

10

Beam Deflections: Second-Order Method

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Note to ASEN 3112 students; the material in §10.1 through §10.3 is largely classical and may be skipped if you remember beams in ASEN 2001. The new stuff starts with the summary in §10.4.

§10.1. Introduction

This Lecture starts the presentation of methods for computed lateral deflections of plane beams undergoing symmetric bending. This topic is covered in great detail in Chapter 9 of the textbook by Beer, Johnston and DeWolf. The lectures posted here summarize some important points and present some examples.

Students are assumed to be familiar with: (1) integration of ODEs, and (2) *statics of plane beams under symmetric bending*. The latter topic is covered in ASEN 2001. Chapters 4 through 6 of the *Mechanics of Materials* textbook by Beer, Johnston and DeWolf deal thoroughly with that subject, which is assumed to be known.

§10.2. What is a Beam?

Beams are the most common type of structural component, particularly in Civil and Mechanical Engineering. A *beam* is a bar-like structural member whose primary function is to support *transverse loading* and carry it to the supports. See Figure 10.1.

By “bar-like” it is meant that one of the dimensions is considerably larger than the other two. This dimension is called the *longitudinal dimension* or *beam axis*. The intersection of planes normal to the longitudinal dimension with the beam member are called *cross sections*. A *longitudinal plane* is one that passes through the beam axis.

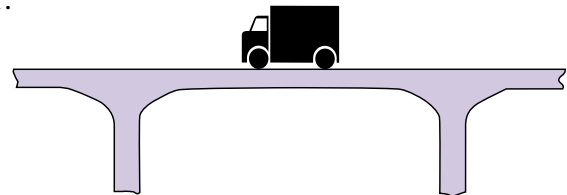


FIGURE 10.1 A beam is a structural member designed to resist transverse loads.

A beam resists transverse loads mainly through *bending action*. Bending produces compressive longitudinal stresses in one side of the beam and tensile stresses in the other.

The two regions are separated by a *neutral surface* of zero stress. The combination of tensile and compressive stresses produces an internal *bending moment*. This moment is the primary mechanism that transports loads to the supports. The mechanism is illustrated in Figure 10.2.

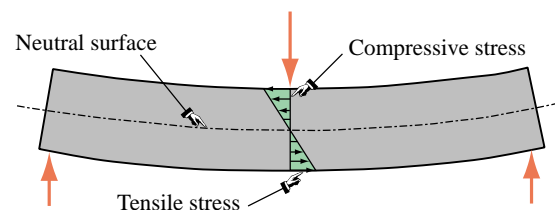


FIGURE 10.2 Beam transverse loads are primarily resisted by bending action.

§10.2.1. Terminology

A *general beam* is a bar-like member designed to resist a combination of loading actions such as biaxial bending, transverse shears, axial stretching or compression, and possibly torsion. If the internal axial force is compressive, the beam has also to be designed to resist buckling. If the beam is subject primarily to bending and axial forces, it is called a *beam-column*. If it is subjected primarily to bending forces, it is called simply a beam. A beam is *straight* if its longitudinal axis is straight. It is *prismatic* if its cross section is constant.

A *spatial beam* supports transverse loads that can act on arbitrary directions along the cross section. A *plane beam* resists primarily transverse loading on a preferred longitudinal plane. This course considers only plane beams undergoing symmetric bending.

§10.2.2. Mathematical Models

One-dimensional mathematical models of structural beams are constructed on the basis of *beam theories*. Because beams are actually three-dimensional bodies, all models necessarily involve some form of approximation to the underlying physics. The simplest and best known models for straight, prismatic beams are based on the *Bernoulli-Euler beam theory* (also called *classical beam theory* and *engineering beam theory*), and the *Timoshenko beam theory*. The Bernoulli-Euler theory is that taught in introductory Mechanics of Materials courses, and is the only one dealt with in this course. The Timoshenko model incorporates a first order kinematic correction for transverse shear effects. This model assumes additional importance in dynamics and vibration.

