. 8.4*b*).†

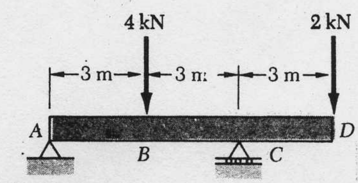


Fig. 8.2

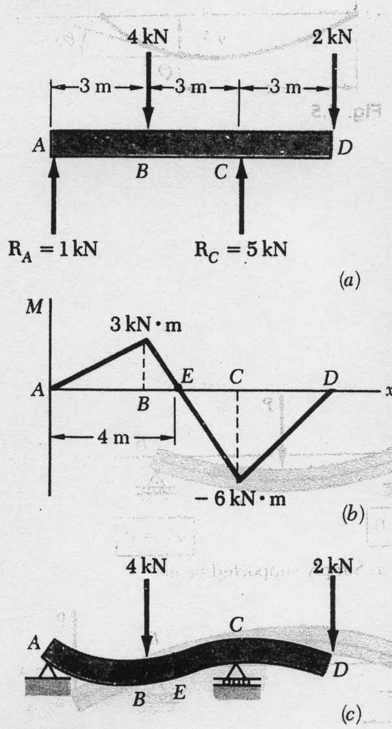


Fig. 8.3

8.2. EQUATION OF THE ELASTIC CURVE

We first recall from elementary calculus that the curvature of a plane curve at a point $Q(x, y)$ of the curve may be expressed as

$$\frac{1}{\rho} = \frac{\dfrac{d^2y}{dx^2}}{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2}} \tag{8.2}$$

where dy/dx and d^2y/dx^2 are the first and second derivatives of the function $y(x)$ represented by that curve. But, in the case of the elastic curve of a beam, the slope dy/dx is very small, and its square is negligible compared to unity. We may write, therefore,

$$\frac{1}{\rho} = \frac{d^2y}{dx^2} \tag{8.3}$$

Substituting for $1/\rho$ from (8.3) into (8.1), we have

$$\frac{d^2y}{dx^2} = \frac{M(x)}{EI} \tag{8.4}$$

The equation obtained is a second-order linear differential equation; it is the governing differential equation for the elastic curve.

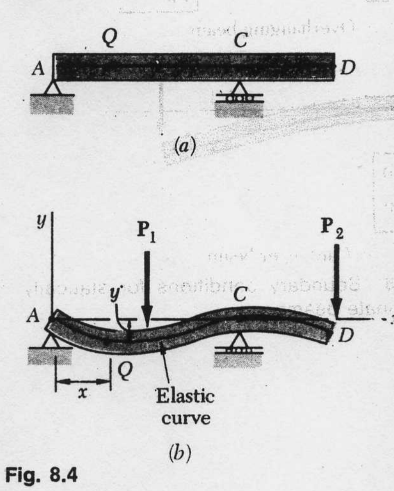


Fig. 8.4

† It should be noted that, in this chapter and the next, y represents a vertical displacement, while it was used in previous chapters to represent the distance of a given point in a transverse section from the neutral axis of that section.

The product EI is known as the *flexural rigidity* and, if it varies along the beam, as in the case of a beam of varying depth, we must express it as a function of x before proceeding to integrate Eq. (8.4). However, in the case of a prismatic beam, which is the case considered here, the flexural rigidity is constant. We may thus multiply both members of Eq. (8.4) by EI and integrate in x. We write

$$EI\frac{dy}{dx} = \int_0^x M(x)\,dx + C_1 \tag{8.5}$$

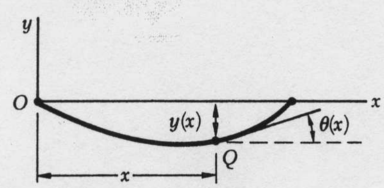


Fig. 8.5

where C_1 is a constant of integration. Denoting by $\theta(x)$ the angle, measured in radians, that the tangent at Q to the elastic curve forms with the horizontal (Fig. 8.5), and recalling that this angle is very small, we have

$$\frac{dy}{dx} = \tan\theta \simeq \theta(x)$$

Thus, we may write Eq. (8.5) in the alternate form

$$EI\,\theta(x) = \int_0^x M(x)\,dx + C_1 \tag{8.5'}$$

Integrating both members of Eq. (8.5) in x, we have

$$EI\,y = \int_0^x \left[\int_0^x M(x)\,dx + C_1\right] dx + C_2$$

$$EI\,y = \int_0^x dx \int_0^x M(x)\,dx + C_1x + C_2 \tag{8.6}$$

where C_2 is a second constant, and where the first term in the right-hand member represents the function of x obtained by integrating twice in x the bending moment $M(x)$. If it were not for the fact that the constants C_1 and C_2 are as yet undetermined, Eq. (8.6) would define the deflection of the beam at any given point Q, and Eq. (8.5) or (8.5′) would similarly define the slope of the beam at Q.

The constants C_1 and C_2 are determined from the *boundary conditions* or, more precisely, from the conditions imposed on the beam by its supports. Limiting our analysis in this section to *statically determinate beams,* i.e., to beams supported in such a way that the reactions at the supports may be obtained by the methods of statics, we note that only three types of beams need to be considered here (Fig. 8.6): (a) the *simply supported beam,* (b) the *overhanging beam,* and (c) the *cantilever beam.*

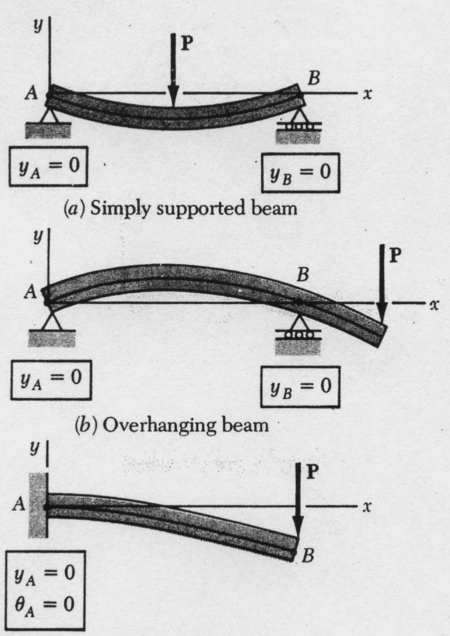


Fig. 8.6 Boundary conditions for statically determinate beams.

In the first two cases, the supports consist of a pin and bracket at A and of a roller at B, and require that the deflection be zero at each of these points. Letting first $x = x_A$, $y = y_A = 0$ in Eq. (8.6), and then $x = x_B$, $y = y_B = 0$ in the same equation, we obtain two equations which may be solved for C_1 and C_2. In the case of the cantilever beam (Fig. 8.6c), we note that both the deflection and the slope at A must be zero. Letting $x = x_A$, $y = y_A = 0$ in Eq. (8.6), and $x = x_A$, $\theta = \theta_A = 0$ in Eq. (8.5′), we obtain again two equations which may be solved for C_1 and C_2.

Example 8.01

The cantilever beam AB is of uniform cross section and carries a load $\mathbf{P}$ at its free end A (Fig. 8.7). Determine the equation of the elastic curve and the deflection and slope at A.

