Systems of Differential Equations

In the introduction to this section we briefly discussed how a system of differential equations can arise from a population problem in which we keep track of the population of both the prey and the predator. It makes sense that the number of prey present will affect the number of the predator present. Likewise, the number of predator present will affect the number of prey present. Therefore the differential equation that governs the population of either the prey or the predator should in some way depend on the population of the other. This will lead to two differential equations that must be solved simultaneously in order to determine the population of the prey and the predator.

The whole point of this is to notice that systems of differential equations can arise quite easily from naturally occurring situations. Developing an effective predator-prey system of differential equations is not the subject of this chapter. However, systems can arise from n^{th} order linear differential equations as well. Before we get into this however, let's write down a system and get some terminology out of the way.

We are going to be looking at first order, linear systems of differential equations. These terms mean the same thing that they have meant up to this point. The largest derivative anywhere in the system will be a first derivative and all unknown functions and their derivatives will only occur to the first power and will not be multiplied by other unknown functions. Here is an example of a system of first order, linear differential equations.

$$\begin{aligned} x_1' &= x_1 + 2x_2 \\ x_2' &= 3x_1 + 2x_2 \end{aligned}$$

We call this kind of system a **coupled** system since knowledge of x_2 is required in order to find x_1 and likewise knowledge of x_1 is required to find x_2. We will worry about how to go about solving these later. At this point we are only interested in becoming familiar with some of the basics of systems.

Now, as mentioned earlier, we can write an n^{th} order linear differential equation as a system. Let's see how that can be done.

Example Write the following 2^{nd} order differential equation as a system of first order, linear differential equations.

$$2y'' - 5y' + y = 0 \qquad y(3) = 6 \qquad y'(3) = -1$$

Solution

We can write higher order differential equations as a system with a very simple change of variable. We'll start by defining the following two new functions.

$$\begin{aligned} x_1(t) &= y(t) \\ x_2(t) &= y'(t) \end{aligned}$$

Now notice that if we differentiate both sides of these we get,

$$x_1' = y' = x_2$$
$$x_2' = y'' = -\frac{1}{2}y + \frac{5}{2}y' = -\frac{1}{2}x_1 + \frac{5}{2}x_2$$

Note the use of the differential equation in the second equation. We can also convert the initial conditions over to the new functions.

$$x_1(3) = y(3) = 6$$
$$x_2(3) = y'(3) = -1$$

Putting all of this together gives the following system of differential equations.

$$x_1' = x_2 \qquad x_1(3) = 6$$
$$x_2' = -\frac{1}{2}x_1 + \frac{5}{2}x_2 \qquad x_2(3) = -1$$

We will call the system in the above example an **Initial Value Problem** just as we did for differential equations with initial conditions.

Let's take a look at another example.

Example ˆ Write the following 4th order differential equation as a system of first order, linear differential equations.

$$y^{(4)} + 3y'' - \sin(t)\,y' + 8y = t^2 \qquad y(0) = 1 \quad y'(0) = 2 \quad y''(0) = 3 \quad y'''(0) = 4$$

Solution

Just as we did in the last example we'll need to define some new functions. This time we'll need 4 new functions.

$$x_1 = y \quad \Rightarrow \quad x_1' = y' = x_2$$
$$x_2 = y' \quad \Rightarrow \quad x_2' = y'' = x_3$$
$$x_3 = y'' \quad \Rightarrow \quad x_3' = y''' = x_4$$
$$x_4 = y''' \quad \Rightarrow \quad x_4' = y^{(4)} = -8y + \sin(t)\,y' - 3y'' + t^2 = -8x_1 + \sin(t)\,x_2 - 3x_3 + t^2$$

The system along with the initial conditions is then,

$$x_1' = x_2 \qquad x_1(0) = 1$$
$$x_2' = x_3 \qquad x_2(0) = 2$$
$$x_3' = x_4 \qquad x_3(0) = 3$$
$$x_4' = -8x_1 + \sin(t)\,x_2 - 3x_3 + t^2 \qquad x_4(0) = 4$$

Now, when we finally get around to solving these we will see that we generally don't solve systems in the form that we've given them in this section. Systems of differential equations can be converted to **matrix form** and this is the form that we usually use in solving systems.

Example Convert the following system to matrix from.

$$x_1' = 4x_1 + 7x_2$$
$$x_2' = -2x_1 - 5x_2$$

Solution

First write the system so that each side is a vector.

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 4x_1 + 7x_2 \\ -2x_1 - 5x_2 \end{pmatrix}$$

Now the right side can be written as a matrix multiplication,

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 4 & 7 \\ -2 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Now, if we define,

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

then,

$$\vec{x}' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}$$

The system can then be written in the matrix form,

$$\vec{x}' = \begin{pmatrix} 4 & 7 \\ -2 & -5 \end{pmatrix} \vec{x}$$

Example Convert the systems from Examples 1 and 2 into matrix form.

