CONVOLUTION

Let f and g be functions, the convolution of f and g, denoted by $f*g$ is defined as

$$f*g = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

Remember that if f and g are functions then it is not true in general that;

$$\mathcal{L}\{f\cdot g\} = \mathcal{L}\{f\}\,\mathcal{L}\{g\}$$

However what is true is that

$$\mathcal{L}\{f*g\} = \mathcal{L}\{f\}\,\mathcal{L}\{g\}$$

Examples

1 Let $f(t) = t$ and $g(t) = t$

$$f*g = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$
$$= \int_0^t \tau\,(t-\tau)\, d\tau$$
$$= \int_0^t \tau t - \tau^2\, d\tau$$
$$= \left[\frac{\tau^2 t}{2} - \frac{\tau^3}{3}\right]_0^t$$
$$= \frac{t^3}{2} - \frac{t^3}{3}$$
$$= \frac{t^3}{6}$$

2 $f(t) = 1$ and $g(t) = t$

$f(\tau) = 1$, $g(t-\tau) = t-\tau$

If $g(t) = t^2$,

$g(t-\tau) = (t-\tau)^2$

$$f*g = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$
$$= \int_0^t t-\tau\, d\tau$$
$$= \left[\tau t - \frac{\tau^2}{2}\right]_0^t$$
$$= t^2 - \frac{t^2}{2}$$
$$= \frac{t^2}{2}$$

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PROPERTIES OF THE CONVOLUTION

Let f, g and h be functions

1. $f * g = g * f$
2. $f * (g * h) = (f * g) * h$
3. $f * 1 \neq f$
4. $f * 0 = 0$
5. $f * (g + h) = f * g + f * h$

PROVE 1

$$f * g = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

Let $s = t - \tau$

$ds = -d\tau$

$\tau = t - s$

Lower limit, $\tau = 0$, $s = t$

Upper limit, $\tau = t$, $s = 0$

$$= -\int_t^0 f(t-s)\, g(s)\, ds$$

$$= \int_0^t g(s)\, f(t-s)\, ds = g * f.$$

Hence proven.

Example.

1 Find the inverse Laplace transform of

$$F(s) = \frac{1}{(s^2+1)^2}$$

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f\}\,\mathcal{L}\{g\}$$

$$\mathcal{L}\{f\} = \frac{1}{s^2+1} \qquad \mathcal{L}\{g\} = \frac{1}{s^2+1}$$

$f(t) = \sin t \qquad g(t) = \sin t$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} = f * g = \sin t * \sin t$$

$$= \int_0^t \sin \tau \, \sin(t-\tau)\, d\tau$$

$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

$\sin(A-B) = \sin A \cos B - \sin B \cos A$

$\cos(A-B) = \cos A \cos B + \sin A \sin B$

$-[\cos(A+B) = \cos A \cos B - \sin A \sin B]$

$\cos(A-B) - \cos(A+B) = 2 \sin A \sin B$

$$= \frac{1}{2}\int_0^t \cos(\tau - t + \tau) - \cos(\tau + t - \tau)\, d\tau$$

$$= \frac{1}{2}\int_0^t \cos(2\tau - t) - \cos t \, d\tau$$

$$= \frac{1}{2}\left[\frac{\sin(2\tau - t)}{2} - \tau \cos t\right]_0^t$$

$$= \frac{1}{2}\left[\left(\frac{\sin t}{2} - t\cos t\right) - \left(\frac{\sin(-t)}{2}\right)\right]$$

$$= \frac{1}{2}\left[\frac{\sin t}{2} - t\cos t + \frac{\sin t}{2}\right] \quad \text{as } \sin(-t) = -\sin t.$$

$$= \frac{1}{2}\left[\sin t + t \cos t\right]$$

$$\therefore L\left\{\frac{1}{(s^2+1)^2}\right\} = \frac{\sin t}{2} + \frac{t\cos t}{2}$$

Ts 2 Solve the following initial value problem

$$y'' + y' - 2y = \begin{cases} 3\sin t - \cos t, & 0 \le t < 2\pi \\ 3\sin 2t - \cos 2t, & t \ge 2\pi \end{cases}$$

$$y(0) = 1, \quad y'(0) = 0$$

R.H.S Let $f(t) = \begin{cases} 3\sin t - \cos t, & 0 \le t < 2\pi \\ 3\sin 2t - \cos 2t, & t \ge 2\pi \end{cases}$

$f(t) = (3\sin t - \cos t)[1 - u(t-2\pi)] + (3\sin 2t - \cos 2t)\, u(t-2\pi)$

$f(t) = 3\sin t - \cos t - (3\sin t - \cos t)\, u(t-2\pi) + (3\sin 2t - \cos 2t)$
$u(t-2\pi)$.

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$$\mathcal{L}\{f(t)\} = \mathcal{L}\{3\sin t\} - \mathcal{L}\{\cos t\} - \mathcal{L}\{(3\sin t - \cos t)\, u(t-2\pi)\}$$
$$+ \mathcal{L}\{(3\sin 2t - \cos 2t)\, u(t-2\pi)\}$$

$$\mathcal{L}\{f(t)\} = \frac{3}{s^2+1} - \frac{s}{s^2+1} - e^{-2\pi s}\,\mathcal{L}\{3\sin(t+2\pi) - \cos(t+2\pi)\}$$
$$+ e^{-2\pi s}\,\mathcal{L}\{3\sin 2(t+2\pi) - \cos[2(t+2\pi)]\}$$

$$\mathcal{L}\{f(t)\} = \frac{3}{s^2+1} - \frac{s}{s^2+1} - e^{-2\pi s}\left[\frac{3}{s^2+1} - \frac{s}{s^2+1}\right] + e^{-2\pi s}\left[\frac{6}{s^2+4} - \frac{s}{s^2+4}\right]$$

