Torsion

MAE 314 – Solid Mechanics

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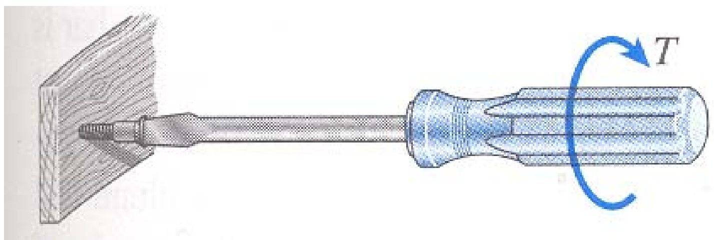
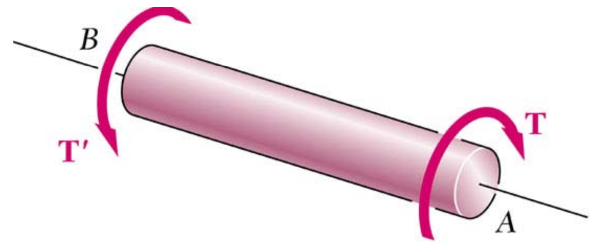
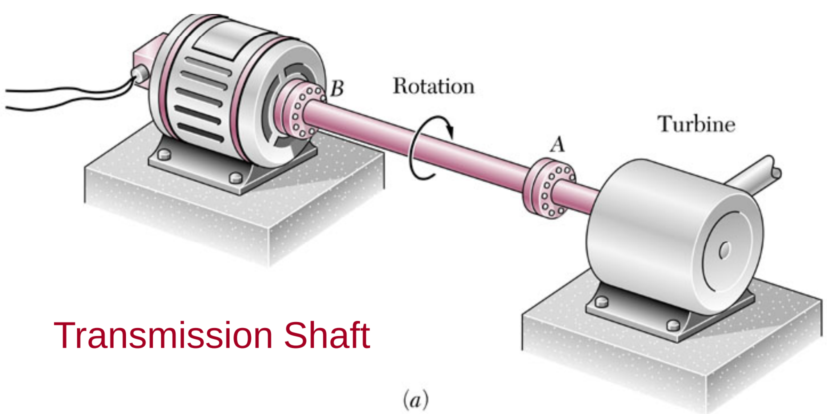
Torsion of Circular Shafts

- In this chapter, we will examine uniaxial bars subject to torque.

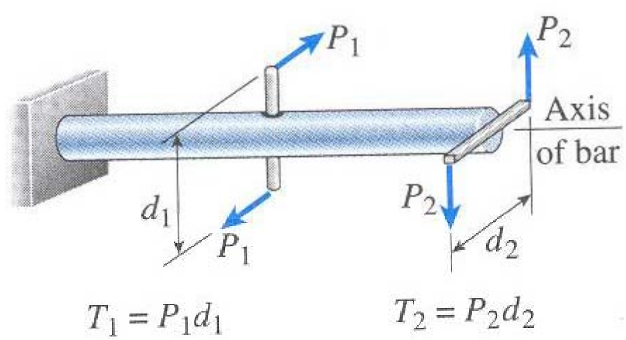




- Where does this occur?



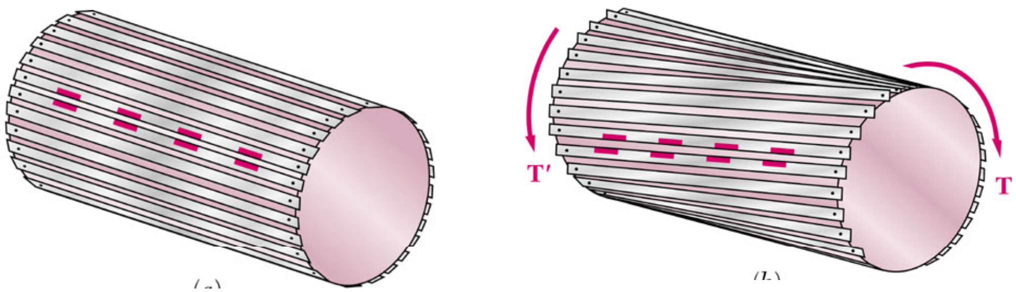
Transmission Shaft



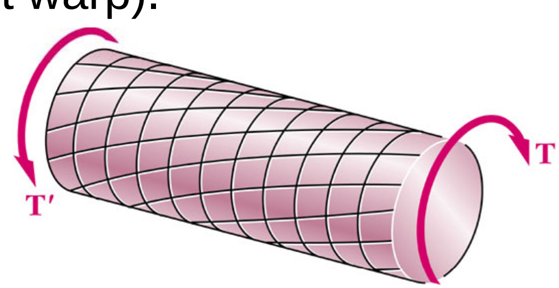
Force Couples

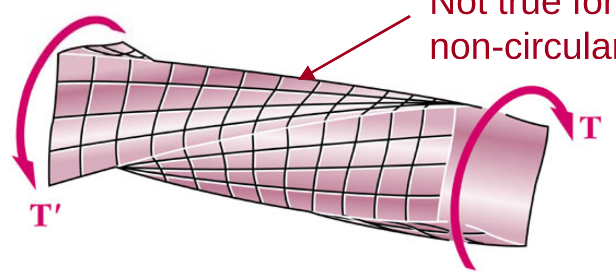
Torsion of Circular Shafts *cont'd*

- We assume
  - Bar is in pure torsion
  - Small rotations (the length and radius will not change)
- How does the bar deform?
  - Cross-section of the bar remains the same shape, bar is simply rotating.



