Mat 3310 Notes : Power Series solution to ODE's

Trevor Chimpinde

February 2021

1 Introduction

Linear Ordinary Differential Equations (ODEs) can be broadly divided into two:

1. Linear ODEs with constant coefficients, and
2. Linear ODEs with variable coefficients.

Linear ODEs with constant coefficients can be solved by algebraic methods, and their solutions are elementary functions known from calculus. For ODEs with variable coefficients the situation is more complicated, and their solutions may be nonelementary functions.

The power series method is the standard method for solving linear ordinary differential equations with variable coefficients. It gives solutions in the form of power series.

2 Power Series

2.1 Definition

A power series (in powers of $x - x_0$) is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \dots$$

In the above, x is a variable and $a_0, a_1, a_2, \dots$ are constants called the coefficients of the series.

x_0 is a constant, called the center of the series. In particular, if $x_0 = 0$, we obtain a power series in powers of x.

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots.$$

We shall assume that all variables and constants are real.

Please note that our use of the term "power series" strictly refers to "infinite sums" that do not include negative or fractional powers of x.

2.2 Analytic Functions

A function is analytic if it can be represented by a power series.

Not every function is analytic, although the majority of the functions you have seen in calculus are.

An analytic function $f(x)$ is equal to its Taylor series near a point x_0. That is, for x near x_0 we have

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(x_0)}{n!}(x - x_0)^n.$$

where $f^{(n)}(x_0)$ denotes the n^{th} derivative of $f(x)$ at the point x_0.

When $x_0 = 0$, we get what is called the Maclaurin series. That is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n.$$

Familiar Power Series are the Maclaurin series

1.
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots .$$

2.
$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots .$$

3.
$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots .$$

4.
$$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots .$$

5.
$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots .$$

2.3 Manipulating Power Series

1. **Termwise Differentiation**. A power series may be differentiated term by term. More precisely: if

$$y = \sum_{n=0}^{\infty} a_n x^n,$$

then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Similarly for the third and further derivatives.

2. **Termwise Addition**. Two power series may be added term by term. More precisely, if:

$$y = \sum_{n=0}^{\infty} a_n x^n \text{ and } z = \sum_{n=0}^{\infty} b_n x^n$$

then

$$y + z = \sum_{n=0}^{\infty} (a_n + b_n) x^n.$$

3. **Shifting summation indices**. It is often convenient or necessary in the power series method to shift the index so that the power under the summation sign is x^n. For example,

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n.$$

How do we see or convert this? In the series

$$\sum_{n=1}^{\infty} n a_n x^{n-1}$$

Let $m = n - 1$, then $n = m + 1$. Therefore, replacing m in that series gives us $\sum_{m=0}^{\infty} (m+1) a_{m+1} x^m$ Since m is just the letter of indexing, we can still go back to n and have

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n.$$

One can verify this by writing down the first few terms of each series explicity.

3 Idea and Technique of the Power Series Method

The idea of the power series method for solving linear ODEs seems natural, once we know that the most important ODEs in applied mathematics have solutions of this form. We explain the idea by an ODE that can readily be solved otherwise.

3.0.1 Example

Find a power series solution to the ordinary differential equation $y' - y = 0$.

Solution:

Assume first that $y = \sum_{n=0}^{\infty} a_n x^n$. Then $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$.

Insert these into the differential equation, we get

$$\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

Manipulating the first series, we get

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

Rewriting under one summation we get

$$\sum_{n=0}^{\infty} \left[(n+1) a_{n+1} - a_n\right] x^n = 0$$

Since the left hand side which is a series equals zero on the right hand side, it implies that the coefficients of the series on the left hand side must all equal zero.