Solution

We'll start with the system from Example 1.

$$x_1' = x_2 \qquad x_1(3) = 6$$
$$x_2' = -\frac{1}{2}x_1 + \frac{5}{2}x_2 \qquad x_2(3) = -1$$

First define,

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The system is then,

$$\vec{x}' = \begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & \frac{5}{2} \end{pmatrix} \vec{x} \qquad \vec{x}(3) = \begin{pmatrix} x_1(3) \\ x_2(3) \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \end{pmatrix}$$

Now, let's do the system from Example 2.

$$\begin{aligned} x_1' &= x_2 & \qquad x_1(0) &= 1 \\ x_2' &= x_3 & x_2(0) &= 2 \\ x_3' &= x_4 & x_3(0) &= 3 \\ x_4' &= -8x_1 + \sin(t)\,x_2 - 3x_3 + t^2 & x_4(0) &= 4 \end{aligned}$$

In this case we need to be careful with the t^2 in the last equation. We'll start by writing the system as a vector again and then break it up into two vectors, one vector that contains the unknown functions and the other that contains any known functions.

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ -8x_1 + \sin(t)\,x_2 - 3x_3 + t^2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ -8x_1 + \sin(t)\,x_2 - 3x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ t^2 \end{pmatrix}$$

Now, the first vector can now be written as a matrix multiplication and we'll leave the second vector alone.

$$\vec{x}' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -8 & \sin(t) & -3 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ t^2 \end{pmatrix} \qquad \vec{x}(0) = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

where,

$$\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix}$$

Note that occasionally for "large" systems such as this we will go one step farther and write the system as,

$$\vec{x}' = A\vec{x} + \vec{g}(t)$$

The last thing that we need to do in this section is get a bit of terminology out of the way. Starting with

$$\vec{x}' = A\vec{x} + \vec{g}(t)$$

we say that the system is **homogeneous** if $\vec{g}(t) = \vec{0}$ and we say the system is **nonhomogeneous** if $\vec{g}(t) \neq \vec{0}$.

Solutions to Systems

Now that we've got some of the basics out of the way for systems of differential equations it's time to start thinking about how to solve a system of differential equations. We will start with the homogeneous system written in matrix form,

$$\vec{x}' = A\,\vec{x} \qquad \textbf{(1)}$$

where, A is an n x n matrix and $\vec{x}$ is a vector whose components are the unknown functions in the system.

Now, if we start with $n = 1$ then the system reduces to a fairly simple linear (or separable) first order differential equation.

$$x' = ax$$

and this has the following solution,

$$x\left(t\right) = c\mathbf{e}^{a\,t}$$

So, let's use this as a guide and for a general n let's see if

$$\vec{x}\left(t\right) = \vec{\eta}\,\mathbf{e}^{r\,t} \qquad \textbf{(2)}$$

will be a solution. Note that the only real difference here is that we let the constant in front of the exponential be a vector. All we need to do then is plug this into the differential equation and see what we get. First notice that the derivative is,

$$\vec{x}'\left(t\right) = r\vec{\eta}\,\mathbf{e}^{r\,t}$$

So, upon plugging the guess into the differential equation we get,

$$\begin{aligned} r\vec{\eta}\mathbf{e}^{r\,t} &= A\vec{\eta}\mathbf{e}^{r\,t} \\ \left(A\vec{\eta} - r\vec{\eta}\right)\mathbf{e}^{r\,t} &= \vec{0} \\ \left(A - rI\right)\vec{\eta}\mathbf{e}^{r\,t} &= \vec{0} \end{aligned}$$

Now, since we know that exponentials are not zero we can drop that portion and we then see that in order for (2) to be a solution to (1) then we must have

$$\left(A - rI\right)\vec{\eta} = \vec{0}$$

Or, in order for (2) to be a solution to (1), r and $\vec{\eta}$ must be an eigenvalue and eigenvector for the matrix A.

Therefore, in order to solve (1) we first find the eigenvalues and eigenvectors of the matrix A and then we can form solutions using (2). There are going to be three cases that we'll need to look at. The cases are real, distinct eigenvalues, complex eigenvalues and repeated eigenvalues.

None of this tells us how to completely solve a system of differential equations. We'll need the following couple of facts to do this.

Fact

1. If $\vec{x}_1(t)$ and $\vec{x}_2(t)$ are two solutions to a homogeneous system, (1), then
$$c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$
is also a solution to the system.
2. Suppose that *A* is an *n* x *n* matrix and suppose that $\vec{x}_1(t)$, $\vec{x}_2(t)$, …, $\vec{x}_n(t)$ are solutions to a homogeneous system, (1). Define,
$$X = \left(\vec{x}_1\ \vec{x}_2\ \cdots\ \vec{x}_n\right)$$
In other words, *X* is a matrix whose i^{th} column is the i^{th} solution. Now define,
$$W = \det(X)$$
We call *W* the **Wronskian**. If $W \neq 0$ then the solutions form a **fundamental set of solutions** and the general solution to the system is,
$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \cdots + c_n\vec{x}_n(t)$$

Note that if we have a fundamental set of solutions then the solutions are also going to be linearly independent. Likewise, if we have a set of linearly independent solutions then they will also be a fundamental set of solutions since the Wronskian will not be zero.