  - Cross-section remains perpendicular to axis of cylinder (cylinder does not warp).





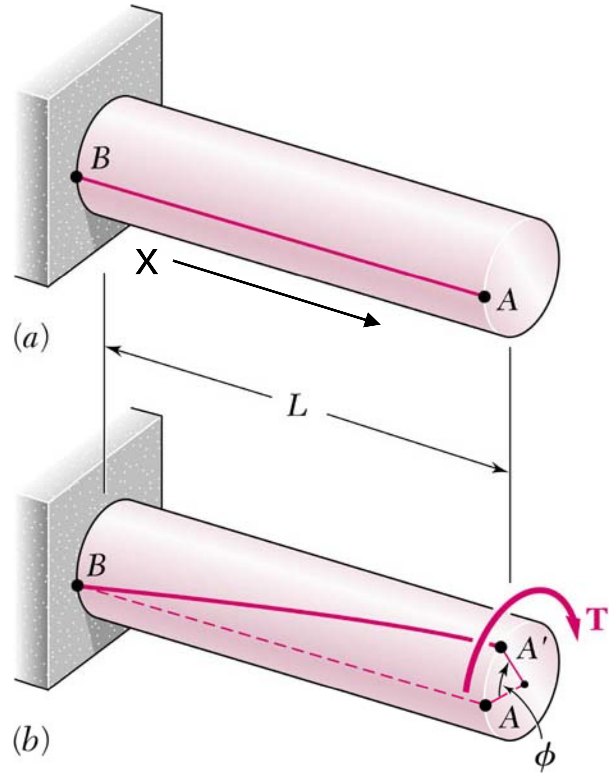
Not true for most non-circular bars

NC STATE achieve! Innovation in Action

Angle of Twist

- Deformation of a circular shaft subjected to pure torsion
  - Fix left end of shaft
  - A moves to A'
  - ϕ = angle of twist (in radians)
- What are the boundary conditions on ϕ?
  - $\phi(x) = 0$ at $x = 0$
  - $\phi(x) = \phi$ at $x = L$
- For pure torsion, ϕ is linear.

$$\phi(x) = \frac{\phi x}{L}$$



Shearing Strain

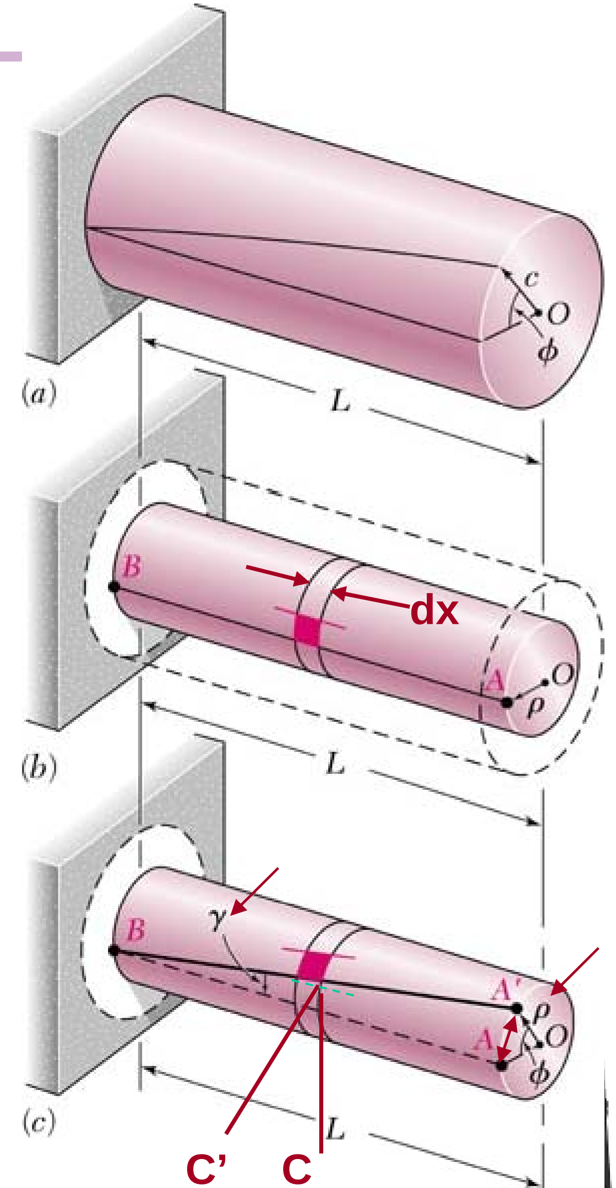
- Calculate the surface shear strain in the cylinder.
- Consider an element of length dx.
- Recall we assume small ϕ & small γ.