* $\sin(t+2\pi) = \sin t$

$\sin 2(t+2\pi) = \sin 2t$

L.H.S

$$y'' + y' - 2y$$
$$\mathcal{L}\{y''\} + \mathcal{L}\{y'\} - \mathcal{L}\{2y\}$$
$$s^2\mathcal{L}\{y\} - s\underbrace{y(0)}_{=1} - \underbrace{y'(0)}_{=0} + s\mathcal{L}\{y\} - \underbrace{y(0)}_{=1} - 2\mathcal{L}\{y\}$$

$$(s^2+s-2)\,\mathcal{L}\{y\} - s - 1$$
$$(s+2)(s-1)\,\mathcal{L}\{y\} - s - 1$$
$$(s+2)(s-1)\,\mathcal{L}\{y\} = s+1+\frac{3-s}{s^2+1} - e^{-2\pi s}\left(\frac{3-s}{s^2+1}\right) + e^{-2\pi s}\left(\frac{6-s}{s^2+4}\right)$$

$$\mathcal{L}\{y\} = \frac{s+1}{(s+2)(s-1)} + \frac{3-s}{(s+2)(s-1)(s^2+1)} - e^{-2\pi s}\left(\frac{3-s}{(s+2)(s-1)(s^2+1)}\right)$$
$$+ e^{-2\pi s}\left(\frac{6-s}{(s+2)(s-1)(s^2+4)}\right)$$

$$\frac{s+1}{(s+2)(s-1)} = \frac{A}{s+2} + \frac{B}{s-1} = \frac{A(s-1)+B(s+2)}{(s+2)(s-1)}$$

$$A(s-1) + B(s+2) = s+1$$

$s=-2$, $-3A = -1$, $A = \frac{1}{3}$

$s=1$, $3B = 2$, $B = \frac{2}{3}$.

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$$\frac{3-s}{(s+2)(s-1)(s^2+1)} = \frac{A}{s+2} + \frac{B}{s-1} + \frac{Cs+D}{s^2+1}$$

$$A(s-1)(s^2+1) + B(s+2)(s^2+1) + (Cs+D)(s+2)(s-1) = 3-s$$

$s=-2$, $A(-3)(5) = 5$

$A = -\frac{1}{3}$

$s=1$, $B(3)(2) = 2$

$B = \frac{1}{3}$

$s=0$, $-\frac{1}{3}(-1)(1) + \frac{1}{3}(2)(1) + D(2)(-1) = 3$

$\frac{1}{3} + \frac{2}{3} - 2D = 3$

$-2D = 2$

$D = -1$

$s=-1$, $-\frac{1}{3}(-2)(2) + \frac{1}{3}(1)(2) + (-C-1)(1)(-2) = 4$

$\frac{4}{3} + \frac{2}{3} + 2C + 2 = 4$

$2C = 0$

$C = 0.$

$$\frac{6-s}{(s+2)(s-1)(s^2+4)} = \frac{A}{s+2} + \frac{B}{s-1} + \frac{Cs+D}{s^2+4}$$

$$A(s-1)(s^2+4) + B(s+2)(s^2+4) + (Cs+D)(s+2)(s-1) = 6-s$$

$s=-2$, $A(-3)(8) = 8$

$A = -\frac{1}{3}.$

$s=1$, $B(3)(5) = 5$

$B = \frac{1}{3}$

$s=0$, $-\frac{1}{3}(-1)(4) + \frac{1}{3}(2)(4) + D(2)(-1) = 6$

$\frac{4}{3} + \frac{8}{3} - 2D = 6$

$D = -1$

$s=-1$, $-\frac{1}{3}(-2)(5) + \frac{1}{3}(1)(5) + (-C-1)(1)(-2) = 7$

$\frac{10}{3} + \frac{5}{3} + 2C + 2 = 7$

$2C = 0$

$C = 0.$

$$\mathcal{L}\{y\} = \frac{\frac{1}{3}}{s+2} + \frac{\frac{2}{3}}{s-1} + \frac{-\frac{1}{3}}{s+2} + \frac{\frac{1}{3}}{s-1} - \frac{1}{s^2+1} - e^{-2\pi s}\Bigg(\frac{-\frac{1}{3}}{s+2}$$
$$+ \frac{\frac{1}{3}}{s-1} - \frac{1}{s^2+1}\Bigg) + e^{-2\pi s}\Bigg(\frac{-\frac{1}{3}}{s+2} + \frac{\frac{1}{3}}{s-1} - \frac{1}{s^2+4}\Bigg)$$

$$\mathcal{L}\{y\} = \frac{1}{s-1} - \frac{1}{s^2+1} + \frac{e^{-2\pi s}}{s^2+1} - \frac{e^{-2\pi s}}{s^2+4}$$

$$y(t)\ \cancel{\mathcal{L}\{y\}} = e^t - \sin t + u(t-2\pi)\cdot\sin(t-2\pi) - \tfrac{1}{2}u(t-2\pi)$$
$$\sin[2(t-2\pi)]$$

$$y(t) = e^t - \sin t + u(t-2\pi)\sin t - \tfrac{1}{2}u(t-2\pi)\sin(2t).$$

$$y(t) = \begin{cases} e^t - \sin t, & 0 \le t < 2\pi \\ e^t - \sin t + \sin t - \frac{1}{2}\sin 2t, & t \ge 2\pi \end{cases}$$

$$y(t) = \begin{cases} e^t - \sin t, & 0 \le t < 2\pi \\ e^t - \frac{1}{2}\sin 2t, & t \ge 2\pi \end{cases}$$

