Chapter 1 : 3-Dimensional Space

In this chapter we will start taking a more detailed look at three dimensional space (3-D space or $\mathbb{R}^3$). This is a very important topic for Calculus III since a good portion of Calculus III is done in three (or higher) dimensional space.

We will be looking at the equations of graphs in 3-D space as well as vector valued functions and how we do calculus with them. We will also be taking a look at a couple of new coordinate systems for 3-D space.

This is the only chapter that exists in two places in the notes. When we originally wrote these notes all of these topics were covered in Calculus II however, we have since moved several of them into Calculus III. So, rather than split the chapter up we kept it in the Calculus II notes and also put a copy in the Calculus III notes. Many of the sections not covered in Calculus III will be used on occasion there anyway and so they serve as a quick reference for when we need them. In addition this allows those that teach the topic in either place to have the notes quickly available to them.

Here is a list of topics in this chapter.

The 3-D Coordinate System – In this section we will introduce the standard three dimensional coordinate system as well as some common notation and concepts needed to work in three dimensions.

Equations of Lines – In this section we will derive the vector form and parametric form for the equation of lines in three dimensional space. We will also give the symmetric equations of lines in three dimensional space. Note as well that while these forms can also be useful for lines in two dimensional space.

Equations of Planes – In this section we will derive the vector and scalar equation of a plane. We also show how to write the equation of a plane from three points that lie in the plane.

Quadric Surfaces – In this section we will be looking at some examples of quadric surfaces. Some examples of quadric surfaces are cones, cylinders, ellipsoids, and elliptic paraboloids.

Functions of Several Variables – In this section we will give a quick review of some important topics about functions of several variables. In particular we will discuss finding the domain of a function of several variables as well as level curves, level surfaces and traces.

Vector Functions – In this section we introduce the concept of vector functions concentrating primarily on curves in three dimensional space. We will however, touch briefly on surfaces as well. We will illustrate how to find the domain of a vector function and how to graph a vector function. We will also show a simple relationship between vector functions and parametric equations that will be very useful at times.

Calculus with Vector Functions – In this section here we discuss how to do basic calculus, *i.e.* limits, derivatives and integrals, with vector functions.

Tangent, Normal and Binormal Vectors – In this section we will define the tangent, normal and binormal vectors.

Arc Length with Vector Functions – In this section we will extend the arc length formula we used early in the material to include finding the arc length of a vector function. As we will see the new formula really is just an almost natural extension of one we've already seen.

Curvature – In this section we give two formulas for computing the curvature (*i.e.* how fast the function is changing at a given point) of a vector function.

Velocity and Acceleration – In this section we will revisit a standard application of derivatives, the velocity and acceleration of an object whose position function is given by a vector function. For the acceleration we give formulas for both the normal acceleration and the tangential acceleration.

Cylindrical Coordinates – In this section we will define the cylindrical coordinate system, an alternate coordinate system for the three dimensional coordinate system. As we will see cylindrical coordinates are really nothing more than a very natural extension of polar coordinates into a three dimensional setting.

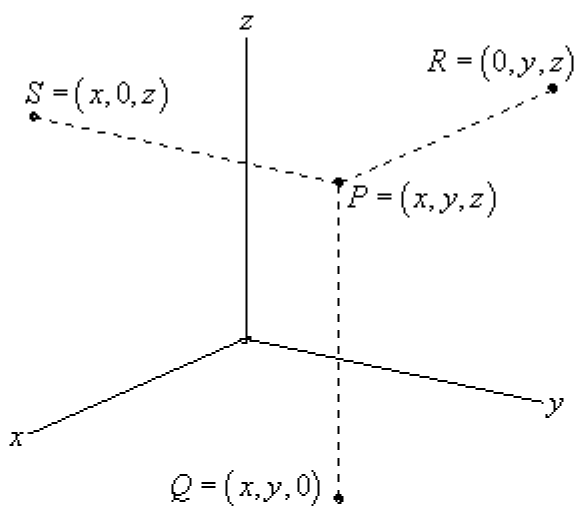
Spherical Coordinates – In this section we will define the spherical coordinate system, yet another alternate coordinate system for the three dimensional coordinate system. This coordinates system is very useful for dealing with spherical objects. We will derive formulas to convert between cylindrical coordinates and spherical coordinates as well as between Cartesian and spherical coordinates (the more useful of the two).

Section 1-1 : The 3-D Coordinate System

We'll start the chapter off with a fairly short discussion introducing the 3-D coordinate system and the conventions that we'll be using. We will also take a brief look at how the different coordinate systems can change the graph of an equation.

Let's first get some basic notation out of the way. The 3-D coordinate system is often denoted by $\mathbb{R}^3$. Likewise, the 2-D coordinate system is often denoted by $\mathbb{R}^2$ and the 1-D coordinate system is denoted by $\mathbb{R}$. Also, as you might have guessed then a general *n* dimensional coordinate system is often denoted by $\mathbb{R}^n$.

Next, let's take a quick look at the basic coordinate system.



This is the standard placement of the axes in this class. It is assumed that only the positive directions are shown by the axes. If we need the negative axes for any reason we will put them in as needed.

Also note the various points on this sketch. The point *P* is the general point sitting out in 3-D space. If we start at *P* and drop straight down until we reach a *z*-coordinate of zero we arrive at the point *Q*. We say that *Q* sits in the *xy*-plane. The *xy*-plane corresponds to all the points which have a zero *z*-coordinate. We can also start at *P* and move in the other two directions as shown to get points in the *xz*-plane (this is *S* with a *y*-coordinate of zero) and the *yz*-plane (this is *R* with an *x*-coordinate of zero).

Collectively, the *xy*, *xz*, and *yz*-planes are sometimes called the coordinate planes. In the remainder of this class you will need to be able to deal with the various coordinate planes so make sure that you can.

Also, the point *Q* is often referred to as the projection of *P* in the *xy*-plane. Likewise, *R* is the projection of *P* in the *yz*-plane and *S* is the projection of *P* in the *xz*-plane.

Many of the formulas that you are used to working with in $\mathbb{R}^2$ have natural extensions in $\mathbb{R}^3$. For instance, the distance between two points in $\mathbb{R}^2$ is given by,

$$d\left(P_1,P_2\right)=\sqrt{\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2}$$

While the distance between any two points in $\mathbb{R}^3$ is given by,

$$d\left(P_1,P_2\right)=\sqrt{\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2+\left(z_2-z_1\right)^2}$$

Likewise, the general equation for a circle with center (h,k) and radius *r* is given by,

$$\left(x-h\right)^2+\left(y-k\right)^2=r^2$$

and the general equation for a sphere with center (h,k,l) and radius *r* is given by,

$$\left(x-h\right)^2+\left(y-k\right)^2+\left(z-l\right)^2=r^2$$

With that said we do need to be careful about just translating everything we know about $\mathbb{R}^2$ into $\mathbb{R}^3$ and assuming that it will work the same way. A good example of this is in graphing to some extent. Consider the following example.

Example 1 Graph $x=3$ in $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^3$.

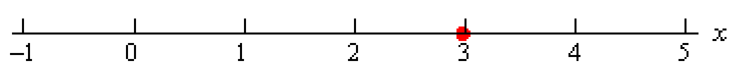
Solution

In $\mathbb{R}$ we have a single coordinate system and so $x=3$ is a point in a 1-D coordinate system.

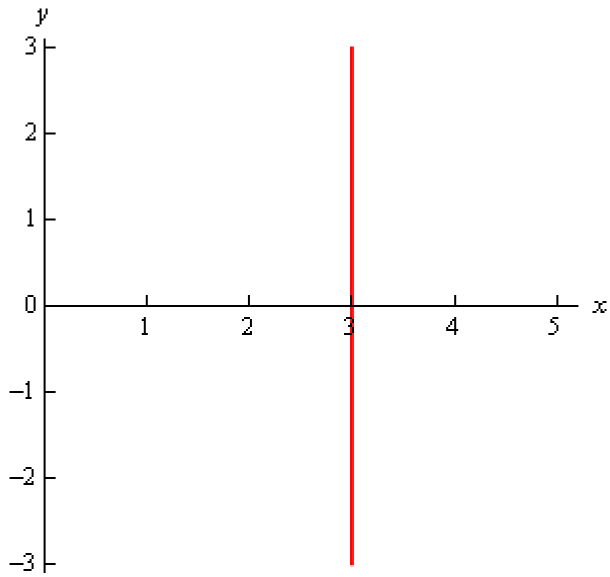
In $\mathbb{R}^2$ the equation $x=3$ tells us to graph all the points that are in the form $(3,y)$. This is a vertical line in a 2-D coordinate system.

In $\mathbb{R}^3$ the equation $x=3$ tells us to graph all the points that are in the form $(3,y,z)$. If you go back and look at the coordinate plane points this is very similar to the coordinates for the *yz*-plane except this time we have $x=3$ instead of $x=0$. So, in a 3-D coordinate system this is a plane that will be parallel to the *yz*-plane and pass through the *x*-axis at $x=3$.

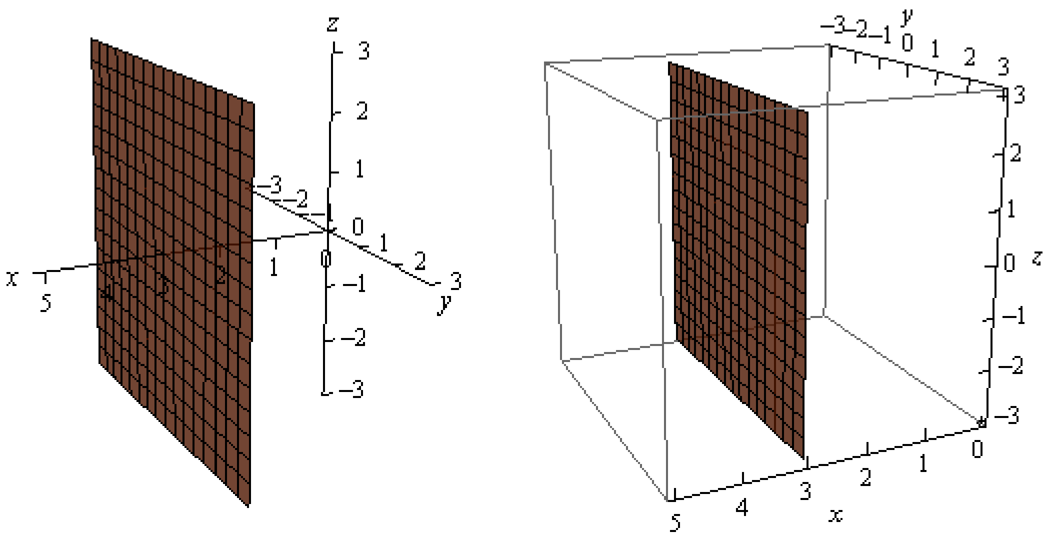
Here is the graph of $x=3$ in $\mathbb{R}$.



Here is the graph of $x=3$ in $\mathbb{R}^2$.



Finally, here is the graph of $x = 3$ in $\mathbb{R}^3$. Note that we've presented this graph in two different styles. On the left we've got the traditional axis system that we're used to seeing and on the right we've put the graph in a box. Both views can be convenient on occasion to help with perspective and so we'll often do this with 3D graphs and sketches.



Note that at this point we can now write down the equations for each of the coordinate planes as well using this idea.

$$\begin{aligned} z &= 0 & \qquad xy-\text{plane} \\ y &= 0 & \qquad xz-\text{plane} \\ x &= 0 & \qquad yz-\text{plane} \end{aligned}$$

Let's take a look at a slightly more general example.

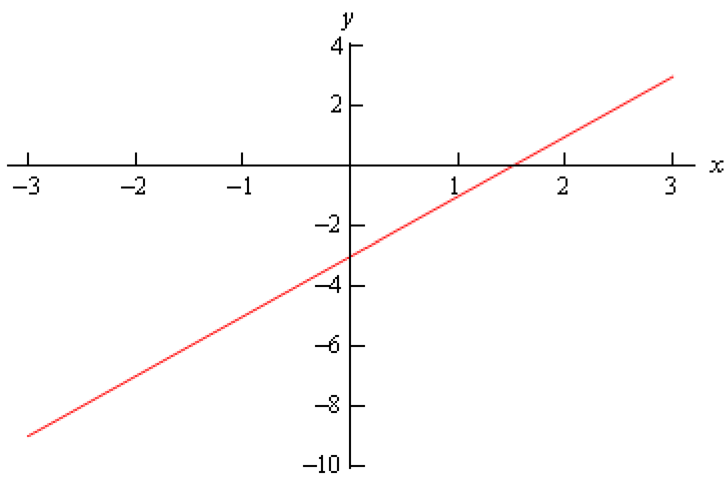
Example 2 Graph $y = 2x - 3$ in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Solution
Note we had to throw out $\mathbb{R}$ for this example since there are two variables which means that we can't be in a 1-D space (1-D space has only one variable!).

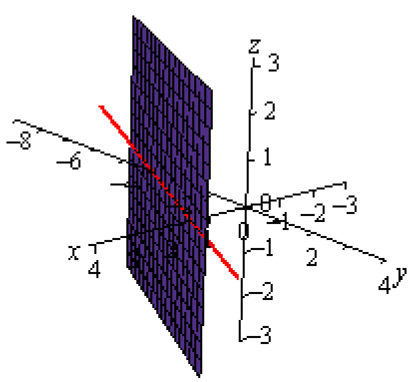
In $\mathbb{R}^2$ this is a line with slope 2 and a *y* intercept of -3.

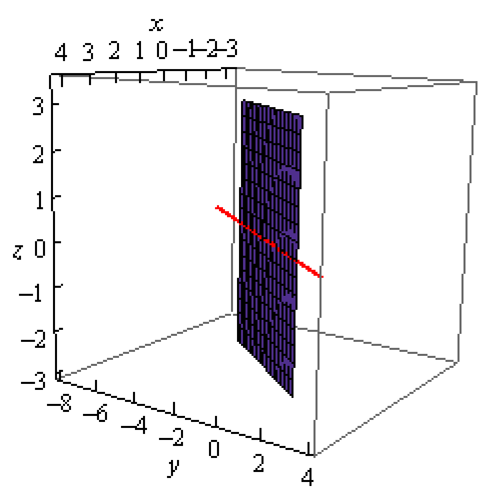
However, in $\mathbb{R}^3$ this is not necessarily a line. Because we have not specified a value of *z* we are forced to let *z* take any value. This means that at any particular value of *z* we will get a copy of this line. So, the graph is then a vertical plane that lies over the line given by $y = 2x - 3$ in the *xy*-plane.

Here is the graph in $\mathbb{R}^2$.



here is the graph in $\mathbb{R}^3$.





Notice that if we look to where the plane intersects the *xy*-plane we will get the graph of the line in $\mathbb{R}^2$ as noted in the above graph by the red line through the plane.

Let's take a look at one more example of the difference between graphs in the different coordinate systems.

Example 3 Graph $x^2 + y^2 = 4$ in $\mathbb{R}^2$ and $\mathbb{R}^3$.

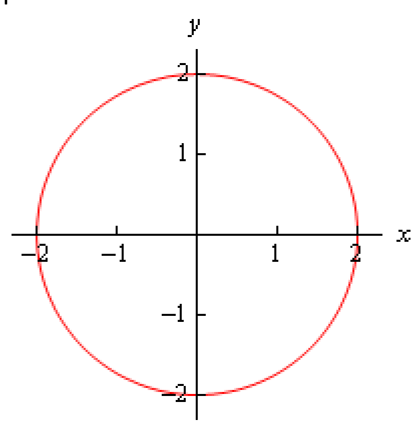
Solution

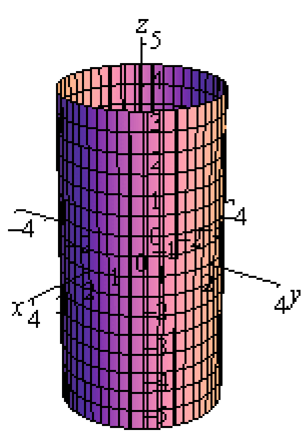
As with the previous example this won't have a 1-D graph since there are two variables.

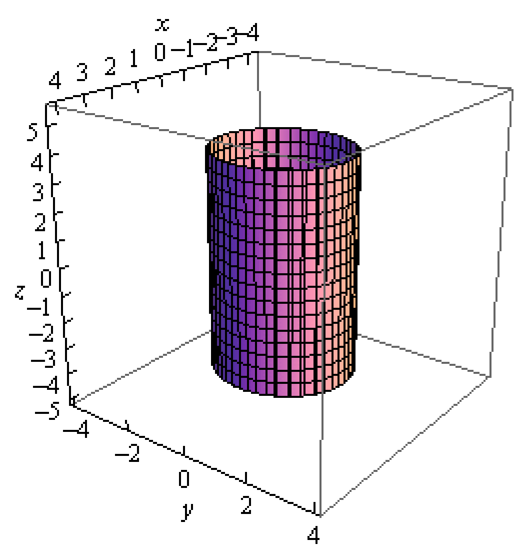
In $\mathbb{R}^2$ this is a circle centered at the origin with radius 2.

In $\mathbb{R}^3$ however, as with the previous example, this may or may not be a circle. Since we have not specified *z* in any way we must assume that *z* can take on any value. In other words, at any value of *z* this equation must be satisfied and so at any value *z* we have a circle of radius 2 centered on the *z*-axis. This means that we have a cylinder of radius 2 centered on the *z*-axis.

Here are the graphs for this example.







Notice that again, if we look to where the cylinder intersects the *xy*-plane we will again get the circle from $\mathbb{R}^2$.

We need to be careful with the last two examples. It would be tempting to take the results of these and say that we can't graph lines or circles in $\mathbb{R}^3$ and yet that doesn't really make sense. There is no reason for there to not be graphs of lines or circles in $\mathbb{R}^3$. Let's think about the example of the circle. To graph a circle in $\mathbb{R}^3$ we would need to do something like $x^2 + y^2 = 4$ at $z = 5$. This would be a circle of radius 2 centered on the *z*-axis at the level of $z = 5$. So, as long as we specify a *z* we will get a circle and not a cylinder. We will see an easier way to specify circles in a later section.

We could do the same thing with the line from the second example. However, we will be looking at lines in more generality in the next section and so we'll see a better way to deal with lines in $\mathbb{R}^3$ there.

The point of the examples in this section is to make sure that we are being careful with graphing equations and making sure that we always remember which coordinate system that we are in.

Another quick point to make here is that, as we've seen in the above examples, many graphs of equations in $\mathbb{R}^3$ are surfaces. That doesn't mean that we can't graph curves in $\mathbb{R}^3$. We can and will graph curves in $\mathbb{R}^3$ as well as we'll see later in this chapter.

Section 1-2 : Equations of Lines

In this section we need to take a look at the equation of a line in $\mathbb{R}^3$. As we saw in the previous section the equation $y = mx + b$ does not describe a line in $\mathbb{R}^3$, instead it describes a plane. This doesn't mean however that we can't write down an equation for a line in 3-D space. We're just going to need a new way of writing down the equation of a curve.

So, before we get into the equations of lines we first need to briefly look at vector functions. We're going to take a more in depth look at vector functions later. At this point all that we need to worry about is notational issues and how they can be used to give the equation of a curve.

The best way to get an idea of what a vector function is and what its graph looks like is to look at an example. So, consider the following vector function.

$$\vec{r}\left(t\right) = \left\langle t, 1\right\rangle$$

A vector function is a function that takes one or more variables, one in this case, and returns a vector. Note as well that a vector function can be a function of two or more variables. However, in those cases the graph may no longer be a curve in space.

The vector that the function gives can be a vector in whatever dimension we need it to be. In the example above it returns a vector in $\mathbb{R}^2$. When we get to the real subject of this section, equations of lines, we'll be using a vector function that returns a vector in $\mathbb{R}^3$

Now, we want to determine the graph of the vector function above. In order to find the graph of our function we'll think of the vector that the vector function returns as a position vector for points on the graph. Recall that a position vector, say $\vec{v} = \left\langle a, b\right\rangle$, is a vector that starts at the origin and ends at the point $\left(a, b\right)$.

So, to get the graph of a vector function all we need to do is plug in some values of the variable and then plot the point that corresponds to each position vector we get out of the function and play connect the dots. Here are some evaluations for our example.

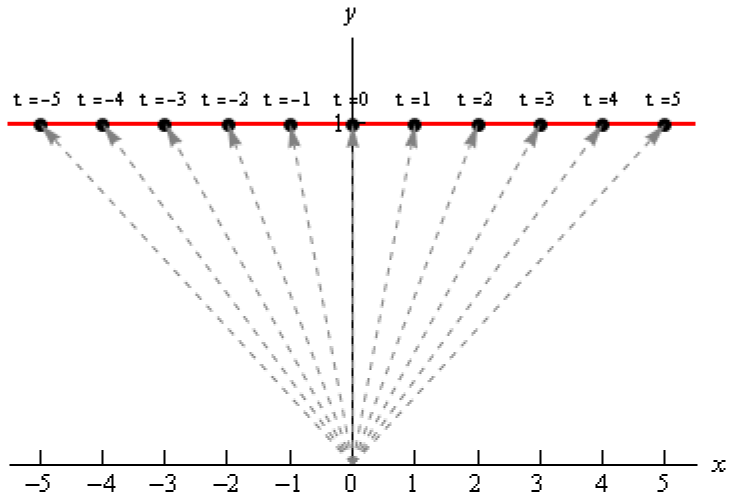
$$\vec{r}\left(-3\right) = \left\langle -3, 1\right\rangle \qquad \vec{r}\left(-1\right) = \left\langle -1, 1\right\rangle \qquad \vec{r}\left(2\right) = \left\langle 2, 1\right\rangle \qquad \vec{r}\left(5\right) = \left\langle 5, 1\right\rangle$$

So, each of these are position vectors representing points on the graph of our vector function. The points,

$$\left(-3, 1\right) \qquad \left(-1, 1\right) \qquad \left(2, 1\right) \qquad \left(5, 1\right)$$

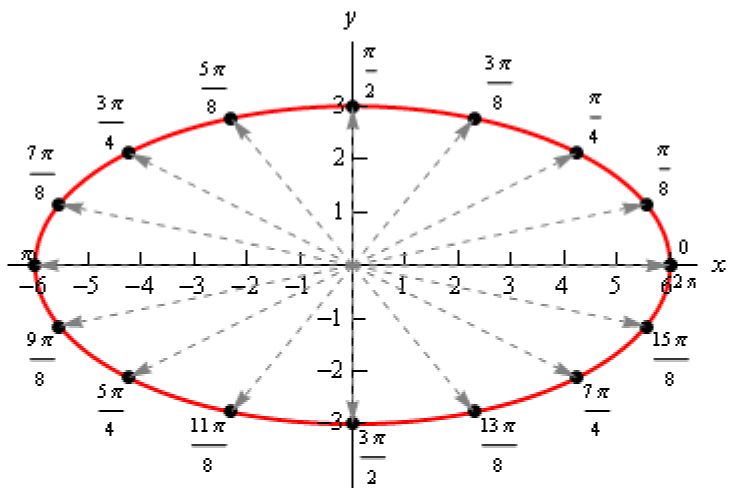
are all points that lie on the graph of our vector function.

If we do some more evaluations and plot all the points we get the following sketch.



In this sketch we've included the position vector (in gray and dashed) for several evaluations as well as the *t* (above each point) we used for each evaluation. It looks like, in this case the graph of the vector equation is in fact the line $y = 1$.

Here's another quick example. Here is the graph of $\vec{r}\left(t\right) = \left\langle 6\cos t, 3\sin t\right\rangle$.



In this case we get an ellipse. It is important to not come away from this section with the idea that vector functions only graph out lines. We'll be looking at lines in this section, but the graphs of vector functions do not have to be lines as the example above shows.

We'll leave this brief discussion of vector functions with another way to think of the graph of a vector function. Imagine that a pencil/pen is attached to the end of the position vector and as we increase the variable the resulting position vector moves and as it moves the pencil/pen on the end sketches out the curve for the vector function.

Okay, we now need to move into the actual topic of this section. We want to write down the equation of a line in $\mathbb{R}^3$ and as suggested by the work above we will need a vector function to do this. To see how we're going to do this let's think about what we need to write down the equation of a line in $\mathbb{R}^2$.

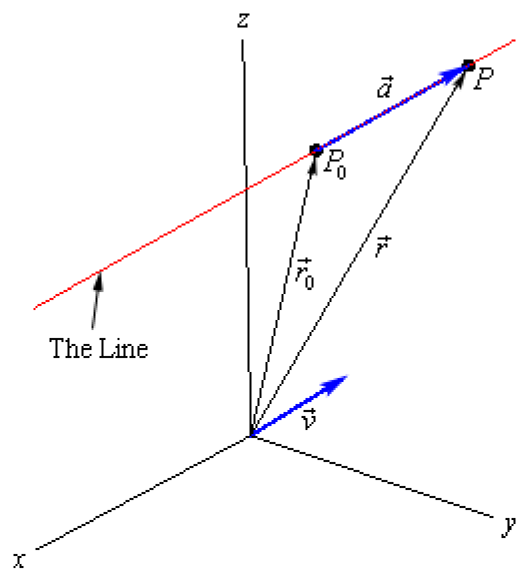
In two dimensions we need the slope (*m*) and a point that was on the line in order to write down the equation.

In $\mathbb{R}^3$ that is still all that we need except in this case the "slope" won't be a simple number as it was in two dimensions. In this case we will need to acknowledge that a line can have a three dimensional slope. So, we need something that will allow us to describe a direction that is potentially in three dimensions. We already have a quantity that will do this for us. Vectors give directions and can be three dimensional objects.

So, let's start with the following information. Suppose that we know a point that is on the line, $P_0 = \left(x_0, y_0, z_0\right)$, and that $\vec{v} = \left\langle a, b, c \right\rangle$ is some vector that is parallel to the line. Note, in all likelihood, $\vec{v}$ will not be on the line itself. We only need $\vec{v}$ to be parallel to the line. Finally, let $P = \left(x, y, z\right)$ be any point on the line.

Now, since our "slope" is a vector let's also represent the two points on the line as vectors. We'll do this with position vectors. So, let $\overrightarrow{r_0}$ and $\vec{r}$ be the position vectors for P_0 and P respectively. Also, for no apparent reason, let's define $\vec{a}$ to be the vector with representation $\overrightarrow{P_0 P}$.

We now have the following sketch with all these points and vectors on it.



Now, we've shown the parallel vector, $\vec{v}$, as a position vector but it doesn't need to be a position vector. It can be anywhere, a position vector, on the line or off the line, it just needs to be parallel to the line.

Next, notice that we can write $\vec{r}$ as follows,

$$\vec{r} = \overrightarrow{r_0} + \vec{a}$$

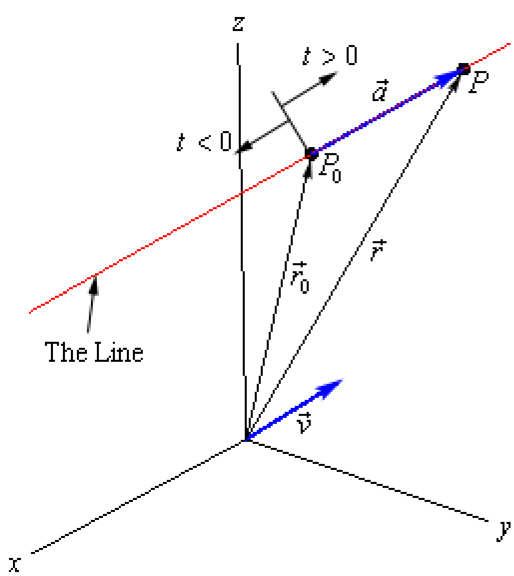
If you're not sure about this go back and check out the sketch for vector addition in the vector arithmetic section. Now, notice that the vectors $\vec{a}$ and $\vec{v}$ are parallel. Therefore there is a number, *t*, such that

$$\vec{a} = t\,\vec{v}$$

We now have,

$$\vec{r} = \vec{r_0} + t\,\vec{v} = \left\langle x_0, y_0, z_0 \right\rangle + t\left\langle a,b,c \right\rangle$$

This is called the **vector form of the equation of a line**. The only part of this equation that is not known is the *t*. Notice that $t\,\vec{v}$ will be a vector that lies along the line and it tells us how far from the original point that we should move. If *t* is positive we move away from the original point in the direction of $\vec{v}$ (right in our sketch) and if *t* is negative we move away from the original point in the opposite direction of $\vec{v}$ (left in our sketch). As *t* varies over all possible values we will completely cover the line. The following sketch shows this dependence on *t* of our sketch.



There are several other forms of the equation of a line. To get the first alternate form let's start with the vector form and do a slight rewrite.

$$\vec{r} = \left\langle x_0, y_0, z_0 \right\rangle + t\left\langle a,b,c \right\rangle$$
$$\left\langle x,y,z \right\rangle = \left\langle x_0 + ta, y_0 + tb, z_0 + tc \right\rangle$$

The only way for two vectors to be equal is for the components to be equal. In other words,

$$x = x_0 + ta$$
$$y = y_0 + tb$$
$$z = z_0 + tc$$

This set of equations is called the **parametric form of the equation of a line**. Notice as well that this is really nothing more than an extension of the parametric equations we've seen previously. The only difference is that we are now working in three dimensions instead of two dimensions.

To get a point on the line all we do is pick a *t* and plug into either form of the line. In the vector form of the line we get a position vector for the point and in the parametric form we get the actual coordinates of the point.

There is one more form of the line that we want to look at. If we assume that *a*, *b*, and *c* are all non-zero numbers we can solve each of the equations in the parametric form of the line for *t*. We can then set all of them equal to each other since *t* will be the same number in each. Doing this gives the following,

$$\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}$$

This is called the **symmetric equations of the line**.

If one of *a*, *b*, or *c* does happen to be zero we can still write down the symmetric equations. To see this let's suppose that $b=0$. In this case *t* will not exist in the parametric equation for *y* and so we will only solve the parametric equations for *x* and *z* for *t*. We then set those equal and acknowledge the parametric equation for *y* as follows,

$$\frac{x-x_0}{a}=\frac{z-z_0}{c} \qquad\qquad y=y_0$$

Let's take a look at an example.

Example 1 Write down the equation of the line that passes through the points $(2,-1,3)$ and $(1,4,-3)$. Write down all three forms of the equation of the line.

Solution

To do this we need the vector $\vec{v}$ that will be parallel to the line. This can be any vector as long as it's parallel to the line. In general, $\vec{v}$ won't lie on the line itself. However, in this case it will. All we need to do is let $\vec{v}$ be the vector that starts at the second point and ends at the first point. Since these two points are on the line the vector between them will also lie on the line and will hence be parallel to the line. So,

$$\vec{v}=\langle 1,-5,6\rangle$$

Note that the order of the points was chosen to reduce the number of minus signs in the vector. We could just have easily gone the other way.

Once we've got $\vec{v}$ there really isn't anything else to do. To use the vector form we'll need a point on the line. We've got two and so we can use either one. We'll use the first point. Here is the vector form of the line.

$$\vec{r}=\langle 2,-1,3\rangle+t\langle 1,-5,6\rangle=\langle 2+t,-1-5t,3+6t\rangle$$

Once we have this equation the other two forms follow. Here are the parametric equations of the line.

$$x = 2 + t$$
$$y = -1 - 5t$$
$$z = 3 + 6t$$

Here is the symmetric form.

$$\frac{x-2}{1} = \frac{y+1}{-5} = \frac{z-3}{6}$$

Example 2 Determine if the line that passes through the point $(0,-3,8)$ and is parallel to the line given by $x = 10 + 3t$, $y = 12t$ and $z = -3 - t$ passes through the *xz*-plane. If it does give the coordinates of that point.

Solution

To answer this we will first need to write down the equation of the line. We know a point on the line and just need a parallel vector. We know that the new line must be parallel to the line given by the parametric equations in the problem statement. That means that any vector that is parallel to the given line must also be parallel to the new line.

Now recall that in the parametric form of the line the numbers multiplied by *t* are the components of the vector that is parallel to the line. Therefore, the vector,

$$\vec{v} = \langle 3, 12, -1 \rangle$$

is parallel to the given line and so must also be parallel to the new line.

The equation of new line is then,

$$\vec{r} = \langle 0,-3,8 \rangle + t\langle 3,12,-1 \rangle = \langle 3t, -3+12t, 8-t \rangle$$

If this line passes through the *xz*-plane then we know that the *y*-coordinate of that point must be zero. So, let's set the *y* component of the equation equal to zero and see if we can solve for *t*. If we can, this will give the value of *t* for which the point will pass through the *xz*-plane.

$$-3 + 12t = 0 \qquad \Rightarrow \qquad t = \frac{1}{4}$$

So, the line does pass through the *xz*-plane. To get the complete coordinates of the point all we need to do is plug $t = \frac{1}{4}$ into any of the equations. We'll use the vector form.

$$\vec{r} = \left\langle 3\left(\frac{1}{4}\right), -3 + 12\left(\frac{1}{4}\right), 8 - \frac{1}{4} \right\rangle = \left\langle \frac{3}{4}, 0, \frac{31}{4} \right\rangle$$

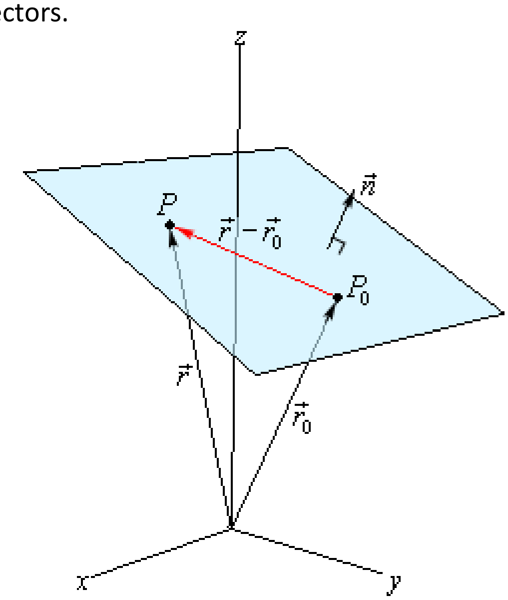
Recall that this vector is the position vector for the point on the line and so the coordinates of the point where the line will pass through the *xz*-plane are $\left(\frac{3}{4}, 0, \frac{31}{4}\right)$.

Section 1-3 : Equations of Planes

In the first section of this chapter we saw a couple of equations of planes. However, none of those equations had three variables in them and were really extensions of graphs that we could look at in two dimensions. We would like a more general equation for planes.

So, let's start by assuming that we know a point that is on the plane, $P_0 = \left(x_0, y_0, z_0\right)$. Let's also suppose that we have a vector that is orthogonal (perpendicular) to the plane, $\vec{n} = \left\langle a, b, c \right\rangle$. This vector is called the **normal vector**. Now, assume that $P = \left(x, y, z\right)$ is any point in the plane. Finally, since we are going to be working with vectors initially we'll let $\vec{r_0}$ and $\vec{r}$ be the position vectors for P_0 and P respectively.

Here is a sketch of all these vectors.



Notice that we added in the vector $\vec{r} - \vec{r_0}$ which will lie completely in the plane. Also notice that we put the normal vector on the plane, but there is actually no reason to expect this to be the case. We put it here to illustrate the point. It is completely possible that the normal vector does not touch the plane in any way.

Now, because $\vec{n}$ is orthogonal to the plane, it's also orthogonal to any vector that lies in the plane. In particular it's orthogonal to $\vec{r} - \vec{r_0}$. Recall from the Dot Product section that two orthogonal vectors will have a dot product of zero. In other words,

$$\vec{n} \bullet \left(\vec{r} - \vec{r_0}\right) = 0 \qquad \Rightarrow \qquad \vec{n} \bullet \vec{r} = \vec{n} \bullet \vec{r_0}$$

This is called the **vector equation of the plane**.

A slightly more useful form of the equations is as follows. Start with the first form of the vector equation and write down a vector for the difference.

$$\langle a,b,c\rangle \bullet \left(\langle x,y,z\rangle - \langle x_0,y_0,z_0\rangle\right) = 0$$
$$\langle a,b,c\rangle \bullet \langle x-x_0, y-y_0, z-z_0\rangle = 0$$

Now, actually compute the dot product to get,

$$a\left(x-x_0\right)+b\left(y-y_0\right)+c\left(z-z_0\right)=0$$

This is called the **scalar equation of plane**. Often this will be written as,

$$ax+by+cz=d$$

where $d = ax_0 + by_0 + cz_0$.

This second form is often how we are given equations of planes. Notice that if we are given the equation of a plane in this form we can quickly get a normal vector for the plane. A normal vector is,

$$\vec{n} = \langle a,b,c\rangle$$

Let's work a couple of examples.

Example 1 Determine the equation of the plane that contains the points $P=\left(1,-2,0\right)$, $Q=\left(3,1,4\right)$ and $R=\left(0,-1,2\right)$.

Solution

In order to write down the equation of plane we need a point (we've got three so we're cool there) and a normal vector. We need to find a normal vector. Recall however, that we saw how to do this in the Cross Product section.

We can form the following two vectors from the given points.

$$\overrightarrow{PQ} = \langle 2,3,4\rangle \qquad \overrightarrow{PR} = \langle -1,1,2\rangle$$

These two vectors will lie completely in the plane since we formed them from points that were in the plane. Notice as well that there are many possible vectors to use here, we just chose two of the possibilities.

Now, we know that the cross product of two vectors will be orthogonal to both of these vectors. Since both of these are in the plane any vector that is orthogonal to both of these will also be orthogonal to the plane. Therefore, we can use the cross product as the normal vector.

$$\vec{n} = \overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 4 \\ -1 & 1 & 2 \end{vmatrix} \begin{matrix} \vec{i} & \vec{j} \\ 2 & 3 \\ -1 & 1 \end{matrix} = 2\vec{i} - 8\vec{j} + 5\vec{k}$$

The equation of the plane is then,

$$2(x-1)-8(y+2)+5(z-0)=0$$
$$2x-8y+5z=18$$

We used *P* for the point but could have used any of the three points.

Example 2 Determine if the plane given by $-x+2z=10$ and the line given by $\vec{r}=\langle 5,2-t,10+4t\rangle$ are orthogonal, parallel or neither.

Solution

This is not as difficult a problem as it may at first appear to be. We can pick off a vector that is normal to the plane. This is $\vec{n}=\langle -1,0,2\rangle$. We can also get a vector that is parallel to the line. This is $v=\langle 0,-1,4\rangle$.

Now, if these two vectors are parallel then the line and the plane will be orthogonal. If you think about it this makes some sense. If $\vec{n}$ and $\vec{v}$ are parallel, then $\vec{v}$ is orthogonal to the plane, but $\vec{v}$ is also parallel to the line. So, if the two vectors are parallel the line and plane will be orthogonal.

Let's check this.

$$\vec{n}\times\vec{v}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -1 & 0 & 2\\ 0 & -1 & 4\end{vmatrix}\begin{matrix}\vec{i} & \vec{j}\\ -1 & 0\\ 0 & -1\end{matrix}=2\vec{i}+4\vec{j}+\vec{k}\neq\vec{0}$$

So, the vectors aren't parallel and so the plane and the line are not orthogonal.

Now, let's check to see if the plane and line are parallel. If the line is parallel to the plane then any vector parallel to the line will be orthogonal to the normal vector of the plane. In other words, if $\vec{n}$ and $\vec{v}$ are orthogonal then the line and the plane will be parallel.

Let's check this.

$$\vec{n}\bullet\vec{v}=0+0+8=8\neq 0$$

The two vectors aren't orthogonal and so the line and plane aren't parallel.

So, the line and the plane are neither orthogonal nor parallel.

Section 1-4 : Quadric Surfaces

In the previous two sections we've looked at lines and planes in three dimensions (or $\mathbb{R}^3$) and while these are used quite heavily at times in a Calculus class there are many other surfaces that are also used fairly regularly and so we need to take a look at those.

In this section we are going to be looking at quadric surfaces. Quadric surfaces are the graphs of any equation that can be put into the general form

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

where *A*, … , *J* are constants.

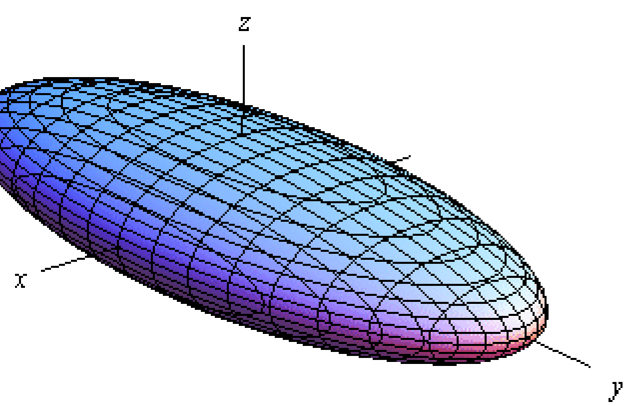
There is no way that we can possibly list all of them, but there are some standard equations so here is a list of some of the more common quadric surfaces.

Ellipsoid

Here is the general equation of an ellipsoid.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Here is a sketch of a typical ellipsoid.



If $a = b = c$ then we will have a sphere.

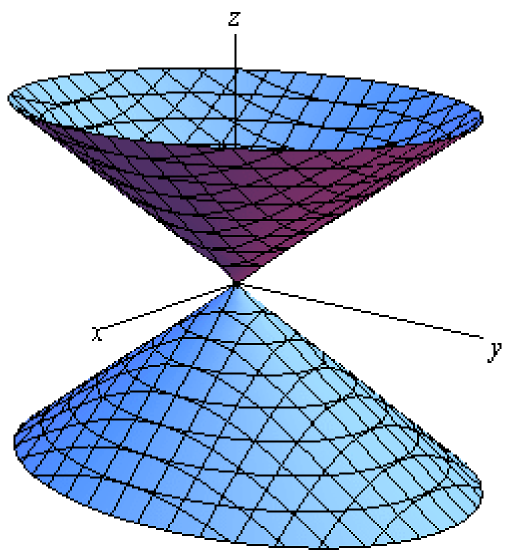
Notice that we only gave the equation for the ellipsoid that has been centered on the origin. Clearly ellipsoids don't have to be centered on the origin. However, in order to make the discussion in this section a little easier we have chosen to concentrate on surfaces that are "centered" on the origin in one way or another.