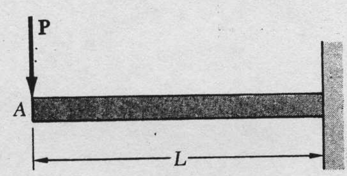


Fig. 8.7

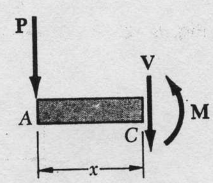


Fig. 8.8

Using the free-body diagram of the portion AC of the beam (Fig. 8.8), where C is located at a distance x from end A, we find

$$M = -Px \tag{8.7}$$

Substituting for M into Eq. (8.4) and multiplying both members by the constant EI, we write

$$EI\frac{d^2y}{dx^2} = -Px$$

Integrating in x, we obtain

$$EI\frac{dy}{dx} = -\tfrac{1}{2}Px^2 + C_1 \tag{8.8}$$

We now observe that at the fixed end B we have $x = L$ and $\theta = dy/dx = 0$ (Fig. 8.9). Substituting these values into (8.8) and solving for C_1 we have

$$C_1 = \tfrac{1}{2}PL^2$$

which we carry back into (8.8):

$$EI\frac{dy}{dx} = -\tfrac{1}{2}Px^2 + \tfrac{1}{2}PL^2 \tag{8.9}$$

Integrating both members of Eq. (8.9), we write

$$EI\,y = -\tfrac{1}{6}Px^3 + \tfrac{1}{2}PL^2x + C_2 \tag{8.10}$$

But, at B we have $x = L$, $y = 0$. Substituting into (8.10), we have

$$0 = -\tfrac{1}{6}PL^3 + \tfrac{1}{2}PL^3 + C_2$$
$$C_2 = -\tfrac{1}{3}PL^3$$

Carrying the value of C_2 back into Eq. (8.10), we obtain the equation of the elastic curve:

$$EI\,y = -\tfrac{1}{6}Px^3 + \tfrac{1}{2}PL^2x - \tfrac{1}{3}PL^3$$

or

$$y = \frac{P}{6EI}(-x^3 + 3L^2x - 2L^3) \tag{8.11}$$

The deflection and slope at A are obtained by letting $x = 0$ in Eqs. (8.11) and (8.9). We find

$$y_A = -\frac{PL^3}{3EI} \quad \text{and} \quad \theta_A = \left(\frac{dy}{dx}\right)_A = \frac{PL^2}{2EI}$$

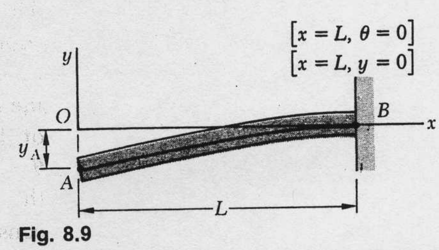


Fig. 8.9

Example 8.02

The simply supported prismatic beam AB carries a uniformly distributed load w per unit length (Fig. 8.10). Determine the equation of the elastic curve and the maximum deflection of the beam.

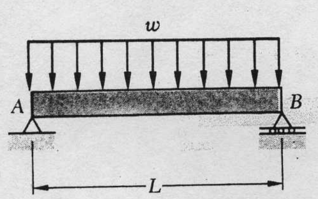


Fig. 8.10

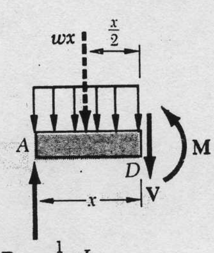


Fig. 8.11

Drawing the free-body diagram of the portion AD of the beam (Fig. 8.11), and taking moments about D, we find that

$$M = \tfrac{1}{2}wLx - \tfrac{1}{2}wx^2 \tag{8.12}$$

Substituting for M into Eq. (8.4) and multiplying both members of this equation by the constant EI, we write

$$EI\frac{d^2y}{dx^2} = -\frac{1}{2}wx^2 + \frac{1}{2}wLx \tag{8.13}$$

Integrating twice in x, we have

$$EI\frac{dy}{dx} = -\frac{1}{6}wx^3 + \frac{1}{4}wLx^2 + C_1 \tag{8.14}$$

$$EI\,y = -\frac{1}{24}wx^4 + \frac{1}{12}wLx^3 + C_1x + C_2 \tag{8.15}$$

Observing that $y = 0$ at both ends of the beam (Fig. 8.12), we first let $x = 0$ and $y = 0$ in Eq. (8.15) and obtain $C_2 = 0$. We then make $x = L$ and $y = 0$ in the same equation and write

$$0 = -\tfrac{1}{24}wL^4 + \tfrac{1}{12}wL^4 + C_1L$$
$$C_1 = -\tfrac{1}{24}wL^3$$

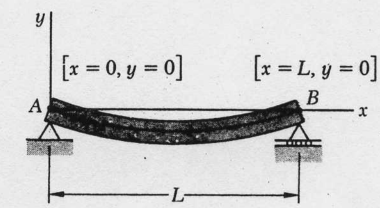


Fig. 8.12

Carrying the values of C_1 and C_2 back into Eq. (8.15), we obtain the equation of the elastic curve:

$$EI\,y = -\frac{1}{24}wx^4 + \frac{1}{12}wLx^3 - \frac{1}{24}wL^3x$$

or

$$y = \frac{w}{24EI}(-x^4 + 2Lx^3 - L^3x) \qquad (8.16)$$

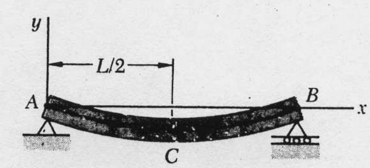


Fig. 8.13

Substituting into Eq. (8.14) the value obtained for C_1, we check that the slope of the beam is zero for $x = L/2$ and that the elastic curve has a minimum at the midpoint C of the beam (Fig. 8.13). Letting $x = L/2$ in Eq. (8.16), we have

$$y_C = \frac{w}{24EI}\left(-\frac{L^4}{16} + 2L\frac{L^3}{8} - L^3\frac{L}{2}\right) = -\frac{5wL^4}{384EI}$$

The maximum deflection or, more precisely, the maximum absolute value of the deflection, is thus

$$|y|_{max} = \frac{5wL^4}{384EI}$$

In each of the two examples considered so far, only one free-body diagram was required to determine the bending moment in the beam. As a result, a single function of x was used to represent M throughout the beam. This, however, is not generally the case. Concentrated loads, reactions at supports, or discontinuities in a distributed load will make it necessary to divide the beam into several portions, and to represent the bending moment by a different function $M(x)$ in each of these portions of beam. Each of the functions $M(x)$ will then lead to a different expression for the slope $\theta(x)$ and for the deflection $y(x)$. Since each of the expressions obtained for the deflection must contain two constants of integration, a large number of constants will have to be determined. As we shall see in the next example, the required additional boundary conditions may be obtained by observing that, while the shear and bending moment can be discontinuous at several points in a beam, the *deflection* and the *slope* of the beam *cannot be discontinuous* at any point.

Example 8.03

For the prismatic beam and the loading shown (Fig. 8.14), determine the slope and deflection at point D.

We must divide the beam into two portions, AD and DB, and determine the function $y(x)$ which defines the elastic curve for each of these portions.