TS 9a) $y(t) + 4\int_0^t y(\tau)(t-\tau)\,d\tau = 2t$

$$\mathcal{L}\{y(t) + 4\int_0^t y(\tau)(t-\tau)\,d\tau\} = 2\mathcal{L}\{t\}$$
$$\mathcal{L}\{y(t)\} + 4\mathcal{L}\{\int_0^t y(\tau)(t-\tau)\,d\tau\} = 2/s^2$$

$$\mathcal{L}\{y(t)\} + 4\mathcal{L}\{y(t)\}\mathcal{L}\{t\} = \frac{2}{s^2}$$
$$\mathcal{L}\{y(t)\} + \frac{4\mathcal{L}\{y(t)\}}{s^2} = \frac{2}{s^2}$$

$$\mathcal{L}\{y(t)\}\left[1 + \frac{4}{s^2}\right] = \frac{2}{s^2}$$

$$\mathcal{L}\{y(t)\}\left(\frac{s^2+4}{s^2}\right) = \frac{2}{s^2}$$

$$\mathcal{L}\{y(t)\} = \frac{2}{s^2+4}$$

$$y(t) = \sin(2t)$$

POWER SERIES METHOD FOR SOLVING ORDINARY DIFFERENTIAL EQUATIONS

INTRO: The power series method for solving ordinary differential equations is one of the more useful methods for solving ODEs. This is because it is specifically useful when solving ODEs with variable coefficients. It assumes the solution has a power series representation.

POWER SERIES

A power series is a series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$$

The exponents of the powers are non-negative integers.

Example

1 $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$

2 $\sum_{n=0}^{\infty} \dfrac{x^n}{n+1} = 1 + \dfrac{x}{2} + \dfrac{x^2}{3} + \dots$

Some function has a power series which we call Taylor series (Mclaurin series).

1 $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$

2 $e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!} = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!}$

3 $\cos x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!} = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \dfrac{x^8}{8!} - \dots$

4 $\sin x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!} = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \dots$

SHIFTING THE INDEX OF A POWER SERIES

Let $y = \sum_{n=0}^{\infty} a_n x^n$

$m = n+1 \quad , \quad n = m-1$

$y = \sum_{m=1}^{\infty} a_{m-1} x^{m-1} = \sum_{n=1}^{\infty} a_{n-1} x^{n-1}$

All the same, index has just been shifted.

DERIVATIVES OF POWER SERIES

Let $y = \sum_{n=0}^{\infty} a_n x^n$

$y = a_0 + a_1 x + a_2 x^2 + \dots$

$y' = a_1 + 2a_2 x + 3a_3 x^2 + \dots$ starting from 2^{nd} term

$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$ as $\frac{dy}{dx}(a_0) = 0$

$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$

Let $m = n-1 \quad , \quad n = m+1$

$y' = \sum_{m=0}^{\infty} (m+1) a_{m+1} x^m$

$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$

Let $y = \sum_{n=0}^{\infty} a_n x^n$

$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$

$y'' = \sum_{n=2}^{\infty} (n-1)n a_n x^{n-2}$

* When increasing index, you subtract the number, when reducing you add.

Is the following true?

$\sum_{n=2}^{\infty} (n-1)n\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n-1)n\, a_n x^{n-2}$

L.H.S $2a_2 + (2)(3)a_3 x + (3)(4)a_4 x^2 + \dots$

R.H.S $0 + 0 + 2a_2 + (2)(3)a_3 x + (3)(4)a_4 x^2 + \dots$

L.H.S = R.H.S $\therefore$ It is true.

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$$y'' = \sum_{n=2}^{\infty} (n-1)\, n\, a_n x^{n-2}$$

Index 1 Let $m = n-1$, $n = m+1$

$$y'' = \sum_{m=1}^{\infty} (m+1-1)(m+1)\, a_{m+1} x^{m+1-2} = \sum_{m=1}^{\infty} m(m+1) a_{m+1} x^{m-1}$$

$$\therefore y'' = \sum_{n=2}^{\infty} (n-1)n\, a_n x^{n-2} = \sum_{n=1}^{\infty} n(n+1)\, a_{n+1} x^{n-1}$$

Index 0 ; $y'' = \sum_{n=0}^{\infty} (n+1)(n+2)\, a_{n+2} x^n$

Examples

1 Let $y = \sum_{n=0}^{\infty} a_n x^n$. Find. $x^2 y''$

$$y'' = \sum_{n=2}^{\infty} (n-1)\, n\, a_n x^{n-2}$$

$$x^2 y'' = \sum_{n=2}^{\infty} (n-1)n\, a_n (x^2)(x^{n-2})$$

Index 2 : $x^2 y'' = \sum_{n=2}^{\infty} \underline{(n-1)n}\, a_n x^n$

Index 0 : $x^2 y'' = \sum_{n=0}^{\infty} \underline{(n+1)(n+2)}\, a_{n+2} x^{n+2}$

