Section 6-4 : Euler Equations

In this section we want to look for solutions to

$$ax^2 y'' + bxy' + cy = 0 \qquad (1)$$

around $x_0 = 0$. These types of differential equations are called **Euler Equations**.

Recall from the previous section that a point is an ordinary point if the quotients,

$$\frac{bx}{ax^2} = \frac{b}{ax} \qquad \text{and} \qquad \frac{c}{ax^2}$$

have Taylor series around $x_0 = 0$. However, because of the *x* in the denominator neither of these will have a Taylor series around $x_0 = 0$ and so $x_0 = 0$ is a singular point. So, the method from the previous section won't work since it required an ordinary point.

However, it is possible to get solutions to this differential equation that aren't series solutions. Let's start off by assuming that *x>0* (the reason for this will be apparent after we work the first example) and that all solutions are of the form,

$$y(x) = x^r \qquad (2)$$

Now plug this into the differential equation to get,

$$\begin{aligned} ax^2(r)(r-1)x^{r-2} + bx(r)x^{r-1} + cx^r &= 0 \\ ar(r-1)x^r + b(r)x^r + cx^r &= 0 \\ \left(ar(r-1) + b(r) + c\right)x^r &= 0 \end{aligned}$$

Now, we assumed that *x>0* and so this will only be zero if,

$$ar(r-1) + b(r) + c = 0 \qquad (3)$$

So solutions will be of the form (2) provided *r* is a solution to (3). This equation is a quadratic in *r* and so we will have three cases to look at : Real, Distinct Roots, Double Roots, and Complex Roots.

Real, Distinct Roots

There really isn't a whole lot to do in this case. We'll get two solutions that will form a fundamental set of solutions (we'll leave it to you to check this) and so our general solution will be,

$$y(x) = c_1 x^{r_1} + c_2 x^{r_2}$$

Example 1 Solve the following IVP

$$2x^2 y'' + 3xy' - 15y = 0, \qquad y(1) = 0 \quad y'(1) = 1$$

Solution

We first need to find the roots to (3).

$$2r(r-1) + 3r - 15 = 0$$

$$2r^2 + r - 15 = (2r-5)(r+3) = 0 \qquad \Rightarrow \qquad r_1 = \frac{5}{2},\; r_2 = -3$$

The general solution is then,

$$y(x) = c_1 x^{\frac{5}{2}} + c_2 x^{-3}$$

To find the constants we differentiate and plug in the initial conditions as we did back in the second order differential equations chapter.

$$y'(x) = \frac{5}{2} c_1 x^{\frac{3}{2}} - 3c_2 x^{-4}$$

$$\left.\begin{aligned} 0 &= y(1) = c_1 + c_2 \\ 1 &= y'(1) = \frac{5}{2} c_1 - 3c_2 \end{aligned}\right\} \qquad \Rightarrow \qquad c_1 = \frac{2}{11},\; c_2 = -\frac{2}{11}$$

The actual solution is then,

$$y(x) = \frac{2}{11} x^{\frac{5}{2}} - \frac{2}{11} x^{-3}$$

With the solution to this example we can now see why we required *x>0*. The second term would have division by zero if we allowed *x=0* and the first term would give us square roots of negative numbers if we allowed *x<0*.

Double Roots

This case will lead to the same problem that we've had every other time we've run into double roots (or double eigenvalues). We only get a single solution and will need a second solution. In this case it can be shown that the second solution will be,

$$y_2(x) = x^r \ln x$$

and so the general solution in this case is,

$$y(x) = c_1 x^r + c_2 x^r \ln x = x^r \left(c_1 + c_2 \ln x\right)$$

We can again see a reason for requiring *x>0*. If we didn't we'd have all sorts of problems with that logarithm.

Example 2 Find the general solution to the following differential equation.

$$x^2 y'' - 7xy' + 16y = 0$$

Solution

First the roots of (3).

$$\begin{aligned} r(r-1) - 7r + 16 &= 0 \\ r^2 - 8r + 16 &= 0 \\ (r-4)^2 &= 0 \qquad \Rightarrow \qquad r = 4 \end{aligned}$$

So, the general solution is then,

$$y(x) = c_1 x^4 + c_2 x^4 \ln x$$

Complex Roots

In this case we'll be assuming that our roots are of the form,

$$r_{1,2} = \lambda \pm \mu i$$

If we take the first root we'll get the following solution.

$$x^{\lambda + \mu i}$$

This is a problem since we don't want complex solutions, we only want real solutions. We can eliminate this by recalling that,

$$x^r = \mathbf{e}^{\ln x^r} = \mathbf{e}^{r \ln x}$$

Plugging the root into this gives,

$$\begin{aligned} x^{\lambda + \mu i} &= \mathbf{e}^{(\lambda + \mu i)\ln x} \\ &= \mathbf{e}^{\lambda \ln x} \mathbf{e}^{\mu i \ln x} \\ &= \mathbf{e}^{\ln x^{\lambda}} \left(\cos(\mu \ln x) + i \sin(\mu \ln x) \right) \\ &= x^{\lambda} \cos(\mu \ln x) + i x^{\lambda} \sin(\mu \ln x) \end{aligned}$$

Note that we had to use Euler formula as well to get to the final step. Now, as we've done every other time we've seen solutions like this we can take the real part and the imaginary part and use those for our two solutions.

So, in the case of complex roots the general solution will be,

$$y(x) = c_1 x^{\lambda} \cos(\mu \ln x) + c_2 x^{\lambda} \sin(\mu \ln x) = x^{\lambda}\left(c_1 \cos(\mu \ln x) + c_2 \sin(\mu \ln x)\right)$$

Once again, we can see why we needed to require $x > 0$.

Example 3 Find the solution to the following differential equation.

$$x^2 y'' + 3xy' + 4y = 0$$

Solution

Get the roots to (3) first as always.

$$r(r-1) + 3r + 4 = 0$$

$$r^2 + 2r + 4 = 0 \qquad \Rightarrow \qquad r_{1,2} = -1 \pm \sqrt{3}\, i$$

The general solution is then,

$$y(x) = c_1 x^{-1} \cos\left(\sqrt{3} \ln x\right) + c_2 x^{-1} \sin\left(\sqrt{3} \ln x\right)$$

Tutorial Questions

1. Find the general solution of the following differential equations.
   (a) $x^2y'' - 6y = 0$
   (b) $x^2y'' + 4y' = 0$
   (c) $x^2y'' - 2xy' + 2y = 0$
   (d) $x^2y'' + 9xy' + 16y = 0$
   (e) $x^2y'' + xy' - y = 0$
   (f) $x^2y'' + 3xy' + y = 0$
   (g) $x^2y'' + 3xy' + 5y = 0$
   (h) $x^2y'' + xy' + y = 0$

2. Solve the following initial value problems.
   (a) $x^2y'' - 4xy' + 4y = 0,\ y(1) = 4,\ y'(1) = 13$
   (b) $4x^2y'' - 4xy' - y = 0,\ y(4) = 2,\ y'(4) = \frac{1}{4}$
   (c) $x^2y'' - 5xy' + 8y = 0,\ y(1) = 5,\ y'(1) = 18$