Real Eigenvalues

It's now time to start solving systems of differential equations. We've seen that solutions to the system,

$$\vec{x}' = A\vec{x}$$

will be of the form

$$\vec{x} = \vec{\eta}\mathbf{e}^{\lambda t}$$

where λ and $\vec{\eta}$ are eigenvalues and eigenvectors of the matrix *A*. We will be working with 2 x 2 systems so this means that we are going to be looking for two solutions, $\vec{x}_1(t)$ and $\vec{x}_2(t)$, where the determinant of the matrix,

$$X = \left(\vec{x}_1 \;\; \vec{x}_2\right)$$

is nonzero.

We are going to start by looking at the case where our two eigenvalues, λ_1 and λ_2 are real and distinct. In other words, they will be real, simple eigenvalues. Recall as well that the eigenvectors for simple eigenvalues are linearly independent. This means that the solutions we get from these will also be linearly independent. If the solutions are linearly independent the matrix *X* must be nonsingular and hence these two solutions will be a fundamental set of solutions. The general solution in this case will then be,

$$\vec{x}(t) = c_1\mathbf{e}^{\lambda_1 t}\vec{\eta}^{(1)} + c_2\mathbf{e}^{\lambda_2 t}\vec{\eta}^{(2)}$$

Note that each of our examples will actually be broken into two examples. The first example will be solving the system and the second example will be sketching the phase portrait for the system. Phase portraits are not always taught in a differential equations course and so we'll strip those out of the solution process so that if you haven't covered them in your class you can ignore the phase portrait example for the system.

Example Solve the following IVP.

$$\vec{x}' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\vec{x}, \qquad \vec{x}(0) = \begin{pmatrix} 0 \\ -4 \end{pmatrix}$$

Solution

So, the first thing that we need to do is find the eigenvalues for the matrix.

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} \\ &= \lambda^2 - 3\lambda - 4 \\ &= (\lambda+1)(\lambda-4) \qquad \Rightarrow \qquad \lambda_1 = -1,\ \lambda_2 = 4 \end{aligned}$$

Now let's find the eigenvectors for each of these.

$\lambda_1 = -1$:

We'll need to solve,

$$\begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \Rightarrow \qquad 2\eta_1 + 2\eta_2 = 0 \qquad \Rightarrow \qquad \eta_1 = -\eta_2$$

The eigenvector in this case is,

$$\vec{\eta} = \begin{pmatrix} -\eta_2 \\ \eta_2 \end{pmatrix} \qquad \Rightarrow \qquad \vec{\eta}^{(1)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad \eta_2 = 1$$

$\lambda_2 = 4$:

We'll need to solve,

$$\begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \Rightarrow \qquad -3\eta_1 + 2\eta_2 = 0 \qquad \Rightarrow \qquad \eta_1 = \frac{2}{3}\eta_2$$

The eigenvector in this case is,

$$\vec{\eta} = \begin{pmatrix} \frac{2}{3}\eta_2 \\ \eta_2 \end{pmatrix} \qquad \Rightarrow \qquad \vec{\eta}^{(2)} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad \eta_2 = 3$$

Then general solution is then,

$$\vec{x}(t) = c_1 \mathbf{e}^{-t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \mathbf{e}^{4t}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Now, we need to find the constants. To do this we simply need to apply the initial conditions.

$$\begin{pmatrix} 0 \\ -4 \end{pmatrix} = \vec{x}(0) = c_1\begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

All we need to do now is multiply the constants through and we then get two equations (one for each row) that we can solve for the constants. This gives,

$$\left.\begin{aligned} -c_1 + 2c_2 &= 0 \\ c_1 + 3c_2 &= -4 \end{aligned}\right\} \qquad \Rightarrow \qquad c_1 = -\frac{8}{5}, \; c_2 = -\frac{4}{5}$$

The solution is then,

$$\vec{x}(t) = -\frac{8}{5}\mathbf{e}^{-t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} - \frac{4}{5}\mathbf{e}^{4t}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

.

Example Convert the following differential equation into a system, solve the system and use this solution to get the solution to the original differential equation.

$$2y'' + 5y' - 3y = 0, \qquad y(0) = -4 \quad y'(0) = 9$$

Solution

So, we first need to convert this into a system. Here's the change of variables,

$$x_1 = y \qquad x_1' = y' = x_2$$

$$x_2 = y' \qquad x_2' = y'' = \frac{3}{2}y - \frac{5}{2}y' = \frac{3}{2}x_1 - \frac{5}{2}x_2$$

The system is then,

$$\vec{x}' = \begin{pmatrix} 0 & 1 \\ \frac{3}{2} & -\frac{5}{2} \end{pmatrix}\vec{x} \qquad \vec{x}(0) = \begin{pmatrix} -4 \\ 9 \end{pmatrix}$$

where,

$$\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}$$

Now we need to find the eigenvalues for the matrix.

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} -\lambda & 1 \\ \frac{3}{2} & -\frac{5}{2} - \lambda \end{vmatrix} \\ &= \lambda^2 + \tfrac{5}{2}\lambda - \tfrac{3}{2} \\ &= \tfrac{1}{2}(\lambda + 3)(2\lambda - 1) \qquad \lambda_1 = -3,\ \lambda_2 = \frac{1}{2} \end{aligned}$$

Now let's find the eigenvectors.