2 Let $y = \sum_{n=0}^{\infty} a_n x^n$. Find $y'' + y$.

Index 2 : $y'' = \sum_{n=2}^{\infty} (n-1)n\, a_n x^{n-2}$

Index 0 : $y'' = \sum_{n=0}^{\infty} (n+1)(n+2)\, a_{n+2} x^n$

$$\therefore y'' + y = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n$$

$$y'' + y = \sum_{n=0}^{\infty} [(n+1)(n+2)\, a_{n+2} + a_n] x^n$$

OR $y = \sum_{n=0}^{\infty} a_n x^n$

Index 2 : $y = \sum_{n=2}^{\infty} a_{n-2} x^{n-2}$

$$y'' + y = \sum_{n=2}^{\infty} (n-1)n\, a_n x^{n-2} + \sum_{n=2}^{\infty} a_{n-2} x^{n-2}$$

$$y'' + y = \sum_{n=2}^{\infty} [(n-1)n\, a_n + a_{n-2}] x^{n-2}$$

3 Let $y = \sum_{n=0}^{\infty} a_n x^n$. Find $y'' + y'$

$$y' = \sum_{n=1}^{\infty} n\, a_n x^{n-1}$$

Index 0 : $y' = \sum_{n=0}^{\infty} (n+1)\, a_{n+1} x^n$

$$y'' = \sum_{n=2}^{\infty} (n-1)\, n\, a_n x^{n-2}$$

Index 1 Let $m = n-1$, $n = m+1$

$$y'' = \sum_{m=1}^{\infty} (m+1-1)(m+1)\, a_{m+1} x^{m+1-2} = \sum_{m=1}^{\infty} m(m+1) a_{m+1} x^{m-1}$$

$$\therefore y'' = \sum_{n=2}^{\infty} (n-1)n\, a_n x^{n-2} = \sum_{n=1}^{\infty} n(n+1)\, a_{n+1} x^{n-1}$$

Index 0 ; $y'' = \sum_{n=0}^{\infty} (n+1)(n+2)\, a_{n+2} x^n$

Examples

1 Let $y = \sum_{n=0}^{\infty} a_n x^n$. Find $x^2 y''$

$$y'' = \sum_{n=2}^{\infty} (n-1)\, n\, a_n x^{n-2}$$

$$x^2 y'' = \sum_{n=2}^{\infty} (n-1)n\, a_n (x^2)(x^{n-2})$$

Index 2 : $x^2 y'' = \sum_{n=2}^{\infty} \underline{(n-1)n}\, a_n x^n$

Index 0 : $x^2 y'' = \sum_{n=0}^{\infty} \underline{(n+1)(n+2)}\, a_{n+2} x^{n+2}$

2 Let $y = \sum_{n=0}^{\infty} a_n x^n$. Find $y'' + y$.

Index 2 : $y'' = \sum_{n=2}^{\infty} (n-1)n\, a_n x^{n-2}$

Index 0 : $y'' = \sum_{n=0}^{\infty} (n+1)(n+2)\, a_{n+2} x^n$

$$\therefore y'' + y = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n$$

$$y'' + y = \sum_{n=0}^{\infty} [(n+1)(n+2)\, a_{n+2} + a_n] x^n$$

OR $y = \sum_{n=0}^{\infty} a_n x^n$

Index 2 : $y = \sum_{n=2}^{\infty} a_{n-2} x^{n-2}$

$$y'' + y = \sum_{n=2}^{\infty} (n-1)n\, a_n x^{n-2} + \sum_{n=2}^{\infty} a_{n-2} x^{n-2}$$

$$y'' + y = \sum_{n=2}^{\infty} [(n-1)n\, a_n + a_{n-2}] x^{n-2}$$

3 Let $y = \sum_{n=0}^{\infty} a_n x^n$. Find $y'' + y'$

$$y' = \sum_{n=1}^{\infty} n\, a_n x^{n-1}$$

Index 0 : $y' = \sum_{n=0}^{\infty} (n+1)\, a_{n+1} x^n$

$$y'' = \sum_{n=2}^{\infty} (n-1)n\, a_n x^{n-2}$$

Index 2 : $$y'' = \sum_{n=0}^{\infty} (n+1)(n+2)\, a_{n+2} x^n$$

$$y''+y' = \sum_{n=0}^{\infty} (n+1)(n+2)\, a_{n+2} x^n + \sum_{n=0}^{\infty} (n+1)\, a_{n+1} x^n$$

$$y''+y' = \sum_{n=0}^{\infty} \underline{[(n+1)(n+2)\, a_{n+2} + (n+1)\, a_{n+1}]}\, x^n$$

4 Let $y = \sum_{n=0}^{\infty} a_n x^n$; Find $y''+xy'$

$$y'' = \sum_{n=0}^{\infty} (n+1)(n+2)\, a_{n+2} x^n$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$xy' = \sum_{n=1}^{\infty} n a_n (x)(x^{n-1}) = \sum_{n=1}^{\infty} n a_n x^n$$

$$y'' = 2a_2 + \sum_{n=1}^{\infty} (n+1)(n+2)\, a_{n+2} x^n$$

$$y''+xy' = 2a_2 + \sum_{n=1}^{\infty} (n+1)(n+2)\, a_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^n$$

$$y''+xy' = 2a_2 + \sum_{n=1}^{\infty} \underline{[(n+1)(n+2)\, a_{n+2} + n a_n]}\, x^n$$

Examples

Use the power series method to find the general ~~formula~~ solution of the following ODE;