§10.2.3. Assumptions of Classical Beam Theory

The classical beam theory for *plane beams* rests on the following assumptions:

1. *Planar symmetry*. The longitudinal axis is straight and the cross section of the beam has a longitudinal plane of symmetry. The resultant of the transverse loads acting on each section lies on that plane. The support conditions are also symmetric about this plane.
2. *Cross section variation*. The cross section is either constant or varies smoothly.
3. *Normality*. Plane sections originally normal to the longitudinal axis of the beam remain plane and normal to the deformed longitudinal axis upon bending.
4. *Strain energy*. The internal strain energy of the member accounts only for bending moment deformations. All other contributions, notably transverse shear and axial force, are ignored.
5. *Linearization*. Transverse deflections, rotations and deformations are considered so small that the assumptions of infinitesimal deformations apply.
6. *Material model*. The material is assumed to be elastic and isotropic. Heterogeneous beams fabricated with several isotropic materials, such as reinforced concrete, are not excluded.

§10.3. The Bernoulli-Euler Beam Theory

§10.3.1. Beam Coordinate Systems

Under transverse loading one of the top surfaces shortens while the other elongates; see Figure 10.2. Therefore a *neutral surface* that undergoes no axial strain exists between the top and the bottom. The intersection of this surface with each cross section defines the *neutral axis* of that cross section. If the beam is homogenous, the neutral axis passes through the centroid of the cross section. If the beam is fabricated of different materials — for example, a reinforced concrete beam — the neutral axis passes through the centroid of an “equivalent” cross section. This topic is covered in Mechanics of Materials textbooks; for example Beer-Johnston-DeWolf’s Chapter 4.

The Cartesian axes for plane beam analysis are chosen as shown in Figure 10.3. Axis x lies along the longitudinal beam axis, at neutral axis height. Axis y lies in the symmetry plane and points upwards. Axis z is directed along the neutral axis, forming a RHS system with x and y . The origin is placed at the leftmost section. The total length (or span) of the beam member is called L .

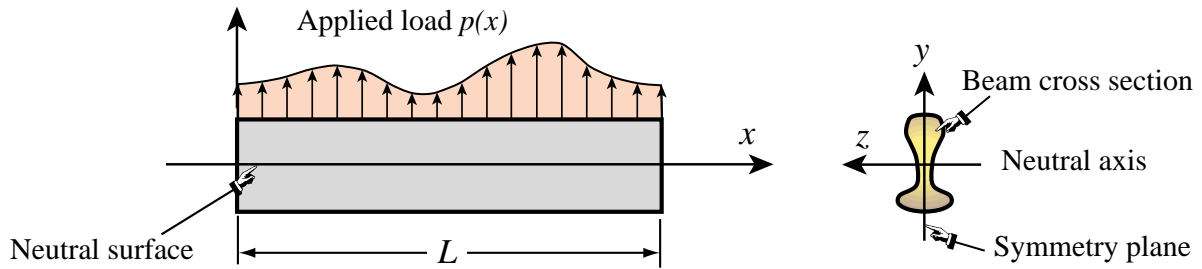


FIGURE 10.3 Terminology and choice of axes for Bernoulli-Euler model of plane beam.

§10.3.2. Beam Motion

The *motion* under loading of a plane beam member in the x, y plane is described by the two dimensional displacement field

$$\begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix}, \tag{10.1}$$

where u and v are the axial and transverse displacement components, respectively, of an arbitrary beam material point. The motion in the z direction, which is primarily due to Poisson’s ratio effects, is of no interest. The normality assumption of the Bernoulli-Euler model can be represented mathematically as

$$u(x, y) = -y \frac{\partial v(x)}{\partial x} = -yv' = -y\theta, \quad v(x, y) = v(x). \tag{10.2}$$

Note that the slope $v' = \partial v / \partial x = dv / dx$ of the deflection curve has been identified with the *rotation* symbol θ . This is permissible because θ represents to first order, according to the kinematic assumptions of this model, the rotation of a cross section about z positive CCW.

§10.3.3. Beam Loading

The transverse force *per unit length* that acts on the beam in the $+y$ direction is denoted by $p(x)$, as illustrated in Figure 10.3. Point loads and moments acting on isolated beam sections can be represented with Discontinuity Functions (DFs), a topic covered in Lecture 12.