$\lambda_1 = -3$:

We'll need to solve,

$$\begin{pmatrix} 3 & 1 \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad 3\eta_1 + \eta_2 = 0 \quad \Rightarrow \quad \eta_2 = -3\eta_1$$

The eigenvector in this case is,

$$\vec{\eta} = \begin{pmatrix} \eta_1 \\ -3\eta_1 \end{pmatrix} \qquad \Rightarrow \qquad \vec{\eta}^{(1)} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}, \quad \eta_1 = 1$$

$\lambda_2 = \frac{1}{2}$:

We'll need to solve,

$$\begin{pmatrix} -\frac{1}{2} & 1 \\ \frac{3}{2} & -3 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \qquad -\frac{1}{2}\eta_1 + \eta_2 = 0 \qquad \Rightarrow \qquad \eta_2 = \frac{1}{2}\eta_1$$

The eigenvector in this case is,

$$\vec{\eta} = \begin{pmatrix} \eta_1 \\ \frac{1}{2}\eta_1 \end{pmatrix} \qquad \Rightarrow \qquad \vec{\eta}^{(2)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \eta_1 = 2$$

The general solution is then,

$$\vec{x}(t) = c_1 \mathbf{e}^{-3t}\begin{pmatrix} 1 \\ -3 \end{pmatrix} + c_2 \mathbf{e}^{\frac{t}{2}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Apply the initial condition.

$$\begin{pmatrix} -4 \\ 9 \end{pmatrix} = \vec{x}(0) = c_1 \begin{pmatrix} 1 \\ -3 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

This gives the system of equations that we can solve for the constants.

$$\left.\begin{aligned} c_1 + 2c_2 &= -4 \\ -3c_1 + c_2 &= 9 \end{aligned}\right\} \qquad \Rightarrow \qquad c_1 = -\frac{22}{7}, \; c_2 = -\frac{3}{7}$$

The actual solution to the system is then,

$$\vec{x}(t) = -\frac{22}{7}\mathbf{e}^{-3t}\begin{pmatrix} 1 \\ -3 \end{pmatrix} - \frac{3}{7}\mathbf{e}^{\frac{t}{2}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Now recalling that,

$$\vec{x}(t) = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}$$

we can see that the solution to the original differential equation is just the top row of the solution to the matrix system. The solution to the original differential equation is then,

$$y(t) = -\frac{22}{7}\mathbf{e}^{-3t} - \frac{6}{7}\mathbf{e}^{\frac{t}{2}}$$

Notice that as a check, in this case, the bottom row should be the derivative of the top row.

Complex Eigenvalues

In this section we will look at solutions to

$$\vec{x}' = A\vec{x}$$

where the eigenvalues of the matrix *A* are complex. With complex eigenvalues we are going to have the same problem that we had back when we were looking at second order differential equations. We want our solutions to only have real numbers in them, however since our solutions to systems are of the form,

$$\vec{x} = \vec{\eta}\mathbf{e}^{\lambda t}$$

we are going to have complex numbers come into our solution from both the eigenvalue and the eigenvector. Getting rid of the complex numbers here will be similar to how we did it back in the second order differential equation case but will involve a little more work this time around. It's easiest to see how to do this in an example.

Example Solve the following IVP.

$$\vec{x}' = \begin{pmatrix} 3 & 9 \\ -4 & -3 \end{pmatrix}\vec{x} \qquad\qquad \vec{x}(0) = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$$

Solution

We first need the eigenvalues and eigenvectors for the matrix.

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 9 \\ -4 & -3-\lambda \end{vmatrix}$$

$$= \lambda^2 + 27 \qquad\qquad \lambda_{1,2} = \pm 3\sqrt{3}\,i$$

So, now that we have the eigenvalues recall that we only need to get the eigenvector for one of the eigenvalues since we can get the second eigenvector for free from the first eigenvector.

$\lambda_1 = 3\sqrt{3}\,i$:

We need to solve the following system.

$$\begin{pmatrix} 3-3\sqrt{3}\,i & 9 \\ -4 & -3-3\sqrt{3}\,i \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Using the first equation we get,

$$\left(3-3\sqrt{3}\,i\right)\eta_1 + 9\eta_2 = 0$$

$$\eta_2 = -\frac{1}{3}\left(1-\sqrt{3}\,i\right)\eta_1$$

So, the first eigenvector is,

$$\vec{\eta} = \begin{pmatrix} \eta_1 \\ -\frac{1}{3}\left(1-\sqrt{3}\,i\right)\eta_1 \end{pmatrix}$$

$$\vec{\eta}^{(1)} = \begin{pmatrix} 3 \\ -1+\sqrt{3}\,i \end{pmatrix} \qquad \eta_1 = 3$$

When finding the eigenvectors in these cases make sure that the complex number appears in the numerator of any fractions since we'll need it in the numerator later on. Also try to clear out any fractions by appropriately picking the constant. This will make our life easier down the road.

Now, the second eigenvector is,

$$\vec{\eta}^{(2)} = \begin{pmatrix} 3 \\ -1-\sqrt{3}\,i \end{pmatrix}$$

However, as we will see we won't need this eigenvector.