Cone

Here is the general equation of a cone.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

Here is a sketch of a typical cone.



Now, note that while we called this a cone it is more of an hour glass shape rather than what most would call a cone. Of course, the upper and the lower portion of the hour glass really are cones as we would normally think of them.

That brings up the question of what if we really did just want the upper or lower portion (*i.e.* a cone in the traditional sense)? That is easy enough to answer. All we need to do is solve the given equation for *z* as follows,

$$z^2 = c^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right) = \frac{c^2}{a^2}x^2 + \frac{c^2}{b^2}y^2 = A^2x^2 + B^2y^2 \quad \to \quad z = \pm\sqrt{A^2x^2+B^2y^2}$$

We simplified the coefficients a little to make it the equation(s) easier to deal with. Now, we know that square roots always return positive numbers and so we can then see that $z = \sqrt{A^2x^2+B^2y^2}$ will always be positive and so be the equation for just the upper portion of the "cone" above. Likewise, $z = -\sqrt{A^2x^2+B^2y^2}$ will always be negative and so be the equation of just the lower portion of the "cone" above.

Also, note that this is the equation of a cone that will open along the *z*-axis. To get the equation of a cone that opens along one of the other axes all we need to do is make a slight modification of the equation. This will be the case for the rest of the surfaces that we'll be looking at in this section as well.

In the case of a cone the variable that sits by itself on one side of the equal sign will determine the axis that the cone opens up along. For instance, a cone that opens up along the *x*-axis will have the equation,

$$\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{x^2}{a^2}$$

For most of the following surfaces we will not give the other possible formulas. We will however acknowledge how each formula needs to be changed to get a change of orientation for the surface.

Cylinder

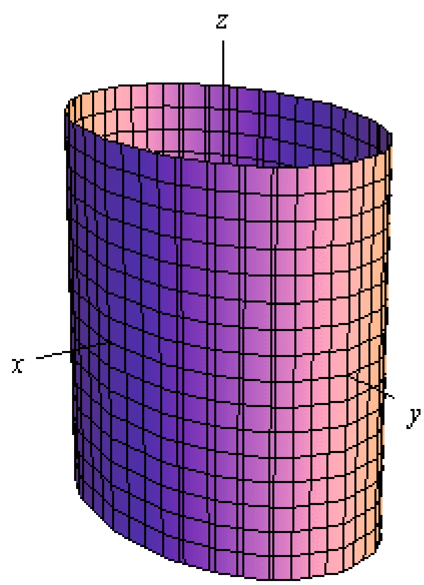
Here is the general equation of a cylinder.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

This is a cylinder whose cross section is an ellipse. If $a = b$ we have a cylinder whose cross section is a circle. We'll be dealing with those kinds of cylinders more than the general form so the equation of a cylinder with a circular cross section is,

$$x^2 + y^2 = r^2$$

Here is a sketch of typical cylinder with an ellipse cross section.



The cylinder will be centered on the axis corresponding to the variable that does not appear in the equation.

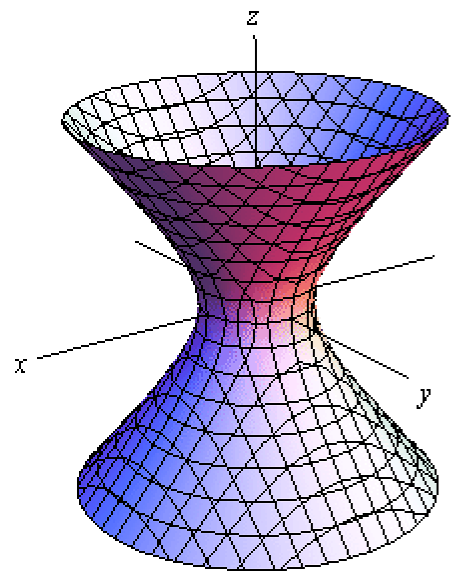
Be careful to not confuse this with a circle. In two dimensions it is a circle, but in three dimensions it is a cylinder.

Hyperboloid of One Sheet

Here is the equation of a hyperboloid of one sheet.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Here is a sketch of a typical hyperboloid of one sheet.



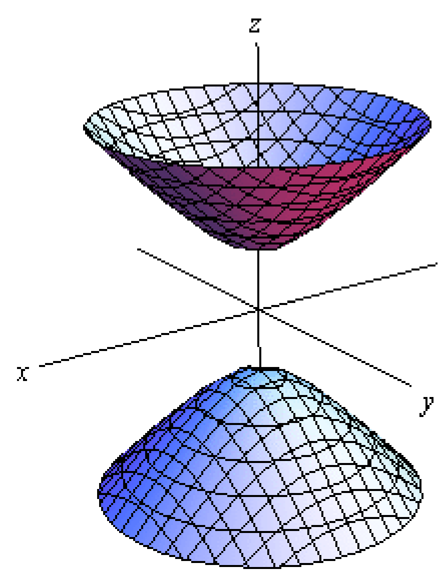
The variable with the negative in front of it will give the axis along which the graph is centered.

Hyperboloid of Two Sheets

Here is the equation of a hyperboloid of two sheets.

$$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

Here is a sketch of a typical hyperboloid of two sheets.



The variable with the positive in front of it will give the axis along which the graph is centered.

Notice that the only difference between the hyperboloid of one sheet and the hyperboloid of two sheets is the signs in front of the variables. They are exactly the opposite signs.

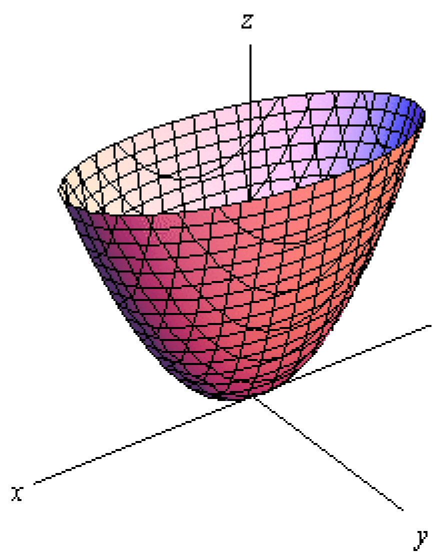
Also note that just as we could do with cones, if we solve the equation for *z* the positive portion will give the equation for the upper part of this while the negative portion will give the equation for the lower part of this.

Elliptic Paraboloid
Here is the equation of an elliptic paraboloid.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z}{c}$$

As with cylinders this has a cross section of an ellipse and if $a=b$ it will have a cross section of a circle. When we deal with these we'll generally be dealing with the kind that have a circle for a cross section.

Here is a sketch of a typical elliptic paraboloid.

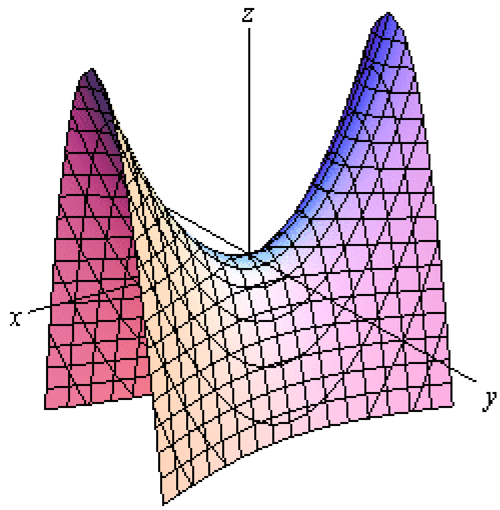


In this case the variable that isn't squared determines the axis upon which the paraboloid opens up. Also, the sign of *c* will determine the direction that the paraboloid opens. If *c* is positive then it opens up and if *c* is negative then it opens down.

Hyperbolic Paraboloid
Here is the equation of a hyperbolic paraboloid.

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{z}{c}$$

Here is a sketch of a typical hyperbolic paraboloid.



These graphs are vaguely saddle shaped and as with the elliptic paraboloid the sign of *c* will determine the direction in which the surface "opens up". The graph above is shown for *c* positive.

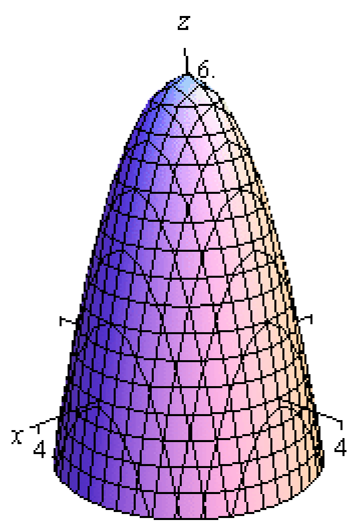
With both of the types of paraboloids discussed above note that the surface can be easily moved up or down by adding/subtracting a constant from the left side.

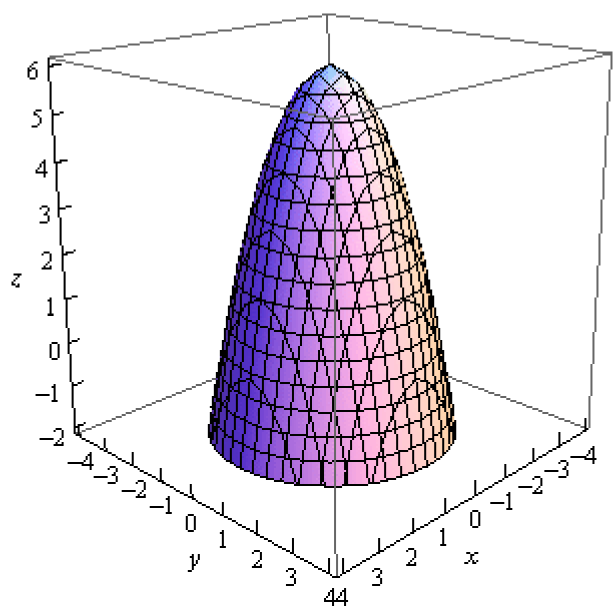
For instance

$$z = -x^2 - y^2 + 6$$

is an elliptic paraboloid that opens downward (be careful, the "-" is on the *x* and *y* instead of the *z*) and starts at $z = 6$ instead of $z = 0$.

Here are a couple of quick sketches of this surface.



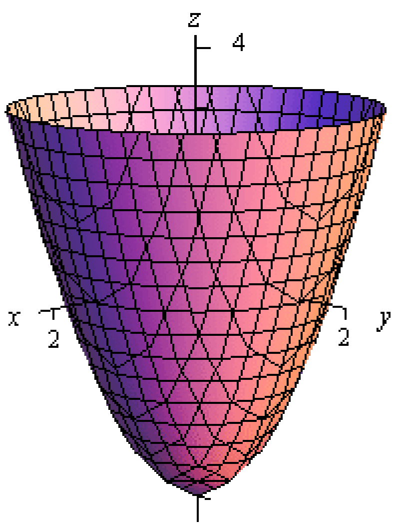


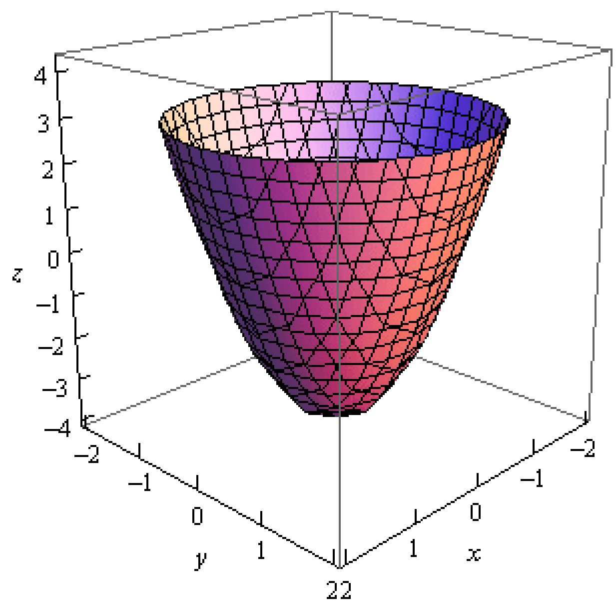
Note that we've given two forms of the sketch here. The sketch on the left has the standard set of axes but it is difficult to see the numbers on the axis. The sketch on the right has been "boxed" and this makes it easier to see the numbers to give a sense of perspective to the sketch. In most sketches that actually involve numbers on the axis system we will give both sketches to help get a feel for what the sketch looks like.

Section 1-5 : Functions of Several Variables

In this section we want to go over some of the basic ideas about functions of more than one variable.

First, remember that graphs of functions of two variables, $z = f\left(x, y\right)$ are surfaces in three dimensional space. For example, here is the graph of $z = 2x^2 + 2y^2 - 4$.





This is an elliptic paraboloid and is an example of a quadric surface. We saw several of these in the previous section. We will be seeing quadric surfaces fairly regularly later on in Calculus III.

Another common graph that we'll be seeing quite a bit in this course is the graph of a plane. We have a convention for graphing planes that will make them a little easier to graph and hopefully visualize.

Recall that the equation of a plane is given by

$$ax + by + cz = d$$

or if we solve this for *z* we can write it in terms of function notation. This gives,

$$f\left(x, y\right) = Ax + By + D$$

To graph a plane we will generally find the intersection points with the three axes and then graph the triangle that connects those three points. This triangle will be a portion of the plane and it will give us a fairly decent idea on what the plane itself should look like. For example, let's graph the plane given by,

$$f\left(x, y\right) = 12 - 3x - 4y$$

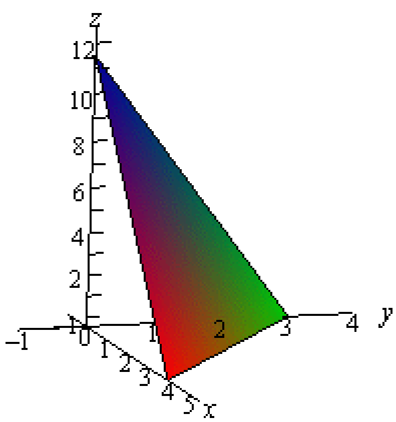
For purposes of graphing this it would probably be easier to write this as,

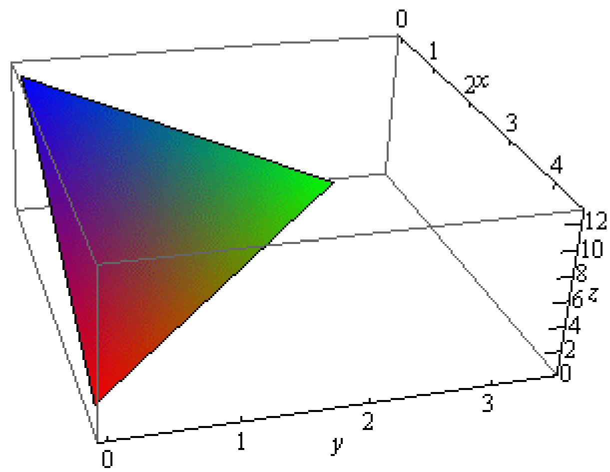
$$z = 12 - 3x - 4y \qquad \Rightarrow \qquad 3x + 4y + z = 12$$

Now, each of the intersection points with the three main coordinate axes is defined by the fact that two of the coordinates are zero. For instance, the intersection with the *z*-axis is defined by $x = y = 0$. So, the three intersection points are,

$$\begin{aligned} x-\text{axis} &: \left(4,0,0\right) \\ y-\text{axis} &: \left(0,3,0\right) \\ z-\text{axis} &: \left(0,0,12\right) \end{aligned}$$

Here is the graph of the plane.





Now, to extend this out, graphs of functions of the form $w = f\left(x, y, z\right)$ would be four dimensional surfaces. Of course, we can't graph them, but it doesn't hurt to point this out.

We next want to talk about the domains of functions of more than one variable. Recall that domains of functions of a single variable, $y = f\left(x\right)$, consisted of all the values of *x* that we could plug into the function and get back a real number. Now, if we think about it, this means that the domain of a function of a single variable is an interval (or intervals) of values from the number line, or one dimensional space.

The domain of functions of two variables, $z = f\left(x, y\right)$, are regions from two dimensional space and consist of all the coordinate pairs, $\left(x, y\right)$, that we could plug into the function and get back a real number.

Example 1 Determine the domain of each of the following.

(a) $f(x,y) = \sqrt{x+y}$

(b) $f(x,y) = \sqrt{x} + \sqrt{y}$

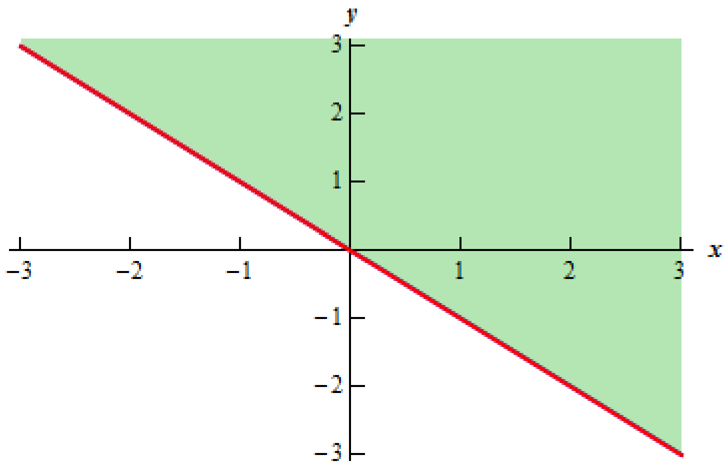
(c) $f(x,y) = \ln\left(9 - x^2 - 9y^2\right)$

Solution

(a) In this case we know that we can't take the square root of a negative number so this means that we must require,

$$x + y \geq 0$$

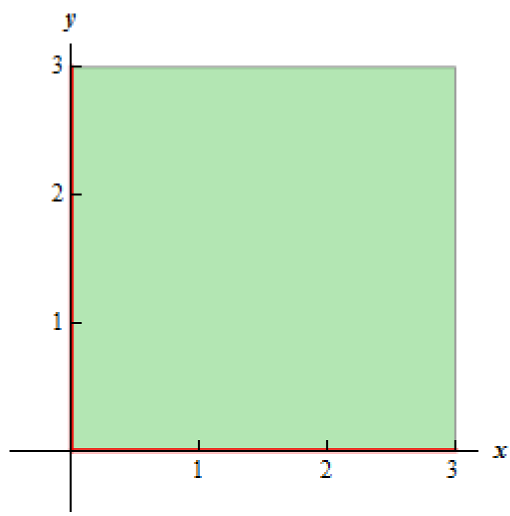
Here is a sketch of the graph of this region.



(b) This function is different from the function in the previous part. Here we must require that,

$$x \geq 0 \quad \text{and} \quad y \geq 0$$

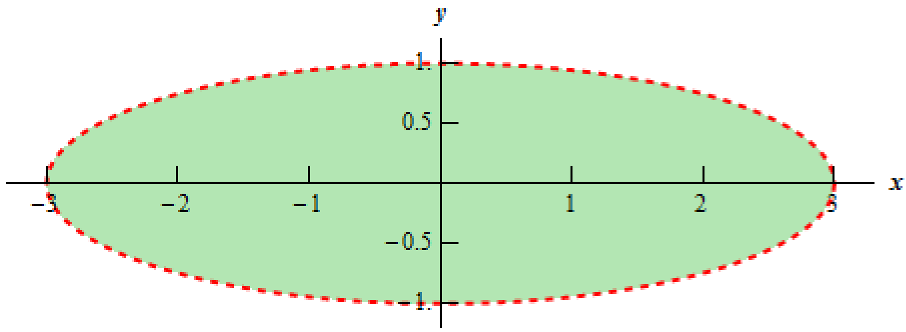
and they really do need to be separate inequalities. There is one for each square root in the function. Here is the sketch of this region.



(c) In this final part we know that we can't take the logarithm of a negative number or zero. Therefore, we need to require that,

$$9 - x^2 - 9y^2 > 0 \qquad \Rightarrow \qquad \frac{x^2}{9} + y^2 < 1$$

and upon rearranging we see that we need to stay interior to an ellipse for this function. Here is a sketch of this region.



Note that domains of functions of three variables, $w = f(x, y, z)$, will be regions in three dimensional space.

Example 2 Determine the domain of the following function,

$$f(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2 - 16}}$$

Solution

In this case we have to deal with the square root and division by zero issues. These will require,

$$x^2 + y^2 + z^2 - 16 > 0 \qquad \Rightarrow \qquad x^2 + y^2 + z^2 > 16$$

So, the domain for this function is the set of points that lies completely outside a sphere of radius 4 centered at the origin.

The next topic that we should look at is that of **level curves** or **contour curves**. The level curves of the function $z = f(x, y)$ are two dimensional curves we get by setting $z = k$, where *k* is any number. So the equations of the level curves are $f(x, y) = k$. Note that sometimes the equation will be in the form $f(x, y, z) = 0$ and in these cases the equations of the level curves are $f(x, y, k) = 0$.

You've probably seen level curves (or contour curves, whatever you want to call them) before. If you've ever seen the elevation map for a piece of land, this is nothing more than the contour curves for the function that gives the elevation of the land in that area. Of course, we probably don't have the function that gives the elevation, but we can at least graph the contour curves.

Let's do a quick example of this.

Example 3 Identify the level curves of $f(x,y) = \sqrt{x^2 + y^2}$. Sketch a few of them.

Solution

First, for the sake of practice, let's identify what this surface given by $f(x,y)$ is. To do this let's rewrite it as,

$$z = \sqrt{x^2 + y^2}$$

Recall from the Quadric Surfaces section that this the upper portion of the "cone" (or hour glass shaped surface).

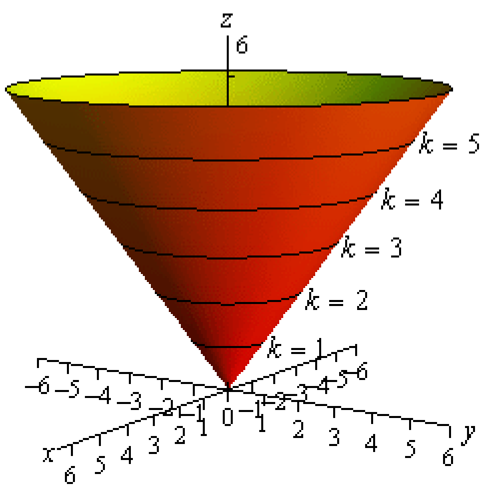
Note that this was not required for this problem. It was done for the practice of identifying the surface and this may come in handy down the road.

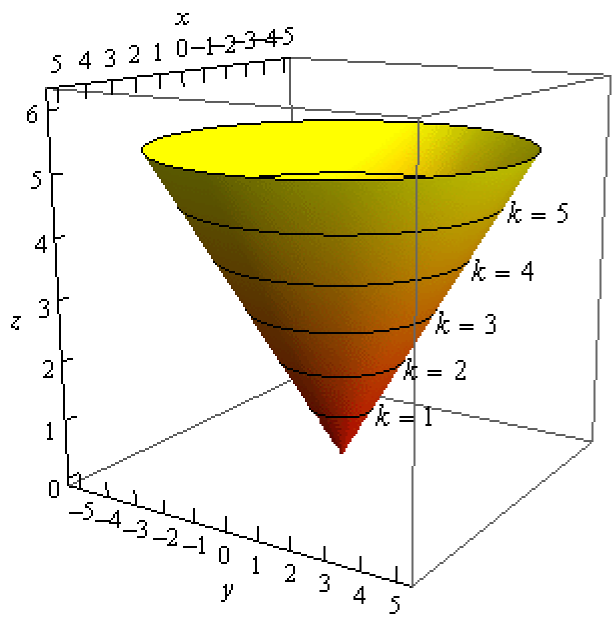
Now on to the real problem. The level curves (or contour curves) for this surface are given by the equation are found by substituting $z = k$. In the case of our example this is,

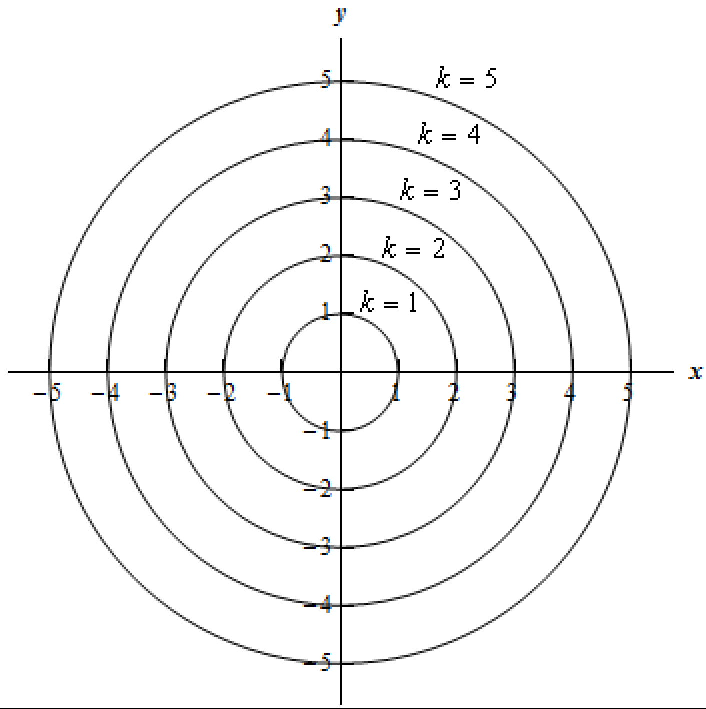
$$k = \sqrt{x^2 + y^2} \qquad \Rightarrow \qquad x^2 + y^2 = k^2$$

where *k* is any number. So, in this case, the level curves are circles of radius *k* with center at the origin.

We can graph these in one of two ways. We can either graph them on the surface itself or we can graph them in a two dimensional axis system. Here is each graph for some values of *k*.







Note that we can think of contours in terms of the intersection of the surface that is given by $z = f(x, y)$ and the plane $z = k$. The contour will represent the intersection of the surface and the plane.

For functions of the form $f(x, y, z)$ we will occasionally look at **level surfaces**. The equations of level surfaces are given by $f(x, y, z) = k$ where *k* is any number.

The final topic in this section is that of **traces**. In some ways these are similar to contours. As noted above we can think of contours as the intersection of the surface given by $z = f(x, y)$ and the plane $z = k$. Traces of surfaces are curves that represent the intersection of the surface and the plane given by $x = a$ or $y = b$.

Let's take a quick look at an example of traces.

Example 4 Sketch the traces of $f(x, y) = 10 - 4x^2 - y^2$ for the plane $x = 1$ and $y = 2$.

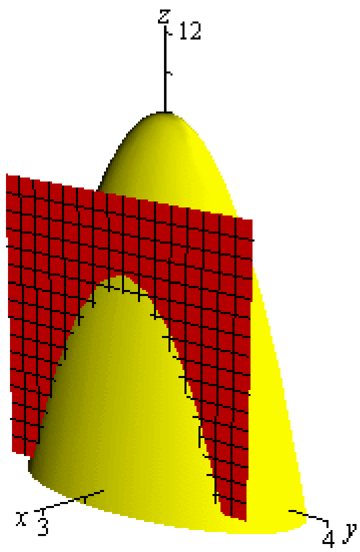
Solution

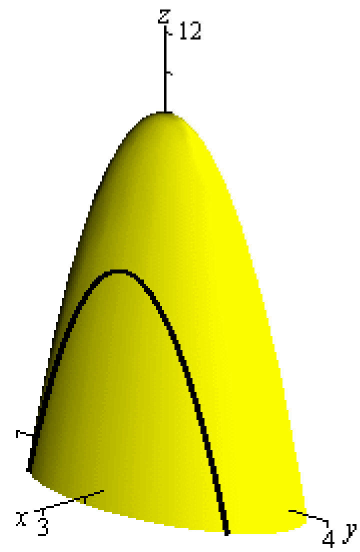
We'll start with $x = 1$. We can get an equation for the trace by plugging $x = 1$ into the equation. Doing this gives,

$$z = f(1, y) = 10 - 4(1)^2 - y^2 \qquad \Rightarrow \qquad z = 6 - y^2$$

and this will be graphed in the plane given by $x = 1$.

Below are two graphs. The graph on the left is a graph showing the intersection of the surface and the plane given by $x = 1$. On the right is a graph of the surface and the trace that we are after in this part.

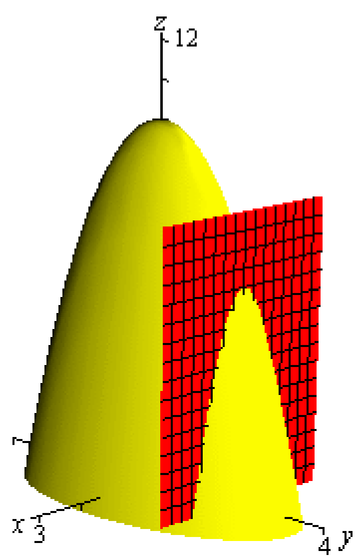


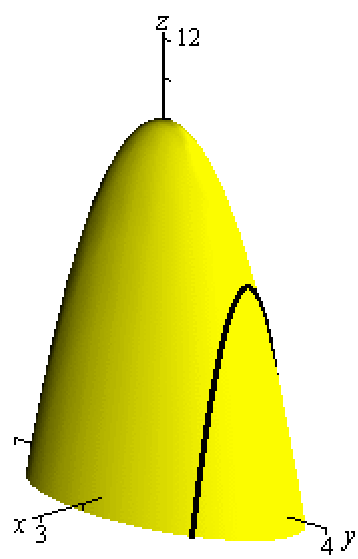


For $y = 2$ we will do pretty much the same thing that we did with the first part. Here is the equation of the trace,

$$z = f\left(x,2\right) = 10 - 4x^2 - \left(2\right)^2 \qquad \Rightarrow \qquad z = 6 - 4x^2$$

and here are the sketches for this case.





Section 1-6 : Vector Functions

We first saw vector functions back when we were looking at the Equation of Lines. In that section we talked about them because we wrote down the equation of a line in $\mathbb{R}^3$ in terms of a **vector function** (sometimes called a **vector-valued function**). In this section we want to look a little closer at them and we also want to look at some vector functions in $\mathbb{R}^3$ other than lines.

A vector function is a function that takes one or more variables and returns a vector. We'll spend most of this section looking at vector functions of a single variable as most of the places where vector functions show up here will be vector functions of single variables. We will however briefly look at vector functions of two variables at the end of this section.

A vector functions of a single variable in $\mathbb{R}^2$ and $\mathbb{R}^3$ have the form,

$$\vec{r}(t) = \langle f(t), g(t) \rangle \qquad\qquad \vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

respectively, where $f(t)$, $g(t)$ and $h(t)$ are called the **component functions**.

The main idea that we want to discuss in this section is that of graphing and identifying the graph given by a vector function. Before we do that however, we should talk briefly about the domain of a vector function. The **domain** of a vector function is the set of all *t*'s for which all the component functions are defined.

Example 1 Determine the domain of the following function.

$$\vec{r}(t) = \left\langle \cos t, \ln(4-t), \sqrt{t+1} \right\rangle$$

Solution

The first component is defined for all *t*'s. The second component is only defined for $t < 4$. The third component is only defined for $t \geq -1$. Putting all of these together gives the following domain.

$$[-1, 4)$$

This is the largest possible interval for which all three components are defined.

Let's now move into looking at the graph of vector functions. In order to graph a vector function all we do is think of the vector returned by the vector function as a position vector for points on the graph. Recall that a position vector, say $\vec{v} = \langle a, b, c \rangle$, is a vector that starts at the origin and ends at the point (a, b, c).

So, in order to sketch the graph of a vector function all we need to do is plug in some values of *t* and then plot points that correspond to the resulting position vector we get out of the vector function.

Because it is a little easier to visualize things we'll start off by looking at graphs of vector functions in $\mathbb{R}^2$.

Example 2 Sketch the graph of each of the following vector functions.

(a) $\vec{r}(t) = \langle t, 1 \rangle$

(b) $\vec{r}(t) = \langle t, t^3 - 10t + 7 \rangle$

Solution

(a) $\vec{r}(t) = \langle t, 1 \rangle$

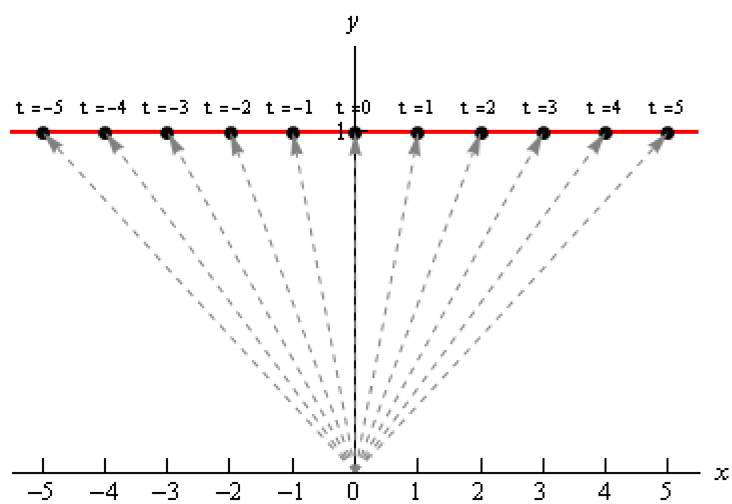
Okay, the first thing that we need to do is plug in a few values of *t* and get some position vectors. Here are a few,

$$\vec{r}(-3) = \langle -3, 1 \rangle \qquad \vec{r}(-1) = \langle -1, 1 \rangle \qquad \vec{r}(2) = \langle 2, 1 \rangle \qquad \vec{r}(5) = \langle 5, 1 \rangle$$

So, what this tells us is that the following points are all on the graph of this vector function.

$$(-3, 1) \qquad (-1, 1) \qquad (2, 1) \qquad (5, 1)$$

Here is a sketch of this vector function.



In this sketch we've included many more evaluations than just those above. Also note that we've put in the position vectors (in gray and dashed) so you can see how all this is working. Note however, that in practice the position vectors are generally not included in the sketch.

In this case it looks like we've got the graph of the line $y = 1$.

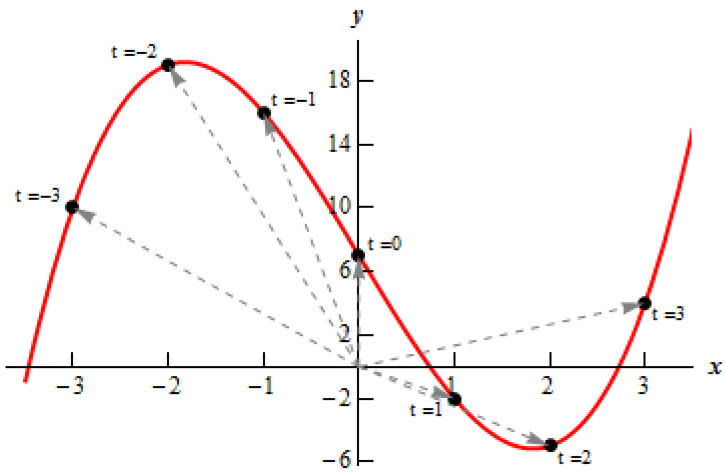
(b) $\vec{r}(t) = \langle t, t^3 - 10t + 7 \rangle$

Here are a couple of evaluations for this vector function.

$$\vec{r}(-3) = \langle -3, 10 \rangle \qquad \vec{r}(-1) = \langle -1, 16 \rangle \qquad \vec{r}(1) = \langle 1, -2 \rangle \qquad \vec{r}(3) = \langle 3, 4 \rangle$$

So, we've got a few points on the graph of this function. However, unlike the first part this isn't really going to be enough points to get a good idea of this graph. In general, it can take quite a few function evaluations to get an idea of what the graph is and it's usually easier to use a computer to do the graphing.

Here is a sketch of this graph. We've put in a few vectors/evaluations to illustrate them, but the reality is that we did have to use a computer to get a good sketch here.



Both of the vector functions in the above example were in the form,

$$\vec{r}(t) = \langle t, g(t) \rangle$$

and what we were really sketching is the graph of $y = g(x)$ as you probably caught onto. Let's graph a couple of other vector functions that do not fall into this pattern.

Example 3 Sketch the graph of each of the following vector functions.

(a) $\vec{r}(t) = \langle 6\cos t, 3\sin t \rangle$

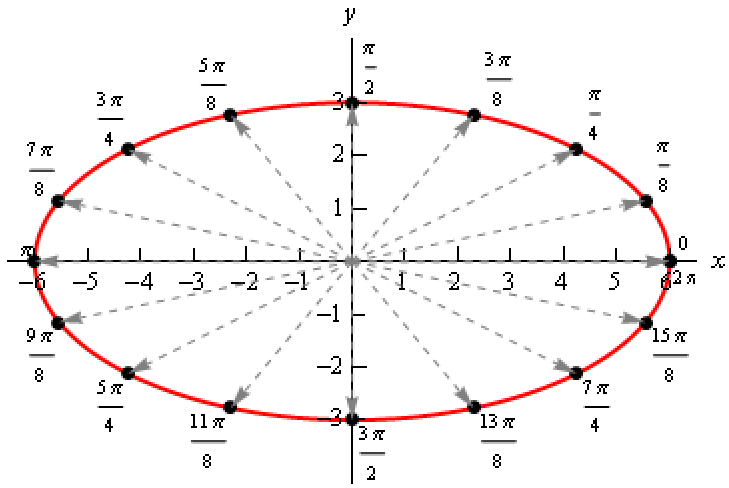
(b) $\vec{r}(t) = \langle t - 2\sin t, t^2 \rangle$

Solution

As we saw in the last part of the previous example it can really take quite a few function evaluations to really be able to sketch the graph of a vector function. Because of that we'll be skipping all the function evaluations here and just giving the graph. The main point behind this set of examples is to not get you too locked into the form we were looking at above. The first part will also lead to an important idea that we'll discuss after this example.

So, with that said here are the sketches of each of these.

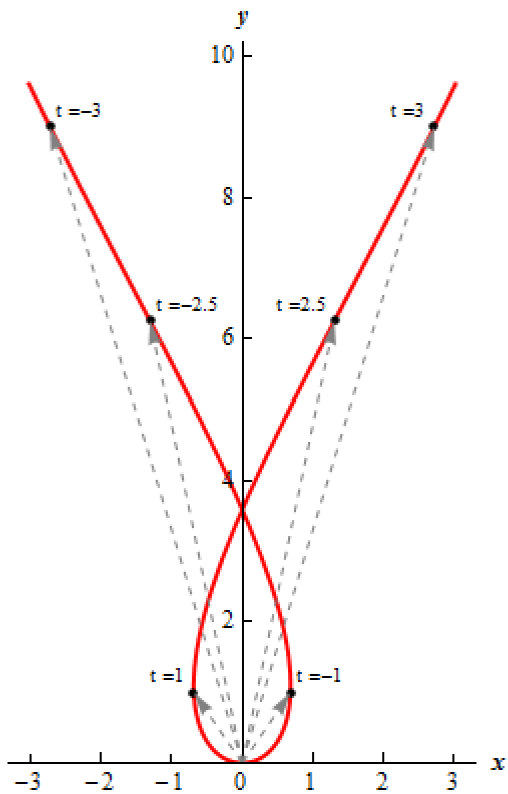
(a) $\vec{r}(t) = \langle 6\cos t, 3\sin t \rangle$



So, in this case it looks like we've got an ellipse.

(b) $\vec{r}(t) = \langle t - 2\sin t, t^2 \rangle$

Here's the sketch for this vector function.



Before we move on to vector functions in $\mathbb{R}^3$ let's go back and take a quick look at the first vector function we sketched in the previous example, $\vec{r}(t) = \langle 6\cos t, 3\sin t \rangle$. The fact that we got an ellipse here should not come as a surprise to you. We know that the first component function gives the *x* coordinate and the second component function gives the *y* coordinates of the point that we graph. If we strip these out to make this clear we get,

$$x = 6\cos t \qquad\qquad y = 3\sin t$$

This should look familiar to you. Back when we were looking at Parametric Equations we saw that this was nothing more than one of the sets of parametric equations that gave an ellipse.

This is an important idea in the study of vector functions. Any vector function can be broken down into a set of parametric equations that represent the same graph. In general, the two dimensional vector function, $\vec{r}\left(t\right) = \left\langle f\left(t\right), g\left(t\right)\right\rangle$, can be broken down into the parametric equations,

$$x = f\left(t\right) \qquad\qquad y = g\left(t\right)$$

Likewise, a three dimensional vector function, $\vec{r}\left(t\right) = \left\langle f\left(t\right), g\left(t\right), h\left(t\right)\right\rangle$, can be broken down into the parametric equations,

$$x = f\left(t\right) \qquad y = g\left(t\right) \qquad z = h\left(t\right)$$

Do not get too excited about the fact that we're now looking at parametric equations in $\mathbb{R}^3$. They work in exactly the same manner as parametric equations in $\mathbb{R}^2$ which we're used to dealing with already. The only difference is that we now have a third component.

Let's take a look at a couple of graphs of vector functions.