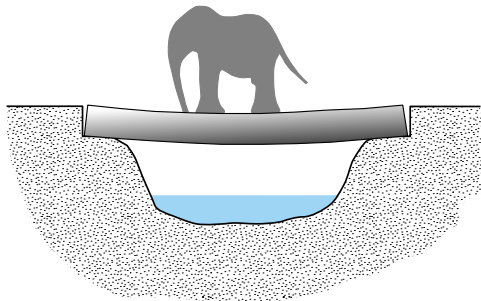


FIGURE 10.4 A simply supported beam has end supports that preclude transverse displacements but permit end rotations.

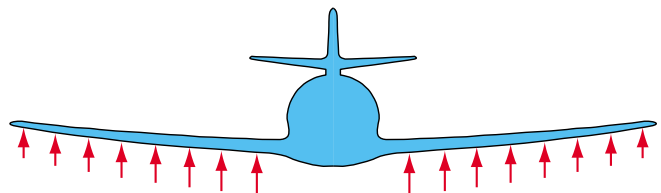


FIGURE 10.5 A cantilever beam is clamped at one end and free at the other. Airplane wings and stabilizers are examples of this configuration.

§10.3.4. Support Conditions

Support conditions for beams exhibit far more variety than for bar members. Two canonical cases are often encountered in engineering practice: simple support and cantilever support. These are illustrated in Figures 10.4 and 10.5, respectively. Beams often appear as components of skeletal structures called frameworks, in which case the support conditions are of more complex type.

§10.3.5. Strains, Stresses and Bending Moments

The Bernoulli-Euler or classical model assumes that the internal energy of beam member is entirely due to bending strains and stresses. Bending produces axial stresses σ_{xx} , which will be abbreviated to σ , and axial strains ϵ_{xx} , which will be abbreviated to ϵ . The strains can be linked to the displacements by differentiating the axial displacement $u(x)$:

$$\epsilon = \epsilon_{xx} = \frac{\partial u}{\partial x} = -y \frac{d^2 v}{dx^2} = -y v'' = -y \kappa. \tag{10.3}$$

Here κ denotes the deformed beam axis curvature, which to first order is $\kappa \approx d^2 v / dx^2 = v''$. The bending stress $\sigma = \sigma_{xx}$ is linked to e through the one-dimensional Hooke's law

$$\sigma = E e = -E y \frac{d^2 v}{dx^2} = -E y \kappa, \tag{10.4}$$

where E is the longitudinal elastic modulus. The most important stress resultant in classical beam theory is the *bending moment* M_z , which is defined as the cross section integral

$$M_z = \int_A -y \sigma dx = E \frac{d^2 v}{dx^2} \int_A y^2 dA = E I_{zz} \kappa. \tag{10.5}$$

Here I_{zz} denotes the moment of inertia $\int_A y^2 dA$ of the cross section with respect to the z (neutral) axis. The bending moment M is considered positive if it compresses the upper portion: $y > 0$, of the beam cross section, as illustrated in Figure 10.6. This convention explains the negative sign of y in the above integral. The product $E I_{zz}$ is called the *bending rigidity* of the beam with respect to flexure about the z axis.

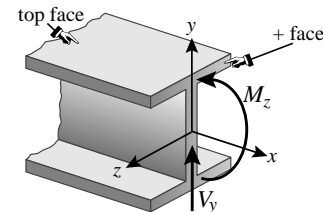


FIGURE 10.6 Positive sign convention for M_z and V_y .

§10.3.6. Beam Kinematics

The kinematic of the classical beam model used in this course is illustrated in Figure 10.7. Additional details may be found in Chapter 9 of Beer-Johnston-DeWolf.

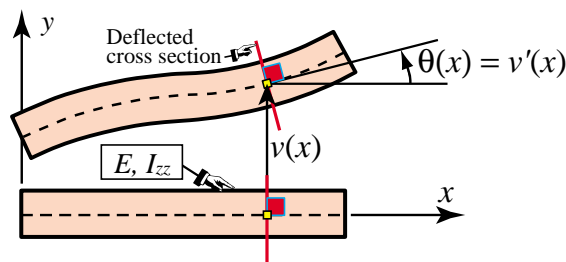


FIGURE 10.7 Beam kinematics.