1 $$y''+y=0$$

Let us assume that $y = \sum_{n=0}^{\infty} a_n x^n$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} (n-1)n\, a_n x^{n-2}$$

$$y'' = \sum_{n=0}^{\infty} (n+1)(n+2)\, a_{n+2} x^n$$

$$\sum_{n=0}^{\infty} (n+1)(n+2)\, a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} [(n+1)(n+2)\, a_{n+2} + a_n]\, x^n = 0$$

$$(n+1)(n+2)\, a_{n+2} + a_n = 0$$

$$(n+1)(n+2)\, a_{n+2} = -a_n$$

$$a_{n+2} = \frac{-a_n}{(n+1)(n+2)}$$

If $n=0$; $a_2 = \frac{-a_0}{2}$

If $n=2$; $a_4 = \frac{-a_2}{12} = \frac{-\frac{a_0}{2}}{12} = \frac{+a_0}{(2)(3)(4)} = \frac{a_0}{4!}$

If $n=1$; $a_3 = \frac{-a_1}{6} = \frac{-a_1}{(2)(3)}$

If $n=3$; $a_5 = \frac{-a_3}{(4)(5)} = \frac{-\left(\frac{-a_1}{(2)(3)}\right)}{(4)(5)} = \frac{+a_1}{(2)(3)(4)(5)} = \frac{a_1}{5!}$

If $n=4$, $a_6 = \frac{-a_0}{6!}$, $a_7 = \frac{-a_1}{7!}$

$y = \sum_{n=0}^{\infty} a_n x^n = \cancel{a_1 x^2 +} a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots$

$y = a_0 + a_1 x + \left(\frac{-a_0}{2!} x^2\right) + \left(-\frac{a_1}{3!} x^3\right) + \frac{a_0}{4!} x^4 + \frac{a_1}{5!} x^5 + \ldots$

$y = a_0 + a_1 x - \frac{a_0}{2!} x^2 - \frac{a_1}{3!} x^3 + \frac{a_0}{4!} x^4 + \frac{a_1}{5!} x^5 + \ldots$

$y = \left(a_0 - \frac{a_0}{2!} x^2 + \frac{a_0}{4!} x^4 - \frac{a_0}{6!} x^6 \ldots\right) + \left(a_1 x - \frac{a_1}{3!} x^3 + \frac{a_1 x^5}{5!} - \frac{a_1 x^7}{7!}\right)$

$y = a_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\right) + a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}\right)$

$y = a_0 \cos x + a_1 \sin x.$

2 $y'' - 2xy' + y = 0 \quad y(0) = 0, \; y'(0) = 1$

Assume $y = \sum_{n=0}^{\infty} a_n x^n$

$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$

$y'' = \sum_{n=2}^{\infty} (n-1) n a_n x^{n-2} = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n$

$\sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n - 2x \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$

$\sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n - \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$

$2a_2 + \sum_{n=1}^{\infty} (n+1)(n+2) a_{n+2} x^n - \sum_{n=1}^{\infty} 2n a_n x^n + a_0 + \sum_{n=1}^{\infty} a_n x^n = 0$

$$a_{n+2} = \frac{-a_n}{(n+1)(n+2)}$$

If $n=0$; $a_2 = \frac{-a_0}{2}$

If $n=2$; $a_4 = \frac{-a_2}{12} = \frac{-\frac{a_0}{2}}{12}$

If $n=1$; $a_3 = \frac{-a_1}{6} = \frac{-a_1}{(2)(3)}$

$= \frac{+a_0}{(2)(3)(4)} = \frac{a_0}{4!}$

If $n=3$; $a_5 = \frac{-a_3}{(4)(5)} = \frac{-\left(\frac{-a_1}{(2)(3)}\right)}{(4)(5)} = \frac{+a_1}{(2)(3)(4)(5)} = \frac{a_1}{5!}$

If $n=4$, $a_6 = \frac{-a_0}{6!}$, $a_7 = \frac{-a_1}{7!}$

$y = \sum_{n=0}^{\infty} a_n x^n = \cancel{a_1 x^2 +} a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots$

$y = a_0 + a_1 x + \left(\frac{-a_0}{2!} x^2\right) + \left(-\frac{a_1}{3!} x^3\right) + \frac{a_0}{4!} x^4 + \frac{a_1}{5!} x^5 + \ldots$

$y = a_0 + a_1 x - \frac{a_0}{2!} x^2 - \frac{a_1}{3!} x^3 + \frac{a_0}{4!} x^4 + \frac{a_1}{5!} x^5 + \ldots$

$y = \left(a_0 - \frac{a_0}{2!} x^2 + \frac{a_0}{4!} x^4 - \frac{a_0}{6!} x^6 \ldots\right) + \left(a_1 x - \frac{a_1}{3!} x^3 + \frac{a_1 x^5}{5!} - \frac{a_1 x^7}{7!} + \ldots\right)$

$y = a_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\right) + a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}\right)$

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$\sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n - 2x \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$

$\sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n - \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$

$2a_2 + \sum_{n=1}^{\infty} (n+1)(n+2) a_{n+2} x^n - \sum_{n=1}^{\infty} 2n a_n x^n + a_0 + \sum_{n=1}^{\infty} a_n x^n = 0$

$$a_{n+2} = \frac{-a_n}{(n+1)(n+2)}$$

If $n=0$; $a_2 = \frac{-a_0}{2}$

If $n=2$; $a_4 = \frac{-a_2}{12} = \frac{-\frac{a_0}{2}}{12} = \frac{+a_0}{(2)(3)(4)} = \frac{a_0}{4!}$

If $n=1$; $a_3 = \frac{-a_1}{6} = \frac{-a_1}{(2)(3)}$

If $n=3$; $a_5 = \frac{-a_3}{(4)(5)} = \frac{-\left(\frac{-a_1}{(2)(3)}\right)}{(4)(5)} = \frac{+a_1}{(2)(3)(4)(5)} = \frac{a_1}{5!}$