$$\gamma = \frac{\overline{C'C}}{dx} \qquad \overline{C'C} = \rho d\phi \qquad \gamma = \rho \frac{d\phi}{dx}$$

$$\frac{d\phi}{dx} = \text{rate of change of angle of twist along the bar}$$

- This equation applies to any function $\phi(x)$.
- For pure torsion $\phi(x) = \phi x / L$, so

$$\boxed{\gamma = \frac{\rho \phi}{L}}$$



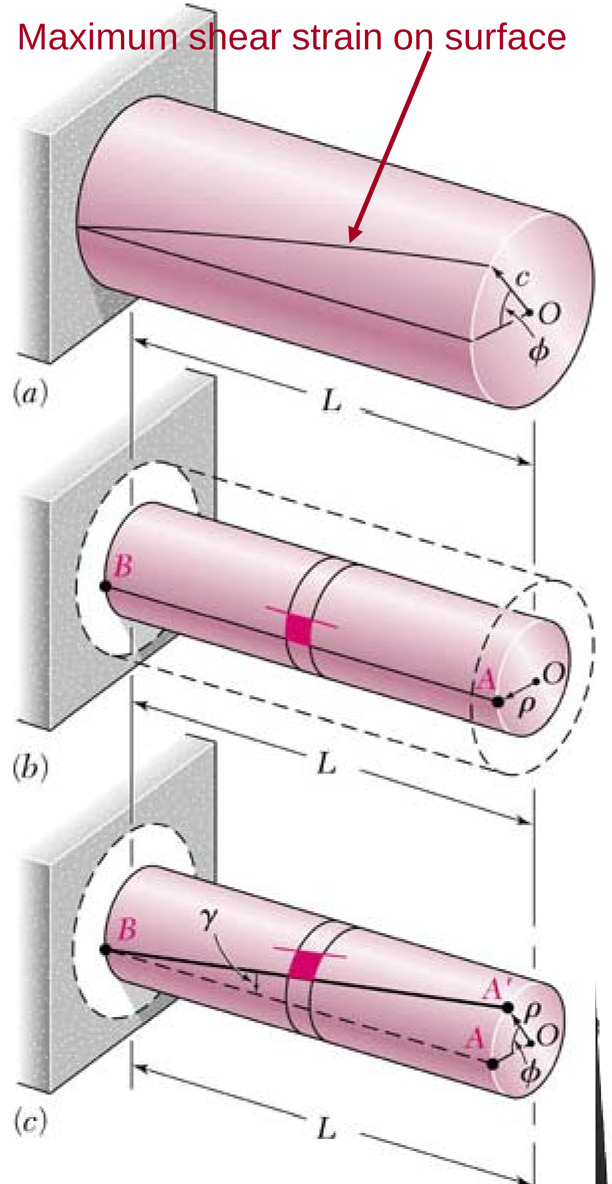
Shearing Strain *cont'd*

- The maximum shear strain on the surface of the cylinder occurs when ρ=c.

$$\gamma_{\max} = \frac{c\phi}{L}$$

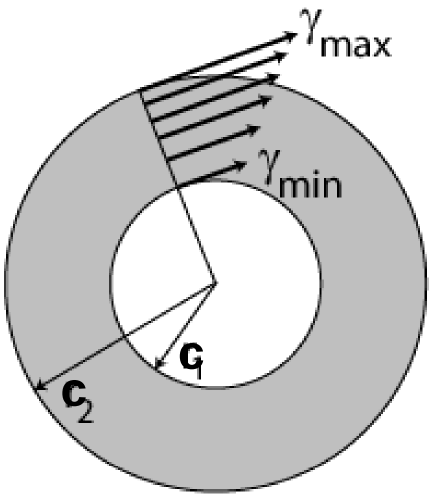
- We can express the shearing strain at any distance from the axis of the shaft as

$$\gamma = \frac{\rho}{c}\gamma_{\max}$$



Shearing Strain *cont'd*

- We can also apply the equation for maximum surface shear strain to a hollow circular tube.