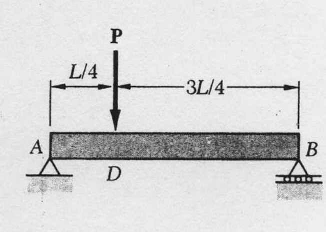


Fig. 8.14

1. *From A to D* ($x < L/4$). We draw the free-body diagram of a portion of beam AE of length $x < L/4$ (Fig. 8.15). Taking moments about E, we have

$$M_1 = \frac{3P}{4}x \tag{8.17}$$

or, recalling Eq. (8.4),

$$EI\frac{d^2y_1}{dx^2} = \frac{3}{4}Px \tag{8.18}$$

where $y_1(x)$ is the function which defines the elastic curve *for portion AD of the beam*. Integrating in x, we write

$$EI\,\theta_1 = EI\frac{dy_1}{dx} = \frac{3}{8}Px^2 + C_1 \tag{8.19}$$

$$EI\,y_1 = \frac{1}{8}Px^3 + C_1x + C_2 \tag{8.20}$$

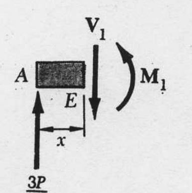


Fig. 8.15

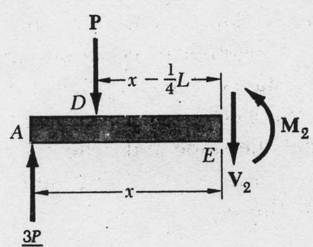


Fig. 8.16

2. *From D to B* ($x > L/4$). We now draw the free-body diagram of a portion of beam AE of length $x > L/4$ (Fig. 8.16) and write

$$M_2 = \frac{3P}{4}x - P\left(x - \frac{L}{4}\right) \tag{8.21}$$

or, recalling Eq. (8.4) and rearranging terms,

$$EI\frac{d^2y_2}{dx^2} = -\frac{1}{4}Px + \frac{1}{4}PL \tag{8.22}$$

where $y_2(x)$ is the function which defines the elastic curve *for portion DB of the beam*. Integrating in x, we write

$$EI\,\theta_2 = EI\frac{dy_2}{dx} = -\frac{1}{8}Px^2 + \frac{1}{4}PLx + C_3 \tag{8.23}$$

$$EI\,y_2 = -\frac{1}{24}Px^3 + \frac{1}{8}PLx^2 + C_3x + C_4 \tag{8.24}$$

Determination of the Constants of Integration. The conditions which must be satisfied by the constants of integration have been summarized in Fig. 8.17. At the support A, where

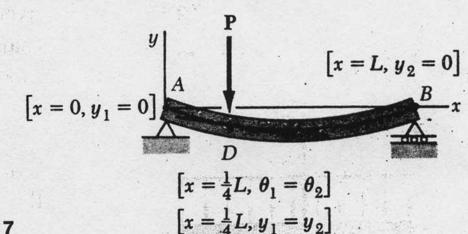


Fig. 8.17

the deflection is defined by Eq. (8.20), we must have $x = 0$ and $y_1 = 0$. At the support B, where the deflection is defined by Eq. (8.24), we must have $x = L$ and $y_2 = 0$. Also, the fact that there can be no sudden change in deflection or in slope at point D requires that $y_1 = y_2$ and $\theta_1 = \theta_2$ when $x = L/4$. We have therefore:

$[x = 0,\ y_1 = 0]$, Eq. (8.20): $\quad 0 = C_2 \quad$ (8.25)

$[x = L,\ y_2 = 0]$, Eq. (8.24): $\quad 0 = \frac{1}{12}PL^3 + C_3L + C_4 \quad$ (8.26)

$[x = L/4,\ \theta_1 = \theta_2]$, Eqs. (8.19) and (8.23):

$$\frac{3}{128}PL^2 + C_1 = \frac{7}{128}PL^2 + C_3 \tag{8.27}$$

$[x = L/4,\ y_1 = y_2]$, Eqs. (8.20) and (8.24):

$$\frac{PL^3}{512} + C_1\frac{L}{4} = \frac{11PL^3}{1536} + C_3\frac{L}{4} + C_4 \tag{8.28}$$

Solving these equations simultaneously, we find

$$C_1 = -\frac{7PL^2}{128},\ C_2 = 0,\ C_3 = -\frac{11PL^2}{128},\ C_4 = \frac{PL^3}{384}$$

Substituting for C_1 and C_2 into Eqs. (8.19) and (8.20), we write that, for $x \le L/4$,

$$EI\,\theta_1 = \frac{3}{8}Px^2 - \frac{7PL^2}{128} \tag{8.29}$$

$$EI\,y_1 = \frac{1}{8}Px^3 - \frac{7PL^2}{128}x \tag{8.30}$$

Letting $x = L/4$ in each of these equations, we find that the slope and deflection at point D are, respectively,

$$\theta_D = -\frac{PL^2}{32EI} \quad \text{and} \quad y_D = -\frac{3PL^3}{256EI}$$

We note that, since $\theta_D \neq 0$, the deflection at D is *not* the maximum deflection of the beam.

*8.4. USE OF SINGULARITY FUNCTIONS

Reviewing the work done in the first three sections of this chapter, we note that the integration method provides a convenient and effective way for determining the slope and deflection at any point of a prismatic beam, *as long as the bending moment may be represented by a single analytical function* $M(x)$. However, when the loading of the beam is such that two different functions are needed to represent the bending moment over the entire length of the beam, as in Example 8.03, four constants of integration are required, and an equal number of equations, expressing continuity conditions at point D, as well as boundary conditions at the supports A and B, must be used to determine these constants. If three or more functions were needed to represent the bending moment, additional constants and a corresponding number of additional equations would be required, resulting in rather lengthy computations. We shall see in this section how the use of *singularity functions* may simplify the computations.

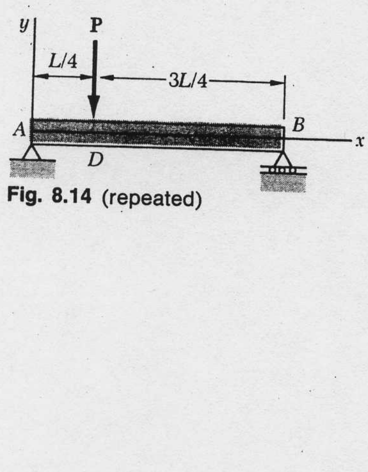


Fig. 8.14 (repeated)

Considering again the beam and loading of Example 8.03 (Fig. 8.14), we recall from Eqs. (8.17) and (8.21) that the bending moment over the portions AD and DB of the beam may be expressed by the functions

$$M_1(x) = \frac{3P}{4}x \qquad \left(0 \le x \le \frac{L}{4}\right) \tag{8.38}$$

$$M_2(x) = \frac{3P}{4}x - P\left(x - \frac{L}{4}\right) \qquad \left(\frac{L}{4} \le x \le L\right) \tag{8.39}$$

where x is the distance measured from end A. The functions $M_1(x)$ and $M_2(x)$ may be represented by the single expression

$$M(x) = \frac{3P}{4}x - P\langle x - \frac{L}{4}\rangle \tag{8.40}$$

if we specify that the second term should be included in our computations when $x \ge L/4$, and ignored when $x < L/4$. In other words, *the brackets* $\langle\ \rangle$ *should be replaced by ordinary parentheses* $(\ )$ *when* $x \ge L/4$, *and by zero when* $x < L/4$.

Substituting for $M(x)$ from (8.40) into Eq. (8.4), we write

$$EI\frac{d^2y}{dx^2} = \frac{3P}{4}x - P\langle x - \frac{L}{4}\rangle \tag{8.41}$$

and, integrating in x,

$$EI\,\theta = EI\frac{dy}{dx} = \frac{3}{8}Px^2 - \frac{1}{2}P\langle x - \frac{L}{4}\rangle^2 + C_1 \tag{8.42}$$

$$EI\,y = \frac{1}{8}Px^3 - \frac{1}{6}P\langle x - \frac{L}{4}\rangle^3 + C_1x + C_2 \tag{8.43}$$

where, again, the brackets should be replaced by parentheses when $x \ge L/4$, and by zero when $x < L/4$.