The solution that we get from the first eigenvalue and eigenvector is,

$$\vec{x}_1(t) = \mathbf{e}^{3\sqrt{3}\,i\,t}\begin{pmatrix} 3 \\ -1+\sqrt{3}\,i \end{pmatrix}$$

So, as we can see there are complex numbers in both the exponential and vector that we will need to get rid of in order to use this as a solution. Recall from the complex roots section of the second order differential equation chapter that we can use Euler's formula to get the complex number out of the exponential. Doing this gives us,

$$\vec{x}_1(t) = \left(\cos\left(3\sqrt{3}t\right) + i\sin\left(3\sqrt{3}t\right)\right)\begin{pmatrix} 3 \\ -1+\sqrt{3}\,i \end{pmatrix}$$

The next step is to multiply the cosines and sines into the vector.

$$\vec{x}_1(t) = \begin{pmatrix} 3\cos\left(3\sqrt{3}t\right) + 3i\sin\left(3\sqrt{3}t\right) \\ -\cos\left(3\sqrt{3}t\right) - i\sin\left(3\sqrt{3}t\right) + \sqrt{3}\,i\cos\left(3\sqrt{3}t\right) - \sqrt{3}\sin\left(3\sqrt{3}t\right) \end{pmatrix}$$

Now combine the terms with an "*i*" in them and split these terms off from those terms that don't contain an "*i*". Also factor the "*i*" out of this vector.

$$\begin{aligned}\vec{x}_1(t) &= \begin{pmatrix} 3\cos\left(3\sqrt{3}t\right) \\ -\cos\left(3\sqrt{3}t\right) - \sqrt{3}\sin\left(3\sqrt{3}t\right) \end{pmatrix} + i\begin{pmatrix} 3\sin\left(3\sqrt{3}t\right) \\ -\sin\left(3\sqrt{3}t\right) + \sqrt{3}\cos\left(3\sqrt{3}t\right) \end{pmatrix} \\ &= \vec{u}(t) + i\,\vec{v}(t)\end{aligned}$$

Now, it can be shown (we'll leave the details to you) that $\vec{u}(t)$ and $\vec{v}(t)$ are two linearly independent solutions to the system of differential equations. This means that we can use them to form a general solution and they are both real solutions.

So, the general solution to a system with complex roots is

$$\vec{x}(t) = c_1\vec{u}(t) + c_2\vec{v}(t)$$

where $\vec{u}(t)$ and $\vec{v}(t)$ are found by writing the first solution as

$$\vec{x}(t) = \vec{u}(t) + i\,\vec{v}(t)$$

For our system then, the general solution is,

$$\vec{x}(t) = c_1\begin{pmatrix} 3\cos\left(3\sqrt{3}t\right) \\ -\cos\left(3\sqrt{3}t\right) - \sqrt{3}\sin\left(3\sqrt{3}t\right)\end{pmatrix} + c_2\begin{pmatrix} 3\sin\left(3\sqrt{3}t\right) \\ -\sin\left(3\sqrt{3}t\right) + \sqrt{3}\cos\left(3\sqrt{3}t\right)\end{pmatrix}$$

We now need to apply the initial condition to this to find the constants.

$$\begin{pmatrix} 2 \\ -4 \end{pmatrix} = \vec{x}(0) = c_1\begin{pmatrix} 3 \\ -1 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ \sqrt{3} \end{pmatrix}$$

This leads to the following system of equations to be solved,

$$\left.\begin{aligned} 3c_1 &= 2 \\ -c_1 + \sqrt{3}c_2 &= -4 \end{aligned}\right\} \qquad \Rightarrow \qquad c_1 = \frac{2}{3},\ c_2 = \frac{-10}{3\sqrt{3}}$$

The actual solution is then,

$$\vec{x}(t) = \frac{2}{3}\begin{pmatrix} 3\cos\left(3\sqrt{3}t\right) \\ -\cos\left(3\sqrt{3}t\right) - \sqrt{3}\sin\left(3\sqrt{3}t\right)\end{pmatrix} - \frac{10}{3\sqrt{3}}\begin{pmatrix} 3\sin\left(3\sqrt{3}t\right) \\ -\sin\left(3\sqrt{3}t\right) + \sqrt{3}\cos\left(3\sqrt{3}t\right)\end{pmatrix}$$

Example Solve the following IVP.

$$\vec{x}\,' = \begin{pmatrix} 3 & -13 \\ 5 & 1 \end{pmatrix} \vec{x} \qquad\qquad \vec{x}\left(0\right) = \begin{pmatrix} 3 \\ -10 \end{pmatrix}$$

Solution

Let's get the eigenvalues and eigenvectors for the matrix.

$$\det\left(A - \lambda I\right) = \begin{vmatrix} 3-\lambda & -13 \\ 5 & 1-\lambda \end{vmatrix}$$

$$= \lambda^2 - 4\lambda + 68 \qquad\qquad \lambda_{1,2} = 2 \pm 8i$$

Now get the eigenvector for the first eigenvalue.