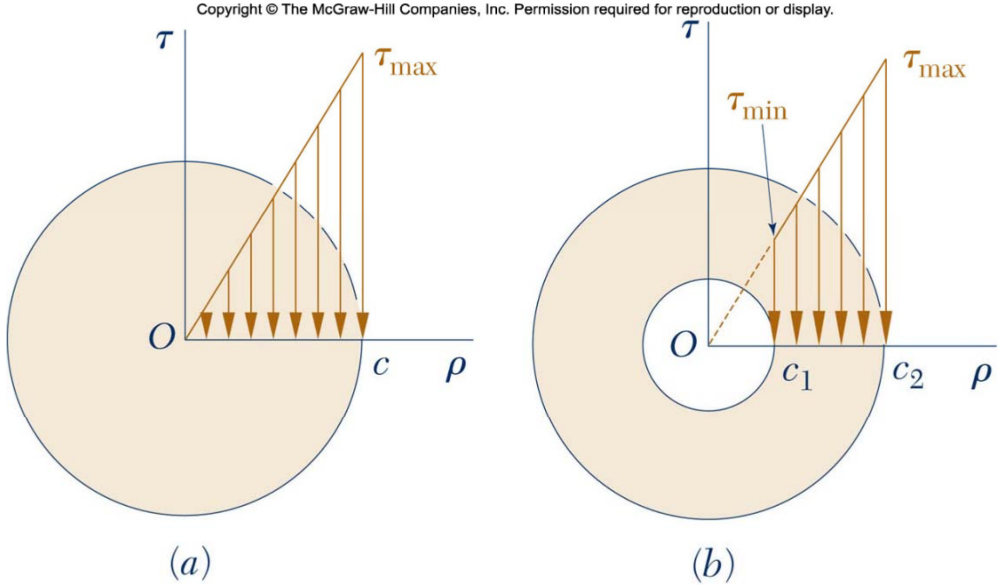


$$\gamma_{\min} = \frac{c_1 \phi}{L}$$

$$\gamma_{\max} = \frac{c_2 \phi}{L}$$

- This applies for all types of materials: elastic, linear, non-linear, plastic, etc.

Elastic Shearing Stress



- Calculate shear stress in a bar made of linearly elastic material.
- Recall Hooke's Law for shearing stress: $\tau = G\gamma$

$$\tau_{max} = G\gamma_{max} = \frac{Gc\phi}{L} \Rightarrow \boxed{\tau = \frac{\rho}{c}\tau_{max}}$$

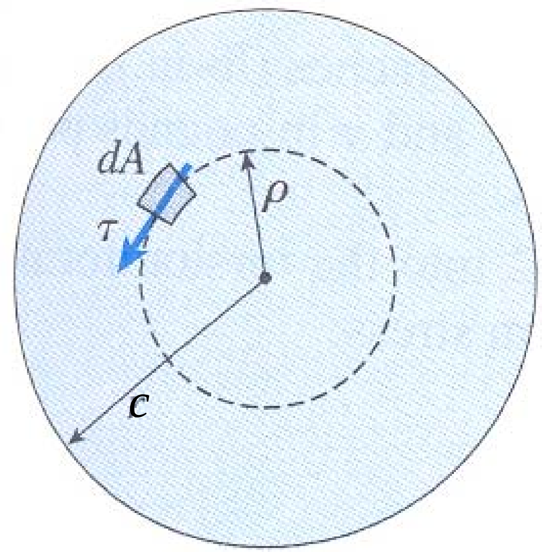
Torque

- We still need to relate τ to the applied torque T, which is generally the known, applied load.
- First, find the resultant moment acting on a cross-section and set this equal to T.

$$\tau = \frac{\rho}{c}\tau_{\max}$$

$$dM = \tau\rho dA = \frac{\rho^2}{c}\tau_{\max} dA$$

$$T = \int_A \frac{\rho^2}{c}\tau_{\max} dA = \frac{\tau_{\max}}{c}\int_A \rho^2 dA$$



Torque *cont'd*

- Continuing from previous slide:

$$T = \frac{\tau_{\max}}{c}\int_A \rho^2 dA = \frac{\tau_{\max}}{c} J \;\Rightarrow\; \boxed{\tau_{\max} = \frac{Tc}{J}}, \quad \boxed{\tau = \frac{T\rho}{J}}$$

  - Where J is the polar moment of inertia of the cross section of the bar (see Appendix in your textbook).

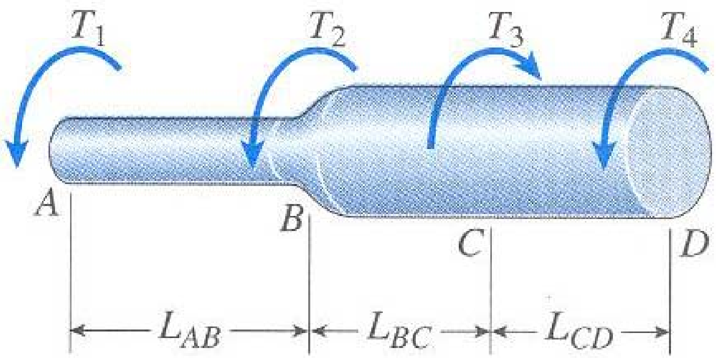
- Plug this into the equation for τ_{max}.

$$\tau_{\max} = \frac{Gc\phi}{L} \;\rightarrow\; \frac{Gc\phi}{L} = \frac{Tc}{J} \;\Rightarrow\; \boxed{\phi = \frac{TL}{GJ}}$$

Torque *cont'd*

- For a non-uniform bar

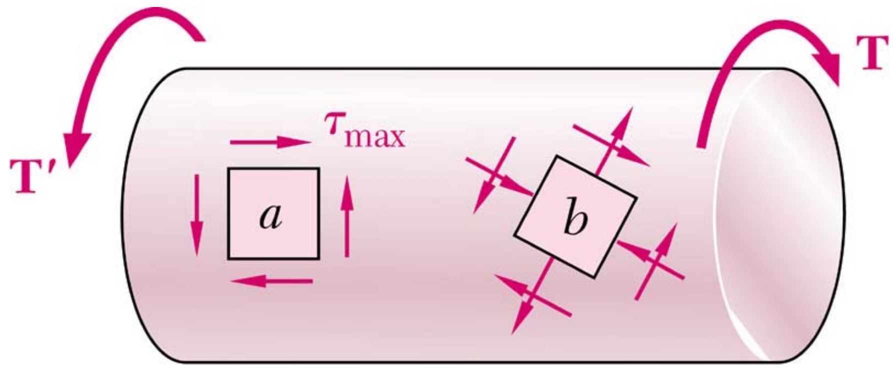
$$\phi = \sum_{i=1}^{n} \phi_i = \sum_{i=1}^{n} \frac{T_i L_i}{G_i J_i}$$