The constants C_1 and C_2 may be determined from the boundary conditions shown in Fig. 8.22. Letting $x = 0$, $y = 0$ in Eq. (8.43) and noting that, since 0 is less than $L/4$, the brackets are equal to zero, we conclude that $C_2 = 0$.† Letting now $x = L$, $y = 0$ in Eq. (8.43) and noting that, since L is greater than $L/4$, the brackets may be replaced by parentheses, we write

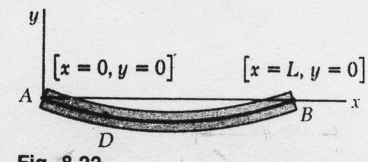


Fig. 8.22

$$0 = \frac{1}{8}PL^3 - \frac{1}{6}P\left(\frac{3L}{4}\right)^3 + C_1L$$

and, solving for C_1,

$$C_1 = -\frac{7PL^2}{128}$$

We check that the expressions obtained for the constants C_1 and C_2 are the same that were found earlier in Sec. 8.2. But the need for additional constants C_3 and C_4 has now been eliminated, and we do not have to write equations expressing that the slope and the deflection are continuous at point D.‡

The expressions $\langle x - \frac{1}{4}L\rangle$, $\langle x - \frac{1}{4}L\rangle^2$, $\langle x - \frac{1}{4}L\rangle^3$ are called *singularity functions*. We have, by definition, and for $n \geq 0$,

$$\langle x - a\rangle^n = \begin{cases} (x - a)^n & \text{when } x \geq a \\ 0 & \text{when } x < a \end{cases} \tag{8.44}$$

While these functions had been introduced in 1862 by the German mathematician A. Clebsch (1833–1872), it is the British engineer W. H. Macaulay who first suggested their use to solve beam problems. The method followed in the preceding example is the same that he presented in a paper published in 1919.§ Three singularity functions, corresponding respectively to $n = 0$, $n = 1$, and $n = 2$, have been plotted in Fig. 8.23. We verify that,

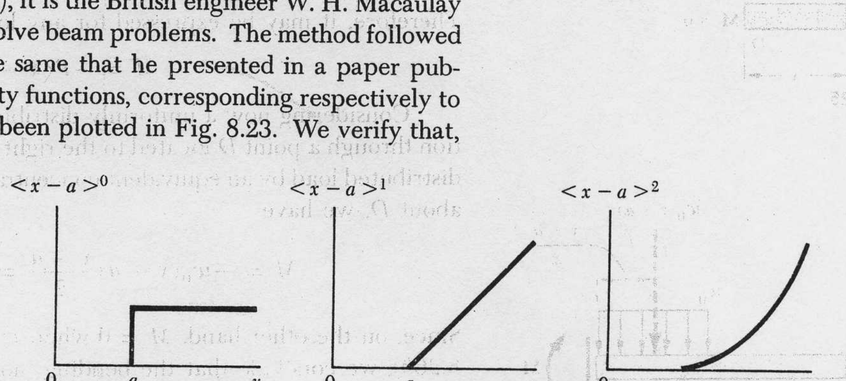


Fig. 8.23 Singularity functions $\langle x - a\rangle^n$.

† We may also note that, whenever the quantity between brackets is negative, the bracket is equal to zero.

‡ The continuity conditions for the slope and deflection at D are "built-in" in Eqs. (8.42) and (8.43). Indeed, the difference between the expressions for the slope θ_1 in AD and the slope θ_2 in DB is represented by the term $-\frac{1}{2}P\langle x - \frac{1}{4}L\rangle^2$ in Eq. (8.42), and this term is equal to zero at D. Similarly, the difference between the expressions for the deflection y_1 in AD and the deflection y_2 in DB is represented by the term $-\frac{1}{6}P\langle x - \frac{1}{4}L\rangle^3$ in Eq. (8.43), and this term is also equal to zero at D.

§ W. H. Macaulay, "Note on the Deflection of Beams," *Messenger of Mathematics*, vol. 48, pp. 129–130, 1919.

since $(x-a)^0 = 1$ for any value of $x - a$, we have

$$\langle x-a\rangle^0 = \begin{cases} 1 \text{ when } x \geq a \\ 0 \text{ when } x < a \end{cases} \tag{8.45}$$

As we may also easily check, it follows from the definition of singularity functions that

$$\int \langle x-a\rangle^n\,dx = \frac{1}{n+1}\langle x-a\rangle^{n+1} \text{ for } n \geq 0 \tag{8.46}$$

and

$$\frac{d}{dx}\langle x-a\rangle^n = n\langle x-a\rangle^{n-1} \text{ for } n \geq 1 \tag{8.47}$$

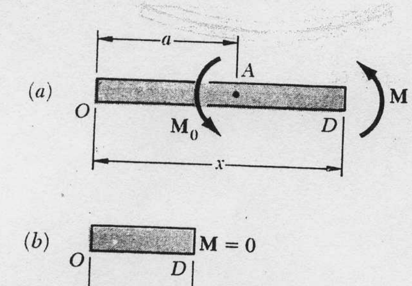


Fig. 8.24

Singularity functions may be used to express the bending moments corresponding to various basic loadings. For example, in the case of a couple $\mathbf{M}_0$ applied at a point A, we note that the bending moment M due to this loading is equal to $-M_0$ when the point D where M is computed is located to the right of A (Fig. 8.24a), and to zero when D is located to the left of A (Fig. 8.24b). Recalling Eq. (8.45), we conclude that M may be expressed for any location of D as

$$M = -M_0\langle x-a\rangle^0$$

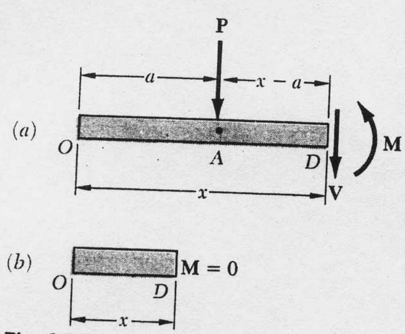


Fig. 8.25

In the case of a concentrated load $\mathbf{P}$ applied at A, we find that the bending moment at a point D is equal to $-P(x-a)$ when D is to the right of A (Fig. 8.25a), and to zero when D is to the left of A (Fig. 8.25b). Therefore, it may be expressed for any location of D as.