Example 4 Sketch the graph of the following vector function.

$$\vec{r}\left(t\right) = \left\langle 2 - 4t, -1 + 5t, 3 + t\right\rangle$$

Solution

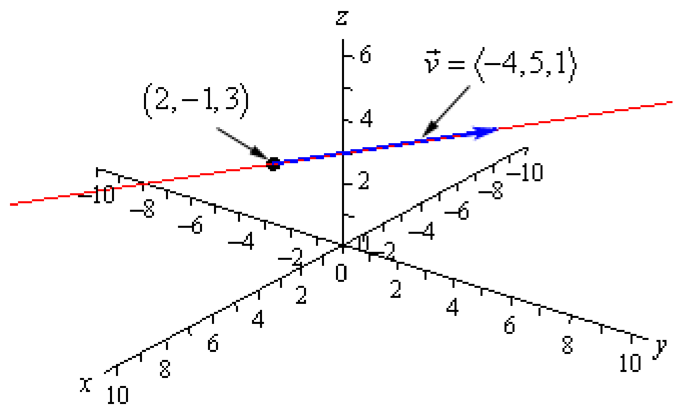
Notice that this is nothing more than a line. It might help if we rewrite it a little.

$$\vec{r}\left(t\right) = \left\langle 2, -1, 3\right\rangle + t\left\langle -4, 5, 1\right\rangle$$

In this form we can see that this is the equation of a line that goes through the point $\left(2, -1, 3\right)$ and is parallel to the vector $\vec{v} = \left\langle -4, 5, 1\right\rangle$.

To graph this line all that we need to do is plot the point and then sketch in the parallel vector. In order to get the sketch will assume that the vector is on the line and will start at the point in the line. To sketch in the line all we do this is extend the parallel vector into a line.

Here is a sketch.



Example 5 Sketch the graph of the following vector function.

$$\vec{r}(t) = \langle 2\cos t, 2\sin t, 3 \rangle$$

Solution

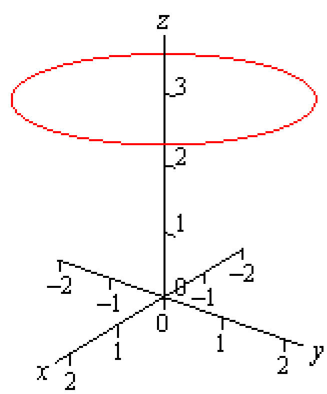
In this case to see what we've got for a graph let's get the parametric equations for the curve.

$$x = 2\cos t \qquad\qquad y = 2\sin t \qquad\qquad z = 3$$

If we ignore the *z* equation for a bit we'll recall (hopefully) that the parametric equations for *x* and *y* give a circle of radius 2 centered on the origin (or about the *z*-axis since we are in $\mathbb{R}^3$).

Now, all the parametric equations here tell us is that no matter what is going on in the graph all the *z* coordinates must be 3. So, we get a circle of radius 2 centered on the *z*-axis and at the level of $z = 3$.

Here is a sketch.



Note that it is very easy to modify the above vector function to get a circle centered on the *x* or *y*-axis as well. For instance,

$$\vec{r}(t) = \langle 10\sin t, -3, 10\cos t \rangle$$

will be a circle of radius 10 centered on the *y*-axis and at $y = -3$. In other words, as long as two of the terms are a sine and a cosine (with the same coefficient) and the other is a fixed number then we will have a circle that is centered on the axis that is given by the fixed number.

Let's take a look at a modification of this.

Example 6 Sketch the graph of the following vector function.

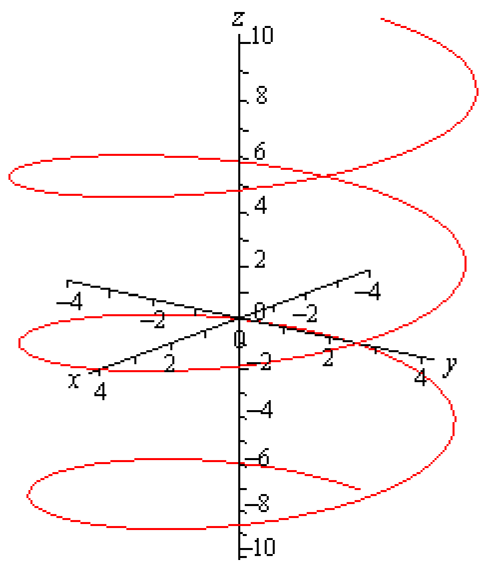
$$\vec{r}(t) = \langle 4\cos t, 4\sin t, t \rangle$$

Solution

If this one had a constant in the *z* component we would have another circle. However, in this case we don't have a constant. Instead we've got a *t* and that will change the curve. However, because the *x* and *y* component functions are still a circle in parametric equations our curve should have a circular nature to it in some way.

In fact, the only change is in the *z* component and as *t* increases the *z* coordinate will increase. Also, as *t* increases the *x* and *y* coordinates will continue to form a circle centered on the *z*-axis. Putting these two ideas together tells us that as we increase *t* the circle that is being traced out in the *x* and *y* directions should also be rising.

Here is a sketch of this curve.



So, we've got a helix (or spiral, depending on what you want to call it) here.

As with circles the component that has the *t* will determine the axis that the helix rotates about. For instance,

$$\vec{r}(t) = \langle t, 6\cos t, 6\sin t \rangle$$

is a helix that rotates around the *x*-axis.

Also note that if we allow the coefficients on the sine and cosine for both the circle and helix to be different we will get ellipses.

For example,

$$\vec{r}(t) = \langle 9\cos t, t, 2\sin t \rangle$$

will be a helix that rotates about the *y*-axis and is in the shape of an ellipse.

There is a nice formula that we should derive before moving onto vector functions of two variables.

Example 7 Determine the vector equation for the line segment starting at the point $P = (x_1, y_1, z_1)$ and ending at the point $Q = (x_2, y_2, z_2)$.

Solution

It is important to note here that we only want the equation of the line segment that starts at *P* and ends at *Q*. We don't want any other portion of the line and we do want the direction of the line segment preserved as we increase *t*. With all that said, let's not worry about that and just find the vector equation of the line that passes through the two points. Once we have this we will be able to get what we're after.

So, we need a point on the line. We've got two and we will use *P*. We need a vector that is parallel to the line and since we've got two points we can find the vector between them. This vector will lie on the line and hence be parallel to the line. Also, let's remember that we want to preserve the starting and ending point of the line segment so let's construct the vector using the same "orientation".

$$\vec{v} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

Using this vector and the point *P* we get the following vector equation of the line.

$$\vec{r}(t) = \langle x_1, y_1, z_1 \rangle + t\langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

While this is the vector equation of the line, let's rewrite the equation slightly.

$$\begin{aligned}\vec{r}(t) &= \langle x_1, y_1, z_1 \rangle + t\langle x_2, y_2, z_2 \rangle - t\langle x_1, y_1, z_1 \rangle \\ &= (1-t)\langle x_1, y_1, z_1 \rangle + t\langle x_2, y_2, z_2 \rangle\end{aligned}$$

This is the equation of the line that contains the points *P* and *Q*. We of course just want the line segment that starts at *P* and ends at *Q*. We can get this by simply restricting the values of *t*.

Notice that

$$\vec{r}(0) = \langle x_1, y_1, z_1 \rangle \qquad\qquad \vec{r}(1) = \langle x_2, y_2, z_2 \rangle$$

So, if we restrict *t* to be between zero and one we will cover the line segment and we will start and end at the correct point.

So, the vector equation of the line segment that starts at $P = (x_1, y_1, z_1)$ and ends at $Q = (x_2, y_2, z_2)$ is,

$$\vec{r}(t) = (1-t)\langle x_1, y_1, z_1 \rangle + t\langle x_2, y_2, z_2 \rangle \qquad\qquad 0 \le t \le 1$$

As noted briefly at the beginning of this section we can also have vector functions of two variables. In these cases the graphs of vector function of two variables are surfaces. So, to make sure that we don't forget that let's work an example with that as well.

Example 8 Identify the surface that is described by $\vec r\left(x,y\right) = x\,\vec i + y\,\vec j + \left(x^2 + y^2\right)\vec k$.

Solution
First, notice that in this case the vector function will in fact be a function of two variables. This will always be the case when we are using vector functions to represent surfaces.

To identify the surface let's go back to parametric equations.

$$x = x \qquad\qquad y = y \qquad\qquad z = x^2 + y^2$$

The first two are really only acknowledging that we are picking *x* and *y* for free and then determining *z* from our choices of these two. The last equation is the one that we want. We should recognize that function from the section on quadric surfaces. The third equation is the equation of an elliptic paraboloid and so the vector function represents an elliptic paraboloid.

As a final topic for this section let's generalize the idea from the previous example and note that given any function of one variable ($y = f\left(x\right)$ or $x = h\left(y\right)$) or any function of two variables ($z = g\left(x,y\right)$, $x = g\left(y,z\right)$, or $y = g\left(x,z\right)$) we can always write down a vector form of the equation.

For a function of one variable this will be,

$$\vec r\left(x\right) = x\vec i + f\left(x\right)\vec j \qquad\qquad \vec r\left(y\right) = h\left(y\right)\vec i + y\,\vec j$$

and for a function of two variables the vector form will be,

$$\vec r\left(x,y\right) = x\vec i + y\vec j + g\left(x,y\right)\vec k \qquad\qquad \vec r\left(y,z\right) = g\left(y,z\right)\vec i + y\vec j + z\,\vec k$$

$$\vec r\left(x,z\right) = x\vec i + g\left(x,z\right)\vec j + z\,\vec k$$

depending upon the original form of the function.

For example, the hyperbolic paraboloid $y = 2x^2 - 5z^2$ can be written as the following vector function.

$$\vec r\left(x,z\right) = x\,\vec i + \left(2x^2 - 5z^2\right)\vec j + z\,\vec k$$

This is a fairly important idea and we will be doing quite a bit of this kind of thing in Calculus III.

Section 1-7 : Calculus with Vector Functions

In this section we need to talk briefly about limits, derivatives and integrals of vector functions. As you will see, these behave in a fairly predictable manner. We will be doing all of the work in $\mathbb{R}^3$ but we can naturally extend the formulas/work in this section to $\mathbb{R}^n$ (*i.e.* *n*-dimensional space).

Let's start with limits. Here is the limit of a vector function.

$$\begin{aligned}\lim_{t\to a}\vec{r}(t) &= \lim_{t\to a}\left\langle f(t), g(t), h(t)\right\rangle \\ &= \left\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t)\right\rangle \\ &= \lim_{t\to a} f(t)\,\vec{i} + \lim_{t\to a} g(t)\,\vec{j} + \lim_{t\to a} h(t)\,\vec{k}\end{aligned}$$

So, all that we do is take the limit of each of the component's functions and leave it as a vector.

Example 1 Compute $\lim\limits_{t\to 1}\vec{r}(t)$ where $\vec{r}(t) = \left\langle t^3, \dfrac{\sin(3t-3)}{t-1}, \mathbf{e}^{2t}\right\rangle$.

Solution

There really isn't all that much to do here.

$$\begin{aligned}\lim_{t\to 1}\vec{r}(t) &= \left\langle \lim_{t\to 1} t^3, \lim_{t\to 1}\frac{\sin(3t-3)}{t-1}, \lim_{t\to 1}\mathbf{e}^{2t}\right\rangle \\ &= \left\langle \lim_{t\to 1} t^3, \lim_{t\to 1}\frac{3\cos(3t-3)}{1}, \lim_{t\to 1}\mathbf{e}^{2t}\right\rangle \\ &= \left\langle 1, 3, \mathbf{e}^2\right\rangle\end{aligned}$$

Notice that we had to use L'Hospital's Rule on the *y* component.

Now let's take care of derivatives and after seeing how limits work it shouldn't be too surprising that we have the following for derivatives.

$$\vec{r}\,'(t) = \left\langle f'(t), g'(t), h'(t)\right\rangle = f'(t)\,\vec{i} + g'(t)\,\vec{j} + h'(t)\,\vec{k}$$

Example 2 Compute $\vec{r}\,'(t)$ for $\vec{r}(t) = t^6\,\vec{i} + \sin(2t)\,\vec{j} - \ln(t+1)\,\vec{k}$.

Solution

There really isn't too much to this problem other than taking the derivatives.

$$\vec{r}\,'(t) = 6t^5\,\vec{i} + 2\cos(2t)\,\vec{j} - \frac{1}{t+1}\vec{k}$$

Most of the basic facts that we know about derivatives still hold however, just to make it clear here are some facts about derivatives of vector functions.

Facts

$$\frac{d}{dt}\left(\vec{u}+\vec{v}\right)=\vec{u}'+\vec{v}'$$
$$\left(c\vec{u}\right)'=c\,\vec{u}'$$
$$\frac{d}{dt}\left(f\left(t\right)\vec{u}\left(t\right)\right)=f'\left(t\right)\vec{u}\left(t\right)+f\left(t\right)\vec{u}'$$
$$\frac{d}{dt}\left(\vec{u}\bullet\vec{v}\right)=\vec{u}'\bullet\vec{v}+\vec{u}\bullet\vec{v}'$$
$$\frac{d}{dt}\left(\vec{u}\times\vec{v}\right)=\vec{u}'\times\vec{v}+\vec{u}\times\vec{v}'$$
$$\frac{d}{dt}\left(\vec{u}\left(f\left(t\right)\right)\right)=f'\left(t\right)\vec{u}'\left(f\left(t\right)\right)$$

There is also one quick definition that we should get out of the way so that we can use it when we need to.

A **smooth curve** is any curve for which $\vec{r}'\left(t\right)$ is continuous and $\vec{r}'\left(t\right)\neq 0$ for any *t* except possibly at the endpoints. A helix is a smooth curve, for example.

Finally, we need to discuss integrals of vector functions. Using both limits and derivatives as a guide it shouldn't be too surprising that we also have the following for integration for indefinite integrals

$$\int \vec{r}\left(t\right)dt=\left\langle \int f\left(t\right)dt,\int g\left(t\right)dt,\int h\left(t\right)dt\right\rangle+\vec{c}$$
$$\int \vec{r}\left(t\right)dt=\int f\left(t\right)dt\;\vec{i}+\int g\left(t\right)dt\;\vec{j}+\int h\left(t\right)dt\;\vec{k}+\vec{c}$$

and the following for definite integrals.

$$\int_a^b \vec{r}\left(t\right)dt=\left\langle \int_a^b f\left(t\right)dt,\int_a^b g\left(t\right)dt,\int_a^b h\left(t\right)dt\right\rangle$$
$$\int_a^b \vec{r}\left(t\right)dt=\int_a^b f\left(t\right)dt\;\vec{i}+\int_a^b g\left(t\right)dt\;\vec{j}+\int_a^b h\left(t\right)dt\;\vec{k}$$

With the indefinite integrals we put in a constant of integration to make sure that it was clear that the constant in this case needs to be a vector instead of a regular constant.

Also, for the definite integrals we will sometimes write it as follows,

$$\int_a^b \vec{r}\left(t\right)dt=\left(\left\langle \int f\left(t\right)dt,\int g\left(t\right)dt,\int h\left(t\right)dt\right\rangle\right)\Big|_a^b$$
$$\int_a^b \vec{r}\left(t\right)dt=\left(\int f\left(t\right)dt\;\vec{i}+\int g\left(t\right)dt\;\vec{j}+\int h\left(t\right)dt\;\vec{k}\right)\Big|_a^b$$

In other words, we will do the indefinite integral and then do the evaluation of the vector as a whole instead of on a component by component basis.

Example 3 Compute $\int \vec{r}\left(t\right)dt$ for $\vec{r}\left(t\right) = \left\langle \sin\left(t\right), 6, 4t \right\rangle$.

Solution

All we need to do is integrate each of the components and be done with it.

$$\int \vec{r}\left(t\right)dt = \left\langle -\cos\left(t\right), 6t, 2t^2 \right\rangle + \vec{c}$$

Example 4 Compute $\int_0^1 \vec{r}\left(t\right)dt$ for $\vec{r}\left(t\right) = \left\langle \sin\left(t\right), 6, 4t \right\rangle$.

Solution

In this case all that we need to do is reuse the result from the previous example and then do the evaluation.

$$\begin{aligned}\int_0^1 \vec{r}\left(t\right)dt &= \left(\left\langle -\cos\left(t\right), 6t, 2t^2 \right\rangle\right)\Big|_0^1 \\ &= \left\langle -\cos\left(1\right), 6, 2 \right\rangle - \left\langle -1, 0, 0 \right\rangle \\ &= \left\langle 1 - \cos\left(1\right), 6, 2 \right\rangle\end{aligned}$$

Section 1-8 : Tangent, Normal and Binormal Vectors

In this section we want to look at an application of derivatives for vector functions. Actually, there are a couple of applications, but they all come back to needing the first one.

In the past we've used the fact that the derivative of a function was the slope of the tangent line. With vector functions we get exactly the same result, with one exception.

Given the vector function, $\vec{r}\left(t\right)$, we call $\vec{r}'\left(t\right)$ the **tangent vector** provided it exists and provided $\vec{r}'\left(t\right) \ne \vec{0}$. The tangent line to $\vec{r}\left(t\right)$ at *P* is then the line that passes through the point *P* and is parallel to the tangent vector, $\vec{r}'\left(t\right)$. Note that we really do need to require $\vec{r}'\left(t\right) \ne \vec{0}$ in order to have a tangent vector. If we had $\vec{r}'\left(t\right) = \vec{0}$ we would have a vector that had no magnitude and so couldn't give us the direction of the tangent.

Also, provided $\vec{r}'\left(t\right) \ne \vec{0}$, the **unit tangent vector** to the curve is given by,

$$\vec{T}\left(t\right) = \frac{\vec{r}'\left(t\right)}{\left\| \vec{r}'\left(t\right) \right\|}$$

While, the components of the unit tangent vector can be somewhat messy on occasion there are times when we will need to use the unit tangent vector instead of the tangent vector.

Example 1 Find the general formula for the tangent vector and unit tangent vector to the curve given by $\vec{r}\left(t\right) = t^2\,\vec{i} + 2\sin t\,\vec{j} + 2\cos t\,\vec{k}$.

Solution

First, by general formula we mean that we won't be plugging in a specific *t* and so we will be finding a formula that we can use at a later date if we'd like to find the tangent at any point on the curve. With that said there really isn't all that much to do at this point other than to do the work.

Here is the tangent vector to the curve.

$$\vec{r}'\left(t\right) = 2t\,\vec{i} + 2\cos t\,\vec{j} - 2\sin t\,\vec{k}$$

To get the unit tangent vector we need the length of the tangent vector.

$$\begin{aligned}\left\| \vec{r}'\left(t\right) \right\| &= \sqrt{4t^2 + 4\cos^2 t + 4\sin^2 t} \\ &= \sqrt{4t^2 + 4}\end{aligned}$$

The unit tangent vector is then,

$$\vec{T}(t) = \frac{1}{\sqrt{4t^2+4}}\left(2t\,\vec{i} + 2\cos t\,\vec{j} - 2\sin t\,\vec{k}\right)$$
$$= \frac{2t}{\sqrt{4t^2+4}}\vec{i} + \frac{2\cos t}{\sqrt{4t^2+4}}\vec{j} - \frac{2\sin t}{\sqrt{4t^2+4}}\vec{k}$$

Example 2 Find the vector equation of the tangent line to the curve given by $\vec{r}(t) = t^2\,\vec{i} + 2\sin t\,\vec{j} + 2\cos t\,\vec{k}$ at $t = \frac{\pi}{3}$.

Solution

First, we need the tangent vector and since this is the function we were working with in the previous example we can just reuse the tangent vector from that example and plug in $t = \frac{\pi}{3}$.

$$\vec{r}'\left(\frac{\pi}{3}\right) = \frac{2\pi}{3}\vec{i} + 2\cos\left(\frac{\pi}{3}\right)\vec{j} - 2\sin\left(\frac{\pi}{3}\right)\vec{k} = \frac{2\pi}{3}\vec{i} + \vec{j} - \sqrt{3}\,\vec{k}$$

We'll also need the point on the line at $t = \frac{\pi}{3}$ so,

$$\vec{r}\left(\frac{\pi}{3}\right) = \frac{\pi^2}{9}\vec{i} + \sqrt{3}\,\vec{j} + \vec{k}$$

The vector equation of the line is then,

$$\vec{r}(t) = \left\langle \frac{\pi^2}{9}, \sqrt{3}, 1 \right\rangle + t\left\langle \frac{2\pi}{3}, 1, -\sqrt{3} \right\rangle$$

Before moving on let's note a couple of things about the previous example. First, we could have used the unit tangent vector had we wanted to for the parallel vector. However, that would have made for a more complicated equation for the tangent line.

Second, notice that we used $\vec{r}(t)$ to represent the tangent line despite the fact that we used that as well for the function. Do not get excited about that. The $\vec{r}(t)$ here is much like *y* is with normal functions. With normal functions, *y* is the generic letter that we used to represent functions and $\vec{r}(t)$ tends to be used in the same way with vector functions.

Next, we need to talk about the **unit normal** and the **binormal** vectors.

The unit normal vector is defined to be,

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\left\|\vec{T}'(t)\right\|}$$

The unit normal is orthogonal (or normal, or perpendicular) to the unit tangent vector and hence to the curve as well. We've already seen normal vectors when we were dealing with Equations of Planes. They will show up with some regularity in several Calculus III topics.

The definition of the unit normal vector always seems a little mysterious when you first see it. It follows directly from the following fact.

Fact

Suppose that $\vec r(t)$ is a vector such that $\left\|\vec r(t)\right\| = c$ for all *t*. Then $\vec r\,'(t)$ is orthogonal to $\vec r(t)$.

To prove this fact is pretty simple. From the fact statement and the relationship between the magnitude of a vector and the dot product we have the following.

$$\vec r(t)\bullet\vec r(t) = \left\|\vec r(t)\right\|^2 = c^2 \qquad \text{for all } t$$

Now, because this is true for all *t* we can see that,

$$\frac{d}{dt}\left(\vec r(t)\bullet\vec r(t)\right) = \frac{d}{dt}\left(c^2\right) = 0$$

Also, recalling the fact from the previous section about differentiating a dot product we see that,

$$\frac{d}{dt}\left(\vec r(t)\bullet\vec r(t)\right) = \vec r\,'(t)\bullet\vec r(t) + \vec r(t)\bullet\vec r\,'(t) = 2\vec r\,'(t)\bullet\vec r(t)$$

Or, upon putting all this together we get,

$$2\vec r\,'(t)\bullet\vec r(t) = 0 \qquad \Rightarrow \qquad \vec r\,'(t)\bullet\vec r(t) = 0$$

Therefore $\vec r\,'(t)$ is orthogonal to $\vec r(t)$.

The definition of the unit normal then falls directly from this. Because $\vec T(t)$ is a unit vector we know that $\left\|\vec T(t)\right\| = 1$ for all *t* and hence by the Fact $\vec T\,'(t)$ is orthogonal to $\vec T(t)$. However, because $\vec T(t)$ is tangent to the curve, $\vec T\,'(t)$ must be orthogonal, or normal, to the curve as well and so be a normal vector for the curve. All we need to do then is divide by $\left\|\vec T\,'(t)\right\|$ to arrive at a unit normal vector.

Next, is the binormal vector. The binormal vector is defined to be,

$$\vec B(t) = \vec T(t)\times\vec N(t)$$

Because the binormal vector is defined to be the cross product of the unit tangent and unit normal vector we then know that the binormal vector is orthogonal to both the tangent vector and the normal vector.

Example 3 Find the normal and binormal vectors for $\vec r(t) = \langle t, 3\sin t, 3\cos t\rangle$.

Solution

We first need the unit tangent vector so first get the tangent vector and its magnitude.

$$\vec{r}'(t) = \langle 1, 3\cos t, -3\sin t \rangle$$
$$\left\| \vec{r}'(t) \right\| = \sqrt{1 + 9\cos^2 t + 9\sin^2 t} = \sqrt{10}$$

The unit tangent vector is then,

$$\vec{T}(t) = \left\langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\cos t, -\frac{3}{\sqrt{10}}\sin t \right\rangle$$

The unit normal vector will now require the derivative of the unit tangent and its magnitude.

$$\vec{T}'(t) = \left\langle 0, -\frac{3}{\sqrt{10}}\sin t, -\frac{3}{\sqrt{10}}\cos t \right\rangle$$
$$\left\| \vec{T}'(t) \right\| = \sqrt{\frac{9}{10}\sin^2 t + \frac{9}{10}\cos^2 t} = \sqrt{\frac{9}{10}} = \frac{3}{\sqrt{10}}$$

The unit normal vector is then,

$$\vec{N}(t) = \frac{\sqrt{10}}{3}\left\langle 0, -\frac{3}{\sqrt{10}}\sin t, -\frac{3}{\sqrt{10}}\cos t \right\rangle = \langle 0, -\sin t, -\cos t \rangle$$

Finally, the binormal vector is,

$$\begin{aligned}
\vec{B}(t) &= \vec{T}(t) \times \vec{N}(t) \\
&= \left| \begin{array}{ccc|cc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}}\cos t & -\frac{3}{\sqrt{10}}\sin t & \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}}\cos t \\ 0 & -\sin t & -\cos t & 0 & -\sin t \end{array} \right. \\
&= -\frac{3}{\sqrt{10}}\cos^2 t\,\vec{i} - \frac{1}{\sqrt{10}}\sin t\,\vec{k} + \frac{1}{\sqrt{10}}\cos t\;\vec{j} - \frac{3}{\sqrt{10}}\sin^2 t\,\vec{i} \\
&= -\frac{3}{\sqrt{10}}\vec{i} + \frac{1}{\sqrt{10}}\cos t\;\vec{j} - \frac{1}{\sqrt{10}}\sin t\;\vec{k}
\end{aligned}$$

Section 1-9 : Arc Length with Vector Functions

In this section we'll recast an old formula into terms of vector functions. We want to determine the length of a vector function,

$$\vec r\left(t \right) = \left\langle {f\left(t \right),g\left(t \right),h\left(t \right)} \right\rangle$$

on the interval $a \le t \le b$.

We actually already know how to do this. Recall that we can write the vector function into the parametric form,

$$x = f\left(t \right) \hspace{0.5in} y = g\left(t \right) \hspace{0.5in} z = h\left(t \right)$$

Also, recall that with two dimensional parametric curves the arc length is given by,

$$L = \int_{a}^{b}{{\sqrt {{{\left[{f'\left(t \right)} \right]}^2} + {{\left[{g'\left(t \right)} \right]}^2}} \,dt}}$$

There is a natural extension of this to three dimensions. So, the length of the curve $\vec r\left(t \right)$ on the interval $a \le t \le b$ is,

$$L = \int_{a}^{b}{{\sqrt {{{\left[{f'\left(t \right)} \right]}^2} + {{\left[{g'\left(t \right)} \right]}^2} + {{\left[{h'\left(t \right)} \right]}^2}} \,dt}}$$

There is a nice simplification that we can make for this. Notice that the integrand (the function we're integrating) is nothing more than the magnitude of the tangent vector,

$$\left\| {\vec r'\left(t \right)} \right\| = \sqrt {{{\left[{f'\left(t \right)} \right]}^2} + {{\left[{g'\left(t \right)} \right]}^2} + {{\left[{h'\left(t \right)} \right]}^2}}$$

Therefore, the arc length can be written as,

$$L = \int_{a}^{b}{{\left\| {\vec r'\left(t \right)} \right\|\,dt}}$$

Let's work a quick example of this.

Example 1 Determine the length of the curve $\vec r\left(t \right) = \left\langle {2t,3\sin \left({2t} \right),3\cos \left({2t} \right)} \right\rangle$ on the interval $0 \le t \le 2\pi$.

Solution
We will first need the tangent vector and its magnitude.

$$\begin{align*}\vec r'\left(t \right) & = \left\langle {2,6\cos \left({2t} \right), - 6\sin \left({2t} \right)} \right\rangle \\ \left\| {\vec r'\left(t \right)} \right\| & = \sqrt {4 + 36{{\cos }^2}\left({2t} \right) + 36{{\sin }^2}\left({2t} \right)} = \sqrt {4 + 36} = 2\sqrt {10} \end{align*}$$

The length is then,

$$\begin{aligned} L &= \int_a^b \left\| \vec{r}'(t) \right\| dt \\ &= \int_0^{2\pi} 2\sqrt{10}\, dt \\ &= 4\pi\sqrt{10} \end{aligned}$$

We need to take a quick look at another concept here. We define the **arc length function** as,

$$s(t) = \int_0^t \left\| \vec{r}'(u) \right\| du$$

Before we look at why this might be important let's work a quick example.

Example 2 Determine the arc length function for $\vec{r}(t) = \langle 2t, 3\sin(2t), 3\cos(2t) \rangle$.

Solution
From the previous example we know that,

$$\left\| \vec{r}'(t) \right\| = 2\sqrt{10}$$

The arc length function is then,

$$s(t) = \int_0^t 2\sqrt{10}\, du = \left(2\sqrt{10}\, u \right)\Big|_0^t = 2\sqrt{10}\, t$$

Okay, just why would we want to do this? Well let's take the result of the example above and solve it for *t*.

$$t = \frac{s}{2\sqrt{10}}$$

Now, taking this and plugging it into the original vector function and we can **reparametrize** the function into the form, $\vec{r}(t(s))$. For our function this is,

$$\vec{r}(t(s)) = \left\langle \frac{s}{\sqrt{10}}, 3\sin\left(\frac{s}{\sqrt{10}}\right), 3\cos\left(\frac{s}{\sqrt{10}}\right) \right\rangle$$

So, why would we want to do this? Well with the reparameterization we can now tell where we are on the curve after we've traveled a distance of *s* along the curve. Note as well that we will start the measurement of distance from where we are at $t = 0$.

Example 3 Where on the curve $\vec{r}(t) = \langle 2t, 3\sin(2t), 3\cos(2t) \rangle$ are we after traveling for a distance of $\frac{\pi\sqrt{10}}{3}$?

Solution
To determine this we need the reparameterization, which we have from above.

$$\vec{r}\left(t\left(s\right)\right) = \left\langle \frac{s}{\sqrt{10}}, 3\sin\left(\frac{s}{\sqrt{10}}\right), 3\cos\left(\frac{s}{\sqrt{10}}\right)\right\rangle$$

Then, to determine where we are all that we need to do is plug in $s = \frac{\pi\sqrt{10}}{3}$ into this and we'll get our location.

$$\vec{r}\left(t\left(\frac{\pi\sqrt{10}}{3}\right)\right) = \left\langle \frac{\pi}{3}, 3\sin\left(\frac{\pi}{3}\right), 3\cos\left(\frac{\pi}{3}\right)\right\rangle = \left\langle \frac{\pi}{3}, \frac{3\sqrt{3}}{2}, \frac{3}{2}\right\rangle$$

So, after traveling a distance of $\frac{\pi\sqrt{10}}{3}$ along the curve we are at the point $\left(\frac{\pi}{3}, \frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$.

Section 1-10 : Curvature

In this section we want to briefly discuss the **curvature** of a smooth curve (recall that for a smooth curve we require $\vec r\,'(t)$ is continuous and $\vec r\,'(t) \neq 0$). The curvature measures how fast a curve is changing direction at a given point.

There are several formulas for determining the curvature for a curve. The formal definition of curvature is,

$$\kappa = \left| \frac{d\vec T}{ds} \right|$$

where $\vec T$ is the unit tangent and *s* is the arc length. Recall that we saw in a previous section how to reparametrize a curve to get it into terms of the arc length.

In general the formal definition of the curvature is not easy to use so there are two alternate formulas that we can use. Here they are.

$$\kappa = \frac{\left\| \vec T\,'(t) \right\|}{\left\| \vec r\,'(t) \right\|} \qquad\qquad \kappa = \frac{\left\| \vec r\,'(t) \times \vec r\,''(t) \right\|}{\left\| \vec r\,'(t) \right\|^3}$$

These may not be particularly easy to deal with either, but at least we don't need to reparametrize the unit tangent.

Example 1 Determine the curvature for $\vec r(t) = \langle t, 3\sin t, 3\cos t \rangle$.

Solution

Back in the section when we introduced the tangent vector we computed the tangent and unit tangent vectors for this function. These were,

$$\vec r\,'(t) = \langle 1, 3\cos t, -3\sin t \rangle$$

$$\vec T(t) = \left\langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\cos t, -\frac{3}{\sqrt{10}}\sin t \right\rangle$$

The derivative of the unit tangent is,

$$\vec T\,'(t) = \left\langle 0, -\frac{3}{\sqrt{10}}\sin t, -\frac{3}{\sqrt{10}}\cos t \right\rangle$$

The magnitudes of the two vectors are,

$$\left\| \vec r\,'(t) \right\| = \sqrt{1 + 9\cos^2 t + 9\sin^2 t} = \sqrt{10}$$

$$\left\| \vec T\,'(t) \right\| = \sqrt{0 + \frac{9}{10}\sin^2 t + \frac{9}{10}\cos^2 t} = \sqrt{\frac{9}{10}} = \frac{3}{\sqrt{10}}$$

The curvature is then,

$$\kappa = \frac{\left\| \vec{T}'(t) \right\|}{\left\| \vec{r}'(t) \right\|} = \frac{{}^{3}/_{\sqrt{10}}}{\sqrt{10}} = \frac{3}{10}$$

In this case the curvature is constant. This means that the curve is changing direction at the same rate at every point along it. Recalling that this curve is a helix this result makes sense.

Example 2 Determine the curvature of $\vec{r}(t) = t^2\,\vec{i} + t\,\vec{k}$.

Solution

In this case the second form of the curvature would probably be easiest. Here are the first couple of derivatives.

$$\vec{r}'(t) = 2t\,\vec{i} + \vec{k} \qquad\qquad \vec{r}''(t) = 2\,\vec{i}$$

Next, we need the cross product.

$$\begin{aligned} \vec{r}'(t) \times \vec{r}''(t) &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2t & 0 & 1 \\ 2 & 0 & 0 \end{vmatrix} \begin{matrix} \vec{i} & \vec{j} \\ 2t & 0 \\ 2 & 0 \end{matrix} \\ &= 2\,\vec{j} \end{aligned}$$

The magnitudes are,

$$\left\| \vec{r}'(t) \times \vec{r}''(t) \right\| = 2 \qquad\qquad \left\| \vec{r}'(t) \right\| = \sqrt{4t^2 + 1}$$

The curvature at any value of *t* is then,

$$\kappa = \frac{2}{\left(4t^2 + 1\right)^{\frac{3}{2}}}$$

There is a special case that we can look at here as well. Suppose that we have a curve given by $y = f(x)$ and we want to find its curvature.

As we saw when we first looked at vector functions we can write this as follows,

$$\vec{r}(x) = x\,\vec{i} + f(x)\,\vec{j}$$

If we then use the second formula for the curvature we will arrive at the following formula for the curvature.

$$\kappa = \frac{\left| f''(x) \right|}{\left(1 + \left[f'(x) \right]^2\right)^{\frac{3}{2}}}$$

Section 1-11 : Velocity and Acceleration

In this section we need to take a look at the velocity and acceleration of a moving object.

From Calculus I we know that given the position function of an object that the velocity of the object is the first derivative of the position function and the acceleration of the object is the second derivative of the position function.

So, given this it shouldn't be too surprising that if the position function of an object is given by the vector function $\vec{r}\left(t\right)$ then the velocity and acceleration of the object is given by,

$$\vec{v}\left(t\right)=\vec{r}'\left(t\right) \qquad\qquad \vec{a}\left(t\right)=\vec{r}''\left(t\right)$$

Notice that the velocity and acceleration are also going to be vectors as well.

In the study of the motion of objects the acceleration is often broken up into a **tangential component**, a_T, and a **normal component**, a_N. The tangential component is the part of the acceleration that is tangential to the curve and the normal component is the part of the acceleration that is normal (or orthogonal) to the curve. If we do this we can write the acceleration as,

$$\vec{a}=a_T\vec{T}+a_N\vec{N}$$

where $\vec{T}$ and $\vec{N}$ are the unit tangent and unit normal for the position function.

If we define $v=\left\|\vec{v}\left(t\right)\right\|$ then the tangential and normal components of the acceleration are given by,

$$a_T=v'=\frac{\vec{r}'\left(t\right)\bullet\vec{r}''\left(t\right)}{\left\|r'\left(t\right)\right\|} \qquad\qquad a_N=\kappa v^2=\frac{\left\|\vec{r}'\left(t\right)\times\vec{r}''\left(t\right)\right\|}{\left\|r'\left(t\right)\right\|}$$

where κ is the curvature for the position function.

There are two formulas to use here for each component of the acceleration and while the second formula may seem overly complicated it is often the easier of the two. In the tangential component, *v*, may be messy and computing the derivative may be unpleasant. In the normal component we will already be computing both of these quantities in order to get the curvature and so the second formula in this case is definitely the easier of the two.

Let's take a quick look at a couple of examples.

Example 1 If the acceleration of an object is given by $\vec{a}=\vec{i}+2\vec{j}+6t\vec{k}$ find the object's velocity and position functions given that the initial velocity is $\vec{v}\left(0\right)=\vec{j}-\vec{k}$ and the initial position is $\vec{r}\left(0\right)=\vec{i}-2\vec{j}+3\vec{k}$.

Solution
We'll first get the velocity. To do this all (well almost all) we need to do is integrate the acceleration.

$$\begin{aligned}\vec{v}(t) &= \int \vec{a}(t)\,dt \\ &= \int \vec{i} + 2\vec{j} + 6t\vec{k}\,dt \\ &= t\,\vec{i} + 2t\,\vec{j} + 3t^2\,\vec{k} + \vec{c}\end{aligned}$$

To completely get the velocity we will need to determine the "constant" of integration. We can use the initial velocity to get this.

$$\vec{j} - \vec{k} = \vec{v}(0) = \vec{c}$$

The velocity of the object is then,

$$\begin{aligned}\vec{v}(t) &= t\,\vec{i} + 2t\,\vec{j} + 3t^2\,\vec{k} + \vec{j} - \vec{k} \\ &= t\,\vec{i} + (2t+1)\,\vec{j} + \left(3t^2 - 1\right)\vec{k}\end{aligned}$$

We will find the position function by integrating the velocity function.

$$\begin{aligned}\vec{r}(t) &= \int \vec{v}(t)\,dt \\ &= \int t\,\vec{i} + (2t+1)\,\vec{j} + \left(3t^2 - 1\right)\vec{k}\,dt \\ &= \frac{1}{2}t^2\,\vec{i} + \left(t^2 + t\right)\vec{j} + \left(t^3 - t\right)\vec{k} + \vec{c}\end{aligned}$$

Using the initial position gives us,

$$\vec{i} - 2\vec{j} + 3\vec{k} = \vec{r}(0) = \vec{c}$$

So, the position function is,

$$\vec{r}(t) = \left(\frac{1}{2}t^2 + 1\right)\vec{i} + \left(t^2 + t - 2\right)\vec{j} + \left(t^3 - t + 3\right)\vec{k}$$

Example 2 For the object in the previous example determine the tangential and normal components of the acceleration.

Solution

There really isn't much to do here other than plug into the formulas. To do this we'll need to notice that,

$$\begin{aligned}\vec{r}\,'(t) &= t\,\vec{i} + (2t+1)\,\vec{j} + \left(3t^2 - 1\right)\vec{k} \\ \vec{r}\,''(t) &= \vec{i} + 2\vec{j} + 6t\vec{k}\end{aligned}$$

Let's first compute the dot product and cross product that we'll need for the formulas.

$$\vec{r}\,'(t) \bullet \vec{r}\,''(t) = t + 2(2t+1) + 6t\left(3t^2 - 1\right) = 18t^3 - t + 2$$

$$\vec r'(t)\times\vec r''(t)=\begin{vmatrix}\vec i & \vec j & \vec k\\ t & 2t+1 & 3t^2-1\\ 1 & 2 & 6t\end{vmatrix}\begin{matrix}\vec i & \vec j\\ t & 2t+1\\ 1 & 2\end{matrix}$$

$$=(6t)(2t+1)\vec i+(3t^2-1)\vec j+2t\vec k-6t^2\vec j-2(3t^2-1)\vec i-(2t+1)\vec k$$

$$=(6t^2+6t+2)\vec i-(3t^2+1)\vec j-\vec k$$

Next, we also need a couple of magnitudes.

$$\left\|\vec r'(t)\right\|=\sqrt{t^2+(2t+1)^2+(3t^2-1)^2}=\sqrt{9t^4-t^2+4t+2}$$

$$\left\|\vec r'(t)\times\vec r''(t)\right\|=\sqrt{(6t^2+6t+2)^2+(3t^2+1)^2+1}=\sqrt{45t^4+72t^3+66t^2+24t+6}$$

The tangential component of the acceleration is then,

$$a_T=\frac{18t^3-t+2}{\sqrt{9t^4-t^2+4t+2}}$$

The normal component of the acceleration is,

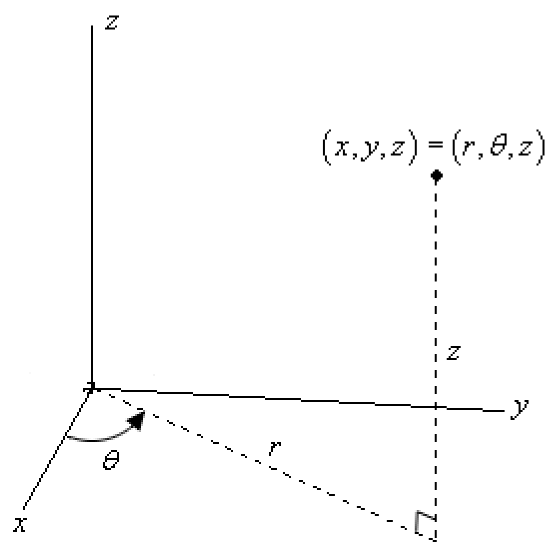
$$a_N=\frac{\sqrt{45t^4+72t^3+66t^2+24t+6}}{\sqrt{9t^4-t^2+4t+2}}=\sqrt{\frac{45t^4+72t^3+66t^2+24t+6}{9t^4-t^2+4t+2}}$$

Section 1-12 : Cylindrical Coordinates

As with two dimensional space the standard (x, y, z) coordinate system is called the Cartesian coordinate system. In the last two sections of this chapter we'll be looking at some alternate coordinate systems for three dimensional space.

We'll start off with the cylindrical coordinate system. This one is fairly simple as it is nothing more than an extension of polar coordinates into three dimensions. Not only is it an extension of polar coordinates, but we extend it into the third dimension just as we extend Cartesian coordinates into the third dimension. All that we do is add a *z* on as the third coordinate. The *r* and θ are the same as with polar coordinates.