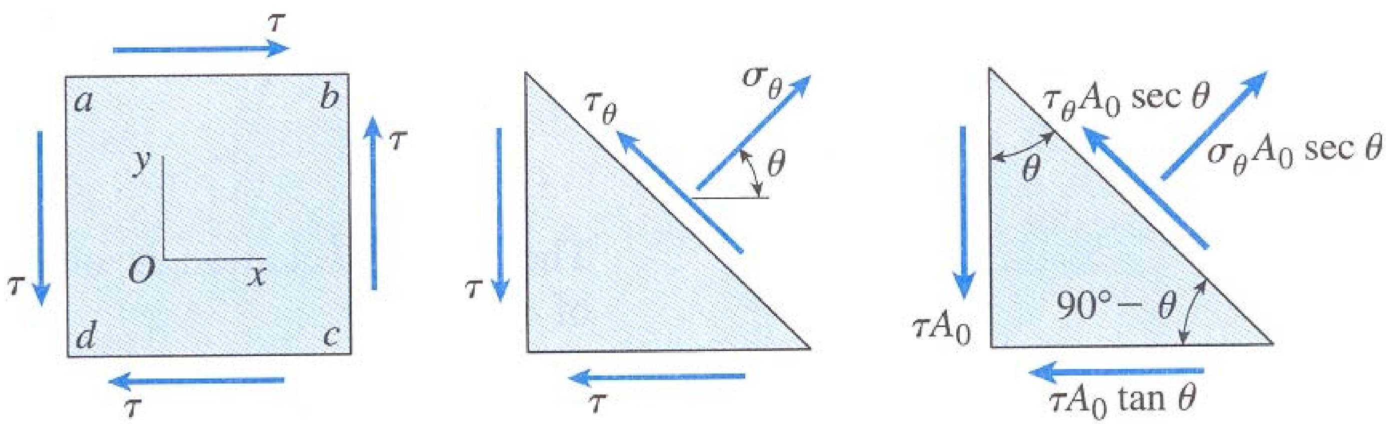
- For a continuously varying bar

$$\phi = \int_0^L \frac{T(x)}{GJ(x)} dx$$

Inclined Plane



- Cut a rectangular element along the plane at an angle θ.



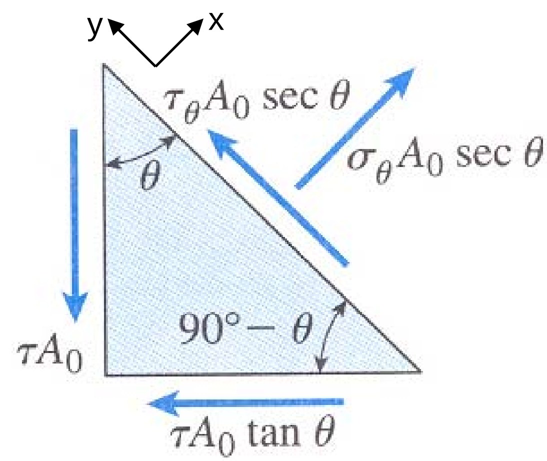
Inclined Plane *cont'd*

- Sum forces in x-direction.

$$\sigma_\theta A_0 \sec\theta - \tau A_0 \sin\theta - \tau A_0 \tan\theta\cos\theta = 0$$

$$\sigma_\theta = \tau\sin\theta\cos\theta + \tau\sin\theta\cos\theta$$

$$\boxed{\sigma_\theta = 2\tau\sin\theta\cos\theta = \tau\sin 2\theta}$$



- Sum forces in y-direction.