§10.4. Notation Summary

Quantity	Symbol	Sign convention(s)
Problem specific load	varies	You pick'em
Generic load for ODE work	$p(x)$	+ if up
Transverse shear force	$V_y(x)$	+ if up on $+x$ face
Bending moment	$M_z(x)$	+ if it produces compression on top face
Slope of deflection curve	$\frac{dv(x)}{dx} = v'(x)$	+ if positive slope, or cross-section rotates CCW
Deflection curve	$v(x)$	+ if beam section moves upward

Note 1: Some textbooks (e.g. Vable and Beer-Johnston-DeWolf) use $V = -V_y$ as alternative transverse shear symbol. This has the advantage of eliminating the $-$ sign in two of the ODEs below. V will only be used occasionally in this course.

Note 2: In our beam model the slope $v' = dv(x)/dx$ is equal to the rotation $\theta(x)$ of the cross section.

§10.5. Differential Equations Summary

The following differential relations assume sufficient differentiability for the derivatives to exist. This requirement can be alleviated by using Discontinuity Functions (DFs), a topic covered in Lecture 13.

Connected quantities	ODEs
From load to transverse shear force	$\frac{dV_y}{dx} = -p$ or $p = -V'_y = V'$
From transverse shear to bending moment	$\frac{dM_z}{dx} = -V_y$ or $M'_z = -V_y = V$
From bending moment to deflection	$E I_{zz} \frac{d^2v}{dx^2} = M_z$ or $v'' = \frac{M_z}{E I_{zz}}$
From load to moment	$M''_z = p$
From load to deflection	$E I_{zz} v^{IV} = p$

§10.6. Second Order Method For Beam Deflections

The second-order method to find beam deflections gets its name from the order of the ODE to be integrated: $E I_{zz} v''(x) = M_z(x)$ (third line in above table) is a *second order* ODE. The procedure can be broken down into the following steps.

1. Find the bending moment $M_z(x)$ directly, for example by statics.
2. Integrate $M_z/(E I_{zz})$ once to get the slope $v'(x) = dv(x)/dx$.
3. Integrate once more to get $v(x)$.
4. If there are no continuity conditions, the foregoing steps will produce two integration constants. Apply kinematic BCs to find those. If there are continuity conditions, more than two integration constants may appear. Apply kinematic BCs *and* continuity conditions to find those.

5. Substitute the constants of integration into the deflection function to get $v(x)$.
6. Evaluate $v(x)$ at specific sections of interest.

Three examples of this method follow.

§10.6.1. Example 1: Cantilever Load Under Tip Point Load

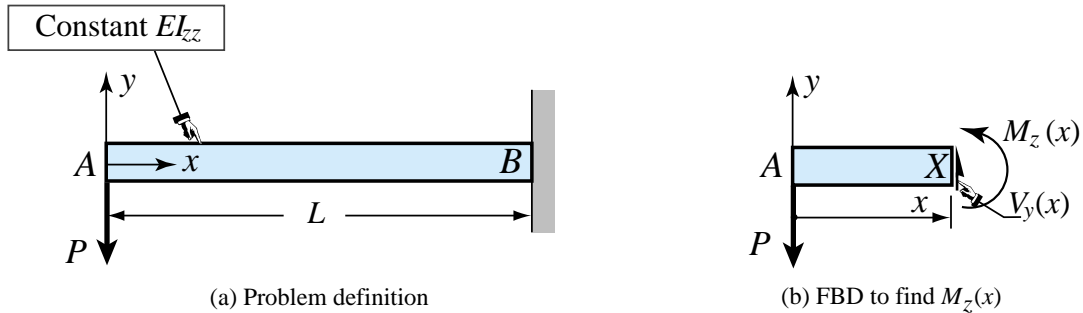


FIGURE 10.8 Beam problem for Example 1.