Here is a sketch of a point in $\mathbb{R}^3$.



The conversions for *x* and *y* are the same conversions that we used back when we were looking at polar coordinates. So, if we have a point in cylindrical coordinates the Cartesian coordinates can be found by using the following conversions.

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

The third equation is just an acknowledgement that the *z*-coordinate of a point in Cartesian and polar coordinates is the same.

Likewise, if we have a point in Cartesian coordinates the cylindrical coordinates can be found by using the following conversions.

$$r = \sqrt{x^2 + y^2} \qquad \text{OR} \qquad r^2 = x^2 + y^2$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$
$$z = z$$

Let's take a quick look at some surfaces in cylindrical coordinates.

Example 1 Identify the surface for each of the following equations.

(a) $r = 5$
(b) $r^2 + z^2 = 100$
(c) $z = r$

Solution

(a) In two dimensions we know that this is a circle of radius 5. Since we are now in three dimensions and there is no *z* in equation this means it is allowed to vary freely. So, for any given *z* we will have a circle of radius 5 centered on the *z*-axis.

In other words, we will have a cylinder of radius 5 centered on the *z*-axis.

(b) This equation will be easy to identify once we convert back to Cartesian coordinates.

$$r^2 + z^2 = 100$$
$$x^2 + y^2 + z^2 = 100$$

So, this is a sphere centered at the origin with radius 10.

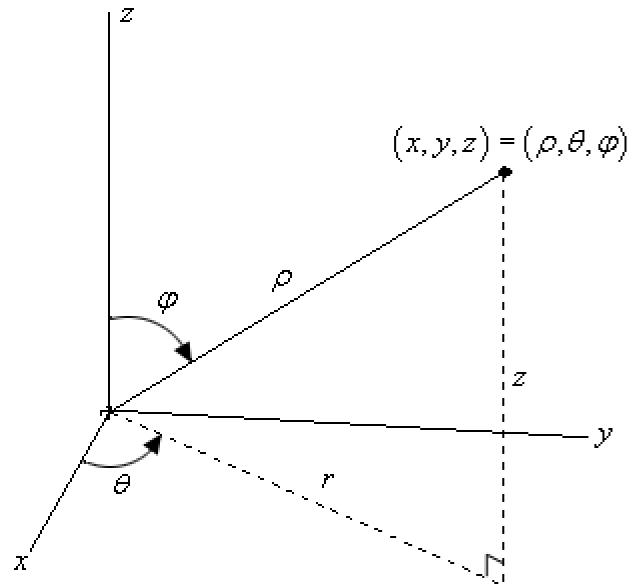
(c) Again, this one won't be too bad if we convert back to Cartesian. For reasons that will be apparent eventually, we'll first square both sides, then convert.

$$z^2 = r^2$$
$$z^2 = x^2 + y^2$$

From the section on quadric surfaces we know that this is the equation of a cone.

Section 1-13 : Spherical Coordinates

In this section we will introduce spherical coordinates. Spherical coordinates can take a little getting used to. It's probably easiest to start things off with a sketch.



Spherical coordinates consist of the following three quantities.

First there is ρ. This is the distance from the origin to the point and we will require $\rho \geq 0$.

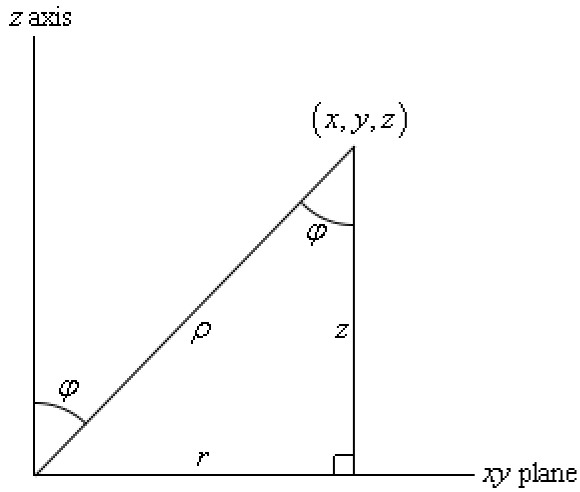
Next there is θ. This is the same angle that we saw in polar/cylindrical coordinates. It is the angle between the positive *x*-axis and the line above denoted by *r* (which is also the same *r* as in polar/cylindrical coordinates). There are no restrictions on θ.

Finally, there is φ. This is the angle between the positive *z*-axis and the line from the origin to the point. We will require $0 \leq \varphi \leq \pi$.

In summary, ρ is the distance from the origin to the point, φ is the angle that we need to rotate down from the positive *z*-axis to get to the point and θ is how much we need to rotate around the *z*-axis to get to the point.

We should first derive some conversion formulas. Let's first start with a point in spherical coordinates and ask what the cylindrical coordinates of the point are. So, we know $\left(\rho, \theta, \varphi\right)$ and want to find $\left(r, \theta, z\right)$. Of course, we really only need to find *r* and *z* since θ is the same in both coordinate systems.

If we look at the sketch above from directly in front of the triangle we get the following sketch,



We know that the angle between the *z*-axis and ρ is φ and with a little geometry we also know that the angle between ρ and the vertical side of the right triangle is also φ.

Then, with a little right triangle trig we get,

$$\begin{aligned} z &= \rho\cos\varphi \\ r &= \rho\sin\varphi \end{aligned}$$

and these are exactly the formulas that we were looking for. So, given a point in spherical coordinates the cylindrical coordinates of the point will be,

$$\begin{aligned} r &= \rho\sin\varphi \\ \theta &= \theta \\ z &= \rho\cos\varphi \end{aligned}$$

Note as well from the Pythagorean theorem we also get,

$$\rho^2 = r^2 + z^2$$

Next, let's find the Cartesian coordinates of the same point. To do this we'll start with the cylindrical conversion formulas from the previous section.

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}$$

Now all that we need to do is use the formulas from above for *r* and *z* to get,

$$x = \rho \sin\varphi \cos\theta$$
$$y = \rho \sin\varphi \sin\theta$$
$$z = \rho \cos\varphi$$

Also note that since we know that $r^2 = x^2 + y^2$ we get,

$$\rho^2 = x^2 + y^2 + z^2$$

Converting points from Cartesian or cylindrical coordinates into spherical coordinates is usually done with the same conversion formulas. To see how this is done let's work an example of each.

Example 1 Perform each of the following conversions.

(a) Convert the point $\left(\sqrt{6}, \frac{\pi}{4}, \sqrt{2}\right)$ from cylindrical to spherical coordinates.

(b) Convert the point $\left(-1, 1, -\sqrt{2}\right)$ from Cartesian to spherical coordinates.

Solution

(a) Convert the point $\left(\sqrt{6}, \frac{\pi}{4}, \sqrt{2}\right)$ from cylindrical to spherical coordinates.

We'll start by acknowledging that θ is the same in both coordinate systems and so we don't need to do anything with that.

Next, let's find ρ.

$$\rho = \sqrt{r^2 + z^2} = \sqrt{6+2} = \sqrt{8} = 2\sqrt{2}$$

Finally, let's get φ. To do this we can use either the conversion for *r* or *z*. We'll use the conversion for *z*.

$$z = \rho\cos\varphi \qquad \Rightarrow \qquad \cos\varphi = \frac{z}{\rho} = \frac{\sqrt{2}}{2\sqrt{2}} \qquad \Rightarrow \qquad \varphi = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

Notice that there are many possible values of φ that will give $\cos\varphi = \frac{1}{2}$, however, we have restricted φ to the range $0 \le \varphi \le \pi$ and so this is the only possible value in that range.

So, the spherical coordinates of this point will are $\left(2\sqrt{2}, \frac{\pi}{4}, \frac{\pi}{3}\right)$.

(b) Convert the point $\left(-1, 1, -\sqrt{2}\right)$ from Cartesian to spherical coordinates.

The first thing that we'll do here is find ρ.

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{1+1+2} = 2$$

Now we'll need to find φ. We can do this using the conversion for *z*.

$$z = \rho\cos\varphi \qquad \Rightarrow \qquad \cos\varphi = \frac{z}{\rho} = \frac{-\sqrt{2}}{2} \qquad \Rightarrow \qquad \varphi = \cos^{-1}\left(\frac{-\sqrt{2}}{2}\right) = \frac{3\pi}{4}$$

As with the last parts this will be the only possible φ in the range allowed.

Finally, let's find θ. To do this we can use the conversion for *x* or *y*. We will use the conversion for *y* in this case.

$$\sin\theta = \frac{y}{\rho\sin\varphi} = \frac{1}{2\left(\frac{\sqrt{2}}{2}\right)} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \qquad \Rightarrow \qquad \theta = \frac{\pi}{4} \text{ or } \theta = \frac{3\pi}{4}$$

Now, we actually have more possible choices for θ but all of them will reduce down to one of the two angles above since they will just be one of these two angles with one or more complete rotations around the unit circle added on.

We will however, need to decide which one is the correct angle since only one will be. To do this let's notice that, in two dimensions, the point with coordinates $x = -1$ and $y = 1$ lies in the second quadrant. This means that θ must be angle that will put the point into the second quadrant. Therefore, the second angle, $\theta = \frac{3\pi}{4}$, must be the correct one.

The spherical coordinates of this point are then $\left(2, \frac{3\pi}{4}, \frac{3\pi}{4}\right)$.

Now, let's take a look at some equations and identify the surfaces that they represent.

Example 2 Identify the surface for each of the following equations.

(a) $\rho = 5$

(b) $\varphi = \frac{\pi}{3}$

(c) $\theta = \frac{2\pi}{3}$

(d) $\rho\sin\varphi = 2$

Solution

(a) $\rho = 5$

There are a couple of ways to think about this one.

First, think about what this equation is saying. This equation says that, no matter what θ and φ are, the distance from the origin must be 5. So, we can rotate as much as we want away from the *z*-axis and around the *z*-axis, but we must always remain at a fixed distance from the origin. This is exactly what a sphere is. So, this is a sphere of radius 5 centered at the origin.

The other way to think about it is to just convert to Cartesian coordinates.

$$\rho = 5$$
$$\rho^2 = 25$$
$$x^2 + y^2 + z^2 = 25$$

Sure enough a sphere of radius 5 centered at the origin.

(b) $\varphi = \dfrac{\pi}{3}$

In this case there isn't an easy way to convert to Cartesian coordinates so we'll just need to think about this one a little. This equation says that no matter how far away from the origin that we move and no matter how much we rotate around the *z*-axis the point must always be at an angle of $\frac{\pi}{3}$ from the *z*-axis.

This is exactly what happens in a cone. All of the points on a cone are a fixed angle from the *z*-axis. So, we have a cone whose points are all at an angle of $\frac{\pi}{3}$ from the *z*-axis.

(c) $\theta = \dfrac{2\pi}{3}$

As with the last part we won't be able to easily convert to Cartesian coordinates here. In this case no matter how far from the origin we get or how much we rotate down from the positive *z*-axis the points must always form an angle of $\frac{2\pi}{3}$ with the *x*-axis.

Points in a vertical plane will do this. So, we have a vertical plane that forms an angle of $\frac{2\pi}{3}$ with the positive *x*-axis.

(d) $\rho \sin\varphi = 2$

In this case we can convert to Cartesian coordinates so let's do that. There are actually two ways to do this conversion. We will look at both since both will be used on occasion.

Solution 1
In this solution method we will convert directly to Cartesian coordinates. To do this we will first need to square both sides of the equation.

$$\rho^2 \sin^2\varphi = 4$$

Now, for no apparent reason add $\rho^2 \cos^2\varphi$ to both sides.

$$\rho^2 \sin^2 \varphi + \rho^2 \cos^2 \varphi = 4 + \rho^2 \cos^2 \varphi$$
$$\rho^2 \left(\sin^2 \varphi + \cos^2 \varphi\right) = 4 + \rho^2 \cos^2 \varphi$$
$$\rho^2 = 4 + \left(\rho \cos \varphi\right)^2$$

Now we can convert to Cartesian coordinates.

$$x^2 + y^2 + z^2 = 4 + z^2$$
$$x^2 + y^2 = 4$$

So, we have a cylinder of radius 2 centered on the *z*-axis.

This solution method wasn't too bad, but it did require some not so obvious steps to complete.

Solution 2
This method is much shorter, but also involves something that you may not see the first time around. In this case instead of going straight to Cartesian coordinates we'll first convert to cylindrical coordinates.

This won't always work, but in this case all we need to do is recognize that $r = \rho \sin \varphi$ and we will get something we can recognize. Using this we get,

$$\rho \sin \varphi = 2$$
$$r = 2$$

At this point we know this is a cylinder (remember that we're in three dimensions and so this isn't a circle!). However, let's go ahead and finish the conversion process out.

$$r^2 = 4$$
$$x^2 + y^2 = 4$$

So, as we saw in the last part of the previous example it will *sometimes* be easier to convert equations in spherical coordinates into cylindrical coordinates before converting into Cartesian coordinates. This won't always be easier, but it can make some of the conversions quicker and easier.

The last thing that we want to do in this section is generalize the first three parts of the previous example.

$\rho = a$	sphere of radius a centered at the origin
$\varphi = \alpha$	cone that makes an angle of α with the positive $z-$axis
$\theta = \beta$	vertical plane that makes an angle of β with the positive $x-$axis

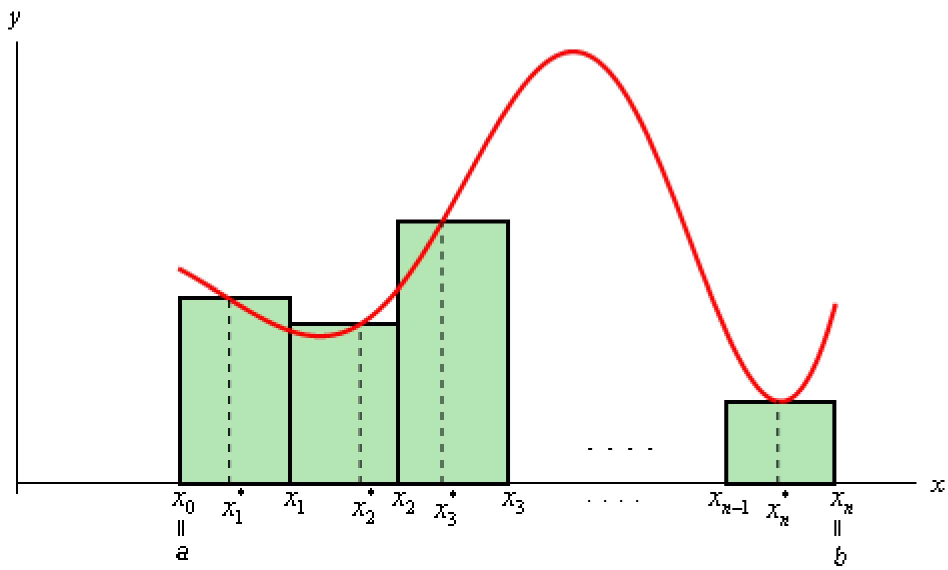
Section 4-1 : Double Integrals

Before starting on double integrals let's do a quick review of the definition of definite integrals for functions of single variables. First, when working with the integral,

$$\int_a^b f(x)\,dx$$

we think of *x*'s as coming from the interval $a \le x \le b$. For these integrals we can say that we are integrating over the interval $a \le x \le b$. Note that this does assume that $a < b$, however, if we have $b < a$ then we can just use the interval $b \le x \le a$.

Now, when we derived the definition of the definite integral we first thought of this as an area problem. We first asked what the area under the curve was and to do this we broke up the interval $a \le x \le b$ into *n* subintervals of width Δx and choose a point, x_i^*, from each interval as shown below,



Each of the rectangles has height of $f\left(x_i^*\right)$ and we could then use the area of each of these rectangles to approximate the area as follows.

$$A \approx f\left(x_1^*\right)\Delta x + f\left(x_2^*\right)\Delta x + \cdots + f\left(x_i^*\right)\Delta x + \cdots + f\left(x_n^*\right)\Delta x$$

To get the exact area we then took the limit as *n* goes to infinity and this was also the definition of the definite integral.

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i^*\right)\Delta x$$

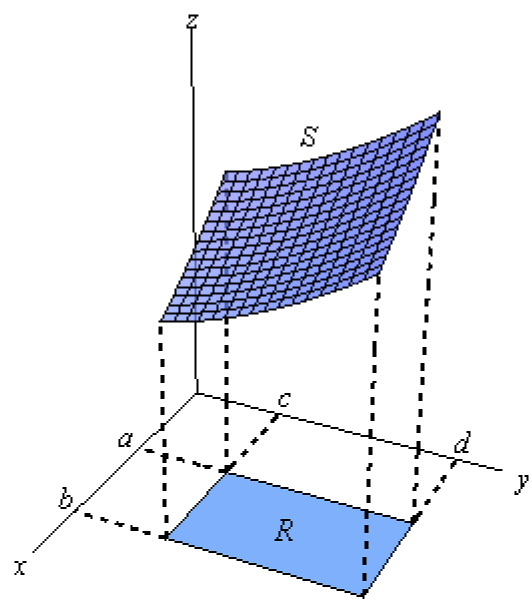
In this section we want to integrate a function of two variables, $f(x,y)$. With functions of one variable we integrated over an interval (*i.e.* a one-dimensional space) and so it makes some sense then that when integrating a function of two variables we will integrate over a region of $\mathbb{R}^2$ (two-dimensional space).

We will start out by assuming that the region in $\mathbb{R}^2$ is a rectangle which we will denote as follows,

$$R = [a,b] \times [c,d]$$

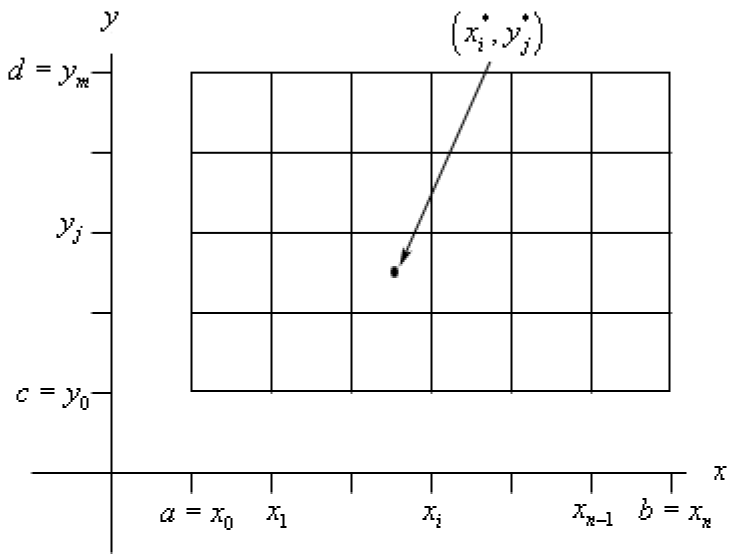
This means that the ranges for *x* and *y* are $a \le x \le b$ and $c \le y \le d$.

Also, we will initially assume that $f(x,y) \ge 0$ although this doesn't really have to be the case. Let's start out with the graph of the surface *S* given by graphing $f(x,y)$ over the rectangle *R*.

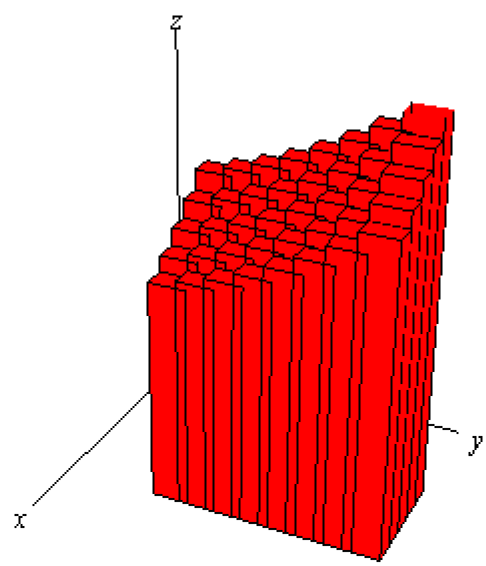


Now, just like with functions of one variable let's not worry about integrals quite yet. Let's first ask what the volume of the region under *S* (and above the *xy*-plane of course) is.

We will approximate the volume much as we approximated the area above. We will first divide up $a \le x \le b$ into *n* subintervals and divide up $c \le y \le d$ into *m* subintervals. This will divide up *R* into a series of smaller rectangles and from each of these we will choose a point (x_i^*, y_j^*). Here is a sketch of this set up.



Now, over each of these smaller rectangles we will construct a box whose height is given by $f\left(x_i^*, y_j^*\right)$. Here is a sketch of that.



Each of the rectangles has a base area of ΔA and a height of $f\left(x_i^*, y_j^*\right)$ so the volume of each of these boxes is $f\left(x_i^*, y_j^*\right)\Delta A$. The volume under the surface *S* is then approximately,

$$V \approx \sum_{i=1}^{n}\sum_{j=1}^{m} f\left(x_i^*, y_j^*\right)\Delta A$$

We will have a *double* sum since we will need to add up volumes in both the *x* and *y* directions.

To get a better estimation of the volume we will take *n* and *m* larger and larger and to get the exact volume we will need to take the limit as both *n* and *m* go to infinity. In other words,

$$V = \lim_{n,\, m \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f\left(x_i^*, y_j^*\right) \Delta A$$

Now, this should look familiar. This looks a lot like the definition of the integral of a function of single variable. In fact, this is also the definition of a double integral, or more exactly an integral of a function of two variables over a rectangle.

Here is the official definition of a double integral of a function of two variables over a rectangular region *R* as well as the notation that we'll use for it.

$$\iint\limits_R f\left(x, y\right) dA = \lim_{n,\, m \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f\left(x_i^*, y_j^*\right) \Delta A$$

Note the similarities and differences in the notation to single integrals. We have two integrals to denote the fact that we are dealing with a two dimensional region and we have a differential here as well. Note that the differential is *dA* instead of the *dx* and *dy* that we're used to seeing. Note as well that we don't have limits on the integrals in this notation. Instead we have the *R* written below the two integrals to denote the region that we are integrating over.

As indicated above one interpretation of the double integral of $f\left(x, y\right)$ over the rectangle *R* is the volume under the function $f\left(x, y\right)$ (and above the *xy*-plane). Or,

$$\text{Volume} = \iint\limits_R f\left(x, y\right) dA$$

We can use this double sum in the definition to estimate the value of a double integral if we need to. We can do this by choosing $\left(x_i^*, y_j^*\right)$ to be the midpoint of each rectangle. When we do this we usually denote the point as $\left(\overline{x}_i, \overline{y}_j\right)$. This leads to the **Midpoint Rule**,

$$\iint\limits_R f\left(x, y\right) dA \approx \sum_{i=1}^{n} \sum_{j=1}^{m} f\left(\overline{x}_i, \overline{y}_j\right) \Delta A$$

In the next section we start looking at how to actually compute double integrals.

Section 4-2 : Iterated Integrals

In the previous section we gave the definition of the double integral. However, just like with the definition of a single integral the definition is very difficult to use in practice and so we need to start looking into how we actually compute double integrals. We will continue to assume that we are integrating over the rectangle

$$R = [a,b] \times [c,d]$$

We will look at more general regions in the next section.

The following theorem tells us how to compute a double integral over a rectangle.

Fubini's Theorem

If $f(x,y)$ is continuous on $R = [a,b] \times [c,d]$ then,

$$\iint\limits_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

These integrals are called **iterated integrals**.

Note that there are in fact two ways of computing a double integral over a rectangle and also notice that the inner differential matches up with the limits on the inner integral and similarly for the outer differential and limits. In other words, if the inner differential is *dy* then the limits on the inner integral must be *y* limits of integration and if the outer differential is *dy* then the limits on the outer integral must be *y* limits of integration.

Now, on some level this is just notation and doesn't really tell us how to compute the double integral. Let's just take the first possibility above and change the notation a little.

$$\iint\limits_R f(x,y)\,dA = \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx$$

We will compute the double integral by first computing

$$\int_c^d f(x,y)\,dy$$

and we compute this by holding *x* constant and integrating with respect to *y* as if this were a single integral. This will give a function involving only *x*'s which we can in turn integrate.

We've done a similar process with partial derivatives. To take the derivative of a function with respect to *y* we treated the *x*'s as constants and differentiated with respect to *y* as if it was a function of a single variable.

Double integrals work in the same manner. We think of all the *x*'s as constants and integrate with respect to *y* or we think of all *y*'s as constants and integrate with respect to *x*.

Let's take a look at some examples.

Example 1 Compute each of the following double integrals over the indicated rectangles.

(a) $\iint\limits_R 6xy^2\,dA$, $R = [2,4]\times[1,2]$

(b) $\iint\limits_R 2x - 4y^3\,dA$, $R = [-5,4]\times[0,3]$

(c) $\iint\limits_R x^2y^2 + \cos(\pi x) + \sin(\pi y)\,dA$, $R = [-2,-1]\times[0,1]$

(d) $\iint\limits_R \frac{1}{(2x+3y)^2}\,dA$, $R = [0,1]\times[1,2]$

(e) $\iint\limits_R x\mathbf{e}^{xy}\,dA$, $R = [-1,2]\times[0,1]$

Solution

(a) $\iint\limits_R 6xy^2\,dA$, $R = [2,4]\times[1,2]$

It doesn't matter which variable we integrate with respect to first, we will get the same answer regardless of the order of integration. To prove that let's work this one with each order to make sure that we do get the same answer.

Solution 1

In this case we will integrate with respect to *y* first. So, the iterated integral that we need to compute is,

$$\iint\limits_R 6xy^2\,dA = \int_2^4\int_1^2 6xy^2\,dy\,dx$$

When setting these up make sure the limits match up to the differentials. Since the *dy* is the inner differential (*i.e.* we are integrating with respect to *y* first) the inner integral needs to have *y* limits.

To compute this we will do the inner integral first and we typically keep the outer integral around as follows,

$$\begin{aligned}\iint\limits_R 6xy^2\,dA &= \int_2^4 \left(2xy^3\right)\Big|_1^2\,dx\\ &= \int_2^4 16x - 2x\,dx\\ &= \int_2^4 14x\,dx\end{aligned}$$

Remember that we treat the *x* as a constant when doing the first integral and we don't do any integration with it yet. Now, we have a normal single integral so let's finish the integral by computing this.

$$\iint\limits_R 6xy^2\,dA = 7x^2\Big|_2^4 = 84$$

Solution 2

In this case we'll integrate with respect to *x* first and then *y*. Here is the work for this solution.

$$\begin{aligned}\iint_R 6xy^2\,dA &= \int_1^2\int_2^4 6xy^2\,dx\,dy\\ &= \int_1^2 \left(3x^2y^2\right)\Big|_2^4\,dy\\ &= \int_1^2 36y^2\,dy\\ &= 12y^3\Big|_1^2\\ &= 84\end{aligned}$$

Sure enough the same answer as the first solution.

So, remember that we can do the integration in any order.

(b) $\displaystyle\iint_R 2x-4y^3\,dA$, $R=\left[-5,4\right]\times\left[0,3\right]$

For this integral we'll integrate with respect to *y* first.

$$\begin{aligned}\iint_R 2x-4y^3\,dA &= \int_{-5}^4\int_0^3 2x-4y^3\,dy\,dx\\ &= \int_{-5}^4 \left(2xy-y^4\right)\Big|_0^3\,dx\\ &= \int_{-5}^4 6x-81\,dx\\ &= \left(3x^2-81x\right)\Big|_{-5}^4\\ &= -756\end{aligned}$$

Remember that when integrating with respect to *y* all *x*'s are treated as constants and so as far as the inner integral is concerned the $2x$ is a constant and we know that when we integrate constants with respect to *y* we just tack on a *y* and so we get $2xy$ from the first term.

(c) $\displaystyle\iint_R x^2y^2+\cos\left(\pi x\right)+\sin\left(\pi y\right)dA$, $R=\left[-2,-1\right]\times\left[0,1\right]$

In this case we'll integrate with respect to *x* first.

$$\iint_R x^2y^2 + \cos(\pi x) + \sin(\pi y)\, dA = \int_0^1 \int_{-2}^{-1} x^2y^2 + \cos(\pi x) + \sin(\pi y)\, dx\, dy$$
$$= \int_0^1 \left(\frac{1}{3}x^3y^2 + \frac{1}{\pi}\sin(\pi x) + x\sin(\pi y) \right)\Big|_{-2}^{-1} dy$$
$$= \int_0^1 \frac{7}{3}y^2 + \sin(\pi y)\, dy$$
$$= \frac{7}{9}y^3 - \frac{1}{\pi}\cos(\pi y)\Big|_0^1$$
$$= \frac{7}{9} + \frac{2}{\pi}$$

Don't forget your basic Calculus I substitutions!

(d) $\displaystyle\iint_R \frac{1}{(2x+3y)^2}\, dA\,,\quad R = [0,1]\times[1,2]$

In this case because the limits for *x* are kind of nice (*i.e.* they are zero and one which are often nice for evaluation) let's integrate with respect to *x* first. We'll also rewrite the integrand to help with the first integration.

$$\iint_R (2x+3y)^{-2}\, dA = \int_1^2 \int_0^1 (2x+3y)^{-2}\, dx\, dy$$
$$= \int_1^2 \left(-\frac{1}{2}(2x+3y)^{-1} \right)\Big|_0^1 dy$$
$$= -\frac{1}{2}\int_1^2 \frac{1}{2+3y} - \frac{1}{3y}\, dy$$
$$= -\frac{1}{2}\left(\frac{1}{3}\ln|2+3y| - \frac{1}{3}\ln|y| \right)\Big|_1^2$$
$$= -\frac{1}{6}(\ln 8 - \ln 2 - \ln 5)$$

(e) $\displaystyle\iint_R x\mathbf{e}^{xy}\, dA\,,\quad R = [-1,2]\times[0,1]$

Now, while we can technically integrate with respect to either variable first sometimes one way is significantly easier than the other way. In this case it will be significantly easier to integrate with respect to *y* first as we will see.

$$\iint_R x\mathbf{e}^{xy}\, dA = \int_{-1}^2 \int_0^1 x\mathbf{e}^{xy}\, dy\, dx$$

The y integration can be done with the quick substitution,

$$u = xy \qquad du = x\,dy$$

which gives

$$\begin{aligned}\iint\limits_R x\mathbf{e}^{xy}\,dA &= \int_{-1}^{2} \mathbf{e}^{xy}\Big|_0^1\,dx\\ &= \int_{-1}^{2} \mathbf{e}^{x} - 1\,dx\\ &= \left(\mathbf{e}^{x} - x\right)\Big|_{-1}^{2}\\ &= \mathbf{e}^{2} - 2 - \left(\mathbf{e}^{-1} + 1\right)\\ &= \mathbf{e}^{2} - \mathbf{e}^{-1} - 3\end{aligned}$$

So, not too bad of an integral there provided you get the substitution. Now let's see what would happen if we had integrated with respect to x first.

$$\iint\limits_R x\mathbf{e}^{xy}\,dA = \int_0^1\int_{-1}^{2} x\mathbf{e}^{xy}\,dx\,dy$$

In order to do this we would have to use integration by parts as follows,

$$u = x \qquad dv = \mathbf{e}^{xy}\,dx$$

$$du = dx \qquad v = \frac{1}{y}\mathbf{e}^{xy}$$

The integral is then,

$$\begin{aligned}\iint\limits_R x\mathbf{e}^{xy}\,dA &= \int_0^1 \left(\frac{x}{y}\mathbf{e}^{xy} - \int \frac{1}{y}\mathbf{e}^{xy}\,dx\right)\Bigg|_{-1}^{2}\,dy\\ &= \int_0^1 \left(\frac{x}{y}\mathbf{e}^{xy} - \frac{1}{y^2}\mathbf{e}^{xy}\right)\Bigg|_{-1}^{2}\,dy\\ &= \int_0^1 \left(\frac{2}{y}\mathbf{e}^{2y} - \frac{1}{y^2}\mathbf{e}^{2y}\right) - \left(-\frac{1}{y}\mathbf{e}^{-y} - \frac{1}{y^2}\mathbf{e}^{-y}\right)dy\end{aligned}$$

We're not even going to continue here as these are very difficult, if not impossible, integrals to do.

[Return to Problems]

As we saw in the previous set of examples we can do the integral in either direction. However, sometimes one direction of integration is significantly easier than the other so make sure that you think about which one you should do first before actually doing the integral.

The next topic of this section is a quick fact that can be used to make some iterated integrals somewhat easier to compute on occasion.

There is a nice special case of this kind of integral. First, let's assume that $f\left(x,y\right)=g\left(x\right)h\left(y\right)$ and let's also assume we are integrating over a rectangle given by $R=\left[a,b\right]\times\left[c,d\right]$. Then, the integral becomes,

$$\iint\limits_R f\left(x,y\right)dA=\iint\limits_R g\left(x\right)h\left(y\right)dA=\int_c^d\int_a^b g\left(x\right)h\left(y\right)dx\,dy$$

Note that it doesn't matter in this case which variable we integrate first as either order will arrive at the same result with the same work.

Next, notice that because the inner integral is with respect to *x* and $h\left(y\right)$ is a function only of *y* it can be considered a "constant" as far as the *x* integration is concerned (changing *x* will not affect the value of *y*!) and because it is also times $g\left(x\right)$ we can factor the $h\left(y\right)$ out of the inner integral. Doing this gives,

$$\iint\limits_R f\left(x,y\right)dA=\iint\limits_R g\left(x\right)h\left(y\right)dA=\int_c^d h\left(y\right)\int_a^b g\left(x\right)dx\,dy$$

Now, $\int_a^b g\left(x\right)dx$ is a standard Calculus I definite integral and we know that its value is just a constant. Therefore, it can be factored out of the *y* integration to get,

$$\iint\limits_R f\left(x,y\right)dA=\iint\limits_R g\left(x\right)h\left(y\right)dA=\int_a^b g\left(x\right)dx\int_c^d h\left(y\right)dy$$

In other words, if we can break up the function into a function only of *x* times a function of only *y* then we can do the two integrals individually and multiply them together.

Here is a quick summary of this idea.

Fact

If $f\left(x,y\right)=g\left(x\right)h\left(y\right)$ and we are integrating over the rectangle $R=\left[a,b\right]\times\left[c,d\right]$ then,

$$\iint\limits_R f\left(x,y\right)dA=\iint\limits_R g\left(x\right)h\left(y\right)dA=\left(\int_a^b g\left(x\right)dx\right)\left(\int_c^d h\left(y\right)dy\right)$$

Let's do a quick example using this integral.

Example 2 Evaluate $\iint\limits_R x\cos^2(y)\,dA$, $R = [-2,3] \times \left[0, \frac{\pi}{2}\right]$.

Solution

Since the integrand is a function of *x* times a function of *y* we can use the fact.

$$\begin{aligned}\iint\limits_R x\cos^2(y)\,dA &= \left(\int_{-2}^{3} x\,dx\right)\left(\int_0^{\frac{\pi}{2}} \cos^2(y)\,dy\right)\\ &= \left(\frac{1}{2}x^2\right)\Big|_{-2}^{3}\left(\frac{1}{2}\int_0^{\frac{\pi}{2}} 1+\cos(2y)\,dy\right)\\ &= \left(\frac{5}{2}\right)\left(\frac{1}{2}\left(y+\frac{1}{2}\sin(2y)\right)\Big|_0^{\frac{\pi}{2}}\right)\\ &= \frac{5\pi}{8}\end{aligned}$$

We have one more topic to discuss in this section. This topic really doesn't have anything to do with iterated integrals, but this is as good a place as any to put it and there are liable to be some questions about it at this point as well so this is as good a place as any.

What we want to do is discuss single indefinite integrals of a function of two variables. In other words, we want to look at integrals like the following.

$$\int x\sec^2(2y) + 4xy\,dy$$

$$\int x^3 - \mathbf{e}^{-\frac{x}{y}}\,dx$$

From Calculus I we know that these integrals are asking what function that we differentiated to get the integrand. However, in this case we need to pay attention to the differential (*dy* or *dx*) in the integral, because that will change things a little.

In the case of the first integral we are asking what function we differentiated with respect to *y* to get the integrand while in the second integral we're asking what function differentiated with respect to *x* to get the integrand. For the most part answering these questions isn't that difficult. The important issue is how we deal with the constant of integration.

Here are the integrals.

$$\int x\sec^2(2y) + 4xy\,dy = \frac{x}{2}\tan(2y) + 2xy^2 + g(x)$$

$$\int x^3 - \mathbf{e}^{-\frac{x}{y}}\,dx = \frac{1}{4}x^4 + y\,\mathbf{e}^{-\frac{x}{y}} + h(y)$$

Notice that the "constants" of integration are now functions of the opposite variable. In the first integral we are differentiating with respect to *y* and we know that any function involving only *x*'s will differentiate to zero and so when integrating with respect to *y* we need to acknowledge that there may have been a function of only *x*'s in the function and so the "constant" of integration is a function of *x*.

Likewise, in the second integral, the "constant" of integration must be a function of *y* since we are integrating with respect to *x*. Again, remember if we differentiate the answer with respect to *x* then any function of only *y*'s will differentiate to zero.

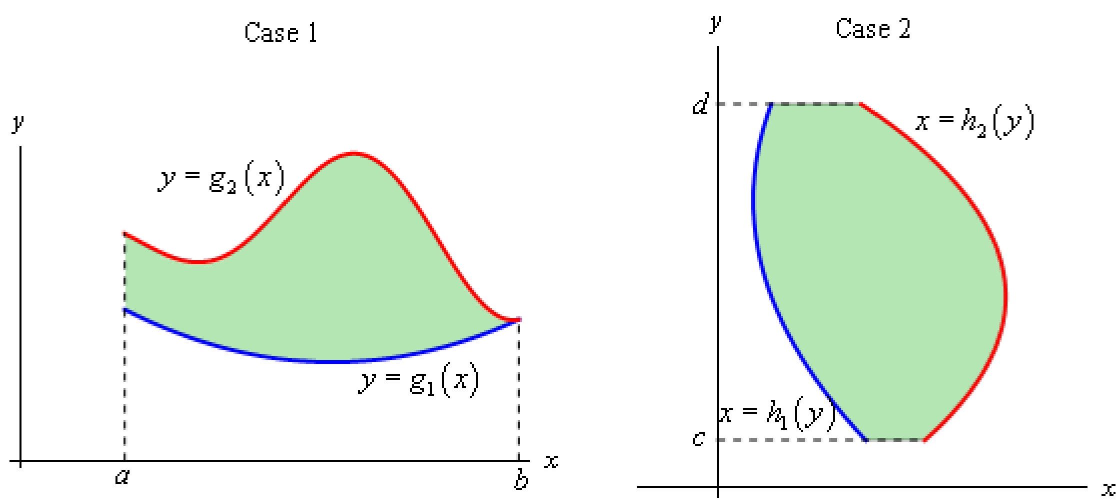
Section 4-3 : Double Integrals over General Regions

In the previous section we looked at double integrals over rectangular regions. The problem with this is that most of the regions are not rectangular so we need to now look at the following double integral,

$$\iint\limits_D f(x,y)\,dA$$

where *D* is any region.

There are two types of regions that we need to look at. Here is a sketch of both of them.



We will often use *set builder notation* to describe these regions. Here is the definition for the region in Case 1

$$D = \left\{ (x,y) \mid a \le x \le b,\; g_1(x) \le y \le g_2(x) \right\}$$

and here is the definition for the region in Case 2.

$$D = \left\{ (x,y) \mid h_1(y) \le x \le h_2(y), c \le y \le d \right\}$$

This notation is really just a fancy way of saying we are going to use all the points, (x,y), in which both of the coordinates satisfy the two given inequalities.

The double integral for both of these cases are defined in terms of iterated integrals as follows.

In Case 1 where $D = \left\{ (x,y) \mid a \le x \le b,\; g_1(x) \le y \le g_2(x) \right\}$ the integral is defined to be,

$$\iint\limits_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

In Case 2 where $D = \left\{ (x,y) \mid h_1(y) \le x \le h_2(y), c \le y \le d \right\}$ the integral is defined to be,

$$\iint\limits_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

Here are some properties of the double integral that we should go over before we actually do some examples. Note that all three of these properties are really just extensions of properties of single integrals that have been extended to double integrals.

Properties

1. $\displaystyle\iint\limits_D f(x,y)+g(x,y)\,dA = \iint\limits_D f(x,y)\,dA + \iint\limits_D g(x,y)\,dA$

2. $\displaystyle\iint\limits_D cf(x,y)\,dA = c\iint\limits_D f(x,y)\,dA$, where *c* is any constant.

3. If the region *D* can be split into two separate regions D_1 and D_2 then the integral can be written as

$$\iint\limits_D f(x,y)\,dA = \iint\limits_{D_1} f(x,y)\,dA + \iint\limits_{D_2} f(x,y)\,dA$$

Let's take a look at some examples of double integrals over general regions.

Example 1 Evaluate each of the following integrals over the given region *D*.

(a) $\displaystyle\iint\limits_D \mathbf{e}^{\frac{x}{y}}\,dA$, $D = \left\{(x,y)\,|\,1 \le y \le 2,\ y \le x \le y^3\right\}$

(b) $\displaystyle\iint\limits_D 4xy - y^3\,dA$, *D* is the region bounded by $y = \sqrt{x}$ and $y = x^3$.

(c) $\displaystyle\iint\limits_D 6x^2 - 40y\,dA$, *D* is the triangle with vertices $(0,3)$, $(1,1)$, and $(5,3)$.

Solution

(a) $\displaystyle\iint\limits_D \mathbf{e}^{\frac{x}{y}}\,dA$, $D = \left\{(x,y)\,|\,1 \le y \le 2,\ y \le x \le y^3\right\}$

Okay, this first one is set up to just use the formula above so let's do that.