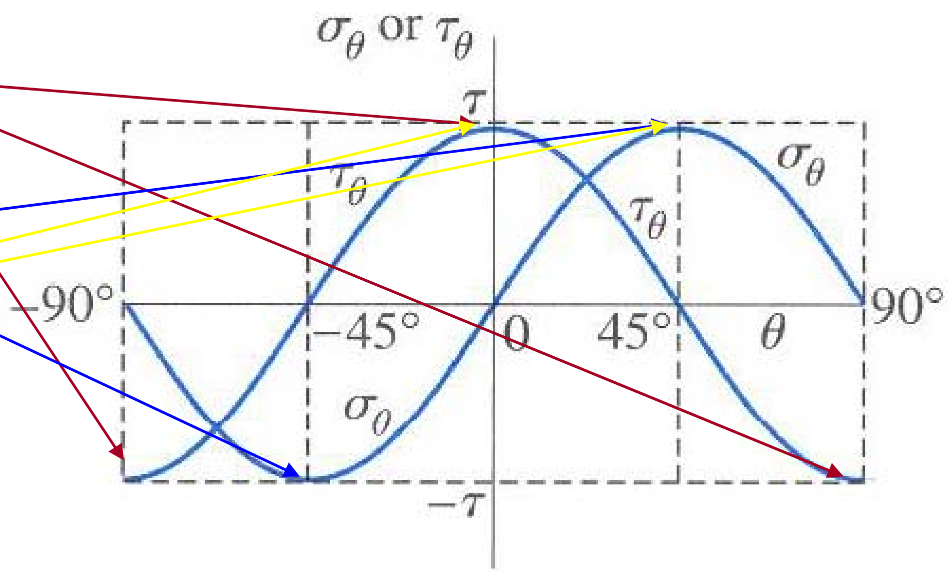
$$\tau_\theta A_0 \sec\theta - \tau A_0 \cos\theta + \tau A_0 \tan\theta\sin\theta = 0$$

$$\tau_\theta = \tau\cos^2\theta - \tau\sin^2\theta$$

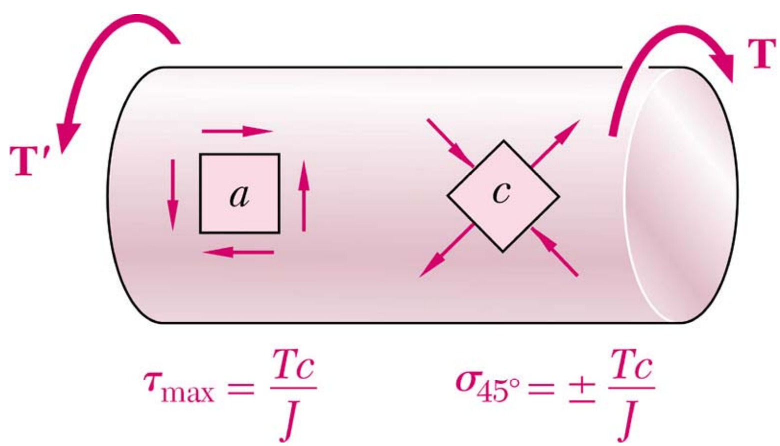
$$\boxed{\tau_\theta = \tau\cos 2\theta}$$

Inclined Plane *cont'd*

- τ_{max} occurs at θ = 0º, ±90º
- σ_{max} occurs at θ = ±45º
- $\tau_{max} = \sigma_{max}$



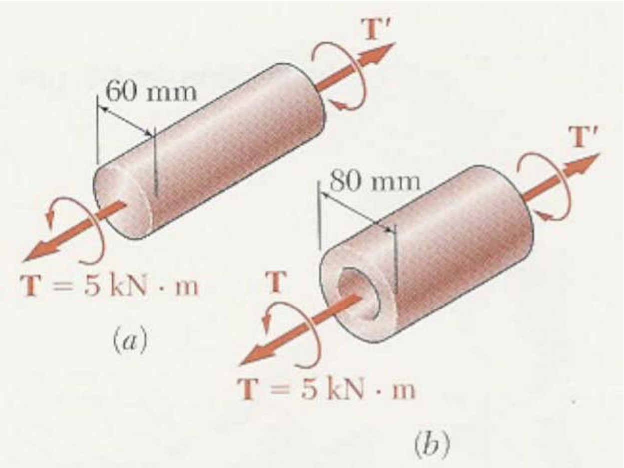
- When σ_θ is max, τ_θ = 0, and when τ_θ is max, σ_θ =0.



Example Problem 1

Part 1. For the 60 mm diameter solid cylinder and loading shown, determine the maximum shearing stress.

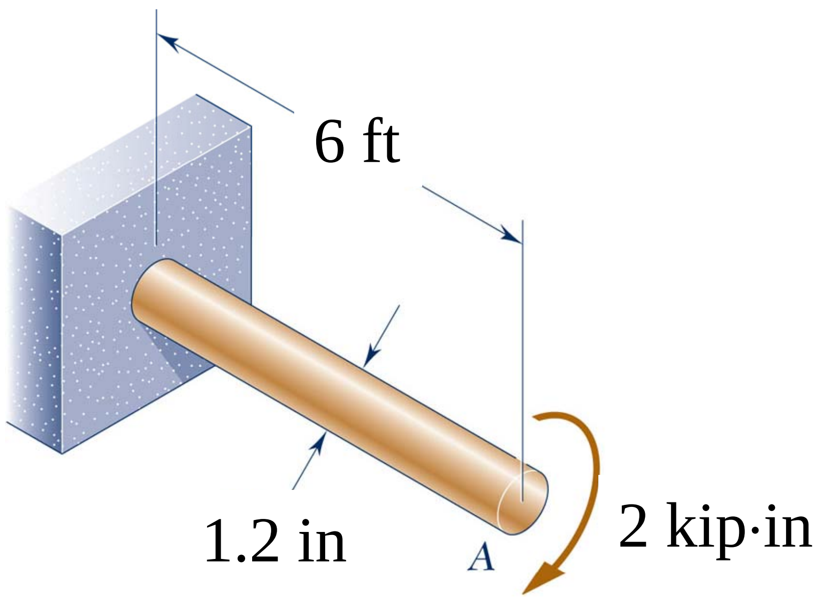
Part 2. Determine the inner diameter of the hollow cylinder , of 80 mm outer diameter, for which the maximum stress is the same as in part 1.