The problem is defined in Figure 10.8(a). Using the FBD pictured in Figure 10.8(b), and stating moment equilibrium with respect to X (to eliminate *ab initio* the effect of the transverse shear force V_y at that section) gives

$$M_z(x) = -Px \quad (10.6)$$

For convenience we scale $v(x)$ by EI_{zz} so that the ODE linking bending moment to deflection is $EI_{zz} v''(x) = M_z(x) = -Px$. Integrating twice:

$$\begin{aligned} EI_{zz} v'(x) &= -\frac{Px^2}{2} + C_1 \\ EI_{zz} v(x) &= -\frac{Px^3}{6} + C_1 x + C_2 \end{aligned} \quad (10.7)$$

The kinematic BCs for the cantilever are $v'(L) = 0$ and $v(L) = 0$ at the fixed end B. The first one gives $EI_{zz} v'(L) = -PL^2/2 + C_1 = 0$, whence $C_1 = PL^2/2$. The second one gives $EI_{zz} v(L) = -PL^3/6 + C_1 L + C_2 = -PL^3/6 + PL^3/2 + C_2 = 0$ whence $C_2 = -PL^3/3$. Substituting into the expression for $v(x)$ gives

$$v(x) = -\frac{P}{6EI_{zz}}(x^3 - 3L^2x + 2L^3) = -\frac{P}{6EI_{zz}}(L-x)^2(2L+x) \quad (10.8)$$

Of particular interest is the tip deflection at free end A, which is the largest one. Setting $x = 0$ yields

$$v(0) = v_A = -\frac{PL^3}{3EI_{zz}} \quad \Downarrow \quad (10.9)$$

The negative sign indicates that the beam deflects downward if $P > 0$.

§10.6.2. Example 2: Cantilever Beam under Triangular Distributed Load

The problem is defined in Figure 10.9(a). Using the FBD pictured in Figure 10.9(b), again doing moment equilibrium with respect to X gives

$$M_z(x) = -\frac{1}{2}w(x)x\left(\frac{1}{3}x\right) = -\frac{w_B x^3}{6L} \quad (10.10)$$

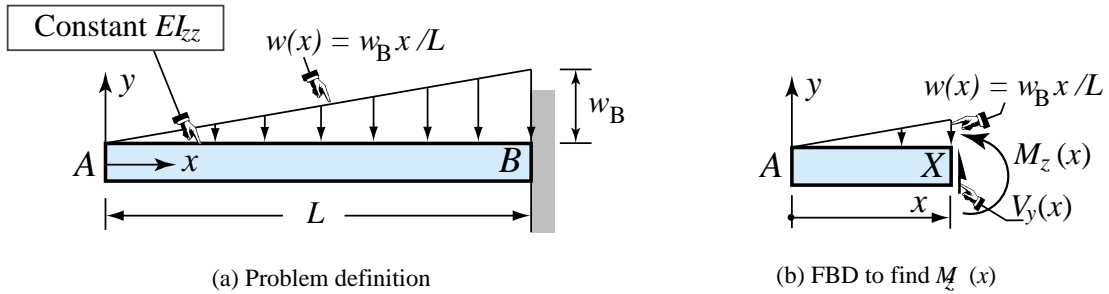


FIGURE 10.9 Beam problem for Example 2.

Integrating $EI_{zz}v''(x) = M_z(x)$ twice yields

$$\begin{aligned} EI_{zz}v'(x) &= -\frac{wx^4}{24L} + C_1, \\ EI_{zz}v(x) &= -\frac{wx^5}{120L} + C_1x + C_2. \end{aligned} \quad (10.11)$$

As in Example 1, the kinematic BCs for the cantilever are $v'(L) = 0$ and $v(L) = 0$ at the fixed end B. The first one gives $EI_{zz}v'(L) = -wL^3/24 + C_1 = 0$, whence $C_1 = w_B L^3/24$. The second one gives $EI_{zz}v(L) = -w_B L^4/120 + C_1 L + C_2 = 0$ whence $C_2 = -w_B L^4/30$. Substituting into the expression for $v(x)$ we obtain the deflection curve as

$$v(x) = \frac{1}{EI_{zz}L} \left(-\frac{w_B x^5}{120} + \frac{w_B L^4 x}{24} - \frac{w_B L^5}{30} \right) = -\frac{w_B}{120EI_{zz}L} (x^5 - 5L^4x + 4L^5) \quad (10.12)$$

Of particular interest is the tip deflection at A, which is the largest one. Setting $x = 0$ yields

$$v(0) = v_A = -\frac{w_B L^4}{30EI_{zz}} \downarrow \quad (10.13)$$

The negative sign indicates that the beam deflects downward if $w_B > 0$.