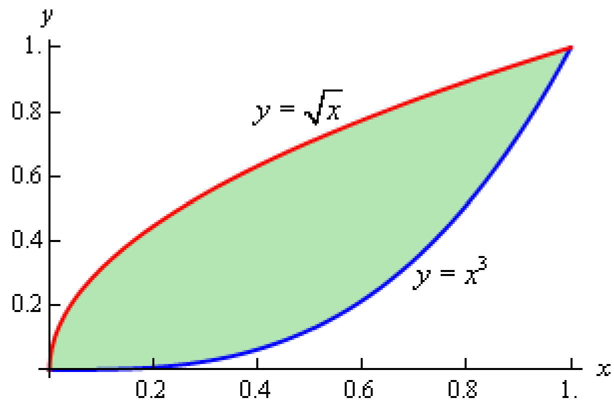
$$\iint\limits_D \mathbf{e}^{\frac{x}{y}}\,dA = \int_1^2 \int_y^{y^3} \mathbf{e}^{\frac{x}{y}}\,dx\,dy = \int_1^2 \left. y\,\mathbf{e}^{\frac{x}{y}} \right|_y^{y^3} dy$$

$$= \int_1^2 y\,\mathbf{e}^{y^2} - y\mathbf{e}^1\,dy$$

$$= \left.\left(\frac{1}{2}\mathbf{e}^{y^2} - \frac{1}{2}y^2\mathbf{e}^1\right)\right|_1^2 = \frac{1}{2}\mathbf{e}^4 - 2\mathbf{e}^1$$

(b) $\iint\limits_D 4xy - y^3\, dA$**,** ***D* is the region bounded by** $y = \sqrt{x}$ **and** $y = x^3$**.**

In this case we need to determine the two inequalities for *x* and *y* that we need to do the integral. The best way to do this is the graph the two curves. Here is a sketch.



So, from the sketch we can see that that two inequalities are,

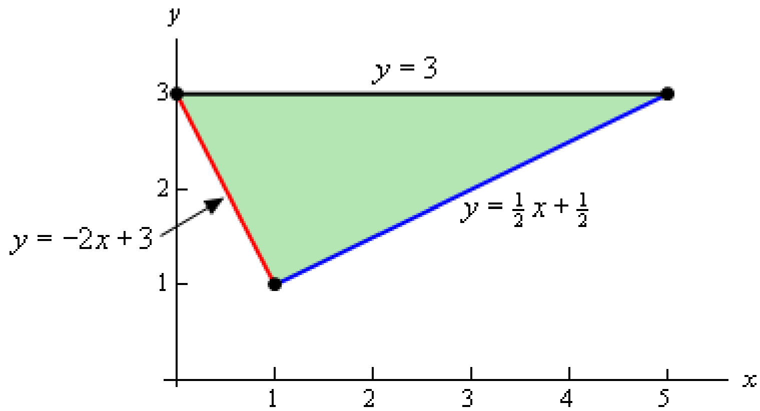
$$0 \le x \le 1 \qquad x^3 \le y \le \sqrt{x}$$

We can now do the integral,

$$\begin{aligned}\iint\limits_D 4xy - y^3\, dA &= \int_0^1 \int_{x^3}^{\sqrt{x}} 4xy - y^3\, dy\, dx \\ &= \int_0^1 \left(2xy^2 - \frac{1}{4}y^4 \right)\Bigg|_{x^3}^{\sqrt{x}} dx \\ &= \int_0^1 \frac{7}{4}x^2 - 2x^7 + \frac{1}{4}x^{12}\, dx \\ &= \left(\frac{7}{12}x^3 - \frac{1}{4}x^8 + \frac{1}{52}x^{13} \right)\Bigg|_0^1 = \frac{55}{156}\end{aligned}$$

(c) $\iint\limits_D 6x^2 - 40y\, dA$**,** ***D* is the triangle with vertices** $(0,3)$**,** $(1,1)$**, and** $(5,3)$**.**

We got even less information about the region this time. Let's start this off by sketching the triangle.



Since we have two points on each edge it is easy to get the equations for each edge and so we'll leave it to you to verify the equations.

Now, there are two ways to describe this region. If we use functions of *x*, as shown in the image we will have to break the region up into two different pieces since the lower function is different depending upon the value of *x*. In this case the region would be given by $D = D_1 \cup D_2$ where,

$$D_1 = \left\{ (x,y) \,|\, 0 \le x \le 1, \; -2x+3 \le y \le 3 \right\}$$

$$D_2 = \left\{ (x,y) \,|\, 1 \le x \le 5, \; \frac{1}{2}x + \frac{1}{2} \le y \le 3 \right\}$$

Note the $\cup$ is the "union" symbol and just means that *D* is the region we get by combing the two regions. If we do this then we'll need to do two separate integrals, one for each of the regions.

To avoid this we could turn things around and solve the two equations for *x* to get,

$$y = -2x + 3 \qquad \Rightarrow \qquad x = -\frac{1}{2}y + \frac{3}{2}$$

$$y = \frac{1}{2}x + \frac{1}{2} \qquad \Rightarrow \qquad x = 2y - 1$$

If we do this we can notice that the same function is always on the right and the same function is always on the left and so the region is,

$$D = \left\{ (x,y) \,|\, -\frac{1}{2}y + \frac{3}{2} \le x \le 2y - 1, \; 1 \le y \le 3 \right\}$$

Writing the region in this form means doing a single integral instead of the two integrals we'd have to do otherwise.

Either way should give the same answer and so we can get an example in the notes of splitting a region up let's do both integrals.

Solution 1

$$\begin{aligned}\iint\limits_D 6x^2 - 40y\,dA &= \iint\limits_{D_1} 6x^2 - 40y\,dA + \iint\limits_{D_2} 6x^2 - 40y\,dA \\ &= \int_0^1 \int_{-2x+3}^{3} 6x^2 - 40y\,dy\,dx + \int_1^5 \int_{\frac{1}{2}x+\frac{1}{2}}^{3} 6x^2 - 40y\,dy\,dx \\ &= \int_0^1 \left(6x^2y - 20y^2\right)\Big|_{-2x+3}^{3} dx + \int_1^5 \left(6x^2y - 20y^2\right)\Big|_{\frac{1}{2}x+\frac{1}{2}}^{3} dx \\ &= \int_0^1 12x^3 - 180 + 20\left(3-2x\right)^2 dx + \int_1^5 -3x^3 + 15x^2 - 180 + 20\left(\tfrac{1}{2}x + \tfrac{1}{2}\right)^2 dx \\ &= \left(3x^4 - 180x - \tfrac{10}{3}\left(3-2x\right)^3\right)\Big|_0^1 + \left(-\tfrac{3}{4}x^4 + 5x^3 - 180x + \tfrac{40}{3}\left(\tfrac{1}{2}x + \tfrac{1}{2}\right)^3\right)\Big|_1^5 \\ &= -\frac{935}{3}\end{aligned}$$

That was a lot of work. Notice however, that after we did the first substitution that we didn't multiply everything out. The two quadratic terms can be easily integrated with a basic Calc I substitution and so we didn't bother to multiply them out. We'll do that on occasion to make some of these integrals a little easier.

Solution 2

This solution will be a lot less work since we are only going to do a single integral.

$$\begin{aligned}\iint\limits_D 6x^2 - 40y\,dA &= \int_1^3 \int_{-\frac{1}{2}y+\frac{3}{2}}^{2y-1} 6x^2 - 40y\,dx\,dy \\ &= \int_1^3 \left(2x^3 - 40xy\right)\Big|_{-\frac{1}{2}y+\frac{3}{2}}^{2y-1} dy \\ &= \int_1^3 100y - 100y^2 + 2\left(2y-1\right)^3 - 2\left(-\tfrac{1}{2}y + \tfrac{3}{2}\right)^3 dy \\ &= \left(50y^2 - \tfrac{100}{3}y^3 + \tfrac{1}{4}\left(2y-1\right)^4 + \left(-\tfrac{1}{2}y + \tfrac{3}{2}\right)^4\right)\Big|_1^3 \\ &= -\frac{935}{3}\end{aligned}$$

So, the numbers were a little messier, but other than that there was much less work for the same result. Also notice that again we didn't cube out the two terms as they are easier to deal with using a Calc I substitution.

As the last part of the previous example has shown us we can integrate these integrals in either order (*i.e. x* followed by *y* or *y* followed by *x*), although often one order will be easier than the other. In fact, there will be times when it will not even be possible to do the integral in one order while it will be possible to do the integral in the other order.

Also, do not forget about Calculus I substitutions. Students often just get in a hurry and multiply everything out after doing the integral evaluation and end up missing a really simple Calculus I substitution that avoids the hassle of multiplying everything out. Calculus I substitutions don't always show up, but then they do they almost always simplify the work for the rest of the problem.

Let's see a couple of examples of these kinds of integrals.

Example 2 Evaluate the following integrals by first reversing the order of integration.

(a) $\displaystyle\int_0^3 \int_{x^2}^9 x^3 \mathbf{e}^{y^3}\, dy\, dx$

(b) $\displaystyle\int_0^8 \int_{\sqrt[3]{y}}^2 \sqrt{x^4+1}\, dx\, dy$

Solution

(a) $\displaystyle\int_0^3 \int_{x^2}^9 x^3 \mathbf{e}^{y^3}\, dy\, dx$

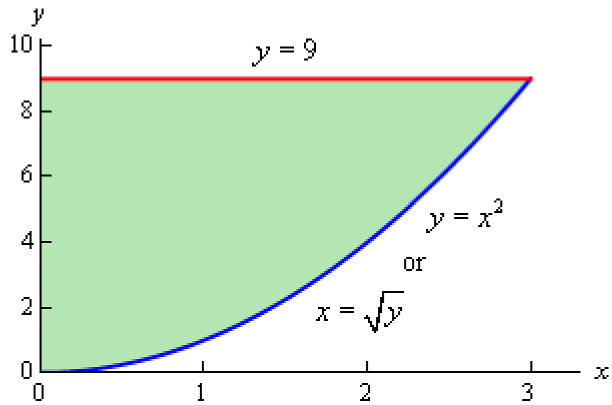
First, notice that if we try to integrate with respect to *y* we can't do the integral because we would need a y^2 in front of the exponential in order to do the *y* integration. We are going to hope that if we reverse the order of integration we will get an integral that we can do.

Now, when we say that we're going to reverse the order of integration this means that we want to integrate with respect to *x* first and then *y*. Note as well that we can't just interchange the integrals, keeping the original limits, and be done with it. This would not fix our original problem and in order to integrate with respect to *x* we can't have *x*'s in the limits of the integrals. Even if we ignored that the answer would not be a constant as it should be.

So, let's see how we reverse the order of integration. The best way to reverse the order of integration is to first sketch the region given by the original limits of integration. From the integral we see that the inequalities that define this region are,

$$0 \le x \le 3$$
$$x^2 \le y \le 9$$

These inequalities tell us that we want the region with $y = x^2$ on the lower boundary and $y = 9$ on the upper boundary that lies between $x = 0$ and $x = 3$. Here is a sketch of that region.



Since we want to integrate with respect to *x* first we will need to determine limits of *x* (probably in terms of *y*) and then get the limits on the *y*'s. Here they are for this region.

$$0 \le x \le \sqrt{y}$$
$$0 \le y \le 9$$

Any horizontal line drawn in this region will start at $x = 0$ and end at $x = \sqrt{y}$ and so these are the limits on the *x*'s and the range of *y*'s for the regions is 0 to 9.

The integral, with the order reversed, is now,

$$\int_0^3 \int_{x^2}^9 x^3 \mathbf{e}^{y^3}\, dy\, dx = \int_0^9 \int_0^{\sqrt{y}} x^3 \mathbf{e}^{y^3}\, dx\, dy$$

and notice that we can do the first integration with this order. We'll also hope that this will give us a second integral that we can do. Here is the work for this integral.

$$\begin{aligned} \int_0^3 \int_{x^2}^9 x^3 \mathbf{e}^{y^3}\, dy\, dx &= \int_0^9 \int_0^{\sqrt{y}} x^3 \mathbf{e}^{y^3}\, dx\, dy \\ &= \int_0^9 \left. \frac{1}{4} x^4 \mathbf{e}^{y^3} \right|_0^{\sqrt{y}} dy \\ &= \int_0^9 \frac{1}{4} y^2 \mathbf{e}^{y^3}\, dy \\ &= \left. \frac{1}{12} \mathbf{e}^{y^3} \right|_0^9 \\ &= \frac{1}{12}\left(\mathbf{e}^{729} - 1\right) \end{aligned}$$

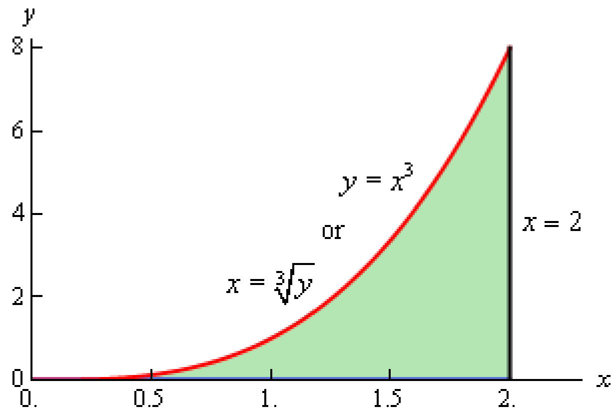
So, as we hoped, we were able to do the integral once we interchanged the order of integration.

(b) $\int_0^8 \int_{\sqrt[3]{y}}^2 \sqrt{x^4+1}\,dx\,dy$

As with the first integral we cannot do this integral by integrating with respect to *x* first so we'll hope that by reversing the order of integration we will get something that we can integrate. Here are the limits for the variables that we get from this integral.

$$\sqrt[3]{y} \le x \le 2$$
$$0 \le y \le 8$$

and here is a sketch of this region.



So, if we reverse the order of integration we get the following limits.

$$0 \le x \le 2$$
$$0 \le y \le x^3$$

The integral is then,

$$\begin{aligned} \int_0^8 \int_{\sqrt[3]{y}}^2 \sqrt{x^4+1}\,dx\,dy &= \int_0^2 \int_0^{x^3} \sqrt{x^4+1}\,dy\,dx \\ &= \int_0^2 y\sqrt{x^4+1}\Big|_0^{x^3}\,dx \\ &= \int_0^2 x^3\sqrt{x^4+1}\,dx = \frac{1}{6}\left(17^{\frac{3}{2}}-1\right) \end{aligned}$$

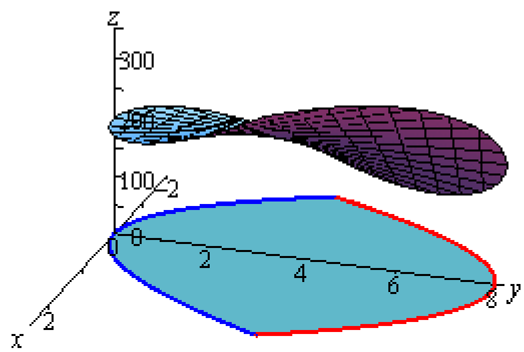
The final topic of this section is two geometric interpretations of a double integral. The first interpretation is an extension of the idea that we used to develop the idea of a double integral in the first section of this chapter. We did this by looking at the volume of the solid that was below the surface of the function $z = f(x,y)$ and over the rectangle *R* in the *xy*-plane. This idea can be extended to more general regions.

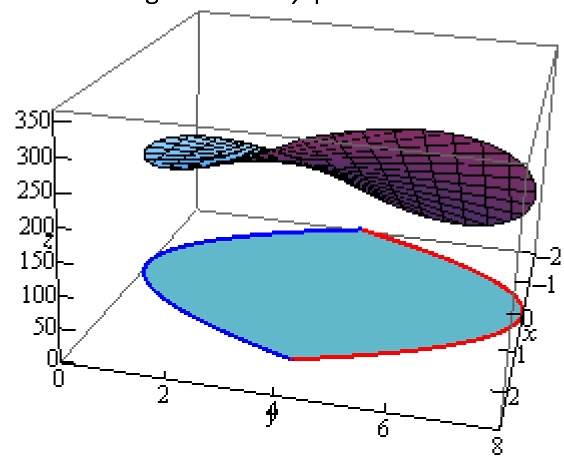
The volume of the solid that lies below the surface given by $z = f(x,y)$ and above the region *D* in the *xy*-plane is given by,

$$V = \iint\limits_D f\left(x,y\right) dA$$

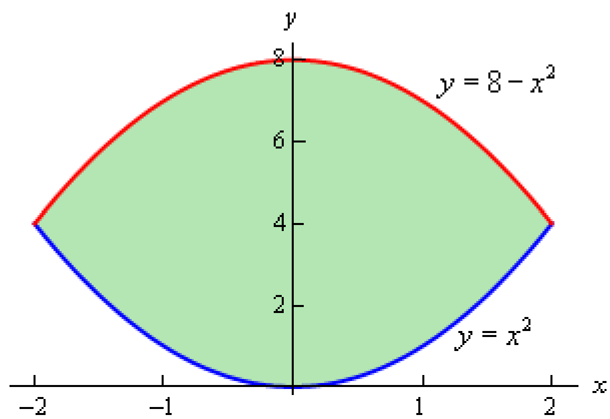
Example 3 Find the volume of the solid that lies below the surface given by $z = 16xy + 200$ and lies above the region in the *xy*-plane bounded by $y = x^2$ and $y = 8 - x^2$.

Solution
Here is the graph of the surface and we've tried to show the region in the *xy*-plane below the surface.





Here is a sketch of the region in the *xy*-plane by itself.



By setting the two bounding equations equal we can see that they will intersect at $x = 2$ and $x = -2$. So, the inequalities that will define the region *D* in the *xy*-plane are,

$$-2 \le x \le 2$$
$$x^2 \le y \le 8 - x^2$$

The volume is then given by,

$$
\begin{aligned}
V &= \iint\limits_D 16xy + 200\,dA \\
&= \int_{-2}^{2}\int_{x^2}^{8-x^2} 16xy + 200\,dy\,dx \\
&= \int_{-2}^{2} \left(8xy^2 + 200y\right)\Big|_{x^2}^{8-x^2}\,dx \\
&= \int_{-2}^{2} -128x^3 - 400x^2 + 512x + 1600\,dx \\
&= \left(-32x^4 - \frac{400}{3}x^3 + 256x^2 + 1600x\right)\Bigg|_{-2}^{2} = \frac{12800}{3}
\end{aligned}
$$

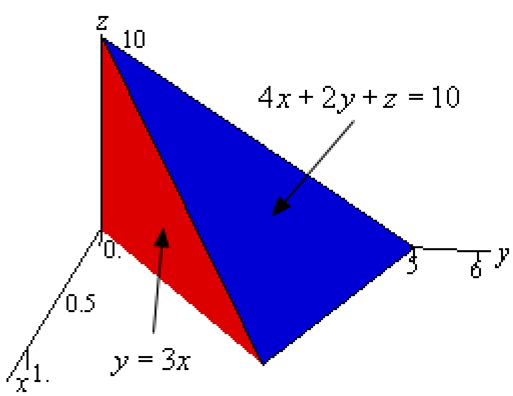
Example 4 Find the volume of the solid enclosed by the planes $4x + 2y + z = 10$, $y = 3x$, $z = 0$, $x = 0$.

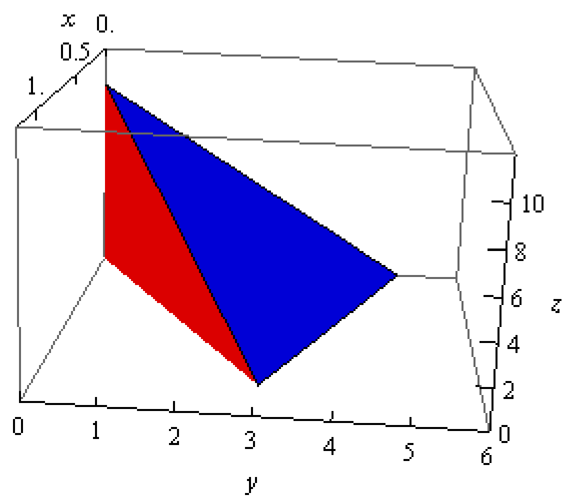
Solution This example is a little different from the previous one. Here the region *D* is not explicitly given so we're going to have to find it. First, notice that the last two planes are really telling us that we won't go past the *xy*-plane and the *yz*-plane when we reach them.

The first plane, $4x + 2y + z = 10$, is the top of the volume and so we are really looking for the volume under,

$$z = 10 - 4x - 2y$$

and above the region *D* in the *xy*-plane. The second plane, $y = 3x$ (yes that is a plane), gives one of the sides of the volume as shown below.

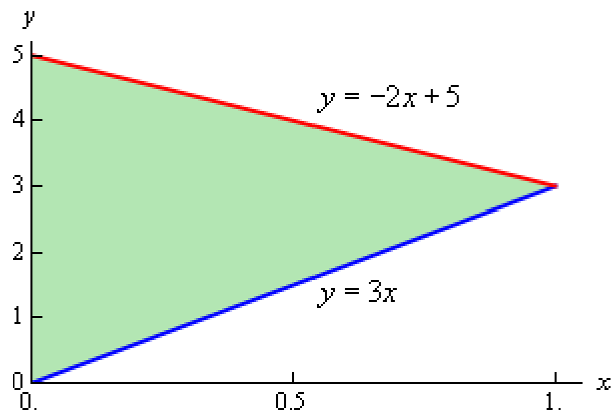




The region *D* will be the region in the *xy*-plane (*i.e.* $z = 0$) that is bounded by $y = 3x$, $x = 0$, and the line where $z + 4x + 2y = 10$ intersects the *xy*-plane. We can determine where $z + 4x + 2y = 10$ intersects the *xy*-plane by plugging $z = 0$ into it.

$$0 + 4x + 2y = 10 \qquad \Rightarrow \qquad 2x + y = 5 \qquad \Rightarrow \qquad y = -2x + 5$$

So, here is a sketch the region *D*.



The region *D* is really where this solid will sit on the *xy*-plane and here are the inequalities that define the region.

$$0 \le x \le 1$$
$$3x \le y \le -2x + 5$$

Here is the volume of this solid.

$$\begin{aligned} V &= \iint_D 10 - 4x - 2y \, dA \\ &= \int_0^1 \int_{3x}^{-2x+5} 10 - 4x - 2y \, dy \, dx \\ &= \int_0^1 \left(10y - 4xy - y^2\right)\Big|_{3x}^{-2x+5} dx \\ &= \int_0^1 25x^2 - 50x + 25 \, dx \\ &= \left(\frac{25}{3}x^3 - 25x^2 + 25x\right)\Big|_0^1 = \frac{25}{3} \end{aligned}$$

Note that more generally,

$$V = \iint_D f(x, y) \, dA$$

gives the net volume between the graph of $z = f(x, y)$ and the region *D* in the *xy*-plane. Regions that are below the *xy*-plane have a negative volume and regions that are above the *xy*-plane have a positive volume.

We saw a similar idea in Calculus I where,

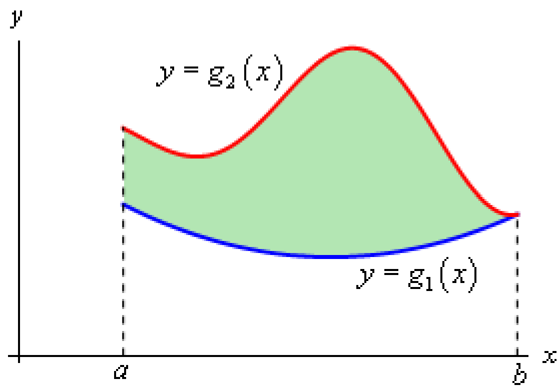
$$A = \int_a^b f(x) \, dx$$

gives the net area between the curve given by $y = f(x)$ and the *x*-axis on the interval $[a,b]$.

The second geometric interpretation of a double integral is the following.

$$\text{Area of } D = \iint\limits_D dA$$

This is easy to see why this is true in general. Let's suppose that we want to find the area of the region shown below.



From Calculus I we know that this area can be found by the integral,

$$A = \int_a^b g_2(x) - g_1(x)\,dx$$

Or in terms of a double integral we have,

$$\begin{aligned}\text{Area of } D &= \iint\limits_D dA \\ &= \int_a^b \int_{g_1(x)}^{g_2(x)} dy\,dx \\ &= \int_a^b y\Big|_{g_1(x)}^{g_2(x)} dx = \int_a^b g_2(x) - g_1(x)\,dx\end{aligned}$$

This is exactly the same formula we had in Calculus I.

Section 4-4 : Double Integrals in Polar Coordinates

To this point we've seen quite a few double integrals. However, in every case we've seen to this point the region *D* could be easily described in terms of simple functions in Cartesian coordinates. In this section we want to look at some regions that are much easier to describe in terms of polar coordinates. For instance, we might have a region that is a disk, ring, or a portion of a disk or ring. In these cases, using Cartesian coordinates could be somewhat cumbersome. For instance, let's suppose we wanted to do the following integral,

$$\iint\limits_D f\left(x,y\right)dA, \quad D \text{ is the disk of radius 2}$$

To this we would have to determine a set of inequalities for *x* and *y* that describe this region. These would be,

$$\begin{gathered} -2 \le x \le 2 \\ -\sqrt{4-x^2} \le y \le \sqrt{4-x^2} \end{gathered}$$

With these limits the integral would become,

$$\iint\limits_D f\left(x,y\right)dA = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f\left(x,y\right)dy\,dx$$

Due to the limits on the inner integral this is liable to be an unpleasant integral to compute.

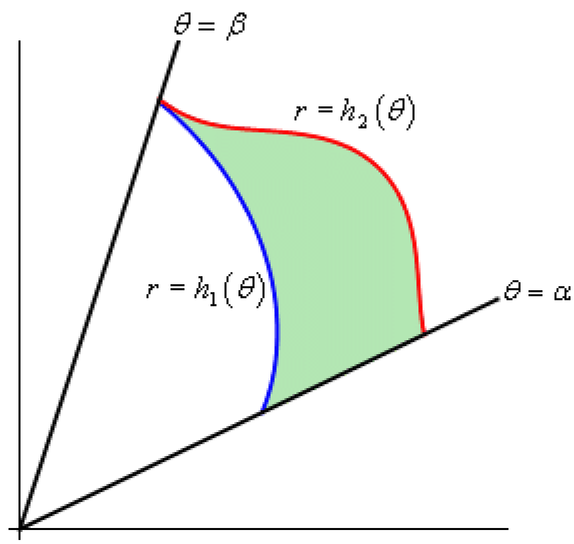
However, a disk of radius 2 can be defined in polar coordinates by the following inequalities,

$$\begin{gathered} 0 \le \theta \le 2\pi \\ 0 \le r \le 2 \end{gathered}$$

These are very simple limits and, in fact, are constant limits of integration which almost always makes integrals somewhat easier.

So, if we could convert our double integral formula into one involving polar coordinates we would be in pretty good shape. The problem is that we can't just convert the *dx* and the *dy* into a *dr* and a $d\theta$. In computing double integrals to this point we have been using the fact that $dA = dx\,dy$ and this really does require Cartesian coordinates to use. Once we've moved into polar coordinates $dA \ne dr\,d\theta$ and so we're going to need to determine just what *dA* is under polar coordinates.

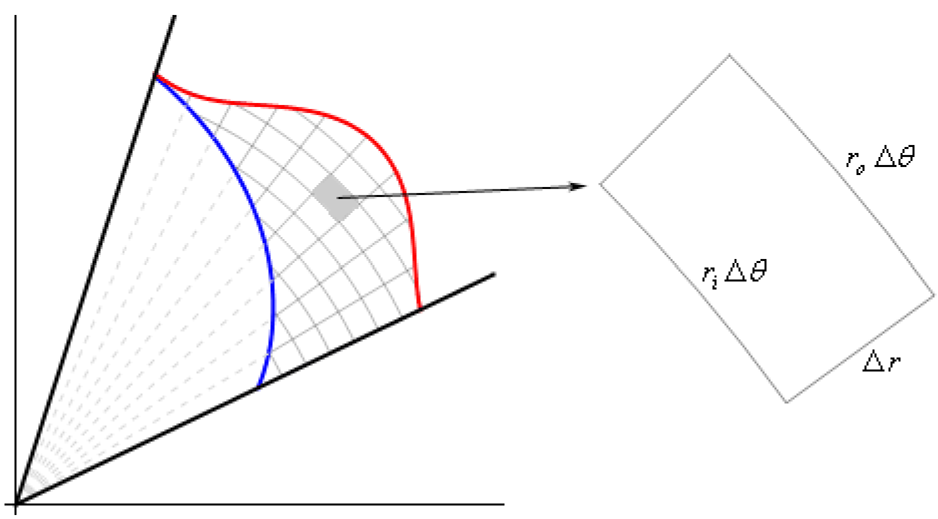
So, let's step back a little bit and start off with a general region in terms of polar coordinates and see what we can do with that. Here is a sketch of some region using polar coordinates.



So, our general region will be defined by inequalities,

$$\alpha \le \theta \le \beta$$
$$h_1(\theta) \le r \le h_2(\theta)$$

Now, to find *dA* let's redo the figure above as follows,



As shown, we'll break up the region into a mesh of radial lines and arcs. Now, if we pull one of the pieces of the mesh out as shown we have something that is almost, but not quite a rectangle. The area of this piece is ΔA. The two sides of this piece both have length $\Delta r = r_o - r_i$ where r_o is the radius of the outer arc and r_i is the radius of the inner arc. Basic geometry then tells us that the length of the inner edge is $r_i\,\Delta\theta$ while the length of the out edge is $r_o\,\Delta\theta$ where $\Delta\theta$ is the angle between the two radial lines that form the sides of this piece.

Now, let's assume that we've taken the mesh so small that we can assume that $r_i \approx r_o = r$ and with this assumption we can also assume that our piece is close enough to a rectangle that we can also then assume that,

$$\Delta A \approx r\,\Delta\theta\,\Delta r$$

Also, if we assume that the mesh is small enough then we can also assume that,

$$dA \approx \Delta A \qquad d\theta \approx \Delta\theta \qquad dr \approx \Delta r$$

With these assumptions we then get $dA \approx r\,dr\,d\theta$.

In order to arrive at this we had to make the assumption that the mesh was very small. This is not an unreasonable assumption. Recall that the definition of a double integral is in terms of two limits and as limits go to infinity the mesh size of the region will get smaller and smaller. In fact, as the mesh size gets smaller and smaller the formula above becomes more and more accurate and so we can say that,

$$dA = r\,dr\,d\theta$$

We'll see another way of deriving this once we reach the Change of Variables section later in this chapter. This second way will not involve any assumptions either and so it maybe a little better way of deriving this.

Before moving on it is again important to note that $dA \ne dr\,d\theta$. The actual formula for *dA* has an *r* in it. It will be easy to forget this *r* on occasion, but as you'll see without it some integrals will not be possible to do.

Now, if we're going to be converting an integral in Cartesian coordinates into an integral in polar coordinates we are going to have to make sure that we've also converted all the *x*'s and *y*'s into polar coordinates as well. To do this we'll need to remember the following conversion formulas,

$$x = r\cos\theta \qquad y = r\sin\theta \qquad r^2 = x^2 + y^2$$

We are now ready to write down a formula for the double integral in terms of polar coordinates.

$$\iint\limits_D f\left(x,y\right)dA = \int_{\alpha}^{\beta}\int_{h_1(\theta)}^{h_2(\theta)} f\left(r\cos\theta, r\sin\theta\right) r\,dr\,d\theta$$

It is important to not forget the added *r* and don't forget to convert the Cartesian coordinates in the function over to polar coordinates.

Let's look at a couple of examples of these kinds of integrals.

Example 1 Evaluate the following integrals by converting them into polar coordinates.

(a) $\iint\limits_D 2x\,y\,dA$, *D* is the portion of the region between the circles of radius 2 and radius 5 centered at the origin that lies in the first quadrant.

(b) $\iint\limits_D \mathbf{e}^{x^2+y^2}\,dA$, *D* is the unit disk centered at the origin.

Solution

(a) $\iint\limits_D 2x\,y\,dA$ **, *D* is the portion of the region between the circles of radius 2 and radius 5 centered at the origin that lies in the first quadrant.**

First let's get *D* in terms of polar coordinates. The circle of radius 2 is given by $r = 2$ and the circle of radius 5 is given by $r = 5$. We want the region between the two circles, so we will have the following inequality for *r*.

$$2 \le r \le 5$$

Also, since we only want the portion that is in the first quadrant we get the following range of θ's.

$$0 \le \theta \le \frac{\pi}{2}$$

Now that we've got these we can do the integral.

$$\iint\limits_D 2x\,y\,dA = \int_0^{\frac{\pi}{2}}\int_2^5 2\left(r\cos\theta\right)\left(r\sin\theta\right)r\,dr\,d\theta$$

Don't forget to do the conversions and to add in the extra *r*. Now, let's simplify and make use of the double angle formula for sine to make the integral a little easier.

$$\begin{aligned}\iint\limits_D 2x\,y\,dA &= \int_0^{\frac{\pi}{2}}\int_2^5 r^3\sin\left(2\theta\right)dr\,d\theta \\ &= \int_0^{\frac{\pi}{2}} \frac{1}{4}r^4\sin\left(2\theta\right)\Big|_2^5\,d\theta \\ &= \int_0^{\frac{\pi}{2}} \frac{609}{4}\sin\left(2\theta\right)d\theta \\ &= -\frac{609}{8}\cos\left(2\theta\right)\Big|_0^{\frac{\pi}{2}} \\ &= \frac{609}{4}\end{aligned}$$

(b) $\iint\limits_D \mathbf{e}^{x^2+y^2}\,dA$ **, *D* is the unit disk centered at the origin.**

In this case we can't do this integral in terms of Cartesian coordinates. We will however be able to do it in polar coordinates. First, the region *D* is defined by,

$$0 \le \theta \le 2\pi$$
$$0 \le r \le 1$$

In terms of polar coordinates the integral is then,

$$\iint\limits_D \mathbf{e}^{x^2+y^2}\,dA = \int_0^{2\pi}\int_0^1 r\,\mathbf{e}^{r^2}\,dr\,d\theta$$

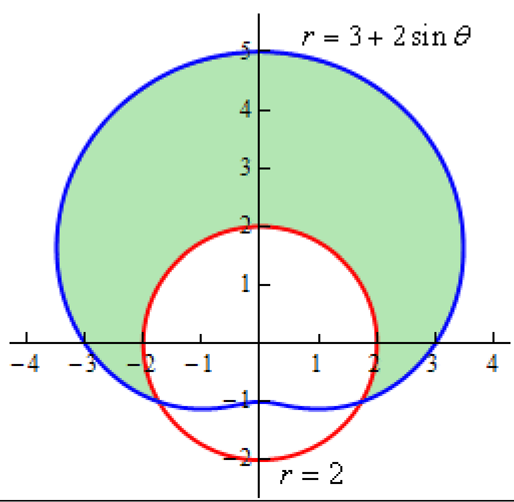
Notice that the addition of the *r* gives us an integral that we can now do. Here is the work for this integral.

$$\begin{aligned}\iint\limits_D \mathbf{e}^{x^2+y^2}\,dA &= \int_0^{2\pi}\int_0^1 r\,\mathbf{e}^{r^2}\,dr\,d\theta \\ &= \int_0^{2\pi} \left.\frac{1}{2}\mathbf{e}^{r^2}\right|_0^1 d\theta \\ &= \int_0^{2\pi} \frac{1}{2}\left(\mathbf{e}-1\right)d\theta \\ &= \pi\left(\mathbf{e}-1\right)\end{aligned}$$

Let's not forget that we still have the two geometric interpretations for these integrals as well.

Example 2 Determine the area of the region that lies inside $r = 3 + 2\sin\theta$ and outside $r = 2$.

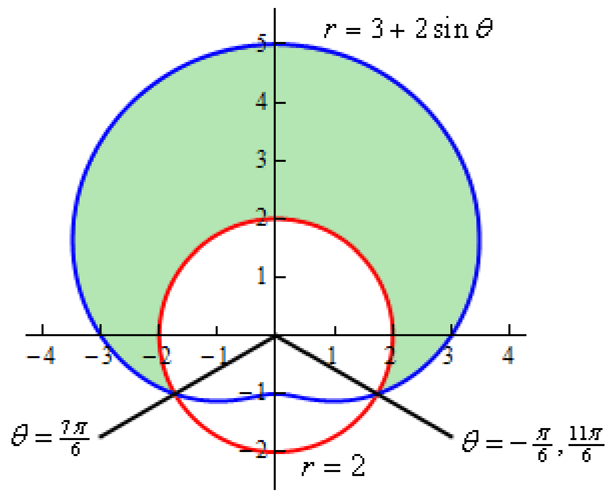
Solution
Here is a sketch of the region, *D*, that we want to determine the area of.



To determine this area we'll need to know that value of θ for which the two curves intersect. We can determine these points by setting the two equations equal and solving.

$$3+2\sin\theta=2$$
$$\sin\theta=-\frac{1}{2} \qquad \Rightarrow \qquad \theta=\frac{7\pi}{6},\frac{11\pi}{6}$$

Here is a sketch of the figure with these angles added.



Note as well that we've acknowledged that $-\frac{\pi}{6}$ is another representation for the angle $\frac{11\pi}{6}$. This is important since we need the range of θ to actually enclose the regions as we increase from the lower limit to the upper limit. If we'd chosen to use $\frac{11\pi}{6}$ then as we increase from $\frac{7\pi}{6}$ to $\frac{11\pi}{6}$ we would be tracing out the lower portion of the circle and that is not the region that we are after.

So, here are the ranges that will define the region.

$$-\frac{\pi}{6}\le\theta\le\frac{7\pi}{6}$$
$$2\le r\le 3+2\sin\theta$$

To get the ranges for *r* the function that is closest to the origin is the lower bound and the function that is farthest from the origin is the upper bound.

The area of the region *D* is then,

$$\begin{aligned} A &= \iint_D dA \\ &= \int_{-\pi/6}^{7\pi/6}\int_2^{3+2\sin\theta} r\,dr\,d\theta \\ &= \int_{-\pi/6}^{7\pi/6} \left.\frac{1}{2}r^2\right|_2^{3+2\sin\theta} d\theta \\ &= \int_{-\pi/6}^{7\pi/6} \frac{5}{2} + 6\sin\theta + 2\sin^2\theta\,d\theta \\ &= \int_{-\pi/6}^{7\pi/6} \frac{7}{2} + 6\sin\theta - \cos\left(2\theta\right)d\theta \\ &= \left.\left(\frac{7}{2}\theta - 6\cos\theta - \frac{1}{2}\sin\left(2\theta\right)\right)\right|_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \\ &= \frac{11\sqrt{3}}{2} + \frac{14\pi}{3} = 24.187 \end{aligned}$$

Example 3 Determine the volume of the region that lies under the sphere $x^2 + y^2 + z^2 = 9$, above the plane $z = 0$ and inside the cylinder $x^2 + y^2 = 5$.

Solution

We know that the formula for finding the volume of a region is,

$$V = \iint_D f\left(x, y\right)dA$$

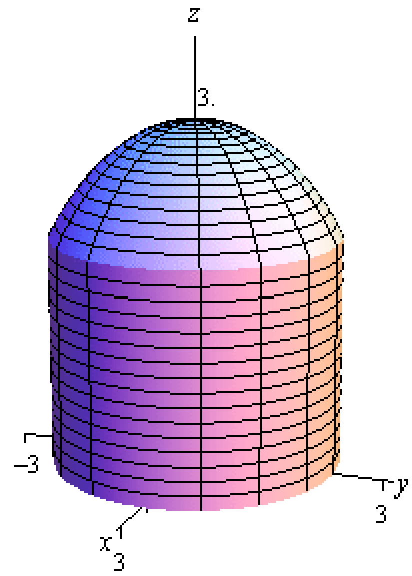
In order to make use of this formula we're going to need to determine the function that we should be integrating and the region *D* that we're going to be integrating over.

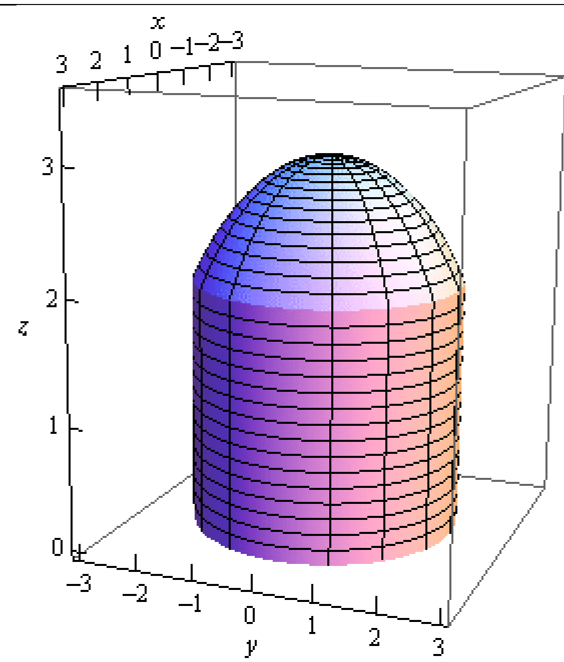
The function isn't too bad. It's just the sphere, however, we do need it to be in the form $z = f\left(x, y\right)$. We are looking at the region that lies under the sphere and above the plane $z = 0$ (just the *xy*-plane right?) and so all we need to do is solve the equation for *z* and when taking the square root we'll take the positive one since we are wanting the region above the *xy*-plane. Here is the function.

$$z = \sqrt{9 - x^2 - y^2}$$

The region *D* isn't too bad in this case either. As we take points, $\left(x, y\right)$, from the region we need to completely graph the portion of the sphere that we are working with. Since we only want the portion of the sphere that actually lies inside the cylinder given by $x^2 + y^2 = 5$ this is also the region *D*. The region *D* is the disk $x^2 + y^2 \le 5$ in the *xy*-plane.

For reference purposes here is a sketch of the region that we are trying to find the volume of.