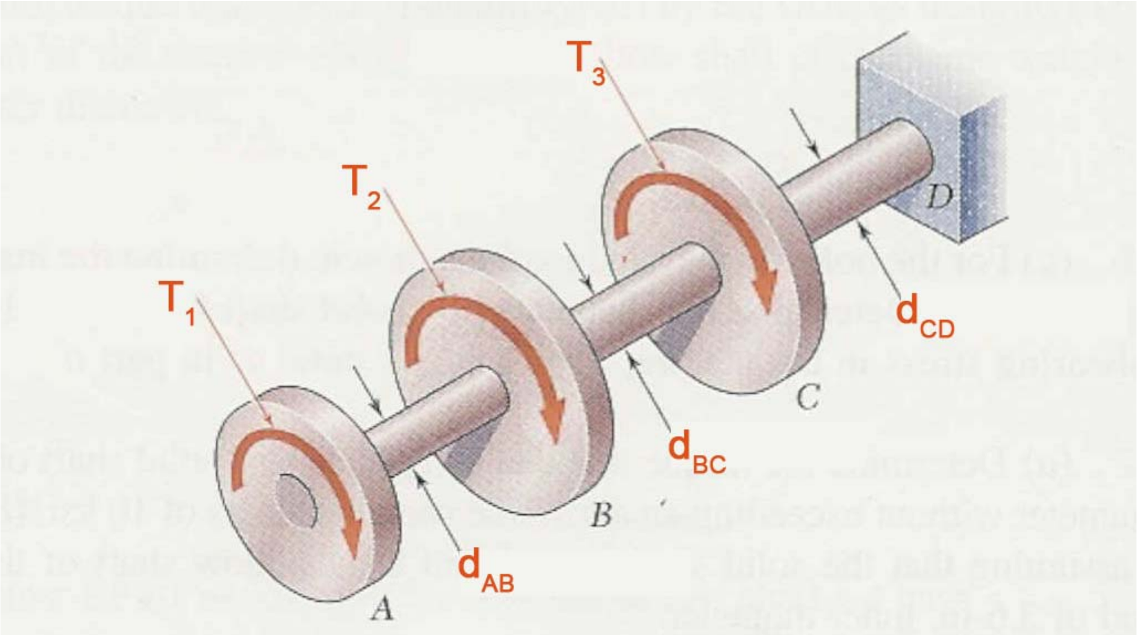
Example Problem 2

Part 1. For the solid steel shaft shown ($G = 11.2 \times 10^6$ psi) determine the angle of twist at A.

Part 2. Determine the angle of twist at A assuming the steel shaft is hollow with a 1.2 inch outer diameter and 0.8 inch inner diameter.



Statically Determinate Problems - Example



Find maximum shearing stress in each bar.

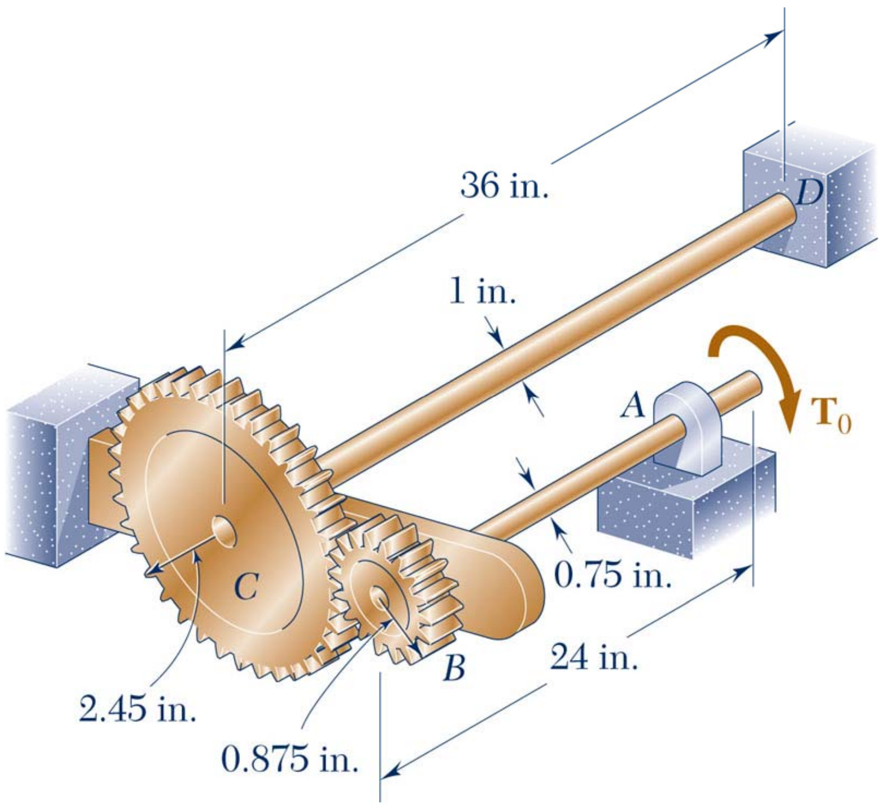
Statically Indeterminate Problems

- Method for torsion is the same as the method for statically indeterminate axial load deflection problems.