If $n=4$, $a_6 = \frac{-a_0}{6!}$, $a_7 = \frac{-a_1}{7!}$

$$y = \sum_{n=0}^{\infty} a_n x^n = \cancel{a_1 x^2 +} a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots$$

$$y = a_0 + a_1 x + \left(\frac{-a_0}{2!} x^2\right) + \left(-\frac{a_1}{3!} x^3\right) + \frac{a_0}{4!} x^4 + \frac{a_1}{5!} x^5 + \dots$$

$$y = a_0 + a_1 x - \frac{a_0}{2!} x^2 - \frac{a_1}{3!} x^3 + \frac{a_0}{4!} x^4 + \frac{a_1}{5!} x^5 + \dots$$

$$y = \left(a_0 - \frac{a_0}{2!} x^2 + \frac{a_0}{4!} x^4 - \frac{a_0}{6!} x^6 \dots\right) + \left(a_1 x - \frac{a_1}{3!} x^3 + \frac{a_1 x^5}{5!} - \frac{a_1 x^7}{7!}\right)$$

$$y = a_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right) + a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}\right)$$

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$y'' = \sum_{n=2}^{\infty} (n-1) n a_n x^{n-2} = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n$

$$\sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n - 2x \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n - \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty} (n+1)(n+2) a_{n+2} x^n - \sum_{n=1}^{\infty} 2n a_n x^n + a_0 + \sum_{n=1}^{\infty} a_n x^n = 0$$

$$(2a_2 + a_0) + \sum_{n=1}^{\infty} [(n+1)(n+2)a_{n+2} - 2na_n + a_n] x^n = 0$$

$$2a_2 + a_0 = 0$$

$$2a_2 = -a_0$$

$$a_2 = \frac{-a_0}{2} \quad \ldots (1)$$

$$(n+1)(n+2)a_{n+2} - 2na_n + a_n = 0 \quad \text{when } n \geq 1$$

$$(n+1)(n+2)a_{n+2} = (2n-1)a_n$$

$$a_{n+2} = \frac{(2n-1)a_n}{(n+1)(n+2)}$$

If $n=1$; $a_3 = \dfrac{a_1}{(2)(3)}$

If $n=2$; $a_4 = \dfrac{\cancel{3}a_2}{(\cancel{3})(4)} = \dfrac{a_2}{(4)} = \dfrac{-\frac{a_0}{2}}{4} = -\dfrac{a_0}{8}$

If $n=3$; $a_5 = \dfrac{\cancel{5}a_3}{(4)(\cancel{5})} = \dfrac{a_3}{4} = \dfrac{\frac{a_1}{(2)(3)}}{4} = \dfrac{a_1}{4!} = \dfrac{a_1}{24}$

If $n=4$, $a_6 = \dfrac{7a_4}{(5)(6)} = \dfrac{7\left(-\frac{a_0}{8}\right)}{(5)(6)} = \dfrac{-7a_0}{240}$

If $n=5$, $a_7 = \dfrac{9a_5}{(6)(7)} = \dfrac{9\left(\frac{a_1}{4!}\right)}{(6)(7)} = \dfrac{a_1}{112}$

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

$$y = a_0 + a_1 x - \frac{a_0}{2} x^2 + \frac{a_1}{6} x^3 - \frac{a_0}{8} x^4 + \frac{a_1}{24} x^5 - \frac{7a_0}{240} x^6$$

$+ \dfrac{a_1}{112} x^7 + \ldots$ as the general solution.

$$y(0) = a_0, \quad a_0 = 0$$

$$y'(x) = a_1 - a_0 x + \frac{a_1}{2} x^2 - \frac{a_0}{2} x^3 + \frac{5a_1}{24} x^4 - \frac{7a_0}{40} x^5 + \frac{7a_1}{112} x^6 + \ldots$$

$y'(0) = a_1$. This implies that $a_1 = 1$

Therefore the particular solution is

$$y(x) = x + \frac{x^3}{6} + \frac{x^5}{24} + \frac{x^7}{112} + \dots$$

NOTE: Not all linear differential equations with variable coefficients have a power series solution.

Consider the ODE $x^2y'' + xy' + y = 0$.

Lets check if it has a power series solution.

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} (n-1)n\, a_n x^{n-2}$$

$$x^2 \sum_{n=2}^{\infty} (n-1)n\, a_n x^{n-2} + x \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} (n-1)n\, a_n x^n + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} (n-1)n\, a_n x^n + a_1 x + \sum_{n=2}^{\infty} n a_n x^n + a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n = 0$$

$$a_0 + 2a_1 x + \sum_{n=2}^{\infty} \left[(n-1)n\, a_n + n a_n + a_n \right] x^n = 0$$

$$a_0 = 0$$

$$2a_1 = 0 \quad , \quad a_1 = 0$$

$$(n-1)n\, a_n + n\, a_n + a_n = 0$$

$$[(n-1)n + n + 1]\, a_n = 0 \quad , \quad a_n = 0 \text{ for } n \geq 2.$$

We see that all the coefficients i.e a_i's are zero, giving us the trivial solution $y = 0$.

THEOREM: EXISTANCE OF A POWER SERIES SOLUTIONS

Let $a(x)y'' + b(x)y' + c(x)y = 0$ be a general second order linear differential equation with variable coefficients. Dividing this ODE by the coefficients of y'' i.e $a(x)$

we get

$$y'' + \frac{b(x)}{a(x)} y' + \frac{c(x)}{a(x)} y = 0.$$

If $\frac{b(x)}{a(x)}$ and $\frac{c(x)}{a(x)}$ are analytic at $x=0$ [have a power series representation about $x=0$], then the ODE has a power series solution.