So, the region that we want the volume for is really a cylinder with a cap that comes from the sphere.

We are definitely going to want to do this integral in terms of polar coordinates so here are the limits (in polar coordinates) for the region,

$$0 \le \theta \le 2\pi$$
$$0 \le r \le \sqrt{5}$$

and we'll need to convert the function to polar coordinates as well.

$$z = \sqrt{9 - \left(x^2 + y^2\right)} = \sqrt{9 - r^2}$$

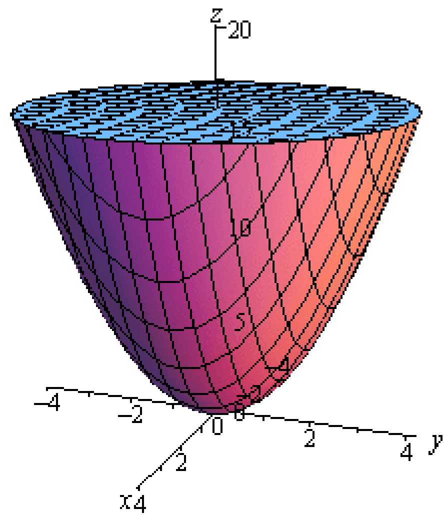
The volume is then,

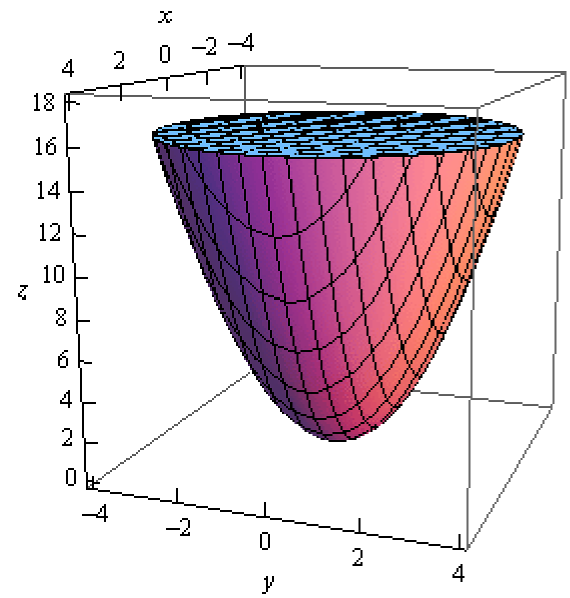
$$\begin{aligned} V &= \iint\limits_D \sqrt{9 - x^2 - y^2}\, dA \\ &= \int_0^{2\pi} \int_0^{\sqrt{5}} r\sqrt{9 - r^2}\, dr\, d\theta \\ &= \int_0^{2\pi} -\frac{1}{3}\left(9 - r^2\right)^{\frac{3}{2}}\Bigg|_0^{\sqrt{5}} d\theta \\ &= \int_0^{2\pi} \frac{19}{3} d\theta \\ &= \frac{38\pi}{3} \end{aligned}$$

Example 4 Find the volume of the region that lies inside $z = x^2 + y^2$ and below the plane $z = 16$.

Solution

Let's start this example off with a quick sketch of the region.





Now, in this case the standard formula is not going to work. The formula

$$V = \iint\limits_D f(x,y)\,dA$$

finds the volume under the function $f(x,y)$ and we're actually after the volume that is above a function. This isn't the problem that it might appear to be however. First, notice that

$$V = \iint\limits_D 16\,dA$$

will be the volume under $z = 16$ (of course we'll need to determine *D* eventually) while

$$V = \iint\limits_D x^2 + y^2\,dA$$

is the volume under $z = x^2 + y^2$, using the same *D*.

The volume that we're after is really the difference between these two or,

$$V = \iint\limits_D 16\,dA - \iint\limits_D x^2 + y^2\,dA = \iint\limits_D 16 - \left(x^2 + y^2\right)dA$$

Now all that we need to do is to determine the region *D* and then convert everything over to polar coordinates.

Determining the region *D* in this case is not too bad. If we were to look straight down the *z*-axis onto the region we would see a circle of radius 4 centered at the origin. This is because the top of the region, where the elliptic paraboloid intersects the plane, is the widest part of the region. We know the *z* coordinate at the intersection so, setting $z = 16$ in the equation of the paraboloid gives,

$$16 = x^2 + y^2$$

which is the equation of a circle of radius 4 centered at the origin.

Here are the inequalities for the region and the function we'll be integrating in terms of polar coordinates.

$$0 \le \theta \le 2\pi \qquad\qquad 0 \le r \le 4 \qquad\qquad z = 16 - r^2$$

The volume is then,

$$\begin{aligned} V &= \iint\limits_D 16 - \left(x^2 + y^2\right) dA \\ &= \int_0^{2\pi} \int_0^4 r\left(16 - r^2\right) dr\, d\theta \\ &= \int_0^{2\pi} \left(8r^2 - \frac{1}{4}r^4\right)\Big|_0^4 \, d\theta \\ &= \int_0^{2\pi} 64\, d\theta \\ &= 128\pi \end{aligned}$$

In both of the previous volume problems we would have not been able to easily compute the volume without first converting to polar coordinates so, as these examples show, it is a good idea to always remember polar coordinates.

There is one more type of example that we need to look at before moving on to the next section. Sometimes we are given an iterated integral that is already in terms of *x* and *y* and we need to convert this over to polar so that we can actually do the integral. We need to see an example of how to do this kind of conversion.

Example 5 Evaluate the following integral by first converting to polar coordinates.

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{0} \cos\left(x^2 + y^2\right) dy\, dx$$

Solution

First, notice that we cannot do this integral in Cartesian coordinates and so converting to polar coordinates may be the only option we have for actually doing the integral. Notice that the function will convert to polar coordinates nicely and so shouldn't be a problem.

Let's first determine the region that we're integrating over and see if it's a region that can be easily converted into polar coordinates. Here are the inequalities that define the region in terms of Cartesian coordinates.

$$\begin{gathered} -1 \le x \le 1 \\ -\sqrt{1-x^2} \le y \le 0 \end{gathered}$$

Now, the lower limit for the *y*'s is,

$$y = -\sqrt{1-x^2}$$

and this looks like the bottom of the circle of radius 1 centered at the origin. Since the upper limit for the *y*'s is $y = 0$ we won't have any portion of the top half of the disk and so it looks like we are going to have a portion (or all) of the bottom of the disk of radius 1 centered at the origin.

The range for the *x*'s in turn, tells us that we are will in fact have the complete bottom part of the disk.

So, we know that the inequalities that will define this region in terms of polar coordinates are then,

$$\pi \le \theta \le 2\pi$$
$$0 \le r \le 1$$

Finally, we just need to remember that,

$$dx\,dy = dA = r\,dr\,d\theta$$

and so the integral becomes,

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{0} \cos\left(x^2 + y^2\right) dy\,dx = \int_{\pi}^{2\pi}\int_{0}^{1} r\cos\left(r^2\right) dr\,d\theta$$

Note that this is an integral that we can do. So, here is the rest of the work for this integral.

$$\begin{aligned}\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{0} \cos\left(x^2 + y^2\right) dy\,dx &= \int_{\pi}^{2\pi} \frac{1}{2}\sin\left(r^2\right)\Big|_{0}^{1} d\theta \\ &= \int_{\pi}^{2\pi} \frac{1}{2}\sin\left(1\right) d\theta \\ &= \frac{\pi}{2}\sin\left(1\right)\end{aligned}$$

Section 4-5 : Triple Integrals

Now that we know how to integrate over a two-dimensional region we need to move on to integrating over a three-dimensional region. We used a double integral to integrate over a two-dimensional region and so it shouldn't be too surprising that we'll use a **triple integral** to integrate over a three dimensional region. The notation for the general triple integrals is,

$$\iiint\limits_E f\left(x,y,z\right)dV$$

Let's start simple by integrating over the box,

$$B = \left[a,b\right]\times\left[c,d\right]\times\left[r,s\right]$$

Note that when using this notation we list the *x*'s first, the *y*'s second and the *z*'s third.

The triple integral in this case is,

$$\iiint\limits_B f\left(x,y,z\right)dV = \int_r^s\int_c^d\int_a^b f\left(x,y,z\right)dx\,dy\,dz$$

Note that we integrated with respect to *x* first, then *y*, and finally *z* here, but in fact there is no reason to the integrals in this order. There are 6 different possible orders to do the integral in and which order you do the integral in will depend upon the function and the order that you feel will be the easiest. We will get the same answer regardless of the order however.

Let's do a quick example of this type of triple integral.

Example 1 Evaluate the following integral.

$$\iiint\limits_B 8xyz\,dV\,,\qquad B = \left[2,3\right]\times\left[1,2\right]\times\left[0,1\right]$$

Solution

Just to make the point that order doesn't matter let's use a different order from that listed above. We'll do the integral in the following order.

$$\begin{aligned}\iiint\limits_B 8xyz\,dV &= \int_1^2\int_2^3\int_0^1 8xyz\,dz\,dx\,dy\\ &= \int_1^2\int_2^3 4xyz^2\Big|_0^1\,dx\,dy\\ &= \int_1^2\int_2^3 4xy\,dx\,dy\\ &= \int_1^2 2x^2y\Big|_2^3\,dy\\ &= \int_1^2 10y\,dy = 15\end{aligned}$$

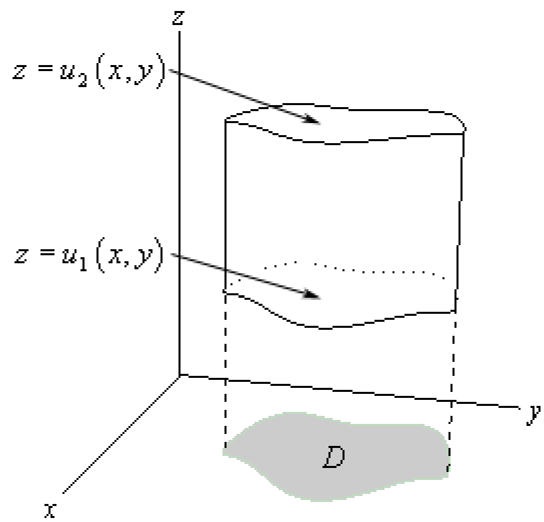
Before moving on to more general regions let's get a nice geometric interpretation about the triple integral out of the way so we can use it in some of the examples to follow.

Fact

The volume of the three-dimensional region *E* is given by the integral,

$$V = \iiint\limits_E dV$$

Let's now move on the more general three-dimensional regions. We have three different possibilities for a general region. Here is a sketch of the first possibility.



In this case we define the region *E* as follows,

$$E = \left\{ (x,y,z) \mid (x,y) \in D,\ u_1(x,y) \le z \le u_2(x,y) \right\}$$

where $(x,y) \in D$ is the notation that means that the point (x,y) lies in the region *D* from the *xy*-plane. In this case we will evaluate the triple integral as follows,

$$\iiint\limits_E f(x,y,z)\,dV = \iint\limits_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right] dA$$

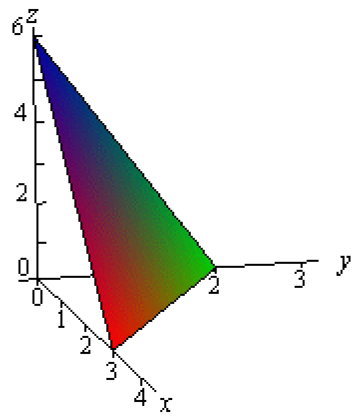
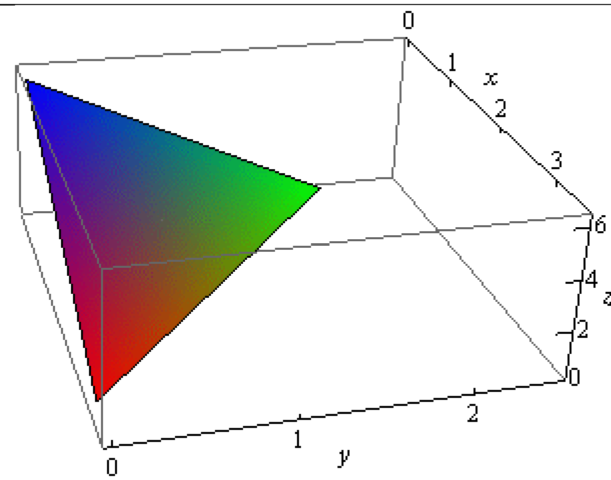
where the double integral can be evaluated in any of the methods that we saw in the previous couple of sections. In other words, we can integrate first with respect to *x*, we can integrate first with respect to *y*, or we can use polar coordinates as needed.

Example 2 Evaluate $\iiint\limits_E 2x\,dV$ where *E* is the region under the plane $2x + 3y + z = 6$ that lies in the first octant.

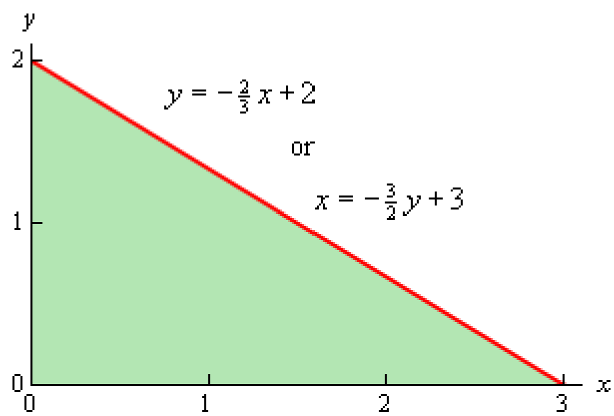
Solution

We should first define *octant*. Just as the two-dimensional coordinates system can be divided into four quadrants the three-dimensional coordinate system can be divided into eight octants. The first octant is the octant in which all three of the coordinates are positive.

Here is a sketch of the plane in the first octant.

We now need to determine the region *D* in the *xy*-plane. We can get a visualization of the region by pretending to look straight down on the object from above. What we see will be the region *D* in the *xy*-plane. So *D* will be the triangle with vertices at $\left(0,0\right)$, $\left(3,0\right)$, and $\left(0,2\right)$. Here is a sketch of *D*.



Now we need the limits of integration. Since we are under the plane and in the first octant (so we're above the plane $z = 0$) we have the following limits for *z*.

$$0 \le z \le 6 - 2x - 3y$$

We can integrate the double integral over *D* using either of the following two sets of inequalities.

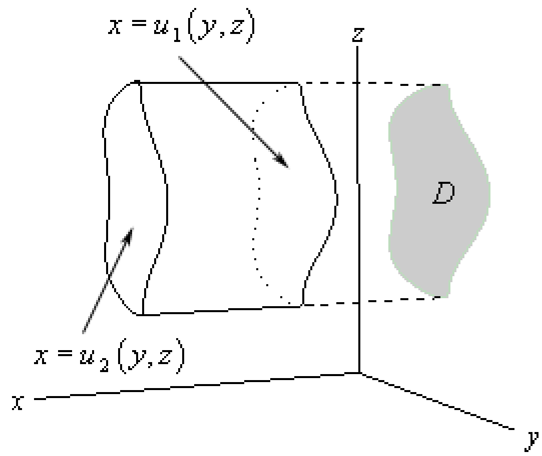
$$0 \le x \le 3 \qquad\qquad 0 \le x \le -\frac{3}{2}y + 3$$

$$0 \le y \le -\frac{2}{3}x + 2 \qquad\qquad 0 \le y \le 2$$

Since neither really holds an advantage over the other we'll use the first one. The integral is then,

$$\begin{aligned}
\iiint\limits_E 2x\,dV &= \iint\limits_D \left[\int_0^{6-2x-3y} 2x\,dz\right]dA \\
&= \iint\limits_D 2xz\Big|_0^{6-2x-3y}\,dA \\
&= \int_0^3 \int_0^{-\frac{2}{3}x+2} 2x\left(6-2x-3y\right)dy\,dx \\
&= \int_0^3 \left(12xy-4x^2y-3xy^2\right)\Big|_0^{-\frac{2}{3}x+2}\,dx \\
&= \int_0^3 \frac{4}{3}x^3-8x^2+12x\,dx \\
&= \left(\frac{1}{3}x^4-\frac{8}{3}x^3+6x^2\right)\Bigg|_0^3 \\
&= 9
\end{aligned}$$

Let's now move onto the second possible three-dimensional region we may run into for triple integrals. Here is a sketch of this region.



For this possibility we define the region *E* as follows,

$$E = \left\{\left(x,y,z\right) \mid \left(y,z\right) \in D,\ u_1\left(y,z\right) \le x \le u_2\left(y,z\right)\right\}$$

So, the region *D* will be a region in the *yz*-plane. Here is how we will evaluate these integrals.

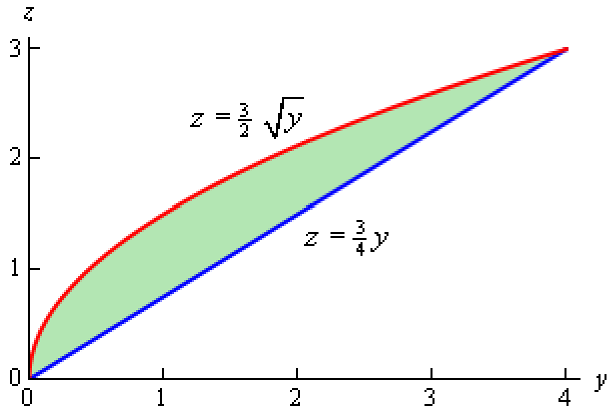
$$\iiint\limits_E f\left(x,y,z\right)dV = \iint\limits_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f\left(x,y,z\right)dx\right]dA$$

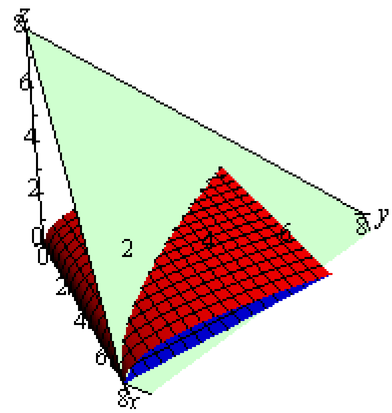
As with the first possibility we will have two options for doing the double integral in the *yz*-plane as well as the option of using polar coordinates if needed.

Example 3 Determine the volume of the region that lies behind the plane $x + y + z = 8$ and in front of the region in the *yz*-plane that is bounded by $z = \frac{3}{2}\sqrt{y}$ and $z = \frac{3}{4}y$.

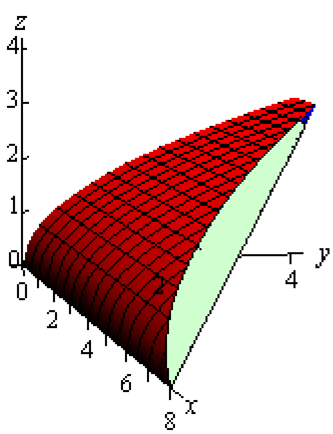
Solution

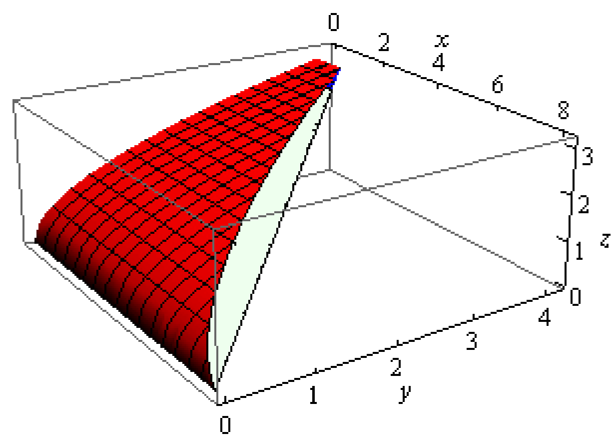
In this case we've been given *D* and so we won't have to really work to find that. Here is a sketch of the region *D* as well as a quick sketch of the plane and the curves defining *D* projected out past the plane so we can get an idea of what the region we're dealing with looks like.





Now, the graph of the region above is all okay, but it doesn't really show us what the region is. So, here is a sketch of the region itself.





Here are the limits for each of the variables.

$$0 \le y \le 4$$

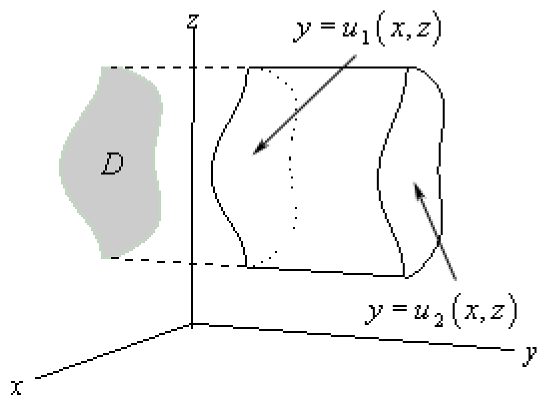
$$\frac{3}{4}y \le z \le \frac{3}{2}\sqrt{y}$$

$$0 \le x \le 8 - y - z$$

The volume is then,

$$\begin{aligned} V &= \iiint\limits_E dV = \iint\limits_D \left[\int_0^{8-y-z} dx \right] dA \\ &= \int_0^4 \int_{3y/4}^{3\sqrt{y}/2} 8 - y - z \, dz \, dy \\ &= \int_0^4 \left(8z - yz - \frac{1}{2} z^2 \right) \Bigg|_{\frac{3y}{4}}^{\frac{3\sqrt{y}}{2}} dy \\ &= \int_0^4 12 y^{\frac{1}{2}} - \frac{57}{8} y - \frac{3}{2} y^{\frac{3}{2}} + \frac{33}{32} y^2 \, dy \\ &= \left(8 y^{\frac{3}{2}} - \frac{57}{16} y^2 - \frac{3}{5} y^{\frac{5}{2}} + \frac{11}{32} y^3 \right) \Bigg|_0^4 = \frac{49}{5} \end{aligned}$$

We now need to look at the third (and final) possible three-dimensional region we may run into for triple integrals. Here is a sketch of this region.



In this final case *E* is defined as,

$$E = \left\{ (x, y, z) \mid (x, z) \in D, \; u_1(x, z) \le y \le u_2(x, z) \right\}$$

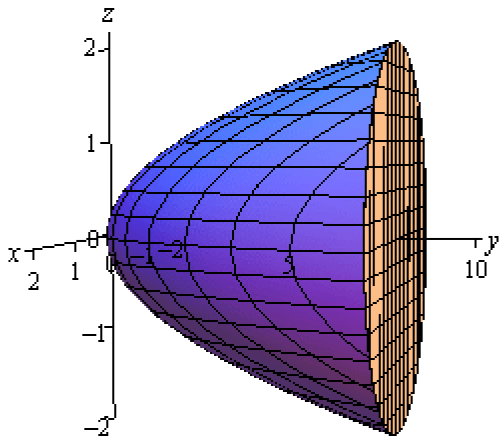
and here the region *D* will be a region in the *xz*-plane. Here is how we will evaluate these integrals.

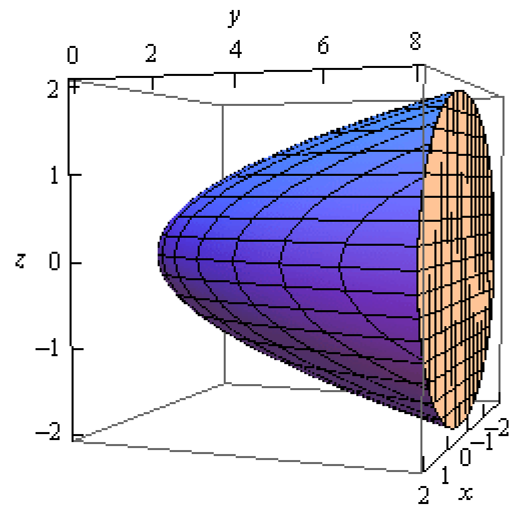
$$\iiint\limits_E f(x, y, z) \, dV = \iint\limits_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z) \, dy \right] dA$$

where we will can use either of the two possible orders for integrating *D* in the *xz*-plane or we can use polar coordinates if needed.

Example 4 Evaluate $\iiint\limits_E \sqrt{3x^2 + 3z^2}\, dV$ where *E* is the solid bounded by $y = 2x^2 + 2z^2$ and the plane $y = 8$.

Solution
Here is a sketch of the solid *E*.





The region *D* in the *xz*-plane can be found by "standing" in front of this solid and we can see that *D* will be a disk in the *xz*-plane. This disk will come from the front of the solid and we can determine the equation of the disk by setting the elliptic paraboloid and the plane equal.

$$2x^2 + 2z^2 = 8 \qquad \Rightarrow \qquad x^2 + z^2 = 4$$

This region, as well as the integrand, both seems to suggest that we should use something like polar coordinates. However, we are in the *xz*-plane and we've only seen polar coordinates in the *xy*-plane. This is not a problem. We can always "translate" them over to the *xz*-plane with the following definition.

$$x = r\cos\theta \qquad\qquad z = r\sin\theta$$

Since the region doesn't have *y*'s we will let *z* take the place of *y* in all the formulas. Note that these definitions also lead to the formula,

$$x^2 + z^2 = r^2$$

With this in hand we can arrive at the limits of the variables that we'll need for this integral.

$$2x^2 + 2z^2 \le y \le 8$$
$$0 \le r \le 2$$
$$0 \le \theta \le 2\pi$$

The integral is then,

$$\iiint\limits_E \sqrt{3x^2+3z^2}\,dV = \iint\limits_D \left[\int_{2x^2+2z^2}^{8} \sqrt{3x^2+3z^2}\,dy\right] dA$$
$$= \iint\limits_D \left(y\sqrt{3x^2+3z^2} \right)\Big|_{2x^2+2z^2}^{8} dA$$
$$= \iint\limits_D \sqrt{3\left(x^2+z^2\right)}\left(8-\left(2x^2+2z^2\right)\right) dA$$

Now, since we are going to do the double integral in polar coordinates let's get everything converted over to polar coordinates. The integrand is,

$$\sqrt{3\left(x^2+z^2\right)}\left(8-\left(2x^2+2z^2\right)\right) = \sqrt{3r^2}\left(8-2r^2\right)$$
$$= \sqrt{3}\ r\left(8-2r^2\right)$$
$$= \sqrt{3}\left(8r-2r^3\right)$$

The integral is then,

$$\iiint\limits_E \sqrt{3x^2+3z^2}\,dV = \iint\limits_D \sqrt{3}\ \left(8r-2r^3\right) dA$$
$$= \sqrt{3}\int_0^{2\pi}\int_0^2 \left(8r-2r^3\right) r\,dr\,d\theta$$
$$= \sqrt{3}\int_0^{2\pi} \left(\frac{8}{3}r^3-\frac{2}{5}r^5\right)\Big|_0^2 d\theta$$
$$= \sqrt{3}\int_0^{2\pi} \frac{128}{15}\,d\theta$$
$$= \frac{256\sqrt{3}\,\pi}{15}$$

Section 4-6 : Triple Integrals in Cylindrical Coordinates

In this section we want do take a look at triple integrals done completely in Cylindrical Coordinates. Recall that cylindrical coordinates are really nothing more than an extension of polar coordinates into three dimensions. The following are the conversion formulas for cylindrical coordinates.

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

In order to do the integral in cylindrical coordinates we will need to know what *dV* will become in terms of cylindrical coordinates. We will be able to show in the Change of Variables section of this chapter that,

$$dV = r\,dz\,dr\,d\theta$$

The region, *E*, over which we are integrating becomes,

$$\begin{aligned} E &= \left\{ (x,y,z) \mid (x,y) \in D,\ u_1(x,y) \le z \le u_2(x,y) \right\} \\ &= \left\{ (r,\theta,z) \mid \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta),\ u_1(r\cos\theta, r\sin\theta) \le z \le u_2(r\cos\theta, r\sin\theta) \right\} \end{aligned}$$

Note that we've only given this for *E*'s in which *D* is in the *xy*-plane. We can modify this accordingly if *D* is in the *yz*-plane or the *xz*-plane as needed.

In terms of cylindrical coordinates a triple integral is,

$$\iiint\limits_E f(x,y,z)\,dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} r\, f(r\cos\theta, r\sin\theta, z)\,dz\,dr\,d\theta$$

Don't forget to add in the *r* and make sure that all the *x*'s and *y*'s also get converted over into cylindrical coordinates.

Let's see an example.

Example 1 Evaluate $\iiint\limits_E y\,dV$ where *E* is the region that lies below the plane $z = x + 2$ above the *xy*-plane and between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Solution

There really isn't too much to do with this one other than do the conversions and then evaluate the integral.

We'll start out by getting the range for *z* in terms of cylindrical coordinates.

$$0 \le z \le x + 2 \qquad \Rightarrow \qquad 0 \le z \le r\cos\theta + 2$$

Remember that we are above the *xy*-plane and so we are above the plane $z = 0$

Next, the region *D* is the region between the two circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ in the *xy*-plane and so the ranges for it are,

$$0 \le \theta \le 2\pi \qquad\qquad 1 \le r \le 2$$

Here is the integral.

$$\begin{aligned}
\iiint\limits_E y\,dV &= \int_0^{2\pi}\int_1^2\int_0^{r\cos\theta+2}\left(r\sin\theta\right)r\,dz\,dr\,d\theta \\
&= \int_0^{2\pi}\int_1^2 r^2\sin\theta\left(r\cos\theta+2\right)dr\,d\theta \\
&= \int_0^{2\pi}\int_1^2 \frac{1}{2}r^3\sin\left(2\theta\right)+2r^2\sin\theta\,dr\,d\theta \\
&= \int_0^{2\pi}\left.\left(\frac{1}{8}r^4\sin\left(2\theta\right)+\frac{2}{3}r^3\sin\theta\right)\right|_1^2 d\theta \\
&= \int_0^{2\pi}\frac{15}{8}\sin\left(2\theta\right)+\frac{14}{3}\sin\theta\,d\theta \\
&= \left.\left(-\frac{15}{16}\cos\left(2\theta\right)-\frac{14}{3}\cos\theta\right)\right|_0^{2\pi} \\
&= 0
\end{aligned}$$

Just as we did with double integral involving polar coordinates we can start with an iterated integral in terms of *x*, *y*, and *z* and convert it to cylindrical coordinates.

Example 2 Convert $\int_{-1}^{1}\int_0^{\sqrt{1-y^2}}\int_{x^2+y^2}^{\sqrt{x^2+y^2}} xyz\,dz\,dx\,dy$ into an integral in cylindrical coordinates.

Solution

Here are the ranges of the variables from this iterated integral.

$$-1 \le y \le 1$$
$$0 \le x \le \sqrt{1-y^2}$$
$$x^2+y^2 \le z \le \sqrt{x^2+y^2}$$

The first two inequalities define the region *D* and since the upper and lower bounds for the *x*'s are $x = \sqrt{1-y^2}$ and $x = 0$ we know that we've got at least part of the right half a circle of radius 1 centered at the origin. Since the range of *y*'s is $-1 \le y \le 1$ we know that we have the complete right half of the disk of radius 1 centered at the origin. So, the ranges for *D* in cylindrical coordinates are,

$$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$
$$0 \le r \le 1$$

All that's left to do now is to convert the limits of the *z* range, but that's not too bad.

$$r^2 \le z \le r$$

On a side note notice that the lower bound here is an elliptic paraboloid and the upper bound is a cone. Therefore, *E* is a portion of the region between these two surfaces.

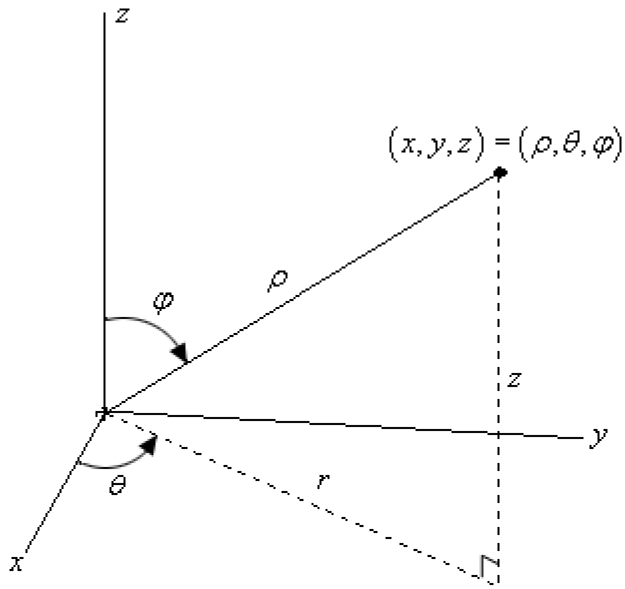
The integral is,

$$\begin{aligned}\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}}\int_{x^2+y^2}^{\sqrt{x^2+y^2}} xyz\,dz\,dx\,dy &= \int_{-\pi/2}^{\pi/2}\int_{0}^{1}\int_{r^2}^{r} r\left(r\cos\theta\right)\left(r\sin\theta\right)z\,dz\,dr\,d\theta \\ &= \int_{-\pi/2}^{\pi/2}\int_{0}^{1}\int_{r^2}^{r} zr^3\cos\theta\sin\theta\,dz\,dr\,d\theta\end{aligned}$$

Section 4-7 : Triple Integrals in Spherical Coordinates

In the previous section we looked at doing integrals in terms of cylindrical coordinates and we now need to take a quick look at doing integrals in terms of spherical coordinates.

First, we need to recall just how spherical coordinates are defined. The following sketch shows the relationship between the Cartesian and spherical coordinate systems.



Here are the conversion formulas for spherical coordinates.

$$x = \rho \sin\varphi \cos\theta \qquad y = \rho \sin\varphi \sin\theta \qquad z = \rho \cos\varphi$$
$$x^2 + y^2 + z^2 = \rho^2$$

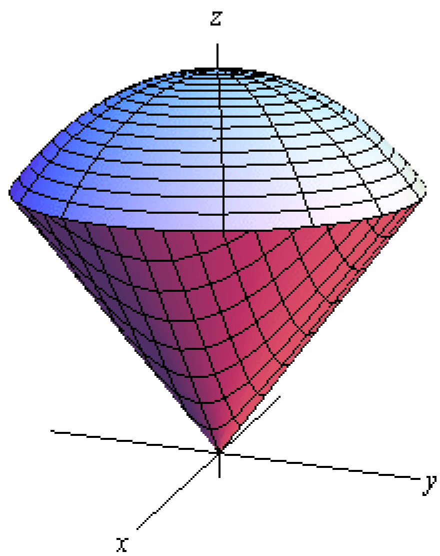
We also have the following restrictions on the coordinates.

$$\rho \ge 0 \qquad 0 \le \varphi \le \pi$$

For our integrals we are going to restrict *E* down to a spherical wedge. This will mean that we are going to take ranges for the variables as follows,

$$a \le \rho \le b$$
$$\alpha \le \theta \le \beta$$
$$\delta \le \varphi \le \gamma$$

Here is a quick sketch of a spherical wedge in which the lower limit for both ρ and φ are zero for reference purposes. Most of the wedges we'll be working with will fit into this pattern.



From this sketch we can see that *E* is nothing more than the intersection of a sphere and a cone and generally will represent a shape that is reminiscent of an ice cream cone.

In the next section we will show that

$$dV = \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

Therefore, the integral will become,

$$\iiint\limits_E f(x,y,z)\,dV = \int_\delta^\gamma \int_\alpha^\beta \int_a^b \rho^2 \sin\varphi \; f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\, d\rho \, d\theta \, d\varphi$$

This looks bad but given that the limits are all constants the integrals here tend to not be too bad.

Example 1 Evaluate $\iiint\limits_E 16z\,dV$ where *E* is the upper half of the sphere $x^2 + y^2 + z^2 = 1$.

Solution
Since we are taking the upper half of the sphere the limits for the variables are,

$$0 \le \rho \le 1$$
$$0 \le \theta \le 2\pi$$
$$0 \le \varphi \le \frac{\pi}{2}$$

The integral is then,

$$\begin{aligned}\iiint\limits_E 16z\,dV &= \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^1 \rho^2\sin\varphi\left(16\rho\cos\varphi\right)d\rho\,d\theta\,d\varphi\\ &= \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^1 8\rho^3\sin\left(2\varphi\right)d\rho\,d\theta\,d\varphi\\ &= \int_0^{\frac{\pi}{2}}\int_0^{2\pi} 2\sin\left(2\varphi\right)d\theta\,d\varphi\\ &= \int_0^{\frac{\pi}{2}} 4\pi\sin\left(2\varphi\right)d\varphi\\ &= -2\pi\cos\left(2\varphi\right)\Big|_0^{\frac{\pi}{2}}\\ &= 4\pi\end{aligned}$$

Example 2 Evaluate $\iiint\limits_E z\,x\,dV$ where *E* is above $x^2+y^2+z^2=4$, inside the cone (pointing upward) that makes an angle of $\frac{\pi}{3}$ with the negative *z*-axis and has $x \le 0$.

Solution

First, we need to take care of the limits. The region *E* is basically an upside down ice cream cone that has been cut in half so that only the portion with $x \le 0$ remains. Therefore, because we are inside a portion of a sphere of radius 2 we must have,

$$0 \le \rho \le 2$$

For φ we need to be careful. The problem statement says that the cone makes an angle of $\frac{\pi}{3}$ with the negative *z*-axis. However, remember that φ is measured from the positive *z*-axis. Therefore, the first angle, as measured from the positive *z*-axis, that will "start" the cone will be $\varphi = \frac{2\pi}{3}$ and it goes to the negative *z*-axis. Therefore, we get the following limits for φ.

$$\frac{2\pi}{3} \le \varphi \le \pi$$

Finally, for the θ we can use the fact that we are also told that $x \le 0$. This means we are to the left of the *y*-axis and so the range of θ must be,

$$\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$$

Now that we have the limits we can evaluate the integral.

$$\begin{aligned}\iiint\limits_E z\,x\,dV &= \int_{\frac{2\pi}{3}}^{\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\int_0^2 \left(\rho\cos\varphi\right)\left(\rho\sin\varphi\cos\theta\right)\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi \\ &= \int_{\frac{2\pi}{3}}^{\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\int_0^2 \rho^4\cos\varphi\sin^2\varphi\cos\theta\,d\rho\,d\theta\,d\varphi \\ &= \int_{\frac{2\pi}{3}}^{\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{32}{5}\cos\varphi\sin^2\varphi\cos\theta\,d\theta\,d\varphi \\ &= \int_{\frac{2\pi}{3}}^{\pi} -\frac{64}{5}\cos\varphi\sin^2\varphi\,d\varphi \\ &= -\frac{64}{15}\sin^3\varphi\Big|_{\frac{2\pi}{3}}^{\pi} \\ &= \frac{8\sqrt{3}}{5}\end{aligned}$$

Example 3 Convert $\int_0^3\int_0^{\sqrt{9-y^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} x^2+y^2+z^2\,dz\,dx\,dy$ into spherical coordinates.

Solution

Let's first write down the limits for the variables.

$$0 \le y \le 3$$

$$0 \le x \le \sqrt{9-y^2}$$

$$\sqrt{x^2+y^2} \le z \le \sqrt{18-x^2-y^2}$$

The range for *x* tells us that we have a portion of the right half of a disk of radius 3 centered at the origin. Since we are restricting *y*'s to positive values it looks like we will have the quarter disk in the first quadrant. Therefore, since *D* is in the first quadrant the region, *E*, must be in the first octant and this in turn tells us that we have the following range for θ (since this is the angle around the *z*-axis).

$$0 \le \theta \le \frac{\pi}{2}$$

Now, let's see what the range for *z* tells us. The lower bound, $z=\sqrt{x^2+y^2}$, is the upper half of a cone. At this point we don't need this quite yet, but we will later. The upper bound, $z=\sqrt{18-x^2-y^2}$, is the upper half of the sphere,

$$x^2+y^2+z^2=18$$

and so from this we now have the following range for ρ

$$0 \le \rho \le \sqrt{18} = 3\sqrt{2}$$

Now all that we need is the range for φ. There are two ways to get this. One is from where the cone and the sphere intersect. Plugging in the equation for the cone into the sphere gives,

$$\left(\sqrt{x^2+y^2}\right)^2+z^2=18$$
$$z^2+z^2=18$$
$$z^2=9$$
$$z=3$$

Note that we can assume *z* is positive here since we know that we have the upper half of the cone and/or sphere. Finally, plug this into the conversion for *z* and take advantage of the fact that we know that $\rho=3\sqrt{2}$ since we are intersecting on the sphere. This gives,

$$\rho\cos\varphi=3$$
$$3\sqrt{2}\cos\varphi=3$$
$$\cos\varphi=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}\qquad\Rightarrow\qquad\varphi=\frac{\pi}{4}$$

So, it looks like we have the following range,

$$0\le\varphi\le\frac{\pi}{4}$$

The other way to get this range is from the cone by itself. By first converting the equation into cylindrical coordinates and then into spherical coordinates we get the following,

$$z=r$$
$$\rho\cos\varphi=\rho\sin\varphi$$
$$1=\tan\varphi\qquad\Rightarrow\qquad\varphi=\frac{\pi}{4}$$

So, recalling that $\rho^2=x^2+y^2+z^2$, the integral is then,

$$\int_0^3\int_0^{\sqrt{9-y^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}}x^2+y^2+z^2\,dz\,dx\,dy=\int_0^{\pi/4}\int_0^{\pi/2}\int_0^{3\sqrt{2}}\rho^4\sin\varphi\,d\rho\,d\theta\,d\varphi$$

Section 4-8 : Change of Variables

Back in Calculus I we had the substitution rule that told us that,

$$\int_a^b f\left(g\left(x\right)\right)g'\left(x\right)dx = \int_c^d f\left(u\right)du \qquad \text{where} \quad u = g\left(x\right)$$

In essence this is taking an integral in terms of *x*'s and changing it into terms of *u*'s. We want to do something similar for double and triple integrals. In fact we've already done this to a certain extent when we converted double integrals to polar coordinates and when we converted triple integrals to cylindrical or spherical coordinates. The main difference is that we didn't actually go through the details of where the formulas came from. If you recall, in each of those cases we commented that we would justify the formulas for *dA* and *dV* eventually. Now is the time to do that justification.