$$M = -P\langle x-a\rangle^1$$

Considering now a uniformly distributed load w_0, we first pass a section through a point D located to the right of A (Fig. 8.26a). Replacing the distributed load by an equivalent concentrated load and summing moments about D, we have

$$M = -w_0(x-a)\frac{x-a}{2} = -\frac{1}{2}w_0(x-a)^2$$

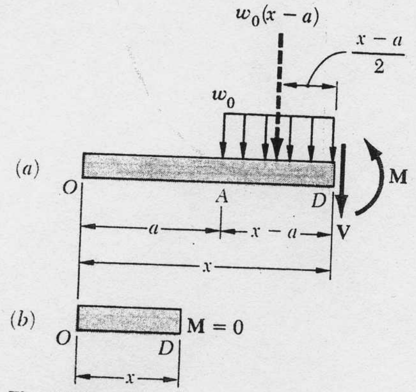


Fig. 8.26

Since, on the other hand, $M = 0$ when D is located to the left of A (Fig. 8.26b), we conclude that the bending moment may be expressed for any location of D as

$$M = -\tfrac{1}{2}w_0\langle x-a\rangle^2$$

The functions obtained for the bending moments associated with the basic loadings of Figs. 8.24 through 8.26 have been represented in Fig. 8.27, together with the bending moments corresponding to two additional loadings. To derive the expression given for $M(x)$ in Fig. 8.27e we shall find it convenient to use Eqs. (7.4) and (7.6) of Sec. 7.3. From Eq. (7.4), we have

$$V = -\int_0^x w\,dx = -\int_0^x k\langle x-a\rangle^n\,dx$$

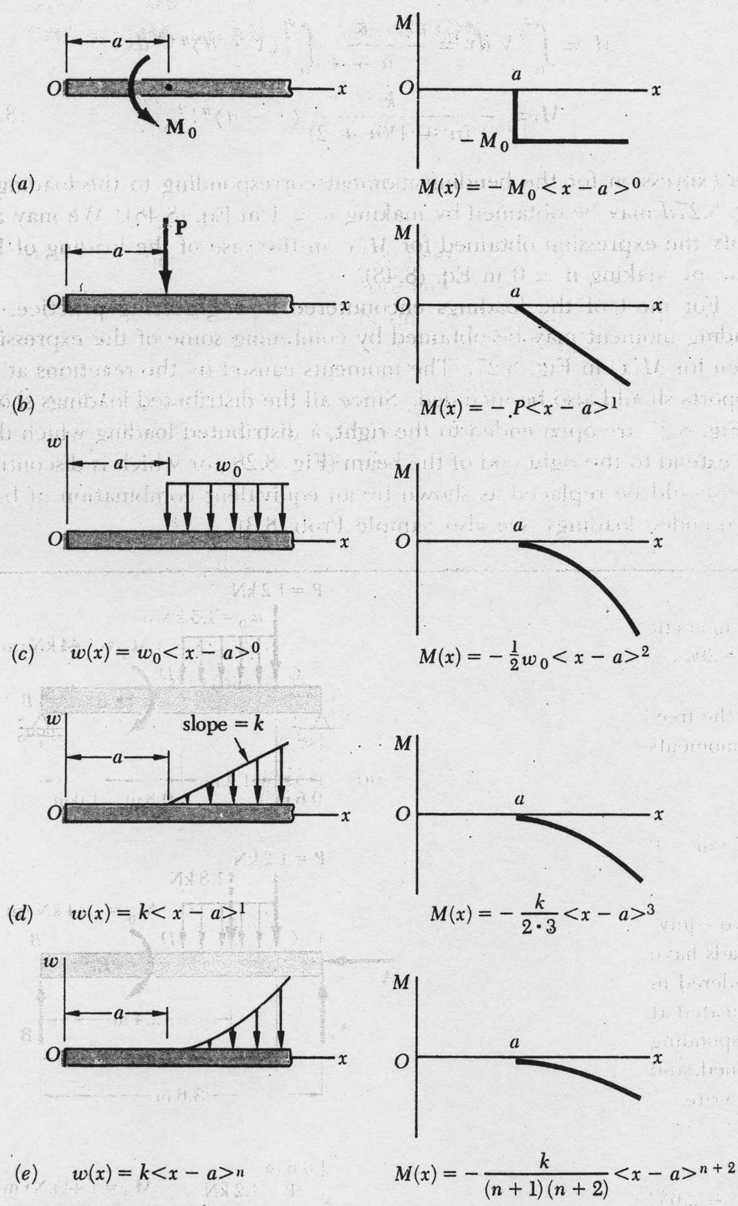
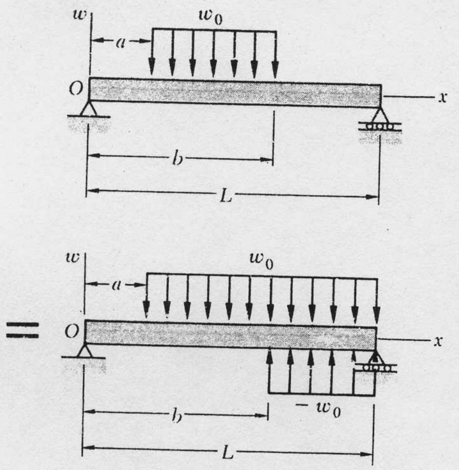


Fig. 8.27 Basic loadings and corresponding bending moments.

or, recalling Eq. (8.46),

$$V = -\frac{k}{n+1}\langle x - a\rangle^{n+1}$$

Using now Eq. (7.6), we write



$w(x) = w_0\langle x-a\rangle^0 - w_0\langle x-b\rangle^0$

Fig. 8.28

$$M = \int_0^x V\,dx = -\frac{k}{n+1}\int_0^x \langle x-a\rangle^{n+1}\,dx$$

$$M = -\frac{k}{(n+1)(n+2)}\langle x-a\rangle^{n+2} \qquad (8.48)$$

The expression for the bending moment corresponding to the loading of Fig. 8.27*d* may be obtained by making $n = 1$ in Eq. (8.48). We may also verify the expression obtained for $M(x)$ in the case of the loading of Fig. 8.27*c* by making $n = 0$ in Eq. (8.48).

For most of the loadings encountered in engineering practice, the bending moment may be obtained by combining some of the expressions given for $M(x)$ in Fig. 8.27. The moments caused by the reactions at the supports should also be included. Since all the distributed loadings shown in Fig. 8.27 are open-ended to the right, a distributed loading which does not extend to the right end of the beam (Fig. 8.28), or which is discontinuous, should be replaced as shown by an equivalent combination of basic open-ended loadings (see also Sample Prob. 8.3).

Example 8.05

Using singularity functions, express the bending moment corresponding to the beam and loading shown (Fig. 8.29*a*).

We first determine the reaction at A by drawing the free-body diagram of the beam (Fig. 8.29*b*) and summing moments about B:

$$+\circlearrowleft\Sigma M_B = 0: \qquad -A_y(3.6\text{ m}) + (1.2\text{ kN})(3\text{ m}) + (1.8\text{ kN})(2.4\text{ m}) + 1.44\text{ kN}\cdot\text{m} = 0$$

$$A_y = 2.60\text{ kN}$$

Next, we replace the given distributed load by two equivalent open-ended loadings (Fig. 8.29*c*). The given loads have thus been reduced to basic loadings of the type considered in Figs. 8.24 through 8.26. Following the procedure indicated at that time, we determine the bending moments corresponding to each of the loadings. Adding the expressions obtained, and including the moment due to the reaction at A, we write

$$M = A_y x - P\langle x-0.6\rangle^1 - \tfrac{1}{2}w_0\langle x-0.6\rangle^2 + \tfrac{1}{2}w_0\langle x-1.8\rangle^2 - M_0\langle x-2.6\rangle^0$$

Substituting the numerical values of the reaction and of the loads, and being careful *not to expand* any of the products or squares involving brackets, we obtain the following expression for the bending moment at any point of the beam:

$$M = 2.6x - 1.2\langle x-0.6\rangle^1 - 0.75\langle x-0.6\rangle^2 + 0.75\langle x-1.8\rangle^2 - 1.44\langle x-2.6\rangle^0$$