For example, the ODE $x^2y''+xy'+y=0$ can be rewritten as $y''+\left(\frac{1}{x}\right)y'+\left(\frac{1}{x^2}\right)y=0$. The functions $\frac{1}{x}$ and $\frac{1}{x^2}$ are not analytic at $x=0$. Therefore, the given ODE does not have a power series solution.

EULER EQUATIONS

An ODE of the form $a(x^2)y''+b(x)y'+cy=0$ where $a,b,c \in \mathbb{R}$ and $a \neq 0$ is called and Euler equation.

It can be shown that to solve an Euler equation we have to assume that

$$y = x^r$$
$$y' = rx^{r-1}$$
$$y'' = (r-1)r\,x^{r-2}$$

$$ax^2y'' + bxy' + cy = 0$$
$$ax^2[(r-1)r\,x^{r-2}] + bx[rx^{r-1}] + cx^r = 0$$
$$a(r-1)r\,x^r + brx^r + cx^r = 0$$
$$[a(r-1)r + br + c]x^r = 0$$

If we assume $x^r \neq 0$ we must have that $a(r-1)r+br+c=0$

The above is a quadratic in r.

Example

Solve the IVP $2x^2y''+3xy'-15y=0$, $y(1)=0$, $y'(1)=1$

$y=x^r$, $y'=rx^{r-1}$, $y''=(r-1)r\,x^{r-2}$

$$2x^2[(r-1)r\,x^{r-2}] + 3x[r\,x^{r-1}] - 15x^r = 0$$

$$2(r-1)r\,x^r + 3r\,x^r - 15x^r = 0$$

$$[2(r-1)r + 3r - 15]\,x^r = 0$$

$$2(r-1)r + 3r - 15 = 0$$

$$2r^2 - 2r + 3r - 15 = 0$$

$$2r^2 + r - 15 = 0$$

$$2r^2 + 6r - 5r - 15 = 0$$

$$2r(r+3) - 5(r+3) = 0$$

$$(r+3)(2r-5) = 0$$

$$r = -3 \quad \text{or} \quad r = 5/2.$$

We get $y_1 = x^{-3}$ and $y_2 = x^{5/2}$.

$\therefore$ General solution ~~are~~ is $y(x) = C_1 x^{-3} + C_2 x^{5/2}$

$y(1) = 0$, $y'(1) = 1$

$y(1) = C_1 + C_2 = 0 \ldots (1)$

$y'(x) = -3C_1 x^{-4} + \frac{5}{2} C_2 x^{3/2}$

$y'(1) = -3C_1 + \frac{5}{2} C_2 = 1 \ldots (2)$

Solving 1 and 2

$C_1 = -C_2$

$-3(-C_2) + \frac{5}{2} C_2 = 1$, $3C_2 + \frac{5}{2} C_2 = 1$

$\frac{11}{2} C_2 = 1$, $C_2 = \frac{2}{11}$

$C_1 = -\frac{2}{11}$

$\therefore$ Particular solution is $y(x) = -\frac{2}{11} x^{-3} + \frac{2}{11} x^{5/2}$

FROBENIUS METHOD

$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n$

$x^3 y'' + x y' + y = 0$

Not there

We have three possibilities for $a(r-1)r + br + c = 0$

1 The quadratic equation has real and distinct solutions. In this case, the general solution to the Euler equation becomes

$$y(x) = c_1 x^{r_1} + c_2 x^{r_2}$$

2 The quadratic equation has complex roots, Let the root be $r_1 = \alpha + \beta i$ and $r_2 = \alpha - \beta i$. Then the solutions are $x^{\alpha+\beta i}$ and $x^{\alpha-\beta i}$. These are complex solutions. What we seek are real solutions.

To remedy this we get the real part of $x^{\alpha+\beta i}$ and imaginary part of $x^{\alpha-\beta i}$. These become our two linearly independent solutions.

$$\begin{aligned} x^{\alpha+\beta i} &= x^{\alpha} x^{\beta i} \\ &= x^{\alpha} \left(e^{\ln x^{\beta i}}\right) \\ &= x^{\alpha}\left(e^{\beta i \ln x}\right) \\ &= x^{\alpha}\left(e^{\beta \ln x\, i}\right) \end{aligned}$$

$$x^a = e^{\ln x^a} = e^{a \ln x}$$

$$e^{ix} = \cos x + i \sin x$$

$$\begin{aligned} x^{\alpha+\beta i} &= x^{\alpha}\left[\cos(\beta \ln x) + i \sin(\beta \ln x)\right] \\ &= x^{\alpha}\cos(\beta \ln x) + i\, x^{\alpha} \sin(\beta \ln x) \end{aligned}$$

$$\therefore y_1(x) = \mathrm{Re}\left(x^{\alpha+\beta i}\right) = x^{\alpha}\cos(\beta \ln x).$$

$$y_2(x) = \mathrm{Im}\left(x^{\alpha+\beta i}\right) = x^{\alpha}\sin(\beta \ln x).$$

The general solution $= y(x) = c_1 y_1(x) + c_2 y_2(x)$.

i.e The general solution is $y(x) = c_1 x^{\alpha}\cos(\beta \ln x) + c_2 x^{\alpha}\sin(\beta \ln x)$