While often the reason for changing variables is to get us an integral that we can do with the new variables, another reason for changing variables is to convert the region into a nicer region to work with. When we were converting the polar, cylindrical or spherical coordinates we didn't worry about this change since it was easy enough to determine the new limits based on the given region. That is not always the case however. So, before we move into changing variables with multiple integrals we first need to see how the region may change with a change of variables.

First, we need a little terminology/notation out of the way. We call the equations that define the change of variables a **transformation**. Also, we will typically start out with a region, *R*, in *xy*-coordinates and transform it into a region in *uv*-coordinates.

Example 1 Determine the new region that we get by applying the given transformation to the region *R*.

(a) *R* is the ellipse $x^2 + \frac{y^2}{36} = 1$ and the transformation is $x = \frac{u}{2}$, $y = 3v$.

(b) *R* is the region bounded by $y = -x + 4$, $y = x + 1$, and $y = \frac{x}{3} - \frac{4}{3}$ and the transformation is $x = \frac{1}{2}\left(u + v\right)$, $y = \frac{1}{2}\left(u - v\right)$.

Solution

(a) *R* is the ellipse $x^2 + \frac{y^2}{36} = 1$ and the transformation is $x = \frac{u}{2}$, $y = 3v$.

There really isn't too much to do with this one other than to plug the transformation into the equation for the ellipse and see what we get.

$$\left(\frac{u}{2}\right)^2 + \frac{\left(3v\right)^2}{36} = 1$$

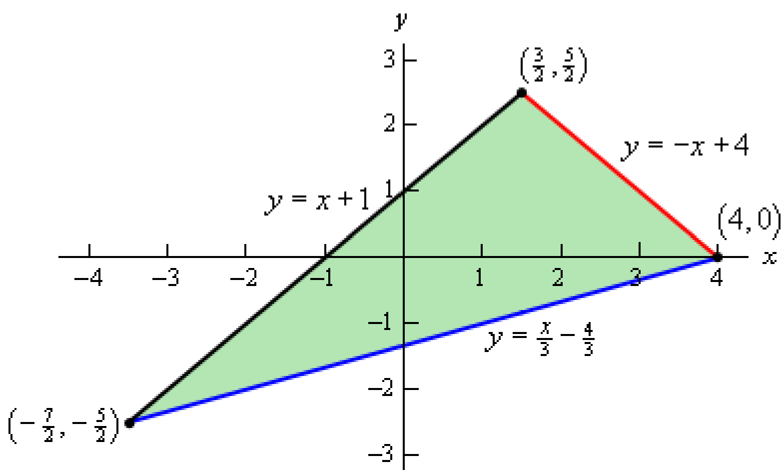
$$\frac{u^2}{4} + \frac{9v^2}{36} = 1$$

$$u^2 + v^2 = 4$$

So, we started out with an ellipse and after the transformation we had a disk of radius 2.

(b) *R* is the region bounded by $y = -x + 4$, $y = x + 1$, and $y = \frac{x}{3} - \frac{4}{3}$ and the transformation is $x = \frac{1}{2}(u + v)$, $y = \frac{1}{2}(u - v)$.

As with the first part we'll need to plug the transformation into the equation, however, in this case we will need to do it three times, once for each equation. Before we do that let's sketch the graph of the region and see what we've got.



So, we have a triangle. Now, let's go through the transformation. We will apply the transformation to each edge of the triangle and see where we get.

Let's do $y = -x + 4$ first. Plugging in the transformation gives,

$$\begin{aligned} \frac{1}{2}(u - v) &= -\frac{1}{2}(u + v) + 4 \\ u - v &= -u - v + 8 \\ 2u &= 8 \\ u &= 4 \end{aligned}$$

The first boundary transforms very nicely into a much simpler equation.

Now let's take a look at $y = x + 1$,

$$\begin{aligned} \frac{1}{2}(u - v) &= \frac{1}{2}(u + v) + 1 \\ u - v &= u + v + 2 \\ -2v &= 2 \\ v &= -1 \end{aligned}$$

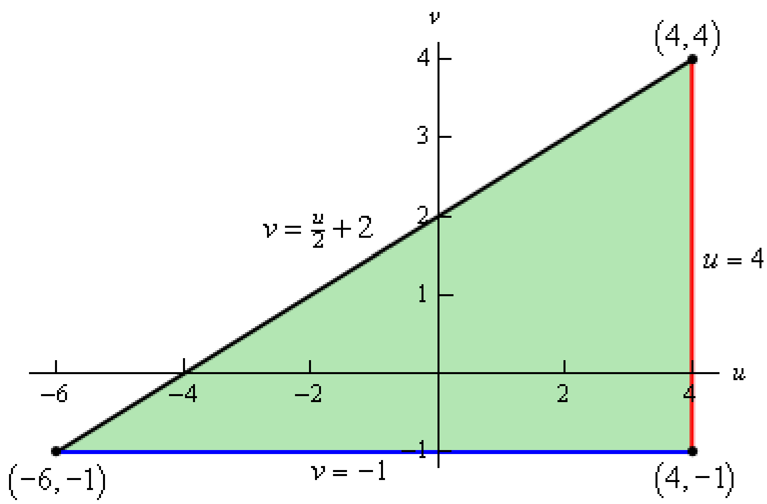
Again, a much nicer equation that what we started with.

Finally, let's transform $y = \frac{x}{3} - \frac{4}{3}$.

$$\begin{aligned} \frac{1}{2}(u-v) &= \frac{1}{3}\left(\frac{1}{2}(u+v)\right) - \frac{4}{3} \\ 3u - 3v &= u + v - 8 \\ 4v &= 2u + 8 \\ v &= \frac{u}{2} + 2 \end{aligned}$$

So, again, we got a somewhat simpler equation, although not quite as nice as the first two.

Let's take a look at the new region that we get under the transformation.



We still get a triangle, but a much nicer one.

Note that we can't always expect to transform a specific type of region (a triangle for example) into the same kind of region. It is completely possible to have a triangle transform into a region in which each of the edges are curved and in no way resembles a triangle.

Notice that in each of the above examples we took a two dimensional region that would have been somewhat difficult to integrate over and converted it into a region that would be much nicer in integrate over. As we noted at the start of this set of examples, that is often one of the points behind the transformation. In addition to converting the integrand into something simpler it will often also transform the region into one that is much easier to deal with.

Before proceeding with the next topic let's address another point. On occasion, we will also need to know the range of *u* and/or *v* for each of the new equations we get from the transformation. We didn't need that for the two examples above and it is not something that we will often need. However, it can help on occasion in determining the new region.

So, let's work a quick example to see how we do that.

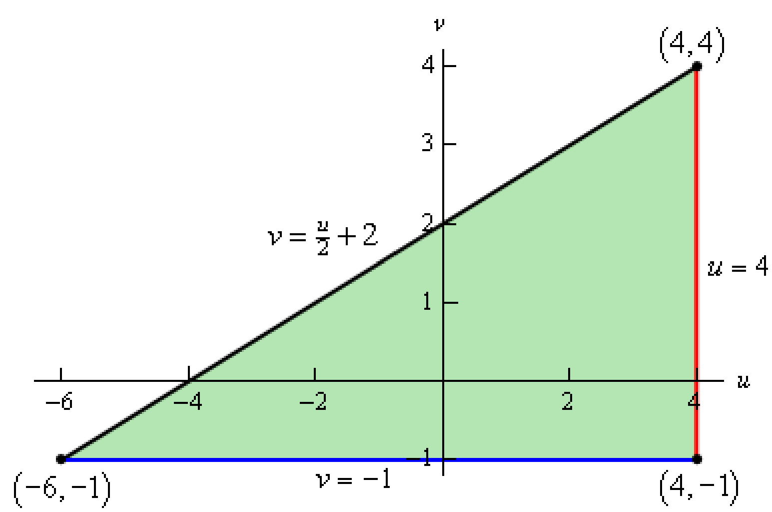
Example 2 For the region bounded by $y=-x+4$, $y=x+1$, and $y=\frac{x}{3}-\frac{4}{3}$ and the transformation $x=\frac{1}{2}(u+v)$, $y=\frac{1}{2}(u-v)$ determine the ranges of *u* and *v* for each of the new equations from the transformation.

Solution

Okay, we already know what the new region looks like and what the new equations are from the previous example. So, here is a quick review of the transformation of each of the original equations.

$$\begin{aligned} y&=-x+4 & \Rightarrow \quad & u=4 \\ y&=x+1 & \Rightarrow \quad & v=-1 \\ y&=\frac{x}{3}-\frac{4}{3} & \Rightarrow \quad & v=\frac{u}{2}+2 \end{aligned}$$

Here is the new region we get under the transformation.



Note, that, in this case, we could determine the range of *u* and *v* for each equation from the sketch above. However, in cases where we might actually need the ranges that is usually not an option as we often need the ranges for *u* and/or *v* to get an accurate sketch of the new region.

So, let's now actually start working the problem.

Let's start with the equation $u=4$. First, we don't need a "range" of *u*'s here as equation makes it pretty clear we have a single value of *u*, namely $u=4$. So, let's determine the range of *v*'s we should get.

Let's start with the *x* transformation and plug in the knows value of *u* for this equation. That gives,

$$x = \frac{1}{2}(u+v) = \frac{1}{2}(4+v)$$

Now, we know that the range of *x*'s for the original equation, $y = -x + 4$, are $\frac{3}{2} \le x \le 4$. We also know from above what *x* is in terms of *v*, so plug that into this range and do a little manipulation as follows,

$$\begin{gathered} \frac{3}{2} \le x \le 4 \\ \frac{3}{2} \le \frac{1}{2}(4+v) \le 4 \\ 3 \le 4 + v \le 8 \\ -1 \le v \le 4 \end{gathered}$$

So, the range of *v*'s for $u = 4$ must be $-1 \le v \le 4$, which nicely matches with what we would expect from the graph of the new region.

Note that we could just as easily used the *y* transformation and *y* range for the original equation and gotten the same result.

Okay, let's now move onto $v = -1$ and we won't put in quite as much explanation for this part.

First, we don't need a range of *v* for this because we clearly have just a single value of *v*. So, to get the range of *u* let's again start with the *x* transformation, plug $v = -1$ in that and then use the range of *x*'s from the original equation, $y = x + 1$.

Here is that work.

$$\begin{gathered} -\frac{7}{2} \le x \le \frac{3}{2} \\ -\frac{7}{2} \le \frac{1}{2}(u-1) \le \frac{3}{2} \\ -7 \le u - 1 \le 3 \\ -6 \le u \le 4 \end{gathered}$$

So, the range of *u* for $v = -1$ is $-6 \le u \le 4$ which, again, matches up with what we see on the graph. Also note that once again, we could have used the *y* ranges to do this work.

Finally, let's find the range of *u* and *v* for $v = \frac{u}{2} + 2$. This time let's use the *y* transformation so we can say we used that in one these. So, we'll start with the range of *y*'s for the original equation, $y = \frac{x}{3} - \frac{4}{3}$, plug in the *y* transformation and then plug in for *v*. Doing this gives,

$$-\frac{5}{2} \le y \le 0$$
$$-\frac{5}{2} \le \frac{1}{2}\left(u - \left(\frac{u}{2} + 2\right)\right) \le 0$$
$$-5 \le \frac{u}{2} - 2 \le 0$$
$$-3 \le \frac{u}{2} \le 2$$
$$-6 \le u \le 4$$

So, again we get the range of *u*'s we expect to get from the graph. Once we have those the appropriate range of *v* can be found from the equation itself as follows,

$$-6 \le u \le 4$$
$$-3 \le \frac{u}{2} \le 2$$
$$-1 \le \frac{u}{2} + 2 \le 4$$
$$-1 \le v \le 4$$

Basically, start with the range of *u*'s and "build up" the equation for the side and we get the range of *v*'s for this side.

So, we now know how to get ranges of *u* and/or *v* for new equations under a transformation. However, this is not something that is done terribly often but it is a useful skill to have in case it does arise somewhere.

Now that we've seen a couple of examples of transforming regions we need to now talk about how we actually do change of variables in the integral. We will start with double integrals. In order to change variables in a double integral we will need the **Jacobian** of the transformation. Here is the definition of the Jacobian.

Definition

The **Jacobian** of the transformation $x = g(u,v)$, $y = h(u,v)$ is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

The Jacobian is defined as a determinant of a 2x2 matrix, if you are unfamiliar with this that is okay. Here is how to compute the determinant.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Therefore, another formula for the determinant is,

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Now that we have the Jacobian out of the way we can give the formula for change of variables for a double integral.

Change of Variables for a Double Integral

Suppose that we want to integrate $f(x,y)$ over the region *R*. Under the transformation $x = g(u,v)$, $y = h(u,v)$ the region becomes *S* and the integral becomes,

$$\iint\limits_R f(x,y)\,dA = \iint\limits_S f\left(g(u,v),h(u,v)\right)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| d\bar{A}$$

Note that we use $d\bar{A}$ in the *u*/*v* integral above to denote that it will be in terms of *du* and *dv* once we convert to two single integrals rather than the *dx* and *dy* we are used to using for *dA*. This is notational only and we generally just use *dA* for both and just make sure to remember that the "new" *dA* is in terms of *du* and *dv*.

Also note that we are taking the absolute value of the Jacobian.

If we look just at the differentials in the above formula we can also say that

$$dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| d\bar{A}$$

Example 3 Show that when changing to polar coordinates we have $dA = r\,dr\,d\theta$

Solution

So, what we are doing here is justifying the formula that we used back when we were integrating with respect to polar coordinates. All that we need to do is use the formula above for *dA*.

The transformation here is the standard conversion formulas,

$$x = r\cos\theta \qquad\qquad y = r\sin\theta$$

The Jacobian for this transformation is,

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$
$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$
$$= r\cos^2\theta - \left(-r\sin^2\theta\right)$$
$$= r\left(\cos^2\theta + \sin^2\theta\right)$$
$$= r$$

We then get,

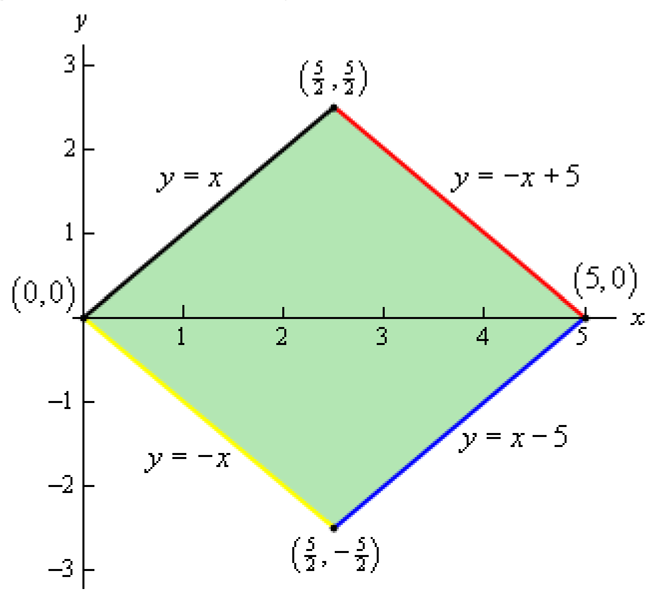
$$dA = \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta = |r|\,dr\,d\theta = r\,dr\,d\theta$$

So, the formula we used in the section on polar integrals was correct.

Now, let's do a couple of integrals.

Example 4 Evaluate $\iint\limits_R x + y\,dA$ where *R* is the trapezoidal region with vertices given by $(0,0)$, $(5,0)$, $\left(\frac{5}{2},\frac{5}{2}\right)$ and $\left(\frac{5}{2},-\frac{5}{2}\right)$ using the transformation $x = 2u + 3v$ and $y = 2u - 3v$.

Solution
First, let's sketch the region *R* and determine equations for each of the sides.



Each of the equations was found by using the fact that we know two points on each line (*i.e.* the two vertices that form the edge).

While we could do this integral in terms of *x* and *y* it would involve two integrals and so would be some work.

Let's use the transformation and see what we get. We'll do this by plugging the transformation into each of the equations above.

Let's start the process off with $y = x$.

$$\begin{aligned} 2u - 3v &= 2u + 3v \\ 6v &= 0 \\ v &= 0 \end{aligned}$$

Transforming $y = -x$ is similar.

$$\begin{aligned} 2u - 3v &= -\left(2u + 3v\right) \\ 4u &= 0 \\ u &= 0 \end{aligned}$$

Next, we'll transform $y = -x + 5$.

$$\begin{aligned} 2u - 3v &= -\left(2u + 3v\right) + 5 \\ 4u &= 5 \\ u &= \frac{5}{4} \end{aligned}$$

Finally, let's transform $y = x - 5$.

$$\begin{aligned} 2u - 3v &= 2u + 3v - 5 \\ -6v &= -5 \\ v &= \frac{5}{6} \end{aligned}$$

The region *S* is then a rectangle whose sides are given by $u = 0$, $v = 0$, $u = \frac{5}{4}$ and $v = \frac{5}{6}$ and so the ranges of *u* and *v* are,

$$0 \le u \le \frac{5}{4} \qquad\qquad 0 \le v \le \frac{5}{6}$$

Next, we need the Jacobian.

$$\frac{\partial\left(x,y\right)}{\partial\left(u,v\right)} = \begin{vmatrix} 2 & 3 \\ 2 & -3 \end{vmatrix} = -6 - 6 = -12$$

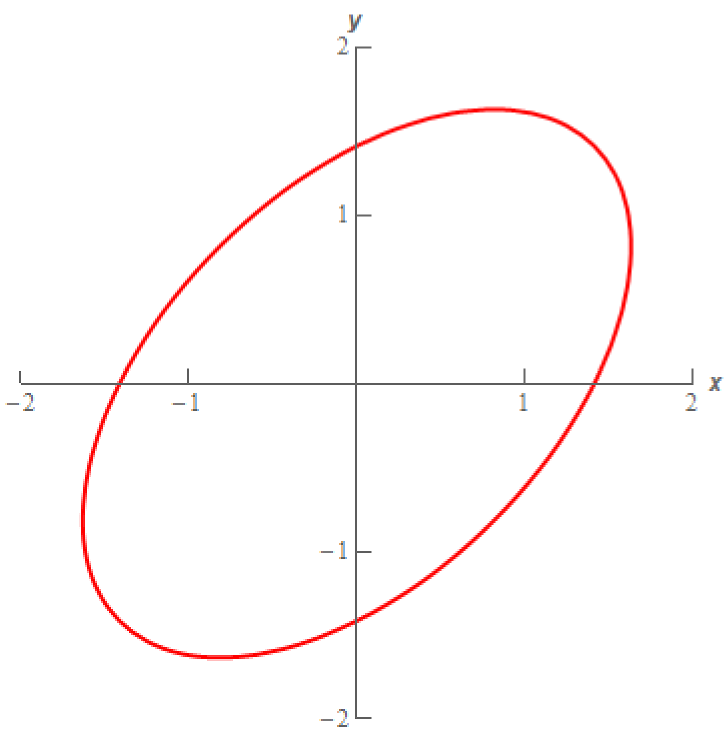
The integral is then,

$$\begin{aligned}
\iint_R x + y\,dA &= \int_0^{\frac{5}{6}} \int_0^{\frac{5}{4}} \left(\left(2u + 3v\right) + \left(2u - 3v\right)\right)\left|-12\right| du\,dv \\
&= \int_0^{\frac{5}{6}} \int_0^{\frac{5}{4}} 48u\,du\,dv \\
&= \int_0^{\frac{5}{6}} 24u^2\Big|_0^{\frac{5}{4}}\,dv \\
&= \int_0^{\frac{5}{6}} \frac{75}{2}\,dv \\
&= \frac{75}{2}v\Big|_0^{\frac{5}{6}} \\
&= \frac{125}{4}
\end{aligned}$$

Example 5 Evaluate $\iint_R x^2 - xy + y^2\,dA$ where *R* is the ellipse given by $x^2 - xy + y^2 \le 2$ and using the transformation $x = \sqrt{2}\,u - \sqrt{\frac{2}{3}}\,v$, $y = \sqrt{2}\,u + \sqrt{\frac{2}{3}}\,v$.

Solution

Before we proceed with this problem. Let's do a quick graph of the boundary of the region *R*. We claimed that it is an ellipse, but is clearly not in "standard" form. Here is the boundary of *R*.



So, it is an ellipse, just one that is at an angle rather than symmetric about the *x* and *y*-axis as we are used to dealing with.

Also, note that we used "≤ 2" when "defining" *R* to make it clear that we are using both the actual ellipse itself as well as the interior of the ellipse for *R*.

Okay, let's proceed with the problem.

The first thing to do is to plug the transformation into the equation for the ellipse to see what the region transforms into.

$$\begin{aligned} 2 &= x^2 - xy + y^2 \\ &= \left(\sqrt{2}\,u - \sqrt{\frac{2}{3}}\,v\right)^2 - \left(\sqrt{2}\,u - \sqrt{\frac{2}{3}}\,v\right)\left(\sqrt{2}\,u + \sqrt{\frac{2}{3}}\,v\right) + \left(\sqrt{2}\,u + \sqrt{\frac{2}{3}}\,v\right)^2 \\ &= 2u^2 - \frac{4}{\sqrt{3}}uv + \frac{2}{3}v^2 - \left(2u^2 - \frac{2}{3}v^2\right) + 2u^2 + \frac{4}{\sqrt{3}}uv + \frac{2}{3}v^2 \\ &= 2u^2 + 2v^2 \end{aligned}$$

Or, upon dividing by 2 we see that the equation describing *R* transforms into

$$u^2 + v^2 = 1$$

or the unit circle. Again, this will be much easier to integrate over than the original region.

Note as well that we've shown that the function that we're integrating is

$$x^2 - xy + y^2 = 2\left(u^2 + v^2\right)$$

in terms of *u* and *v* so we won't have to redo that work when the time to do the integral comes around.

Finally, we need to find the Jacobian.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \sqrt{2} & -\sqrt{\frac{2}{3}} \\ \sqrt{2} & \sqrt{\frac{2}{3}} \end{vmatrix} = \frac{2}{\sqrt{3}} + \frac{2}{\sqrt{3}} = \frac{4}{\sqrt{3}}$$

The integral is then,

$$\iint\limits_R x^2 - xy + y^2 \, dA = \iint\limits_S 2\left(u^2 + v^2\right)\left|\frac{4}{\sqrt{3}}\right| d\bar{A}$$

Before proceeding a word of caution is in order. Do not make the mistake of substituting $x^2 - xy + y^2 = 2$ or $u^2 + v^2 = 1$ in for the integrand. These equations are only valid on the boundary of the region and we are looking at all the points interior to the boundary as well and for those points neither of these equations will be true!

At this point we'll note that this integral will be much easier in terms of polar coordinates and so to finish the integral out will convert to polar coordinates.

$$\begin{aligned}\iint\limits_R x^2 - xy + y^2\, dA &= \iint\limits_S 2\left(u^2+v^2\right)\left|\frac{4}{\sqrt{3}}\right| d\overline{A}\\ &= \frac{8}{\sqrt{3}}\int_0^{2\pi}\int_0^1 \left(r^2\right) r\, dr\, d\theta\\ &= \frac{8}{\sqrt{3}}\int_0^{2\pi} \frac{1}{4}r^4\Big|_0^1\, d\theta\\ &= \frac{8}{\sqrt{3}}\int_0^{2\pi}\frac{1}{4}\, d\theta\\ &= \frac{4\pi}{\sqrt{3}}\end{aligned}$$

Let's now briefly look at triple integrals. In this case we will again start with a region *R* and use the transformation $x = g\left(u,v,w\right)$, $y = h\left(u,v,w\right)$, and $z = k\left(u,v,w\right)$ to transform the region into the new region *S*. To do the integral we will need a Jacobian, just as we did with double integrals. Here is the definition of the Jacobian for this kind of transformation.

$$\frac{\partial\left(x,y,z\right)}{\partial\left(u,v,w\right)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w}\\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}\end{vmatrix}$$

In this case the Jacobian is defined in terms of the determinant of a 3x3 matrix. We saw how to evaluate these when we looked at cross products back in Calculus II. If you need a refresher on how to compute them you should go back and review that section.

The integral under this transformation is,

$$\iiint\limits_R f\left(x,y,z\right) dV = \iiint\limits_S f\left(g\left(u,v,w\right),h\left(u,v,w\right),k\left(u,v,w\right)\right)\left|\frac{\partial\left(x,y,z\right)}{\partial\left(u,v,w\right)}\right| d\overline{V}$$

As with double integrals we used $d\overline{V}$ in the *u*/*v*/*w* integral above to remind ourselves that we will need to use *du*, *dv* and *dw* when converting to single integrals. Again, this is just notation and is usually written as just *dV*.

We can look at just the differentials and note that we must have

$$dV = \left|\frac{\partial\left(x,y,z\right)}{\partial\left(u,v,w\right)}\right| d\overline{V}$$

We're not going to do any integrals here, but let's verify the formula for *dV* for spherical coordinates.

Example 6 Verify that $dV = \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$ when using spherical coordinates.

Solution

Here the transformation is just the standard conversion formulas.

$$x = \rho \sin\varphi \cos\theta \qquad y = \rho \sin\varphi \sin\theta \qquad z = \rho \cos\varphi$$

The Jacobian is,

$$\begin{aligned}
\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} &= \begin{vmatrix} \sin\varphi\cos\theta & -\rho\sin\varphi\sin\theta & \rho\cos\varphi\cos\theta \\ \sin\varphi\sin\theta & \rho\sin\varphi\cos\theta & \rho\cos\varphi\sin\theta \\ \cos\varphi & 0 & -\rho\sin\varphi \end{vmatrix} \\
&= -\rho^2\sin^3\varphi\cos^2\theta - \rho^2\sin\varphi\cos^2\varphi\sin^2\theta + 0 \\
&\qquad -\rho^2\sin^3\varphi\sin^2\theta - 0 - \rho^2\sin\varphi\cos^2\varphi\cos^2\theta \\
&= -\rho^2\sin^3\varphi\left(\cos^2\theta + \sin^2\theta\right) - \rho^2\sin\varphi\cos^2\varphi\left(\sin^2\theta + \cos^2\theta\right) \\
&= -\rho^2\sin^3\varphi - \rho^2\sin\varphi\cos^2\varphi \\
&= -\rho^2\sin\varphi\left(\sin^2\varphi + \cos^2\varphi\right) \\
&= -\rho^2\sin\varphi
\end{aligned}$$

Finally, *dV* becomes,

$$dV = \left|-\rho^2\sin\varphi\right| d\rho \, d\theta \, d\varphi = \rho^2\sin\varphi \, d\rho \, d\theta \, d\varphi$$

Recall that we restricted φ to the range $0 \le \varphi \le \pi$ for spherical coordinates and so we know that $\sin\varphi \ge 0$ and so we don't need the absolute value bars on the sine.

We will leave it to you to check the formula for *dV* for cylindrical coordinates if you'd like to. It is a much easier formula to check.

Section 4-9 : Surface Area

In this section we will look at the lone application (aside from the area and volume interpretations) of multiple integrals in this material. This is not the first time that we've looked at surface area We first saw surface area in Calculus II, however, in that setting we were looking at the surface area of a solid of revolution. In other words, we were looking at the surface area of a solid obtained by rotating a function about the *x* or *y* axis. In this section we want to look at a much more general setting although you will note that the formula here is very similar to the formula we saw back in Calculus II.

Here we want to find the surface area of the surface given by $z = f\left(x, y\right)$ where $\left(x, y\right)$ is a point from the region *D* in the *xy*-plane. In this case the surface area is given by,

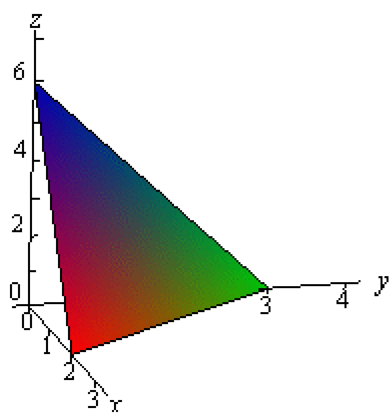
$$S = \iint\limits_D \sqrt{\left[f_x\right]^2 + \left[f_y\right]^2 + 1}\, dA$$

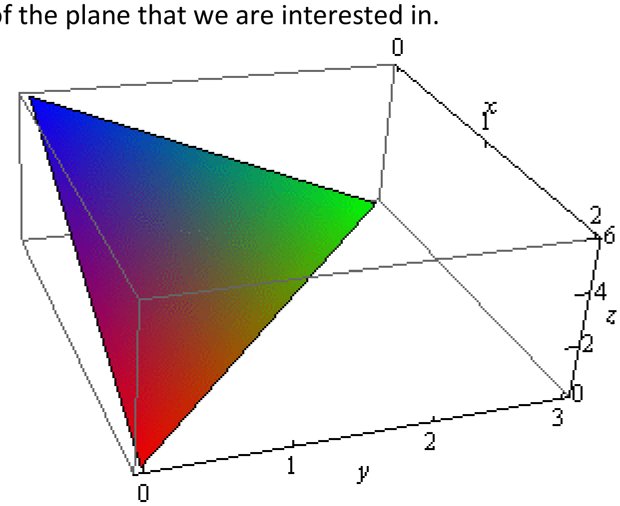
Let's take a look at a couple of examples.

Example 1 Find the surface area of the part of the plane $3x + 2y + z = 6$ that lies in the first octant.

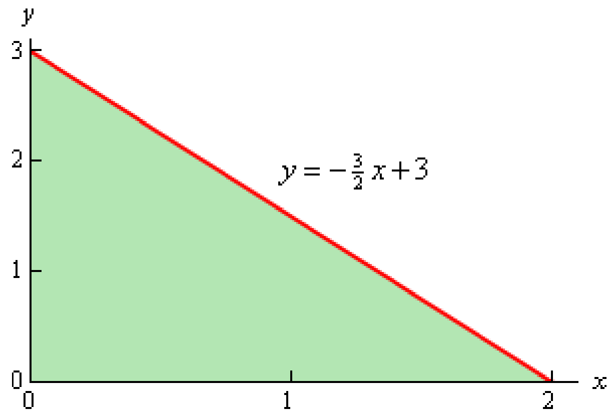
Solution

Remember that the first octant is the portion of the *xyz*-axis system in which all three variables are positive. Let's first get a sketch of the part of the plane that we are interested in.





We'll also need a sketch of the region *D*.



Remember that to get the region *D* we can pretend that we are standing directly over the plane and what we see is the region *D*. We can get the equation for the hypotenuse of the triangle by realizing that this is nothing more than the line where the plane intersects the *xy*-plane and we also know that $z = 0$ on the *xy*-plane. Plugging $z = 0$ into the equation of the plane will give us the equation for the hypotenuse.

Notice that in order to use the surface area formula we need to have the function in the form $z = f\left(x, y\right)$ and so solving for *z* and taking the partial derivatives gives,

$$z = 6 - 3x - 2y \qquad\qquad f_x = -3 \qquad\qquad f_y = -2$$

The limits defining *D* are,

$$0 \le x \le 2 \qquad\qquad 0 \le y \le -\frac{3}{2}x + 3$$

The surface area is then,

$$\begin{aligned} S &= \iint\limits_D \sqrt{\left[-3\right]^2 + \left[-2\right]^2 + 1}\, dA \\ &= \int_0^2 \int_0^{-\frac{3}{2}x+3} \sqrt{14}\, dy\, dx \\ &= \sqrt{14} \int_0^2 -\frac{3}{2}x + 3\, dx \\ &= \sqrt{14}\left(-\frac{3}{4}x^2 + 3x\right)\Big|_0^2 \\ &= 3\sqrt{14} \end{aligned}$$

Example 2 Determine the surface area of the part of $z = xy$ that lies in the cylinder given by $x^2 + y^2 = 1$.

Solution

In this case we are looking for the surface area of the part of $z = xy$ where $\left(x, y\right)$ comes from the disk of radius 1 centered at the origin since that is the region that will lie inside the given cylinder.

Here are the partial derivatives,

$$f_x = y \qquad\qquad f_y = x$$

The integral for the surface area is,

$$S = \iint\limits_D \sqrt{x^2 + y^2 + 1}\, dA$$

Given that *D* is a disk it makes sense to do this integral in polar coordinates.

$$\begin{aligned} S &= \iint\limits_D \sqrt{x^2 + y^2 + 1}\, dA \\ &= \int_0^{2\pi}\int_0^1 r\sqrt{1+r^2}\, dr\, d\theta \\ &= \int_0^{2\pi} \frac{1}{2}\left(\frac{2}{3}\right)\left(1+r^2\right)^{\frac{3}{2}}\Bigg|_0^1\, d\theta \\ &= \int_0^{2\pi} \frac{1}{3}\left(2^{\frac{3}{2}} - 1\right) d\theta \\ &= \frac{2\pi}{3}\left(2^{\frac{3}{2}} - 1\right) \end{aligned}$$

Section 4-10 : Area and Volume Revisited

This section is here only so we can summarize the geometric interpretations of the double and triple integrals that we saw in this chapter. Since the purpose of this section is to summarize these formulas we aren't going to be doing any examples in this section.

We'll first look at the area of a region. The area of the region *D* is given by,

$$\text{Area of } D = \iint\limits_D dA$$

Now let's give the two volume formulas. First the volume of the region *E* is given by,

$$\text{Volume of } E = \iiint\limits_E dV$$

Finally, if the region *E* can be defined as the region under the function $z = f\left(x, y\right)$ and above the region *D* in *xy*-plane then,

$$\text{Volume of } E = \iint\limits_D f\left(x, y\right)\, dA$$

Note as well that there are similar formulas for the other planes. For instance, the volume of the region behind the function $y = f\left(x, z\right)$ and in front of the region *D* in the *xz*-plane is given by,

$$\text{Volume of } E = \iint\limits_D f\left(x, z\right)\, dA$$

Likewise, the the volume of the region behind the function $x = f\left(y, z\right)$ and in front of the region *D* in the *yz*-plane is given by,

$$\text{Volume of } E = \iint\limits_D f\left(y, z\right)\, dA$$

Chapter 5 : Line Integrals

In this section we are going to start looking at Calculus with vector fields (which we'll define in the first section). In particular we will be looking at a new type of integral, the line integral and some of the interpretations of the line integral. We will also take a look at one of the more important theorems involving line integrals, Green's Theorem.

Here is a listing of the topics covered in this chapter.

Section 5-1 : Vector Fields

We need to start this chapter off with the definition of a vector field as they will be a major component of both this chapter and the next. Let's start off with the formal definition of a vector field.

Definition

A vector field on two (or three) dimensional space is a function $\vec{F}$ that assigns to each point (x, y) (or (x, y, z)) a two (or three dimensional) vector given by $\vec{F}(x, y)$ (or $\vec{F}(x, y, z)$).

That may not make a lot of sense, but most people do know what a vector field is, or at least they've seen a sketch of a vector field. If you've seen a current sketch giving the direction and magnitude of a flow of a fluid or the direction and magnitude of the winds then you've seen a sketch of a vector field.

The standard notation for the function $\vec{F}$ is,

$$\vec{F}(x, y) = P(x, y)\vec{i} + Q(x, y)\vec{j}$$

$$\vec{F}(x, y, z) = P(x, y, z)\vec{i} + Q(x, y, z)\vec{j} + R(x, y, z)\vec{k}$$

depending on whether or not we're in two or three dimensions. The function *P*, *Q*, *R* (if it is present) are sometimes called **scalar functions**.

Let's take a quick look at a couple of examples.

Example 1 Sketch each of the following vector fields.

(a) $\vec{F}(x, y) = -y\vec{i} + x\vec{j}$

(b) $\vec{F}(x, y, z) = 2x\vec{i} - 2y\vec{j} - 2x\vec{k}$

Solution

(a) $\vec{F}(x, y) = -y\vec{i} + x\vec{j}$

Okay, to graph the vector field we need to get some "values" of the function. This means plugging in some points into the function. Here are a couple of evaluations.

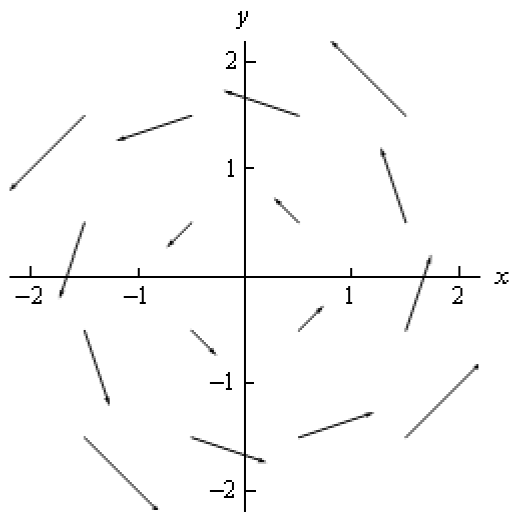
$$\vec{F}\left(\frac{1}{2}, \frac{1}{2}\right) = -\frac{1}{2}\vec{i} + \frac{1}{2}\vec{j}$$

$$\vec{F}\left(\frac{1}{2}, -\frac{1}{2}\right) = -\left(-\frac{1}{2}\right)\vec{i} + \frac{1}{2}\vec{j} = \frac{1}{2}\vec{i} + \frac{1}{2}\vec{j}$$

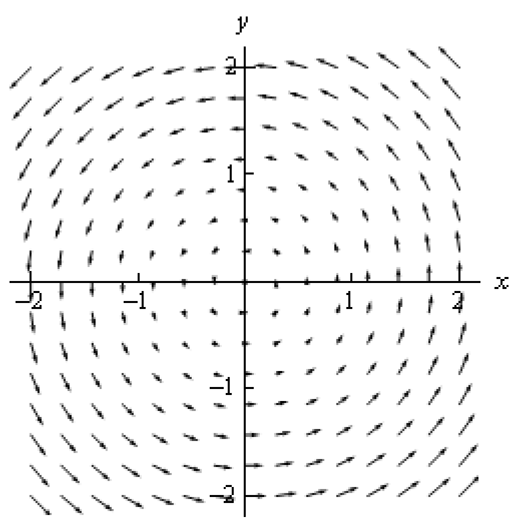
$$\vec{F}\left(\frac{3}{2}, \frac{1}{4}\right) = -\frac{1}{4}\vec{i} + \frac{3}{2}\vec{j}$$

So, just what do these evaluations tell us? Well the first one tells us that at the point $\left(\frac{1}{2}, \frac{1}{2}\right)$ we will plot the vector $-\frac{1}{2}\vec{i} + \frac{1}{2}\vec{j}$. Likewise, the third evaluation tells us that at the point $\left(\frac{3}{2}, \frac{1}{4}\right)$ we will plot the vector $-\frac{1}{4}\vec{i} + \frac{3}{2}\vec{j}$.

We can continue in this fashion plotting vectors for several points and we'll get the following sketch of the vector field.



If we want significantly more points plotted, then it is usually best to use a computer aided graphing system such as Maple or Mathematica. Here is a sketch with many more vectors included that was generated with Mathematica.



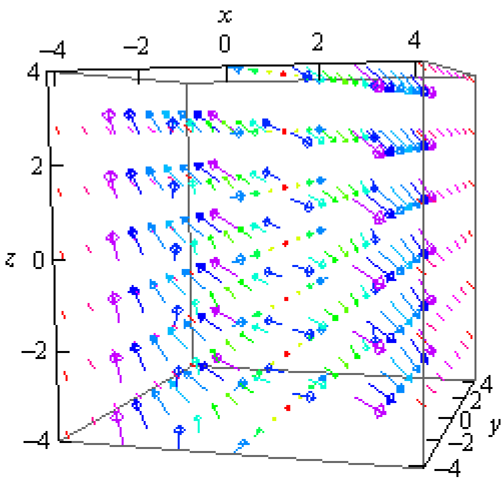
(b) $\vec{F}\left(x,y,z\right)=2x\,\vec{i}-2y\,\vec{j}-2x\,\vec{k}$

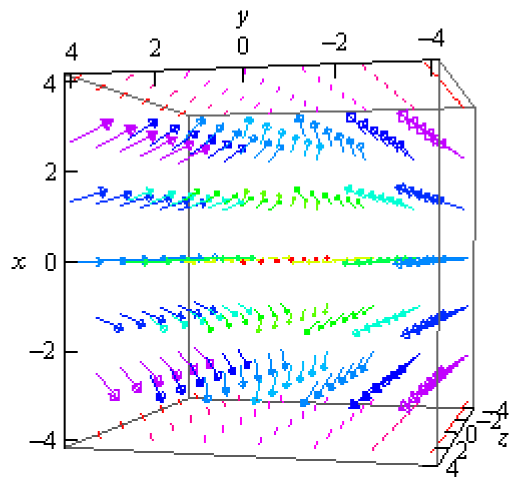
In the case of three dimensional vector fields it is almost always better to use Maple, Mathematica, or some other such tool. Despite that let's go ahead and do a couple of evaluations anyway.

$$\vec{F}\left(1,-3,2\right)=2\,\vec{i}+6\,\vec{j}-2\,\vec{k}$$
$$\vec{F}\left(0,5,3\right)=-10\,\vec{j}$$

Notice that *z* only affects the placement of the vector in this case and does not affect the direction or the magnitude of the vector. Sometimes this will happen so don't get excited about it when it does.

Here is a couple of sketches generated by Mathematica. The sketch on the left is from the "front" and the sketch on the right is from "above".





Now that we've seen a couple of vector fields let's notice that we've already seen a vector field function. In the second chapter we looked at the gradient vector. Recall that given a function $f(x,y,z)$ the gradient vector is defined by,

$$\nabla f = \langle f_x, f_y, f_z \rangle$$

This is a vector field and is often called a **gradient vector field**.

In these cases, the function $f(x,y,z)$ is often called a scalar function to differentiate it from the vector field.