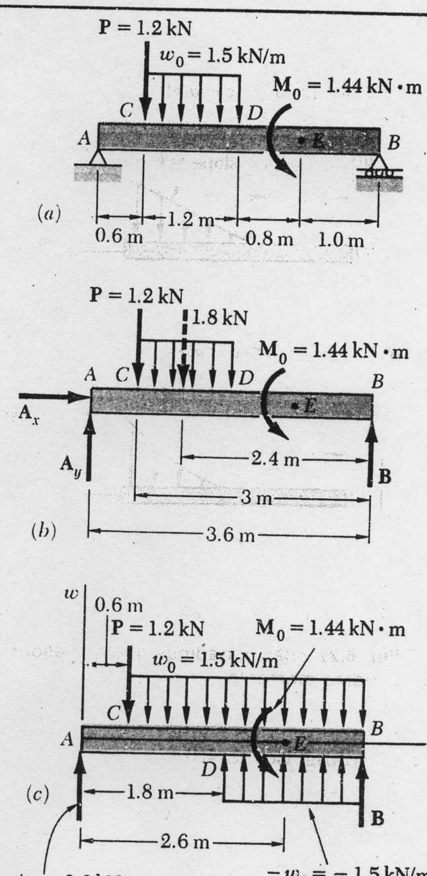


Fig. 8.29

*8.5. DIRECT DETERMINATION OF THE ELASTIC CURVE FROM THE LOAD DISTRIBUTION

We saw in Sec. 8.2 that the equation of the elastic curve may be obtained by integrating twice the differential equation

$$\frac{d^2y}{dx^2} = \frac{M(x)}{EI} \tag{8.4}$$

where $M(x)$ is the bending moment in the beam. We now recall from Sec. 7.3 that, when a beam supports a distributed load $w(x)$, we have $dM/dx = V$ and $dV/dx = -w$ at any point of the beam. Differentiating both members of Eq. (8.4) with respect to x and assuming EI to be constant, we have therefore

$$\frac{d^3y}{dx^3} = \frac{1}{EI}\frac{dM}{dx} = \frac{V(x)}{EI} \tag{8.49}$$

and, differentiating again,

$$\frac{d^4y}{dx^4} = \frac{1}{EI}\frac{dV}{dx} = -\frac{w(x)}{EI}$$

We conclude that, when a prismatic beam supports a distributed load $w(x)$, its elastic curve is governed by the fourth-order linear differential equation

$$\boxed{\frac{d^4y}{dx^4} = -\frac{w(x)}{EI}} \tag{8.50}$$

Multiplying both members of Eq. (8.50) by the constant EI and integrating four times, we write

$$\begin{aligned}
EI\frac{d^4y}{dx^4} &= -w(x)\\
EI\frac{d^3y}{dx^3} &= V(x) = -\int w(x)\,dx + C_1\\
EI\frac{d^2y}{dx^2} &= M(x) = -\int dx\int w(x)\,dx + C_1x + C_2\\
EI\frac{dy}{dx} &= EI\,\theta(x) = -\int dx\int dx\int w(x)\,dx + \frac{1}{2}C_1x^2 + C_2x + C_3\\
EI\,y(x) &= -\int dx\int dx\int dx\int w(x)\,dx + \frac{1}{6}C_1x^3 + \frac{1}{2}C_2x^2 + C_3x + C_4
\end{aligned} \tag{8.51}$$

The four constants of integration may be determined from the boundary conditions. These conditions include (*a*) the conditions imposed on the deflection or slope of the beam by its supports (cf. Sec. 8.2), and (*b*) the condition that V and M be zero at the free end of a cantilever beam, or that M be zero at both ends of a simply supported beam (cf. Sec. 7.3). This has been illustrated in Fig. 8.30.

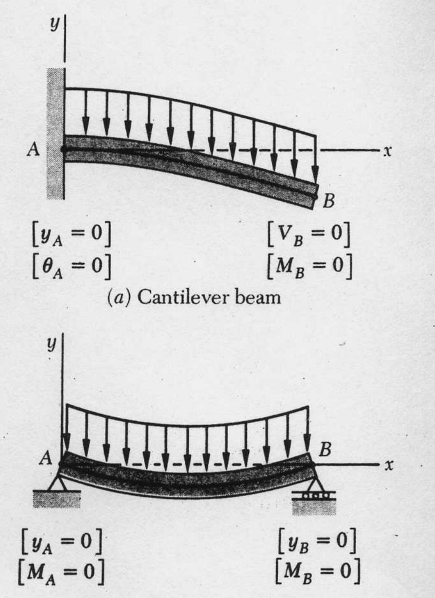


Fig. 8.30 Boundary conditions for beams carrying a distributed load.

The method presented here may be used effectively with cantilever or simply supported beams carrying a distributed load. In the case of overhanging beams, the reactions at the supports will cause discontinuities in the shear, i.e., in the third derivative of y, which may be taken into account through the use of singularity functions. By expressing the distributed load $w(x)$ itself in terms of singularity functions, the method may be extended to the solution of problems involving beams subjected to discontinuous distributed loads (see Sample Prob. 8.7).

Example 8.06

The simply supported prismatic beam AB carries a uniformly distributed load w per unit length (Fig. 8.31). Determine the equation of the elastic curve and the maximum deflection of the beam. (This is the same beam and loading as in Example 8.02.)

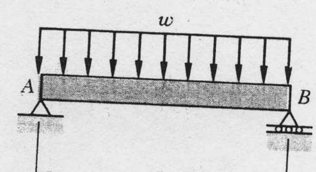


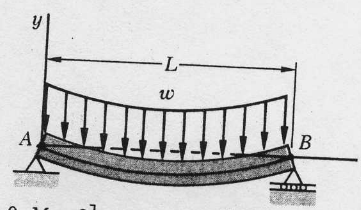
Fig. 8.31

Since w = constant, the first three of Eqs. (8.51) yield

$$EI\frac{d^4y}{dx^4} = -w$$

$$EI\frac{d^3y}{dx^3} = V(x) = -wx + C_1$$

$$EI\frac{d^2y}{dx^2} = M(x) = -\frac{1}{2}wx^2 + C_1x + C_2 \qquad (8.52)$$



$[x = 0, M = 0]$ $[x = 0, y = 0]$ $[x = L, M = 0]$ $[x = L, y = 0]$

Fig. 8.32

Noting that the boundary conditions require that $M = 0$ at both ends of the beam (Fig. 8.32), we first let $x = 0$ and $M = 0$ in Eq. (8.52) and obtain $C_2 = 0$. We then make $x = L$ and $M = 0$ in the same equation and obtain $C_1 = \frac{1}{2}wL$.

Carrying the values of C_1 and C_2 back into Eq. (8.52), and integrating twice, we write

$$EI\frac{d^2y}{dx^2} = -\frac{1}{2}wx^2 + \frac{1}{2}wLx$$

$$EI\frac{dy}{dx} = -\frac{1}{6}wx^3 + \frac{1}{4}wLx^2 + C_3$$

$$EI\,y = -\frac{1}{24}wx^4 + \frac{1}{12}wLx^3 + C_3x + C_4 \qquad (8.53)$$

But the boundary conditions also require that $y = 0$ at both ends of the beam. Letting $x = 0$ and $y = 0$ in Eq. (8.53), we obtain $C_4 = 0$; letting $x = L$ and $y = 0$ in the same equation, we write

$$0 = -\tfrac{1}{24}wL^4 + \tfrac{1}{12}wL^4 + C_3L$$

$$C_3 = -\tfrac{1}{24}wL^3$$

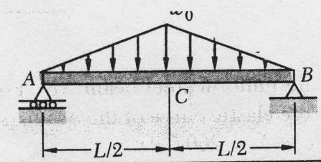
Carrying the values of C_3 and C_4 back into Eq. (8.53), and dividing both members by EI, we obtain the equation of the elastic curve:

$$y = \frac{w}{24EI}(-x^4 + 2Lx^3 - L^3x) \qquad (8.54)$$

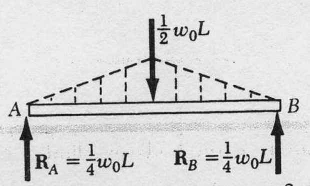
The value of the maximum deflection is obtained by making $x = L/2$ in Eq. (8.54). We have

$$|y|_{\max} = \frac{5wL^4}{384EI}$$

SAMPLE PROBLEM 8.3



For the prismatic beam and loading shown, determine (a) the equation of the elastic curve, (b) the slope at A, (c) the maximum deflection.