Example

Find the general solution of $x^2 y'' + 5xy' + 12y = 0$

$a(r-1)r + br + c = 0$

$a = 1 \quad , \quad b = 5 \quad , \quad c = 12$

$r(r-1) + 5r + 12 = 0$

$r^2 - r + 5r + 12 = 0$

$r^2 + 4r + 12 = 0$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-4 \pm \sqrt{(4)^2 - 4(1)(12)}}{2(1)} = \frac{-4 \pm \sqrt{16-48}}{2}$$

$$r = \frac{-4 \pm \sqrt{-32}}{2} = \frac{-4 \pm 4\sqrt{2}\, i}{2}$$

$$\underline{\underline{r = -2 \pm 2\sqrt{2}\, i}}$$

$\alpha = -2 \qquad \beta = 2\sqrt{2}$

$y(x) = C_1 x^{\alpha} \cos(\beta \ln x) + C_2 x^{\alpha} \sin(\beta \ln x)$

$$y(x) = C_1 x^{-2} \cos(2\sqrt{2} \ln x) + C_2 x^{-2} \sin(2\sqrt{2} \ln x).$$

3 The quadratic equation has only one real solution, say r. Thus we have that x^r is a solution of the given Euler equation.

It can be shown that the other linear independent solution is $x^r \ln x$. so that the general solution will be $y(x) = C_1 x^r + C_2 x^r \ln x$.

Example

Find the particular solution of $x^2 y'' + 3xy' + y = 0$ $y(1) = 0$ and $y'(1) = 1$

$a(r-1)r + br + c = 0$

$a = 1 \quad , b = 3, \quad c = 1$

$r(r-1) + 3r + 1 = 0 \quad , \quad r^2 + 2r + 1 = 0$

$(r+1)^2 = 0$, $r = -1$ is the only solution.

$y(x) = c_1 x^r + c_2 x^r \ln x$

$y(x) = c_1 x^{-1} + c_2 x^{-1} \ln x$ as the general solution.

$y(1) = c_1 (1)^{-1} + c_2 (1)^{-1} \ln(1) = 0$

$y(1) = c_1 = 0$, $c_1 = 0.$

$y'(x) = -c_1 x^{-2} + c_2 \left[-x^{-2} \ln x + x^{-1}\left(\frac{1}{x}\right)\right] = 1$

$y'(1) = -c_1 (1)^{-2} + c_2 \left[-1^{-2} \ln(1) + (1)^{-1} \left(\frac{1}{1}\right)\right] = 1$

$y'(1) = -c_1 + c_2 = 1$, $y'(1) = 0 + c_2 = 1$

$c_2 = 1$

$\therefore$ Particular solution is $y(x) = \dfrac{\ln x}{x}$

SYSTEMS OF FIRST LINEAR DIFFERENTIAL EQUATIONS

A system of first order linear differential equations takes the form

$$x_1' = ax_1 + bx_2 = ax_1(t) + bx_2(t)$$

$$x_2' = cx_1 + dx_2 = cx_1(t) + dx_2(t).$$

We seek the function $x_1(t)$ and $x_2(t)$ that satisfy the two given equations. a, b, c & $d \in \mathbb{R}$:

Consider the following example of a system of first order linear differential equation.

$$x_1'(t) = x_1(t) + 2x_2(t)$$

$$x_2'(t) = 3x_1(t) + 2x_2(t)$$

This system has solution

$$x_1(t) = -e^{-t} + 2e^{4t}$$

$$x_2(t) = e^{-t} + 3e^{4t}$$

We show that $x_1(t)$ and $x_2(t)$ are indeed solutions of the given system by;

$$x_1'(t) = e^{-t} + 8e^{4t}$$

$$x_2'(t) = -e^{-t} + 12e^{4t}$$

$$RHS = x_1(t) + 2x_2(t) = (-e^{-t} + 2e^{4t}) + 2(e^{-t} + 3e^{4t})$$

$$x_1(t) + 2x_2(t) = -e^{-t} + 2e^{4t} + 2e^{-t} + 6e^{4t}$$

$$x_1(t) + 2x_2(t) = e^{-t} + 8e^{4t} = x_1'(t) = L.H.S = RHS$$

$$RHS = 3x_1(t) + 2x_2(t) = 3(-e^{-t} + 2e^{4t}) + 2(e^{-t} + 3e^{4t})$$

$$RHS = -3e^{-t} + 6e^{4t} + 2e^{-t} + 6e^{4t} = -e^{-t} + 12e^{4t} = LHS$$

Indeed $x_1(t)$ and $x_2(t)$ are solutions.

SYSTEMS OF FIRST LINEAR DIFFERENTIAL EQUATIONS

A system of first order linear differential equations takes the form

$$x_1' = ax_1 + bx_2 = ax_1(t) + bx_2(t)$$

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We seek the function $x_1(t)$ and $x_2(t)$ that satisfy the two given equations. a, b, c & $d \in \mathbb{R}$.

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$$x_1(t) + 2x_2(t) = -e^{-t} + 2e^{4t} + 2e^{-t} + 6e^{4t}$$

$$x_1(t) + 2x_2(t) = e^{-t} + 8e^{4t} = x_1'(t) = L.H.S = RHS$$

$$RHS = 3x_1(t) + 2x_2(t) = 3(-e^{-t} + 2e^{4t}) + 2(e^{-t} + 3e^{4t})$$

$$RHS = -3e^{-t} + 6e^{4t} + 2e^{-t} + 6e^{4t} = -e^{-t} + 12e^{4t} = LHS$$

Indeed $x_1(t)$ and $x_2(t)$ are solutions.

A general ~~so~~ system

$$x_1'(t) = a x_1(t) + b\, x_2(t)$$

$$x_2'(t) = c x_1(t) + d\, x_2(t)$$

can be written in matrix form;

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

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