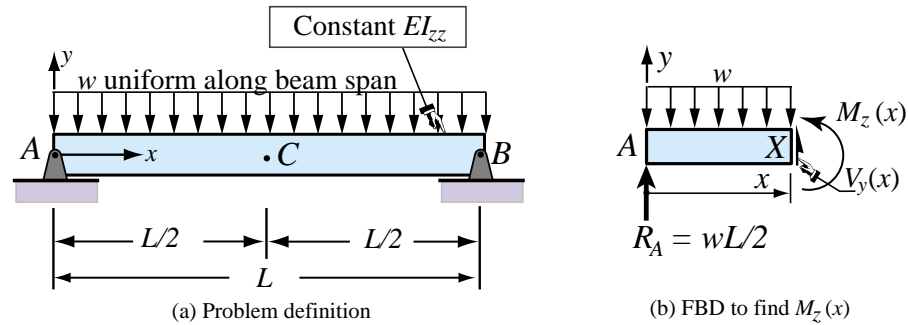


FIGURE 10.10 Beam problem for Example 3.

§10.6.3. Example 3: Simply Supported Beam Under Uniform Load

The problem is defined in Figure 10.10(a). Using the FBD pictured in Figure 10.10(b), and writing down moment equilibrium with respect to X gives the bending moment

$$M_z(x) = R_A x - \frac{wx^2}{2} = \frac{wL}{2} x - \frac{wx^2}{2} = \frac{wx}{2}(L-x)$$

Integrating $EI_{zz}v''(x) = M_z(x)$ twice yields

$$EI_{zz}v'(x) = \frac{wLx^2}{4} - \frac{wx^3}{6} + C_1 = \frac{wx^2}{12}(3L - 2x) + C_1, \tag{10.14}$$

$$EI_{zz}v(x) = \frac{wLx^3}{12} - \frac{wx^4}{24} + C_1x + C_2 = \frac{wx^3}{24}(2L - x) + C_1x + C_2.$$

The kinematic BCs for a SS beam are $v_A = v(0) = 0$ and $v_B = v(L) = 0$. The first one gives $C_2 = 0$ and the second one $C_1 = -wL^3/24$. Substituting into the expression for $v(x)$ gives, after some algebraic simplifications,

$$v(x) = -\frac{w}{24EI_{zz}}x(L-x)(L^2 + Lx - x^2) \tag{10.15}$$

Note that since $v(x) = v(L-x)$ the deflection curve is symmetric about the midspan C. The midspan deflection is the largest one:

$$v_C = v(L/2) = -\frac{5wL^4}{384EI_{zz}} \downarrow \tag{10.16}$$

The negative sign indicates that the beam deflects downward if $w > 0$.

§10.6.4. Assessment of The Second Order Method

Good Points. The governing ODE $v''(x) = M_z(x)/EI_{zz}$ is of second order, so it only needs to be integrated twice, and only two constants of integration appear. Finding $M_z(x)$ directly from statics is particularly straightforward for cantilever beams since the reactions at the free end are zero.

Beam Boundary Conditions for Shear, Moment, Slope & Deflection				
Condition	Shear force $V_y(x)$	Bending moment $M_z(x)$	Slope (= rotation) $v'(x) = \theta(x)$	Deflection $v(x)$
Simple support		$0^{\&}$		0
Fixed end			0	0
Free end	0^*	$0^{\#}$		
Symmetry	0		0	
Antisymmetry		0		0

* Unless a point force is applied at the free end
 & Unless a point moment is applied at the simple support
 # Unless a point moment is applied at the free end
 Blank entry means that value is unknown and has to be determined by solving problem

FIGURE 10.11 Beam boundary conditions for some common support configurations.

Bad Points. Clumsy if continuity conditions appear within the beam span, unless Discontinuity Functions (DFs) are used to represent $M_z(x)$ as a single expression. But this is prone to errors in human calculations. This disadvantage is shared by the fourth-order method covered in the next Lecture, but the representation of $p(x)$ in terms of DFs (to be covered in the lecture of Tu October 5*) is simpler.

§10.7. Addendum. Boundary Condition Table

The BC table provided in Figure 10.11 for some common support configurations is part of a Supplementary Crib Sheet allowed for Midterm Exams 2 and 3, in addition to the student-prepared two-sided crib sheet.

* This topic (DFs) will not be in midterm exam #2 but in #3.