$\lambda_1 = 2 + 8i$:

We need to solve the following system.

$$\begin{pmatrix} 1-8i & -13 \\ 5 & -1-8i \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Using the second equation we get,

$$5\eta_1 + \left(-1-8i\right)\eta_2 = 0$$

$$\eta_1 = \frac{1}{5}\left(1+8i\right)\eta_2$$

So, the first eigenvector is,

$$\vec{\eta} = \begin{pmatrix} \frac{1}{5}\left(1+8i\right)\eta_2 \\ \eta_2 \end{pmatrix}$$

$$\vec{\eta}^{(1)} = \begin{pmatrix} 1+8i \\ 5 \end{pmatrix} \qquad\qquad \eta_2 = 5$$

The solution corresponding to this eigenvalue and eigenvector is

$$\vec{x}_1\left(t\right) = \mathbf{e}^{(2+8i)t} \begin{pmatrix} 1+8i \\ 5 \end{pmatrix}$$

$$= \mathbf{e}^{2t}\mathbf{e}^{8it} \begin{pmatrix} 1+8i \\ 5 \end{pmatrix}$$

$$= \mathbf{e}^{2t}\left(\cos\left(8t\right) + i\sin\left(8t\right)\right) \begin{pmatrix} 1+8i \\ 5 \end{pmatrix}$$

As with the first example multiply cosines and sines into the vector and split it up. Don't forget about the exponential that is in the solution this time.

$$\vec{x}_1(t) = \mathbf{e}^{2t}\begin{pmatrix} \cos(8t) - 8\sin(8t) \\ 5\cos(8t) \end{pmatrix} + i\mathbf{e}^{2t}\begin{pmatrix} 8\cos(8t) + \sin(8t) \\ 5\sin(8t) \end{pmatrix}$$
$$= \vec{u}(t) + i\,\vec{v}(t)$$

The general solution to this system then,

$$\vec{x}(t) = c_1\mathbf{e}^{2t}\begin{pmatrix} \cos(8t) - 8\sin(8t) \\ 5\cos(8t) \end{pmatrix} + c_2\mathbf{e}^{2t}\begin{pmatrix} 8\cos(8t) + \sin(8t) \\ 5\sin(8t) \end{pmatrix}$$

Now apply the initial condition and find the constants.

$$\begin{pmatrix} 3 \\ -10 \end{pmatrix} = \vec{x}(0) = c_1\begin{pmatrix} 1 \\ 5 \end{pmatrix} + c_2\begin{pmatrix} 8 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} c_1 + 8c_2 &= 3 \\ 5c_1 &= -10 \end{aligned}\right\} \qquad \Rightarrow \qquad c_1 = -2,\ c_2 = \frac{5}{8}$$

The actual solution is then,

$$\vec{x}(t) = -2\mathbf{e}^{2t}\begin{pmatrix} \cos(8t) - 8\sin(8t) \\ 5\cos(8t) \end{pmatrix} + \frac{5}{8}\mathbf{e}^{2t}\begin{pmatrix} 8\cos(8t) + \sin(8t) \\ 5\sin(8t) \end{pmatrix}$$

Repeated Eigenvalues

This is the final case that we need to take a look at. In this section we are going to look at solutions to the system,

$$\vec{x}' = A\vec{x}$$

where the eigenvalues are repeated eigenvalues. Since we are going to be working with systems in which *A* is a 2 x 2 matrix we will make that assumption from the start. So, the system will have a double eigenvalue, λ.

This presents us with a problem. We want two linearly independent solutions so that we can form a general solution. However, with a double eigenvalue we will have only one,

$$\vec{x}_1 = \vec{\eta}\mathbf{e}^{\lambda t}$$

So, we need to come up with a second solution. Recall that when we looked at the double root case with the second order differential equations we ran into a similar problem. In that section we simply added a *t* to the solution and were able to get a second solution. Let's see if the same thing will work in this case as well. We'll see if

$$\vec{x} = t\,\mathbf{e}^{\lambda t}\vec{\eta}$$

will also be a solution.

To check all we need to do is plug into the system. Don't forget to product rule the proposed solution when you differentiate!

$$\vec{\eta}\mathbf{e}^{\lambda t} + \lambda\vec{\eta}t\mathbf{e}^{\lambda t} = A\vec{\eta}t\mathbf{e}^{\lambda t}$$

Now, we got two functions here on the left side, an exponential by itself and an exponential times a *t*. So, in order for our guess to be a solution we will need to require,

$$A\vec{\eta} = \lambda\vec{\eta} \qquad \Rightarrow \qquad \left(A - \lambda I\right)\vec{\eta} = \vec{0}$$
$$\vec{\eta} = \vec{0}$$

The first requirement isn't a problem since this just says that λ is an eigenvalue and it's eigenvector is $\vec{\eta}$. We already knew this however so there's nothing new there. The second however is a problem. Since $\vec{\eta}$ is an eigenvector we know that it can't be zero, yet in order to satisfy the second condition it would have to be.

So, our guess was incorrect. The problem seems to be that there is a lone term with just an exponential in it so let's see if we can't fix up our guess to correct that. Let's try the following guess.

$$\vec{x} = t\,\mathbf{e}^{\lambda t}\vec{\eta} + \mathbf{e}^{\lambda t}\vec{\rho}$$

where $\vec{\rho}$ is an unknown vector that we'll need to determine.