That is $(n+1)a_{n+1} - a_n = 0$, implying that $a_{n+1} = \frac{a_n}{n+1}$.

value of n	implication
$n = 0$	$a_1 = a_0$
$n = 1$	$a_2 = \frac{a_1}{2} = \frac{a_0}{2}$
$n = 2$	$a_3 = \frac{a_2}{3} = \frac{a_0}{2 \cdot 3} = \frac{a_0}{3!}$
$n = 3$	$a_4 = \frac{a_3}{4} = \frac{a_0}{4 \cdot 3!} = \frac{a_0}{4!}$
$\cdots$	$\cdots$

From this, we see that all the coefficients are equal. That is,

$$\begin{aligned} y = \sum_{n=0}^{\infty} a_n x^n &= a_0 + a_0 x + \frac{a_0}{2!} x^2 + \frac{a_0}{3!} x^3 + \cdots \\ &= a_0 \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) \\ &= a_0 e^x. \end{aligned}$$

4 Series Solution of Linear Second Order ODEs

Suppose we have a linear second order homogeneous ODE of the form

$$p(x)y'' + q(x)y' + r(x)y = s(x)$$

We divide the ODE by $p(x)$ to get

$$y'' + \frac{q(x)}{p(x)}y' + \frac{r(x)}{[p(x)}y = \frac{s(x)}{p(x)}$$

The above ODE has a series solution of the form $\sum_{n=0}^{\infty} a_0(x - x_0)^n$ if the functions $\frac{q(x)}{p(x)}$, $\frac{r(x)}{[p(x)}$ and $\frac{s(x)}{p(x)}$ are analytic about $x = x_0$.

If $p(x)$, $q(x)$, $r(x)$ and $s(x)$ are polynomials, the condition that $\frac{q(x)}{p(x)}$, $\frac{r(x)}{[p(x)}$ and $\frac{s(x)}{p(x)}$ are analytic about $x = x_0$ is equivalent to requiring that $p(x_0) \neq 0$.

For simplicity, all the series we will be considering will have center $x_0 = 0$.

Lets start with a simple example.

4.0.1 Example 1

Find a power series solution to the ODE below.

$$y'' - y = 0.$$

Solution:

Let

$$y = \sum_{n=0}^{\infty} a_n x^n$$

, then

$$y'' = \sum_{n=2}^{\infty} (n-1)n a_n x^{n-2}.$$

Then $y'' - y = 0$ becomes

$$\sum_{n=2}^{\infty} (n-1)n a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^n = 0.$$

Shifting the index of the first series, we get

$$\sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2}x^n - \sum_{n=0}^{\infty} a_n x^n = 0.$$

Writing under one summation symbol, we get

$$\sum_{n=0}^{\infty} \left[(n+1)(n+2)a_{n+2} - a_n\right] x^n = 0.$$

Therefore, $(n+1)(n+2)a_{n+2} - a_n = 0$. Thus, $a_{n+2} = \frac{a_n}{(n+1)(n+2)}$.

$$n = 0, \quad \Rightarrow \quad a_2 = \frac{a_0}{2} \tag{1}$$
$$n = 1, \quad \Rightarrow \quad a_3 = \frac{a_1}{2 \times 3} = \frac{a_1}{3!} \tag{2}$$
$$n = 2, \quad \Rightarrow \quad a_4 = \frac{a_2}{4 \times 3} = \frac{a_0}{4 \times 3 \times 2} = \frac{a_0}{4!} \tag{3}$$
$$n = 3, \quad \Rightarrow \quad a_5 = \frac{a_3}{5 \times 4} \tag{4}$$
$$\tag{5}$$

And so on.

Therefore,

$$\begin{aligned} y = \sum_{n=0}^{\infty} a_n x^n &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \\ &= a_0 + a_1 x + \frac{a_0}{2} x^2 + \frac{a_1}{3!} x^3 + \cdots \\ &= a_0 \left(1 + \frac{x^2}{2} + \frac{x^4}{4!} x^4 + \cdots \right) + a_1 \left(x + \frac{x^3}{3!} + \frac{x^5}{5} + \cdots \right) \\ &= a_0 \cosh(x) + a_2 \sinh(x) \end{aligned}$$

We now do a more complex example.