- Apply what you've already learned:
  - **M = R – N**
  - M = number of compatibility equations needed
  - R = number of unknown reactions (or internal stresses)
  - N = number of equilibrium equations

- Compatibility equations for a torsion problem are based on angle of twist.

Statically Indeterminate Problems - Example



Given:

L_{CD} = 36 in, d_{CD} = 1 in
r_C = 2.45 in, r_B = 0.875 in
L_{AB} = 24 in, d_{AB} = 0.75 in

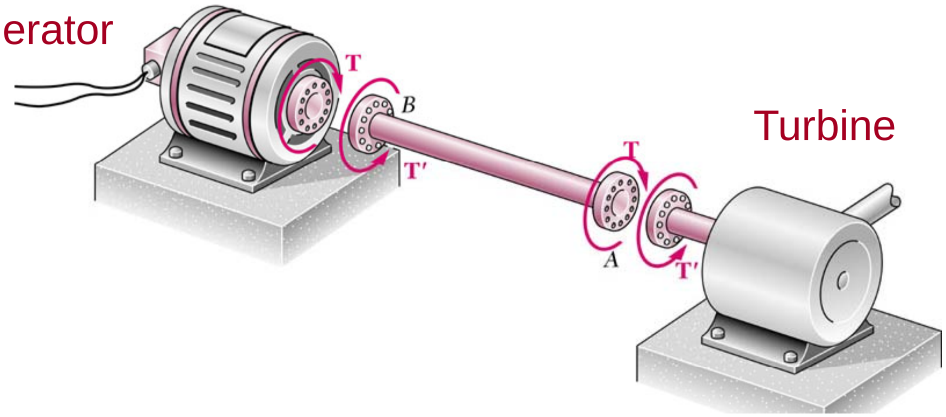
G = 11.2×10^6 psi

The allowable shear stress is 8 ksi

Find the largest torque T_0 that can be applied to the end of shaft AB and the angle of rotation of the end A of shaft AB.

Transmission Shafts

- In a transmission, a circular shaft transmits mechanical power from one device to another.



  - ω = angular speed of rotation of the shaft
  - The shaft applies a torque T to another device
  - To satisfy equilibrium the other device applies torque T to the shaft.
  - The power transmitted by the shaft is

$$P = T\omega$$

Transmission Shafts *cont'd*

- Units for P=Tω
  - ω = rad/s
  - T = N·m (SI)
  - T = ft·lb (English)
  - P = Watts (1 W = 1 N·m/s) (SI)
  - P = ft·lb/s (1 horsepower = hp = 550 ft·lb/s) (English)

- We can also express power in terms of frequency.

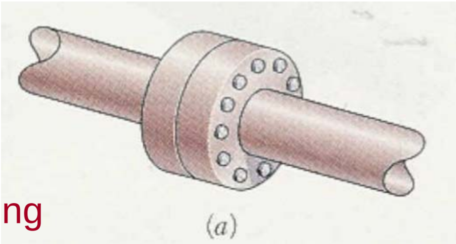
$$\omega = 2\pi f \qquad f = Hz = s^{-1}$$

$$\boxed{P = 2\pi f T}$$

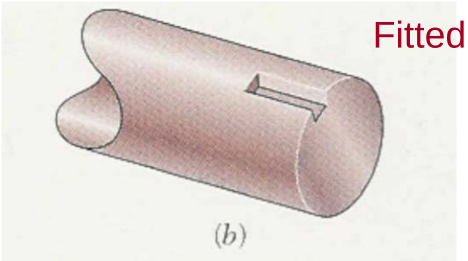
Stress Concentrations in Circular Shafts

- Up to now, we assumed that transmission shafts are loaded at the ends through solidly attached, rigid end plates.

- In practice, torques are applied through flange couplings and fitted keyways, which produce high stress concentrations.



Flange coupling



Fitted keyway

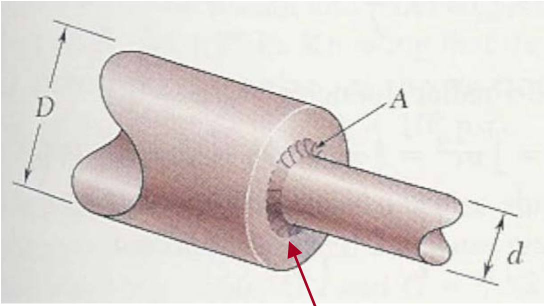
- One way to reduce stress concentrations is through the use of a fillet.

Stress Concentrations in Circular Shafts *cont'd*

- Maximum shear stress at the fillet

$$\tau_{\max} = K\frac{Tc}{J}$$

  - Tc/J is calculated for the smaller-diameter shaft
  - K = stress concentration factor



Fillet