As with the first guess let's plug this into the system and see what we get.

$$\vec{\eta}\mathbf{e}^{\lambda t} + \lambda\vec{\eta}t\mathbf{e}^{\lambda t} + \lambda\vec{\rho}\mathbf{e}^{\lambda t} = A\left(\vec{\eta}t\mathbf{e}^{\lambda t} + \vec{\rho}\mathbf{e}^{\lambda t}\right)$$
$$\left(\vec{\eta} + \lambda\vec{\rho}\right)\mathbf{e}^{\lambda t} + \lambda\vec{\eta}t\mathbf{e}^{\lambda t} = A\vec{\eta}t\mathbf{e}^{\lambda t} + A\vec{\rho}\mathbf{e}^{\lambda t}$$

Now set coefficients equal again,

$$\lambda\vec{\eta} = A\vec{\eta} \qquad \Rightarrow \qquad (A-\lambda I)\vec{\eta} = \vec{0}$$

$$\vec{\eta} + \lambda\vec{\rho} = A\vec{\rho} \qquad \Rightarrow \qquad (A-\lambda I)\vec{\rho} = \vec{\eta}$$

As with our first guess the first equation tells us nothing that we didn't already know. This time the second equation is not a problem. All the second equation tells us is that $\vec{\rho}$ must be a solution to this equation.

It looks like our second guess worked. Therefore,

$$\vec{x}_2 = t\,\mathbf{e}^{\lambda t}\vec{\eta} + \mathbf{e}^{\lambda t}\vec{\rho}$$

will be a solution to the system provided $\vec{\rho}$ is a solution to

$$(A-\lambda I)\vec{\rho} = \vec{\eta}$$

Also, this solution and the first solution are linearly independent and so they form a fundamental set of solutions and so the general solution in the double eigenvalue case is,

$$\vec{x} = c_1\mathbf{e}^{\lambda t}\vec{\eta} + c_2\left(t\,\mathbf{e}^{\lambda t}\vec{\eta} + \mathbf{e}^{\lambda t}\vec{\rho}\right)$$

Let's work an example.

Example Solve the following IVP.

$$\vec{x}' = \begin{pmatrix} 7 & 1 \\ -4 & 3 \end{pmatrix}\vec{x} \qquad \qquad \vec{x}(0) = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$$

Solution

First find the eigenvalues for the system.

$$\det(A-\lambda I) = \begin{vmatrix} 7-\lambda & 1 \\ -4 & 3-\lambda \end{vmatrix}$$

$$= \lambda^2 - 10\lambda + 25$$

$$= (\lambda-5)^2 \qquad \Rightarrow \qquad \lambda_{1,2} = 5$$

So, we got a double eigenvalue. Of course, that shouldn't be too surprising given the section that we're in. Let's find the eigenvector for this eigenvalue.

$$\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \Rightarrow \qquad 2\eta_1 + \eta_2 = 0 \qquad \eta_2 = -2\eta_1$$

The eigenvector is then,

$$\vec{\eta} = \begin{pmatrix} \eta_1 \\ -2\eta_1 \end{pmatrix} \qquad \eta_1 \neq 0$$

$$\vec{\eta}^{(1)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \qquad \eta_1 = 1$$

The next step is find $\vec{\rho}$. To do this we'll need to solve,

$$\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}\begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \qquad \Rightarrow \qquad 2\rho_1 + \rho_2 = 1 \qquad \rho_2 = 1 - 2\rho_1$$

Note that this is almost identical to the system that we solve to find the eigenvalue. The only difference is the right hand side. The most general possible $\vec{\rho}$ is

$$\vec{\rho} = \begin{pmatrix} \rho_1 \\ 1 - 2\rho_1 \end{pmatrix} \qquad \Rightarrow \qquad \vec{\rho} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \text{if } \rho_1 = 0$$

In this case, unlike the eigenvector system we can choose the constant to be anything we want, so we might as well pick it to make our life easier. This usually means picking it to be zero.

We can now write down the general solution to the system.

$$\vec{x}(t) = c_1 \mathbf{e}^{5t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2\left(\mathbf{e}^{5t} t \begin{pmatrix} 1 \\ -2 \end{pmatrix} + \mathbf{e}^{5t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

Applying the initial condition to find the constants gives us,

$$\begin{pmatrix} 2 \\ -5 \end{pmatrix} = \vec{x}(0) = c_1\begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\left.\begin{aligned} c_1 &= 2 \\ -2c_1 + c_2 &= -5 \end{aligned}\right\} \qquad \Rightarrow \qquad c_1 = 2, \; c_2 = -1$$

The actual solution is then,

$$\begin{aligned} \vec{x}(t) &= 2\mathbf{e}^{5t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} - \left(t\mathbf{e}^{5t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + \mathbf{e}^{5t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \\ &= \mathbf{e}^{5t}\begin{pmatrix} 2 \\ -4 \end{pmatrix} - \mathbf{e}^{5t} t\begin{pmatrix} 1 \\ -2 \end{pmatrix} - \mathbf{e}^{5t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \mathbf{e}^{5t}\begin{pmatrix} 2 \\ -5 \end{pmatrix} - \mathbf{e}^{5t} t\begin{pmatrix} 1 \\ -2 \end{pmatrix} \end{aligned}$$

Note that we did a little combining here to simplify the solution up a little.