Reactions $\qquad \mathbf{R}_A = \frac{1}{4}w_0L\uparrow \qquad \mathbf{R}_B = \frac{1}{4}w_0L\uparrow$

Modified Loading Diagram. We note that the given distributed loading is equivalent to the two open-ended loadings shown (Fig. a). Each loading and each reaction (Fig. b) corresponds to one of the basic loadings of Fig. 8.27.

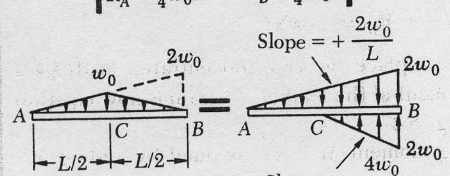


Fig. a

Bending Moment. The bending moment $M(x)$ is obtained by adding, for each reaction and loading in Fig. b, the bending moment given in Fig. 8.27.

$$M(x) = R_A x - \frac{1}{6}k_1 x^3 - \frac{1}{6}k_2\langle x - \tfrac{1}{2}L\rangle^3$$

$$M(x) = +\frac{1}{4}w_0Lx - \frac{1}{6}\left(\frac{2w_0}{L}\right)x^3 - \frac{1}{6}\left(-\frac{4w_0}{L}\right)\langle x - \tfrac{1}{2}L\rangle^3$$

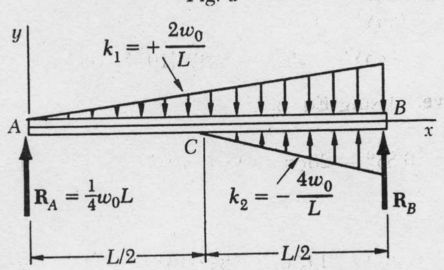


Fig. b

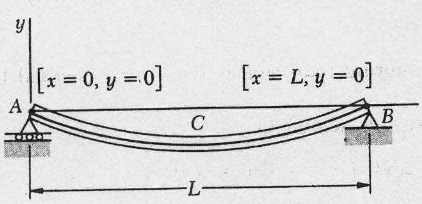
(a) Equation of the Elastic Curve. Using Eq. 8.4, we have

$$EI\frac{d^2y}{dx^2} = \frac{1}{4}w_0Lx - \frac{w_0}{3L}x^3 + \frac{2w_0}{3L}\langle x - \tfrac{1}{2}L\rangle^3 \qquad (1)$$

Integrating twice in x,

$$EI\,\theta = \frac{w_0L}{8}x^2 - \frac{w_0}{12L}x^4 + \frac{w_0}{6L}\langle x - \tfrac{1}{2}L\rangle^4 + C_1 \qquad (2)$$

$$EI\,y = \frac{w_0L}{24}x^3 - \frac{w_0}{60L}x^5 + \frac{w_0}{30L}\langle x - \tfrac{1}{2}L\rangle^5 + C_1x + C_2 \qquad (3)$$



Boundary Conditions.

$[x = 0, y = 0]$: Substituting $x = 0$ into Eq. (3), we note that the quantity between brackets is negative; thus the bracket is equal to zero. Therefore, $C_2 = 0$.

$[x = L, y = 0]$: Again using Eq. (3), we write

$$0 = +\frac{w_0L^4}{24} - \frac{w_0L^4}{60} + \frac{w_0}{30L}\left(\frac{L}{2}\right)^5 + C_1L \qquad C_1 = -\frac{5}{192}w_0L^3$$

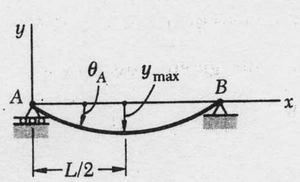
Substituting for C_1 and C_2 into Eqs. (2) and (3), we have

$$EI\,\theta = \frac{w_0L}{8}x^2 - \frac{w_0}{12L}x^4 + \frac{w_0}{6L}\langle x - \tfrac{1}{2}L\rangle^4 - \frac{5}{192}w_0L^3 \qquad (4)$$

$$EI\,y = \frac{w_0L}{24}x^3 - \frac{w_0}{60L}x^5 + \frac{w_0}{30L}\langle x - \tfrac{1}{2}L\rangle^5 - \frac{5}{192}w_0L^3x \quad \blacktriangleleft$$

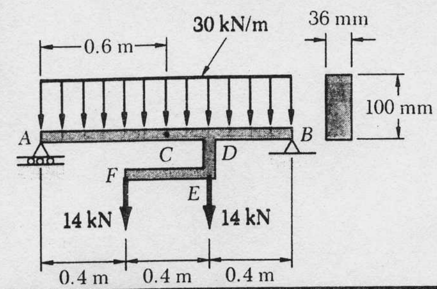
(b) Slope at A. Substituting $x = 0$ into Eq. (4), we find

$$EI\,\theta_A = -\frac{5}{192}w_0L^3 \qquad \theta_A = \frac{5w_0L^3}{192EI} \quad \blacktriangleleft$$



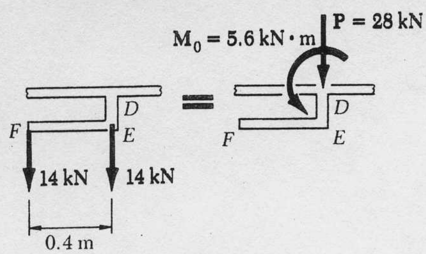
(c) Maximum Deflection. At point C, where $x = \frac{1}{2}L$, we have

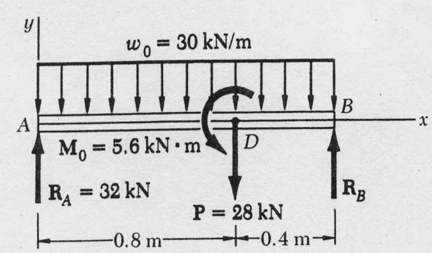
$$EI\,y_{max} = w_0L^4\left[\frac{1}{24(8)} - \frac{1}{60(32)} + 0 - \frac{5}{192(2)}\right] = -\frac{w_0L^4}{120EI} \qquad y_{max} = \frac{w_0L^4}{120EI}\downarrow \quad \blacktriangleleft$$

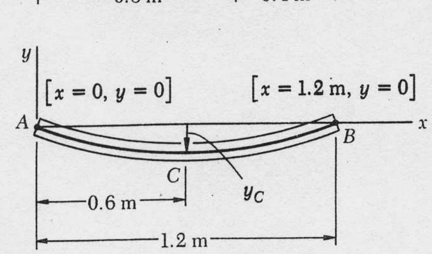


SAMPLE PROBLEM 8.4

The rigid bar DEF is welded at point D to the uniform steel beam AB. For the loading shown, determine (*a*) the equation of the elastic curve of the beam, (*b*) the deflection at the midpoint C of the beam. Use $E = 200$ GPa.