4.0.2 Example 2

Find a power series solution to the ODE below.

$$y'' - xy = 0.$$

Solution:

Let

$$y = \sum_{n=0}^{\infty} a_n x^n$$

, then

$$y'' = \sum_{n=2}^{\infty} (n-1) n a_n x^{n-2}$$

and $y'' - xy = 0$ becomes

$$\sum_{n=2}^{\infty} (n-1) n a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0.$$

Shifting the index of the first and second series, we get

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n - \sum_{n=1}^{\infty} a_{n-1}x^n = 0.$$

We now write out one term of the first series so that its indexing starts from one, which is the same as the second series.

$$2a_0 + \sum_{n=1}^{\infty}(n+1)(n+2)a_{n+2}x^n - \sum_{n=1}^{\infty} a_{n-1}x^n = 0.$$

Writing under one summation symbol, we get

$$2a_2 + \sum_{n=1}^{\infty}\left[(n+1)(n+2)a_{n+2} - a_{n-1}\right]x^n = 0.$$

Therefore, $2a_0 = 0$ and $(n+1)(n+2)a_{n+2} - a_n = 0$. Thus, $a_2 = 0$ and $a_{n+2} = \frac{a_{n-1}}{(n+2)(n+1)}$.

$$n = 1, \quad \Rightarrow \quad a_3 = \frac{a_0}{3\times 2} \tag{6}$$

$$n = 2, \quad \Rightarrow \quad a_4 = \frac{a_1}{4\times 3} \tag{7}$$

$$n = 3, \quad \Rightarrow \quad a_5 = \frac{a_2}{5\times 4} = 0 \tag{8}$$

$$n = 4, \quad \Rightarrow \quad a_6 = \frac{a_3}{6\times 5} = \frac{a_0}{6\times 5\times 3\times 2} \tag{9}$$

$$n = 5, \quad \Rightarrow \quad a_7 = \frac{a_4}{7\times 6} = \frac{a_1}{7\times 6\times 4\times 3} \tag{10}$$

$$n = 6, \quad \Rightarrow \quad a_8 = \frac{a_5}{8\times 7} = \frac{a_2}{8\times 7\times 5\times 4} \tag{11}$$

$$\tag{12}$$

And so on.

Therefore,

$$\begin{aligned} y = \sum_{n=0}^{\infty} a_n x^n &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \\ &= a_0 + a_1 x + \frac{a_0}{3\times 2}x^3 + \frac{a_1}{4\times 3}x^4 + \frac{a_0}{6\times 5\times 3\times 2}x^6 + \frac{a_1}{7\times 6\times 4\times 3}x^7 + \cdots \end{aligned}$$

5 Tutorial Questions

1. Apply the power series method to solve the following differential equations.

   (a) $y' = 2y$ (b) $y'' + y = 0$ (c) $y' = ky \quad k$ us a constant
   (d) $(1-x)y' = y$ (e) $(x+1)y' = 3y$ (f) $(1+x)y' + y = 0$
   (g) $y' + 2xy = 0$ (h) $y' = 3x^2y$ (i) $y'' - y = 0$
   (j) $y'' + 2xy = 0$ (k) $y'' - y' = 0$ (l) $y'' - 9y = 0$

2. Show that

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} = \sum_{j=1}^{\infty}(j+1)ja_{j+1}x^{j-1} = \sum_{m=2}^{\infty}(s+1)(s+1)a_{s+2}x^s$$

3. For each of the series below, shift the index so that the power under the summation sign is x^m.

   (a) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3n} x^{n+2}$ (b) $\sum_{s=1}^{\infty} \frac{s(s+1)}{s^2+1} x^{s-1}$ (c) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{6^k} x^{k-3}$

4. Find the general solution of the following differential equations.

   (a) $xy' = 3y + 3$ (b) $(x-3)y' - xy = 0$ (c) $y' = 2xy$
   (d) $(1-x^4)y' = 4x^3y$ (e) $(x+1)y' - (2x+3)y = 0$ (f) $(1+x)y'' - y = x$
   (g) $y'' - 3y' + 2y = 0$ (h) $y'' - 4xy' + (4x^2-2)y = 0$ (i) $(1-x^2)y'' - 2xy + 2y = 0$
   (j) $y'' - xy + y = 0$