Example Solve the following IVP.

$$\vec{x}' = \begin{pmatrix} -1 & \frac{3}{2} \\ -\frac{1}{6} & -2 \end{pmatrix} \vec{x} \qquad \vec{x}(2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Solution

First the eigenvalue for the system.

$$\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & \frac{3}{2} \\ -\frac{1}{6} & -2-\lambda \end{vmatrix}$$

$$= \lambda^2 + 3\lambda + \frac{9}{4}$$

$$= \left(\lambda + \frac{3}{2}\right)^2 \qquad \Rightarrow \qquad \lambda_{1,2} = -\frac{3}{2}$$

Now let's get the eigenvector.

$$\begin{pmatrix} \frac{1}{2} & \frac{3}{2} \\ -\frac{1}{6} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \Rightarrow \qquad \frac{1}{2}\eta_1 + \frac{3}{2}\eta_2 = 0 \qquad \eta_1 = -3\eta_2$$

$$\vec{\eta} = \begin{pmatrix} -3\eta_2 \\ \eta_2 \end{pmatrix} \qquad \eta_2 \neq 0$$

$$\vec{\eta}^{(1)} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \qquad \eta_2 = 1$$

Now find $\vec{\rho}$,

$$\begin{pmatrix} \frac{1}{2} & \frac{3}{2} \\ -\frac{1}{6} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \qquad \Rightarrow \qquad \frac{1}{2}\rho_1 + \frac{3}{2}\rho_2 = -3 \qquad \rho_1 = -6 - 3\rho_2$$

$$\vec{\rho} = \begin{pmatrix} -6 - 3\rho_2 \\ \rho_2 \end{pmatrix} \qquad \Rightarrow \qquad \vec{\rho} = \begin{pmatrix} -6 \\ 0 \end{pmatrix} \qquad \text{if } \rho_2 = 0$$

The general solution for the system is then,

$$\vec{x}(t) = c_1 \mathbf{e}^{-\frac{3t}{2}} \begin{pmatrix} -3 \\ 1 \end{pmatrix} + c_2 \left(t\mathbf{e}^{-\frac{3t}{2}} \begin{pmatrix} -3 \\ 1 \end{pmatrix} + \mathbf{e}^{-\frac{3t}{2}} \begin{pmatrix} -6 \\ 0 \end{pmatrix} \right)$$

Applying the initial condition gives,

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \vec{x}(2) = c_1 \mathbf{e}^{-3} \begin{pmatrix} -3 \\ 1 \end{pmatrix} + c_2 \left(2\mathbf{e}^{-3} \begin{pmatrix} -3 \\ 1 \end{pmatrix} + \mathbf{e}^{-3} \begin{pmatrix} -6 \\ 0 \end{pmatrix} \right)$$

Note that we didn't use *t=0* this time! We now need to solve the following system,

$$\left.\begin{aligned} -3\mathbf{e}^{-3} c_1 - 12\mathbf{e}^{-3} c_2 &= 1 \\ \mathbf{e}^{-3} c_1 + 2\mathbf{e}^{-3} c_2 &= 0 \end{aligned}\right\} \qquad \Rightarrow \qquad c_1 = \frac{\mathbf{e}^3}{3}, \; c_2 = -\frac{\mathbf{e}^3}{6}$$

The actual solution is then,

$$\begin{aligned} \vec{x}(t) &= \frac{\mathbf{e}^3}{3} \mathbf{e}^{-\frac{3t}{2}} \begin{pmatrix} -3 \\ 1 \end{pmatrix} - \frac{\mathbf{e}^3}{6} \left(t\mathbf{e}^{-\frac{3t}{2}} \begin{pmatrix} -3 \\ 1 \end{pmatrix} + \mathbf{e}^{-\frac{3t}{2}} \begin{pmatrix} -6 \\ 0 \end{pmatrix} \right) \\ &= \mathbf{e}^{-\frac{3t}{2}+3} \begin{pmatrix} 0 \\ \frac{1}{3} \end{pmatrix} + t\mathbf{e}^{-\frac{3t}{2}+3} \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{6} \end{pmatrix} \end{aligned}$$

Tutorial Questions

1. Convert each of the following linear ordinary differential equation into a system of first linear ordinary differential equations
   (a) $y'' - 4y' + 5y = 0$
   (b) $y''' - 5y'' + 9y = t\cos 2t$
   (c) $y'''' = 3y''' - \pi y'' + 2\pi y' - 6y = 11$
2. Rewrite each of the systems you found in the question above into a matrix-vector form
3. Find the general solution of each system below.
   (a) $\mathbf{x}' = \begin{pmatrix} 2 & 7 \\ -5 & -10 \end{pmatrix}\mathbf{x}$ (b) $\mathbf{x}' = \begin{pmatrix} -3 & 6 \\ -3 & 3 \end{pmatrix}\mathbf{x}$ (c) $\mathbf{x}' = \begin{pmatrix} 8 & -4 \\ 1 & 4 \end{pmatrix}\mathbf{x}$
   (d) $\mathbf{x}' = \begin{pmatrix} -3 & 2 \\ -1 & -5 \end{pmatrix}\mathbf{x}$ (e) $\mathbf{x}' = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\mathbf{x}$ (f) $\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -4 \end{pmatrix}\mathbf{x}$