Reactions. Considering the entire beam as a free body, we find

$$\mathbf{R}_A = 32 \text{ kN}\uparrow \qquad \mathbf{R}_B = 32 \text{ kN}\uparrow$$

Modified Loading Diagram. We replace the two concentrated loads by a force and couple at point D. Each element of the loading diagram is now equal to one of the basic loadings given in Fig. 8.27.

Bending Moment. The bending moment $M(x)$ is obtained by adding, for each reaction and loading, the bending moment given in Fig. 8.27.

$$M(x) = R_A x - \tfrac{1}{2}w_0 x^2 - M_0\langle x - 0.8\rangle^0 - P\langle x - 0.8\rangle^1$$

$$M(x) = [32x - \tfrac{1}{2}(30)x^2 - 5.6\langle x - 0.8\rangle^0 - 28\langle x - 0.8\rangle^1]10^3 \text{ N}\cdot\text{m}$$

(*a*) Equation of the Elastic Curve. Using Eq. 8.4,

$$EI\frac{dy^2}{dx^2} = [32x - 15x^2 - 5.6\langle x - 0.8\rangle^0 - 28\langle x - 0.8\rangle^1]10^3 \qquad (1)$$

Integrating twice in x,

$$EI\,\theta = [16x^2 - 5x^3 - 5.6\langle x - 0.8\rangle^1 - 14\langle x - 0.8\rangle^2]10^3 + C_1 \qquad (2)$$

$$EI\,y = [5.33x^3 - 1.25x^4 - 2.8\langle x - 0.8\rangle^2 - 4.67\langle x - 0.8\rangle^3]10^3 + C_1x + C_2 \qquad (3)$$

Boundary Conditions.

$[x = 0,\ y = 0]$: Using Eq. (3), and noting that each bracket $\langle\ \rangle$ is equal to zero, we find $C_2 = 0$.

$[x = 1.2 \text{ m},\ y = 0]$: Again using Eq. (3), we write

$$0 = [5.33(1.2)^3 - 1.25(1.2)^4 - 2.8(0.4)^2 - 4.67(0.4)^3]10^3 + C_1(1.2)$$

$$C_1 = -4.89 \times 10^3$$

Substituting for C_1 and C_2 into Eq. (3), we have

$$\boldsymbol{EI\,y = [5.33x^3 - 1.25x^4 - 2.8\langle x - 0.8\rangle^2 - 4.67\langle x - 0.8\rangle^3 - 4.89x]10^3} \blacktriangleleft$$

(*b*) Deflection at Midpoint C. For $x = 0.6$ m, we write

$$EI\,y_C = [5.33(0.6)^3 - 1.25(0.6)^4 - 0 - 0 - 4.89(0.6)]10^3 = -1.945 \times 10^3$$

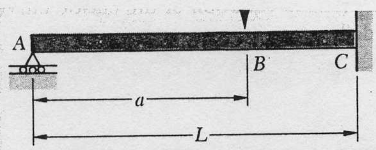
We recall that $E = 200$ GPa and note that, for the given cross section,

$$I = \frac{bh^3}{12} = \frac{(0.036 \text{ m})(0.100 \text{ m})^3}{12} = 3 \times 10^{-6} \text{ m}^4$$

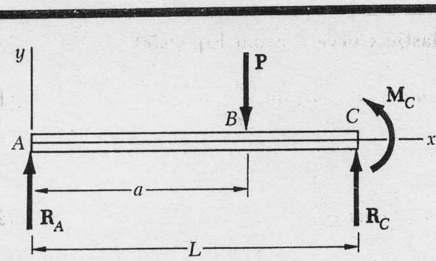
We then write

$$(200 \times 10^9)(3 \times 10^{-6})y_C = -1.945 \times 10^3$$

$$\boldsymbol{y_C = 3.24 \text{ mm}\downarrow} \blacktriangleleft$$



I, (*b*) determine the reaction at A and the deflection under the load when $a = L/2$.



Reactions. For the given vertical load **P** the reactions are as shown. We note that they are statically indeterminate.

Bending Moment. The bending moment $M(x)$ is obtained by adding the moments of the reaction R_A and of the load P. Recalling Fig. 8.27, we write

$$M(x) = R_A x - P\langle x - a\rangle^1$$

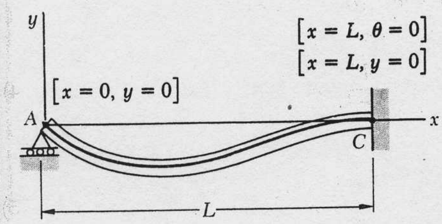
Equation of the Elastic Curve. Using Eq. (8.4),

$$EI\frac{d^2y}{dx^2} = R_A x - P\langle x - a\rangle^1$$

Integrating twice in x,

$$EI\frac{dy}{dx} = EI\,\theta = \frac{1}{2}R_A x^2 - \frac{1}{2}P\langle x - a\rangle^2 + C_1$$

$$EI\,y = \frac{1}{6}R_A x^3 - \frac{1}{6}P\langle x - a\rangle^3 + C_1 x + C_2$$



Boundary Conditions

$[x = 0, y = 0]$: $\quad C_2 = 0 \qquad (1)$

$[x = L, \theta = 0]$: $\quad \frac{1}{2}R_A L^2 - \frac{1}{2}P(L - a)^2 + C_1 = 0 \qquad (2)$

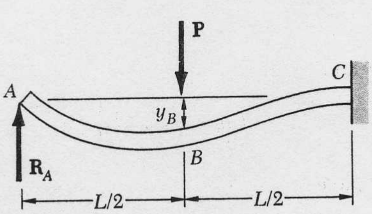
$[x = L, y = 0]$: $\quad \frac{1}{6}R_A L^3 - \frac{1}{6}P(L - a)^3 + C_1 L + C_2 = 0 \qquad (3)$

(*a*) **Reaction at *A*.** Multiplying Eq. (2) by L, subtracting Eq. (3) member by member from the equation obtained, and noting that $C_2 = 0$, we have

$$\frac{1}{3}R_A L^3 - \frac{1}{6}P(L - a)^2[3L - (L - a)] = 0 \qquad \mathbf{R}_A = P\left(1 - \frac{a}{L}\right)^2\left(1 + \frac{a}{2L}\right)\uparrow \blacktriangleleft$$

We note that the reaction is independent of E and I.

(*b*) **Reaction at *A* and Deflection at *B* when $a = \frac{1}{2}L$.** Making $a = \frac{1}{2}L$ in the expression obtained for R_A, we have



$$R_A = P(1 - \tfrac{1}{2})^2(1 + \tfrac{1}{4}) = 5P/16 \qquad \mathbf{R}_A = \tfrac{5}{16}P\uparrow \blacktriangleleft$$

Substituting $a = L/2$ and $R_A = 5P/16$ into Eq. (2) and solving for C_1, we find $C_1 = -PL^2/32$. Making $x = L/2$, $C_1 = -PL^2/32$, and $C_2 = 0$ in the expression obtained for y, we have

$$y_B = -\frac{7PL^3}{768EI} \qquad y_B = \frac{7PL^3}{768EI}\downarrow \blacktriangleleft$$

Note that the deflection obtained is *not* the maximum deflection.