Example 2 Find the gradient vector field of the following functions.

(a) $f(x,y) = x^2 \sin(5y)$

(b) $f(x,y,z) = z\mathbf{e}^{-xy}$

Solution

(a) $f(x,y) = x^2 \sin(5y)$

Note that we only gave the gradient vector definition for a three dimensional function, but don't forget that there is also a two dimension definition. All that we need to drop off the third component of the vector.

Here is the gradient vector field for this function.

$$\nabla f = \langle 2x\sin(5y), 5x^2\cos(5y) \rangle$$

(b) $f(x,y,z) = z\mathbf{e}^{-xy}$

There isn't much to do here other than take the gradient.

$$\nabla f = \left\langle -yz\mathbf{e}^{-xy}, -xz\mathbf{e}^{-xy}, \mathbf{e}^{-xy} \right\rangle$$

Let's do another example that will illustrate the relationship between the gradient vector field of a function and its contours.

Example 3 Sketch the gradient vector field for $f(x,y) = x^2 + y^2$ as well as several contours for this function.

Solution

Recall that the contours for a function are nothing more than curves defined by,

$$f(x,y) = k$$

for various values of *k*. So, for our function the contours are defined by the equation,

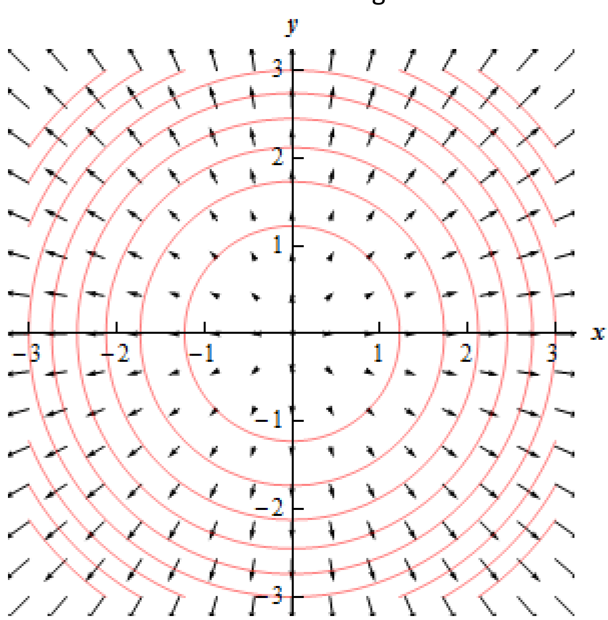
$$x^2 + y^2 = k$$

and so they are circles centered at the origin with radius $\sqrt{k}$.

Here is the gradient vector field for this function.

$$\nabla f(x,y) = 2x\vec{i} + 2y\vec{j}$$

Here is a sketch of several of the contours as well as the gradient vector field.



Notice that the vectors of the vector field are all orthogonal (or perpendicular) to the contours. This will always be the case when we are dealing with the contours of a function as well as its gradient vector field.

The *k*'s we used for the graph above were 1.5, 3, 4.5, 6, 7.5, 9, 10.5, 12, and 13.5. Now notice that as we increased *k* by 1.5 the contour curves get closer together and that as the contour curves get closer together the larger the vectors become. In other words, the closer the contour curves are (as *k* is increased by a fixed amount) the faster the function is changing at that point. Also recall that the direction of fastest change for a function is given by the gradient vector at that point. Therefore, it should make sense that the two ideas should match up as they do here.

The final topic of this section is that of conservative vector fields. A vector field $\vec{F}$ is called a **conservative vector field** if there exists a function f such that $\vec{F} = \nabla f$. If $\vec{F}$ is a conservative vector field then the function, *f*, is called a **potential function** for $\vec{F}$.

All this definition is saying is that a vector field is conservative if it is also a gradient vector field for some function.

For instance the vector field $\vec{F} = y\,\vec{i} + x\,\vec{j}$ is a conservative vector field with a potential function of $f\left(x,y\right) = xy$ because $\nabla f = \left\langle y,x \right\rangle$.

On the other hand, $\vec{F} = -y\,\vec{i} + x\,\vec{j}$ is not a conservative vector field since there is no function *f* such that $\vec{F} = \nabla f$. If you're not sure that you believe this at this point be patient, we will be able to prove this in a couple of sections. In that section we will also show how to find the potential function for a conservative vector field.

Section 5-2 : Line Integrals - Part I

In this section we are now going to introduce a new kind of integral. However, before we do that it is important to note that you will need to remember how to parameterize equations, or put another way, you will need to be able to write down a set of parametric equations for a given curve. You should have seen some of this in your Calculus II course. If you need some review you should go back and review some of the basics of parametric equations and curves.

Here are some of the more basic curves that we'll need to know how to do as well as limits on the parameter if they are required.

Curve	Parametric Equations	
	<u>Counter-Clockwise</u>	<u>Clockwise</u>
$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ (Ellipse)	$x=a\cos(t)$ $y=b\sin(t)$ $0\le t\le 2\pi$	$x=a\cos(t)$ $y=-b\sin(t)$ $0\le t\le 2\pi$
	<u>Counter-Clockwise</u>	<u>Clockwise</u>
$x^2+y^2=r^2$ (Circle)	$x=r\cos(t)$ $y=r\sin(t)$ $0\le t\le 2\pi$	$x=r\cos(t)$ $y=-r\sin(t)$ $0\le t\le 2\pi$
$y=f(x)$	$x=t$ $y=f(t)$	
$x=g(y)$	$x=g(t)$ $y=t$	
Line Segment From (x_0,y_0,z_0) to (x_1,y_1,z_1)	$\vec r(t)=(1-t)\langle x_0,y_0,z_0\rangle+t\langle x_1,y_1,z_1\rangle\ ,\ 0\le t\le 1$ or $x=(1-t)x_0+t\,x_1$ $y=(1-t)y_0+t\,y_1\quad,\quad 0\le t\le 1$ $z=(1-t)z_0+t\,z_1$	

With the final one we gave both the vector form of the equation as well as the parametric form and if we need the two-dimensional version then we just drop the *z* components. In fact, we will be using the two-dimensional version of this in this section.

For the ellipse and the circle we've given two parameterizations, one tracing out the curve clockwise and the other counter-clockwise. As we'll eventually see the direction that the curve is traced out can, on occasion, change the answer. Also, both of these "start" on the positive *x*-axis at $t = 0$.

Now let's move on to line integrals. In Calculus I we integrated $f(x)$, a function of a single variable, over an interval $[a,b]$. In this case we were thinking of *x* as taking all the values in this interval starting at *a* and ending at *b*. With line integrals we will start with integrating the function $f(x,y)$, a function of two variables, and the values of *x* and *y* that we're going to use will be the points, (x,y), that lie on a curve *C*. Note that this is different from the double integrals that we were working with in the previous chapter where the points came out of some two-dimensional region.

Let's start with the curve *C* that the points come from. We will assume that the curve is *smooth* (defined shortly) and is given by the parametric equations,

$$x = h(t) \qquad y = g(t) \qquad a \le t \le b$$

We will often want to write the parameterization of the curve as a vector function. In this case the curve is given by,

$$\vec{r}(t) = h(t)\vec{i} + g(t)\vec{j} \qquad a \le t \le b$$

The curve is called **smooth** if $\vec{r}'(t)$ is continuous and $\vec{r}'(t) \ne 0$ for all *t*.

The **line integral** of $f(x,y)$ along *C* is denoted by,

$$\int_C f(x,y)\,ds$$

We use a *ds* here to acknowledge the fact that we are moving along the curve, *C*, instead of the *x*-axis (denoted by *dx*) or the *y*-axis (denoted by *dy*). Because of the *ds* this is sometimes called the **line integral of *f* with respect to arc length**.

We've seen the notation *ds* before. If you recall from Calculus II when we looked at the arc length of a curve given by parametric equations we found it to be,

$$L = \int_a^b ds, \qquad \text{where} \quad ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

It is no coincidence that we use *ds* for both of these problems. The *ds* is the same for both the arc length integral and the notation for the line integral.

So, to compute a line integral we will convert everything over to the parametric equations. The line integral is then,

$$\int_C f(x,y)\,ds = \int_a^b f\left(h(t), g(t)\right)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

Don't forget to plug the parametric equations into the function as well.

If we use the vector form of the parameterization we can simplify the notation up somewhat by noticing that,

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \left\| \vec{r}'(t) \right\|$$

where $\left\| \vec{r}'(t) \right\|$ is the magnitude or norm of $\vec{r}'(t)$. Using this notation, the line integral becomes,

$$\int_C f(x,y)\,ds = \int_a^b f(h(t), g(t)) \left\| \vec{r}'(t) \right\| dt$$

Note that as long as the parameterization of the curve *C* is traced out exactly once as *t* increases from *a* to *b* the value of the line integral will be independent of the parameterization of the curve.

Let's take a look at an example of a line integral.

Example 1 Evaluate $\int_C xy^4\,ds$ where *C* is the right half of the circle, $x^2 + y^2 = 16$ traced out in a counter clockwise direction.

Solution

We first need a parameterization of the circle. This is given by,

$$x = 4\cos t \qquad y = 4\sin t$$

We now need a range of *t*'s that will give the right half of the circle. The following range of *t*'s will do this.

$$-\frac{\pi}{2} \le t \le \frac{\pi}{2}$$

Now, we need the derivatives of the parametric equations and let's compute *ds*.

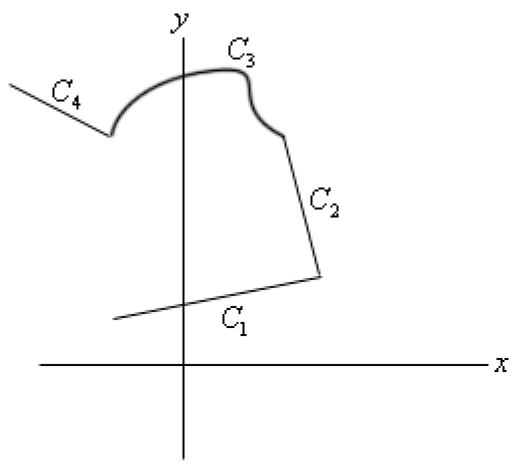
$$\frac{dx}{dt} = -4\sin t \qquad \frac{dy}{dt} = 4\cos t$$

$$ds = \sqrt{16\sin^2 t + 16\cos^2 t}\,dt = 4\,dt$$

The line integral is then,

$$\begin{aligned}\int_C xy^4\,ds &= \int_{-\pi/2}^{\pi/2} 4\cos t\left(4\sin t\right)^4\left(4\right)dt\\ &= 4096\int_{-\pi/2}^{\pi/2}\cos t\,\sin^4 t\,dt\\ &= \frac{4096}{5}\sin^5 t\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\\ &= \frac{8192}{5}\end{aligned}$$

Next we need to talk about line integrals over **piecewise smooth curves**. A piecewise smooth curve is any curve that can be written as the union of a finite number of smooth curves, $C_1,\ldots,C_n$ where the end point of C_i is the starting point of C_{i+1}. Below is an illustration of a piecewise smooth curve.

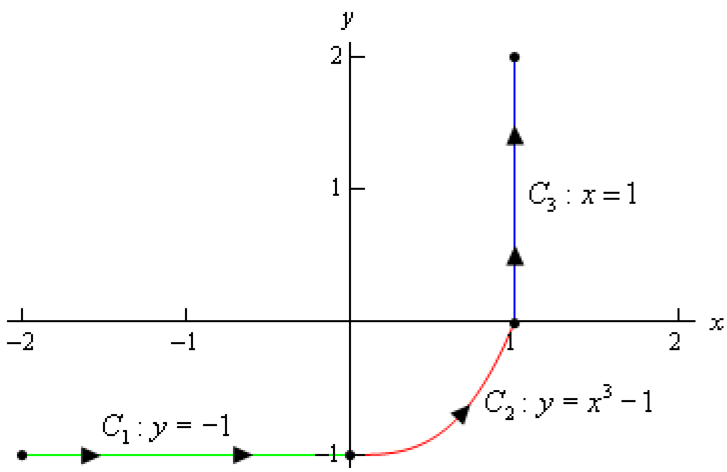


Evaluation of line integrals over piecewise smooth curves is a relatively simple thing to do. All we do is evaluate the line integral over each of the pieces and then add them up. The line integral for some function over the above piecewise curve would be,

$$\int_C f\left(x,y\right)ds = \int_{C_1} f\left(x,y\right)ds + \int_{C_2} f\left(x,y\right)ds + \int_{C_3} f\left(x,y\right)ds + \int_{C_4} f\left(x,y\right)ds$$

Let’s see an example of this.

Example 2 Evaluate $\int_C 4x^3\,ds$ where *C* is the curve shown below.



Solution

So, first we need to parameterize each of the curves.

$$C_1 \;:\; x = t,\; y = -1, \qquad -2 \le t \le 0$$
$$C_2 \;:\; x = t,\; y = t^3 - 1, \qquad 0 \le t \le 1$$
$$C_3 \;:\; x = 1,\; y = t, \qquad 0 \le t \le 2$$

Now let's do the line integral over each of these curves.

$$\int_{C_1} 4x^3\,ds = \int_{-2}^{0} 4t^3\sqrt{(1)^2 + (0)^2}\,dt = \int_{-2}^{0} 4t^3\,dt = t^4\Big|_{-2}^{0} = -16$$

$$\begin{aligned}\int_{C_2} 4x^3\,ds &= \int_0^1 4t^3\sqrt{(1)^2 + \left(3t^2\right)^2}\,dt \\ &= \int_0^1 4t^3\sqrt{1 + 9t^4}\,dt \\ &= \frac{1}{9}\left(\frac{2}{3}\right)\left(1 + 9t^4\right)^{\frac{3}{2}}\Big|_0^1 = \frac{2}{27}\left(10^{\frac{3}{2}} - 1\right) = 2.268\end{aligned}$$

$$\int_{C_3} 4x^3\,ds = \int_0^2 4(1)^3\sqrt{(0)^2 + (1)^2}\,dt = \int_0^2 4\,dt = 8$$

Finally, the line integral that we were asked to compute is,

$$\begin{aligned}\int_C 4x^3\,ds &= \int_{C_1} 4x^3\,ds + \int_{C_2} 4x^3\,ds + \int_{C_3} 4x^3\,ds \\ &= -16 + 2.268 + 8 \\ &= -5.732\end{aligned}$$

Notice that we put direction arrows on the curve in the above example. The direction of motion along a curve *may* change the value of the line integral as we will see in the next section. Also note that the curve can be thought of a curve that takes us from the point $\left(-2,-1\right)$ to the point $\left(1,2\right)$. Let's first see what happens to the line integral if we change the path between these two points.

Example 3 Evaluate $\int_C 4x^3\,ds$ where *C* is the line segment from $\left(-2,-1\right)$ to $\left(1,2\right)$.

Solution

From the parameterization formulas at the start of this section we know that the line segment starting at $\left(-2,-1\right)$ and ending at $\left(1,2\right)$ is given by,

$$\begin{aligned}\vec r\left(t\right) &= \left(1-t\right)\left\langle -2,-1\right\rangle + t\left\langle 1,2\right\rangle \\ &= \left\langle -2+3t,-1+3t\right\rangle\end{aligned}$$

for $0 \le t \le 1$. This means that the individual parametric equations are,

$$x = -2+3t \qquad\qquad y = -1+3t$$

Using this path the line integral is,

$$\begin{aligned}\int_C 4x^3\,ds &= \int_0^1 4\left(-2+3t\right)^3\sqrt{9+9}\,dt \\ &= 12\sqrt{2}\left(\tfrac{1}{12}\right)\left(-2+3t\right)^4\Big|_0^1 \\ &= 12\sqrt{2}\left(-\frac{5}{4}\right) \\ &= -15\sqrt{2} = -21.213\end{aligned}$$

When doing these integrals don't forget simple Calc I substitutions to avoid having to do things like cubing out a term. Cubing it out is not that difficult, but it is more work than a simple substitution.

So, the previous two examples seem to suggest that if we change the path between two points then the value of the line integral (with respect to arc length) will change. While this will happen fairly regularly we can't assume that it will always happen. In a later section we will investigate this idea in more detail.

Next, let's see what happens if we change the direction of a path.

Example 4 Evaluate $\int_C 4x^3\,ds$ where *C* is the line segment from $\left(1,2\right)$ to $\left(-2,-1\right)$.

Solution

This one isn't much different, work wise, from the previous example. Here is the parameterization of the curve.

$$\begin{aligned}\vec r\left(t\right) &= \left(1-t\right)\left\langle 1,2\right\rangle + t\left\langle -2,-1\right\rangle \\ &= \left\langle 1-3t,2-3t\right\rangle\end{aligned}$$

for $0 \le t \le 1$. Remember that we are switching the direction of the curve and this will also change the parameterization so we can make sure that we start/end at the proper point.

Here is the line integral.

$$\begin{aligned}\int_C 4x^3\,ds &= \int_0^1 4\left(1-3t\right)^3\sqrt{9+9}\,dt \\ &= 12\sqrt{2}\left(-\tfrac{1}{12}\right)\left(1-3t\right)^4\Big|_0^1 \\ &= 12\sqrt{2}\left(-\frac{5}{4}\right) \\ &= -15\sqrt{2} = -21.213\end{aligned}$$

So, it looks like when we switch the direction of the curve the line integral (with respect to arc length) will not change. This will always be true for these kinds of line integrals. However, there are other kinds of line integrals in which this won't be the case. We will see more examples of this in the next couple of sections so don't get it into your head that changing the direction will never change the value of the line integral.

Before working another example let's formalize this idea up somewhat. Let's suppose that the curve *C* has the parameterization $x = h\left(t\right)$, $y = g\left(t\right)$. Let's also suppose that the initial point on the curve is *A* and the final point on the curve is *B*. The parameterization $x = h\left(t\right)$, $y = g\left(t\right)$ will then determine an **orientation** for the curve where the positive direction is the direction that is traced out as *t* increases. Finally, let $-C$ be the curve with the same points as *C*, however in this case the curve has *B* as the initial point and *A* as the final point, again *t* is increasing as we traverse this curve. In other words, given a curve *C*, the curve $-C$ is the same curve as *C* except the direction has been reversed.

We then have the following fact about line integrals with respect to arc length.

Fact

$$\int_C f\left(x,y\right)ds = \int_{-C} f\left(x,y\right)ds$$

So, for a line integral with respect to arc length we can change the direction of the curve and not change the value of the integral. This is a useful fact to remember as some line integrals will be easier in one direction than the other.

Now, let's work another example

Example 5 Evaluate $\int_C x\,ds$ for each of the following curves.

(a) $C_1 : y = x^2, \; -1 \le x \le 1$

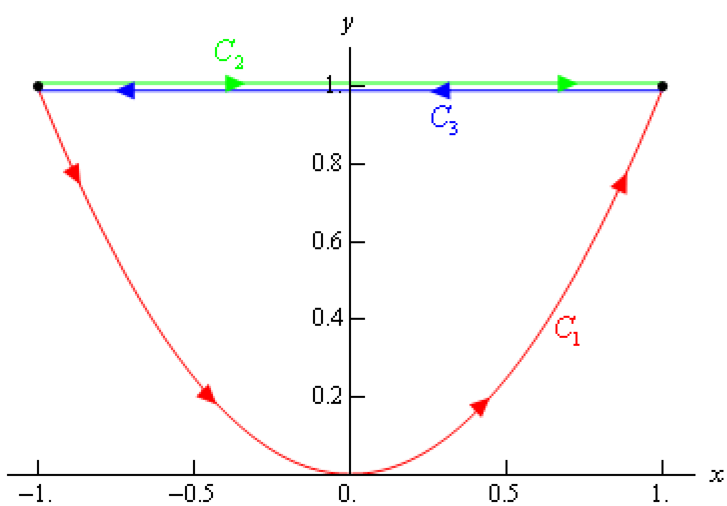
(b) C_2 : The line segment from $(-1,1)$ to $(1,1)$.

(c) C_3 : The line segment from $(1,1)$ to $(-1,1)$.

Solution

Before working any of these line integrals let's notice that all of these curves are paths that connect the points $(-1,1)$ and $(1,1)$. Also notice that $C_3 = -C_2$ and so by the fact above these two should give the same answer.

Here is a sketch of the three curves and note that the curves illustrating C_2 and C_3 have been separated a little to show that they are separate curves in some way even though they are the same line.



(a) $C_1 : y = x^2, \; -1 \le x \le 1$

Here is a parameterization for this curve.

$$C_1 : x = t, \; y = t^2, \; -1 \le t \le 1$$

Here is the line integral.

$$\int_{C_1} x\,ds = \int_{-1}^{1} t\sqrt{1+4t^2}\,dt = \frac{1}{12}\left(1+4t^2\right)^{\frac{3}{2}}\Bigg|_{-1}^{1} = 0$$

(b) C_2 : The line segment from $\left(-1,1\right)$ to $\left(1,1\right)$.

There are two parameterizations that we could use here for this curve. The first is to use the formula we used in the previous couple of examples. That parameterization is,

$$\begin{aligned} C_2 : \vec r\left(t\right) &= \left(1-t\right)\left\langle -1,1\right\rangle + t\left\langle 1,1\right\rangle \\ &= \left\langle 2t-1,1\right\rangle \end{aligned}$$

for $0 \le t \le 1$.

Sometimes we have no choice but to use this parameterization. However, in this case there is a second (probably) easier parameterization. The second one uses the fact that we are really just graphing a portion of the line $y = 1$. Using this the parameterization is,

$$C_2 : x = t,\ y = 1,\ -1 \le t \le 1$$

This will be a much easier parameterization to use so we will use this. Here is the line integral for this curve.

$$\int\limits_{C_2} x\,ds = \int_{-1}^{1} t\sqrt{1+0}\,dt = \frac{1}{2}t^2\Big|_{-1}^{1} = 0$$

Note that this time, unlike the line integral we worked with in Examples 2, 3, and 4 we got the same value for the integral despite the fact that the path is different. This will happen on occasion. We should also not expect this integral to be the same for all paths between these two points. At this point all we know is that for these two paths the line integral will have the same value. It is completely possible that there is another path between these two points that will give a different value for the line integral.

(c) C_3 : The line segment from $\left(1,1\right)$ to $\left(-1,1\right)$.

Now, according to our fact above we really don't need to do anything here since we know that $C_3 = -C_2$. The fact tells us that this line integral should be the same as the second part (*i.e.* zero). However, let's verify that, plus there is a point we need to make here about the parameterization.

Here is the parameterization for this curve.

$$\begin{aligned} C_3 : \vec r\left(t\right) &= \left(1-t\right)\left\langle 1,1\right\rangle + t\left\langle -1,1\right\rangle \\ &= \left\langle 1-2t,1\right\rangle \end{aligned}$$

for $0 \le t \le 1$.

Note that this time we can't use the second parameterization that we used in part (b) since we need to move from right to left as the parameter increases and the second parameterization used in the previous part will move in the opposite direction.

Here is the line integral for this curve.

$$\int_{C_3} x\,ds = \int_0^1 (1-2t)\sqrt{4+0}\,dt = 2\left(t-t^2\right)\Big|_0^1 = 0$$

Sure enough we got the same answer as the second part.

To this point in this section we've only looked at line integrals over a two-dimensional curve. However, there is no reason to restrict ourselves like that. We can do line integrals over three-dimensional curves as well.

Let's suppose that the three-dimensional curve *C* is given by the parameterization,

$$x = x(t), \qquad y = y(t) \qquad z = z(t) \qquad a \le t \le b$$

then the line integral is given by,

$$\int_C f(x,y,z)\,ds = \int_a^b f\left(x(t),y(t),z(t)\right)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

Note that often when dealing with three-dimensional space the parameterization will be given as a vector function.

$$\vec{r}(t) = \left\langle x(t), y(t), z(t)\right\rangle$$

Notice that we changed up the notation for the parameterization a little. Since we rarely use the function names we simply kept the *x*, *y*, and *z* and added on the (t) part to denote that they may be functions of the parameter.

Also notice that, as with two-dimensional curves, we have,

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \left\|\vec{r}\,'(t)\right\|$$

and the line integral can again be written as,

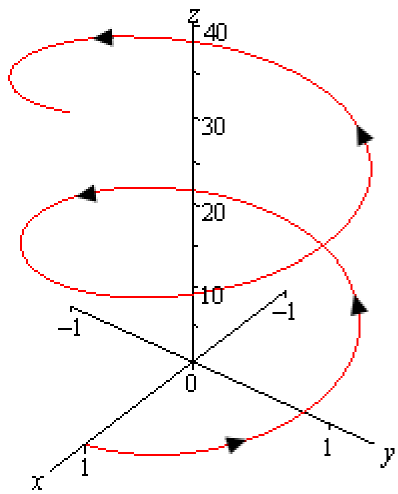
$$\int_C f(x,y,z)\,ds = \int_a^b f\left(x(t),y(t),z(t)\right)\left\|\vec{r}\,'(t)\right\|dt$$

So, outside of the addition of a third parametric equation line integrals in three-dimensional space work the same as those in two-dimensional space. Let's work a quick example.

Example 6 Evaluate $\int_C xyz\,ds$ where *C* is the helix given by, $\vec{r}(t) = \langle \cos(t), \sin(t), 3t \rangle$, $0 \le t \le 4\pi$
.

Solution

Note that we first saw the vector equation for a helix back in the Vector Functions section. Here is a quick sketch of the helix.



Here is the line integral.

$$\begin{aligned}\int_C xyz\,ds &= \int_0^{4\pi} 3t\cos(t)\sin(t)\sqrt{\sin^2 t + \cos^2 t + 9}\,dt \\ &= \int_0^{4\pi} 3t\left(\frac{1}{2}\sin(2t)\right)\sqrt{1+9}\,dt \\ &= \frac{3\sqrt{10}}{2}\int_0^{4\pi} t\sin(2t)\,dt \\ &= \frac{3\sqrt{10}}{2}\left(\frac{1}{4}\sin(2t) - \frac{t}{2}\cos(2t)\right)\Bigg|_0^{4\pi} \\ &= -3\sqrt{10}\,\pi\end{aligned}$$

You were able to do that integral right? It required integration by parts.

So, as we can see there really isn't too much difference between two- and three-dimensional line integrals.

Section 5-3 : Line Integrals - Part II

In the previous section we looked at line integrals with respect to arc length. In this section we want to look at line integrals with respect to *x* and/or *y*.

As with the last section we will start with a two-dimensional curve *C* with parameterization,

$$x = x(t) \qquad y = y(t) \qquad a \le t \le b$$

The **line integral of *f* with respect to *x*** is,

$$\int\limits_C f(x,y)\,dx = \int_a^b f(x(t),y(t))\,x'(t)\,dt$$

The **line integral of *f* with respect to *y*** is,

$$\int\limits_C f(x,y)\,dy = \int_a^b f(x(t),y(t))\,y'(t)\,dt$$

Note that the only notational difference between these two and the line integral with respect to arc length (from the previous section) is the differential. These have a *dx* or *dy* while the line integral with respect to arc length has a *ds*. So, when evaluating line integrals be careful to first note which differential you've got so you don't work the wrong kind of line integral.

These two integral often appear together and so we have the following shorthand notation for these cases.

$$\int\limits_C P\,dx + Q\,dy = \int\limits_C P(x,y)\,dx + \int\limits_C Q(x,y)\,dy$$

Let's take a quick look at an example of this kind of line integral.

Example 1 Evaluate $\int\limits_C \sin(\pi y)\,dy + yx^2\,dx$ where *C* is the line segment from $(0,2)$ to $(1,4)$.

Solution

Here is the parameterization of the curve.

$$\vec r(t) = (1-t)\langle 0,2\rangle + t\langle 1,4\rangle = \langle t, 2+2t\rangle \qquad 0 \le t \le 1$$

The line integral is,

$$\int_C \sin(\pi y)\,dy + yx^2\,dx = \int_C \sin(\pi y)\,dy + \int_C yx^2\,dx$$
$$= \int_0^1 \sin(\pi(2+2t))(2)\,dt + \int_0^1 (2+2t)(t)^2(1)\,dt$$
$$= -\frac{1}{\pi}\cos(2\pi + 2\pi t)\Big|_0^1 + \left(\frac{2}{3}t^3 + \frac{1}{2}t^4\right)\Big|_0^1$$
$$= \frac{7}{6}$$

In the previous section we saw that changing the direction of the curve for a line integral with respect to arc length doesn't change the value of the integral. Let's see what happens with line integrals with respect to *x* and/or *y*.

Example 2 Evaluate $\int_C \sin(\pi y)\,dy + yx^2\,dx$ where *C* is the line segment from $(1,4)$ to $(0,2)$.

Solution

So, we simply changed the direction of the curve. Here is the new parameterization.

$$\vec{r}(t) = (1-t)\langle 1,4\rangle + t\langle 0,2\rangle = \langle 1-t, 4-2t\rangle \qquad 0 \le t \le 1$$

The line integral in this case is,

$$\int_C \sin(\pi y)\,dy + yx^2\,dx = \int_C \sin(\pi y)\,dy + \int_C yx^2\,dx$$
$$= \int_0^1 \sin(\pi(4-2t))(-2)\,dt + \int_0^1 (4-2t)(1-t)^2(-1)\,dt$$
$$= -\frac{1}{\pi}\cos(4\pi - 2\pi t)\Big|_0^1 - \left(-\frac{1}{2}t^4 + \frac{8}{3}t^3 - 5t^2 + 4t\right)\Big|_0^1$$
$$= -\frac{7}{6}$$

So, switching the direction of the curve got us a different value or at least the opposite sign of the value from the first example. In fact this will always happen with these kinds of line integrals.

Fact

If *C* is any curve then,

$$\int_{-C} f(x,y)\,dx = -\int_C f(x,y)\,dx \qquad \text{and} \qquad \int_{-C} f(x,y)\,dy = -\int_C f(x,y)\,dy$$

With the combined form of these two integrals we get,

$$\int_{-C} P\,dx + Q\,dy = -\int_C P\,dx + Q\,dy$$

We can also do these integrals over three-dimensional curves as well. In this case we will pick up a third integral (with respect to *z*) and the three integrals will be.

$$\int_C f(x,y,z)\,dx = \int_a^b f\left(x(t),y(t),z(t)\right)x'(t)\,dt$$
$$\int_C f(x,y,z)\,dy = \int_a^b f\left(x(t),y(t),z(t)\right)y'(t)\,dt$$
$$\int_C f(x,y,z)\,dz = \int_a^b f\left(x(t),y(t),z(t)\right)z'(t)\,dt$$

where the curve *C* is parameterized by

$$x = x(t) \qquad y = y(t) \qquad z = z(t) \qquad a \le t \le b$$

As with the two-dimensional version these three will often occur together so the shorthand we'll be using here is,

$$\int_C P\,dx + Q\,dy + R\,dz = \int_C P(x,y,z)\,dx + \int_C Q(x,y,z)\,dy + \int_C R(x,y,z)\,dz$$

Let's work an example.

Example 3 Evaluate $\int_C y\,dx + x\,dy + z\,dz$ where *C* is given by $x = \cos t$, $y = \sin t$, $z = t^2$, $0 \le t \le 2\pi$.

Solution

So, we already have the curve parameterized so there really isn't much to do other than evaluate the integral.

$$\begin{aligned}
\int_C y\,dx + x\,dy + z\,dz &= \int_C y\,dx + \int_C x\,dy + \int_C z\,dz \\
&= \int_0^{2\pi} \sin t\left(-\sin t\right)dt + \int_0^{2\pi} \cos t\left(\cos t\right)dt + \int_0^{2\pi} t^2\left(2t\right)dt \\
&= -\int_0^{2\pi} \sin^2 t\,dt + \int_0^{2\pi} \cos^2 t\,dt + \int_0^{2\pi} 2t^3\,dt \\
&= -\frac{1}{2}\int_0^{2\pi}\left(1-\cos(2t)\right)dt + \frac{1}{2}\int_0^{2\pi}\left(1+\cos(2t)\right)dt + \int_0^{2\pi} 2t^3\,dt \\
&= \left.\left(-\frac{1}{2}\left(t-\frac{1}{2}\sin(2t)\right)+\frac{1}{2}\left(t+\frac{1}{2}\sin(2t)\right)+\frac{1}{2}t^4\right)\right|_0^{2\pi} \\
&= 8\pi^4
\end{aligned}$$

Section 5-4 : Line Integrals of Vector Fields

In the previous two sections we looked at line integrals of functions. In this section we are going to evaluate line integrals of vector fields. We'll start with the vector field,

$$\vec{F}\left(x,y,z\right)=P\left(x,y,z\right)\vec{i}+Q\left(x,y,z\right)\vec{j}+R\left(x,y,z\right)\vec{k}$$

and the three-dimensional, smooth curve given by

$$\vec{r}\left(t\right)=x\left(t\right)\vec{i}+y\left(t\right)\vec{j}+z\left(t\right)\vec{k} \qquad a\le t\le b$$

The **line integral of $\vec{F}$ along *C*** is

$$\int\limits_C \vec{F}\bullet d\,\vec{r}=\int_a^b \vec{F}\left(\vec{r}\left(t\right)\right)\bullet\vec{r}\,'\left(t\right)dt$$

Note the notation in the integral on the left side. That really is a dot product of the vector field and the differential really is a vector. Also, $\vec{F}\left(\vec{r}\left(t\right)\right)$ is a shorthand for,

$$\vec{F}\left(\vec{r}\left(t\right)\right)=\vec{F}\left(x\left(t\right),y\left(t\right),z\left(t\right)\right)$$

We can also write line integrals of vector fields as a line integral with respect to arc length as follows,

$$\int\limits_C \vec{F}\bullet d\,\vec{r}=\int\limits_C \vec{F}\bullet\vec{T}\,ds$$

where $\vec{T}\left(t\right)$ is the unit tangent vector and is given by,

$$\vec{T}\left(t\right)=\frac{\vec{r}\,'\left(t\right)}{\left\|\vec{r}\,'\left(t\right)\right\|}$$

If we use our knowledge on how to compute line integrals with respect to arc length we can see that this second form is equivalent to the first form given above.

$$\begin{aligned}\int\limits_C \vec{F}\bullet d\,\vec{r}&=\int\limits_C \vec{F}\bullet\vec{T}\,ds\\&=\int_a^b \vec{F}\left(\vec{r}\left(t\right)\right)\bullet\frac{\vec{r}\,'\left(t\right)}{\left\|\vec{r}\,'\left(t\right)\right\|}\left\|\vec{r}\,'\left(t\right)\right\|dt\\&=\int_a^b \vec{F}\left(\vec{r}\left(t\right)\right)\bullet\vec{r}\,'\left(t\right)dt\end{aligned}$$

In general, we use the first form to compute these line integral as it is usually much easier to use. Let's take a look at a couple of examples.

Example 1 Evaluate $\int_C \vec{F} \cdot d\,\vec{r}$ where $\vec{F}(x,y,z) = 8x^2 y z\,\vec{i} + 5z\,\vec{j} - 4x\,y\,\vec{k}$ and *C* is the curve given by $\vec{r}(t) = t\,\vec{i} + t^2\,\vec{j} + t^3\,\vec{k}$, $0 \le t \le 1$.

Solution
Okay, we first need the vector field evaluated along the curve.

$$\vec{F}(\vec{r}(t)) = 8t^2(t^2)(t^3)\vec{i} + 5t^3\,\vec{j} - 4t(t^2)\vec{k} = 8t^7\,\vec{i} + 5t^3\,\vec{j} - 4t^3\,\vec{k}$$

Next, we need the derivative of the parameterization.

$$\vec{r}'(t) = \vec{i} + 2t\,\vec{j} + 3t^2\,\vec{k}$$

Finally, let's get the dot product taken care of.

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = 8t^7 + 10t^4 - 12t^5$$

The line integral is then,

$$\begin{aligned} \int_C \vec{F} \cdot d\,\vec{r} &= \int_0^1 8t^7 + 10t^4 - 12t^5\,dt \\ &= \left(t^8 + 2t^5 - 2t^6\right)\Big|_0^1 \\ &= 1 \end{aligned}$$

Example 2 Evaluate $\int_C \vec{F} \cdot d\,\vec{r}$ where $\vec{F}(x,y,z) = x\,z\,\vec{i} - y\,z\,\vec{k}$ and *C* is the line segment from $(-1,2,0)$ to $(3,0,1)$.

Solution
We'll first need the parameterization of the line segment. We saw how to get the parameterization of line segments in the first section on line integrals. We've been using the two dimensional version of this over the last couple of sections. Here is the parameterization for the line.

$$\begin{aligned} \vec{r}(t) &= (1-t)\langle -1,2,0\rangle + t\langle 3,0,1\rangle \\ &= \langle 4t-1, 2-2t, t\rangle, \qquad\qquad 0 \le t \le 1 \end{aligned}$$

So, let's get the vector field evaluated along the curve.

$$\begin{aligned} \vec{F}(\vec{r}(t)) &= (4t-1)(t)\vec{i} - (2-2t)(t)\vec{k} \\ &= \left(4t^2 - t\right)\vec{i} - \left(2t - 2t^2\right)\vec{k} \end{aligned}$$

Now we need the derivative of the parameterization.

$$\vec{r}'(t) = \langle 4,-2,1\rangle$$

The dot product is then,

$$\vec{F}\left(\vec{r}\left(t\right)\right)\bullet\vec{r}'\left(t\right)=4\left(4t^2-t\right)-\left(2t-2t^2\right)=18t^2-6t$$

The line integral becomes,

$$\begin{aligned}\int_C \vec{F}\bullet d\,\vec{r} &= \int_0^1 18t^2-6t\,dt\\ &=\left(6t^3-3t^2\right)\Big|_0^1\\ &=3\end{aligned}$$

Let's close this section out by doing one of these in general to get a nice relationship between line integrals of vector fields and line integrals with respect to *x*, *y*, and *z*.

Given the vector field $\vec{F}\left(x,y,z\right)=P\,\vec{i}+Q\,\vec{j}+R\,\vec{k}$ and the curve *C* parameterized by $\vec{r}\left(t\right)=x\left(t\right)\vec{i}+y\left(t\right)\vec{j}+z\left(t\right)\vec{k}$, $a\le t\le b$ the line integral is,

$$\begin{aligned}\int_C \vec{F}\bullet d\,\vec{r} &= \int_a^b\left(P\,\vec{i}+Q\,\vec{j}+R\,\vec{k}\right)\bullet\left(x'\,\vec{i}+y'\,\vec{j}+z'\,\vec{k}\right)dt\\ &=\int_a^b Px'+Qy'+Rz'\,dt\\ &=\int_a^b Px'\,dt+\int_a^b Qy'\,dt+\int_a^b Rz'\,dt\\ &=\int_C P\,dx+\int_C Q\,dy+\int_C R\,dz\\ &=\int_C P\,dx+Q\,dy+R\,dz\end{aligned}$$

So, we see that,

$$\int_C \vec{F}\bullet d\,\vec{r}=\int_C P\,dx+Q\,dy+R\,dz$$

Note that this gives us another method for evaluating line integrals of vector fields.

This also allows us to say the following about reversing the direction of the path with line integrals of vector fields.

Fact

$$\int_{-C} \vec{F}\bullet d\,\vec{r}=-\int_C \vec{F}\bullet d\,\vec{r}$$

This should make some sense given that we know that this is true for line integrals with respect to *x*, *y*, and/or *z* and that line integrals of vector fields can be defined in terms of line integrals with respect to *x*, *y*, and *z*.

Section 5-5 : Fundamental Theorem for Line Integrals

In Calculus I we had the Fundamental Theorem of Calculus that told us how to evaluate definite integrals. This told us,

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

It turns out that there is a version of this for line integrals over certain kinds of vector fields. Here it is.

Theorem

Suppose that *C* is a smooth curve given by $\vec{r}(t)$, $a \le t \le b$. Also suppose that *f* is a function whose gradient vector, ∇f, is continuous on *C*. Then,

$$\int_C \nabla f \cdot d\,\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

Note that $\vec{r}(a)$ represents the initial point on *C* while $\vec{r}(b)$ represents the final point on *C*. Also, we did not specify the number of variables for the function since it is really immaterial to the theorem. The theorem will hold regardless of the number of variables in the function.

Proof

This is a fairly straight forward proof.

For the purposes of the proof we'll assume that we're working in three dimensions, but it can be done in any dimension.

Let's start by just computing the line integral.

$$\begin{aligned}\int_C \nabla f \cdot d\,\vec{r} &= \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)\,dt \\ &= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\right)dt\end{aligned}$$

Now, at this point we can use the Chain Rule to simplify the integrand as follows,

$$\begin{aligned}\int_C \nabla f \cdot d\,\vec{r} &= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\right)dt \\ &= \int_a^b \frac{d}{dt}\left[f(\vec{r}(t))\right]dt\end{aligned}$$

To finish this off we just need to use the Fundamental Theorem of Calculus for single integrals.

$$\int_C \nabla f \cdot d\,\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

Let's take a quick look at an example of using this theorem.

Example 1 Evaluate $\int_C \nabla f \bullet d\vec{r}$ where $f(x,y,z) = \cos(\pi x) + \sin(\pi y) - xyz$ and *C* is *any* path that starts at $\left(1, \frac{1}{2}, 2\right)$ and ends at $(2,1,-1)$.

Solution
First let's notice that we didn't specify the path for getting from the first point to the second point. The reason for this is simple. The theorem above tells us that all we need are the initial and final points on the curve in order to evaluate this kind of line integral.

So, let $\vec{r}(t)$, $a \le t \le b$ be any path that starts at $\left(1, \frac{1}{2}, 2\right)$ and ends at $(2,1,-1)$. Then,

$$\vec{r}(a) = \left\langle 1, \frac{1}{2}, 2 \right\rangle \qquad\qquad \vec{r}(b) = \langle 2,1,-1 \rangle$$

The integral is then,

$$\begin{aligned}\int_C \nabla f \bullet d\vec{r} &= f(2,1,-1) - f\left(1, \frac{1}{2}, 2\right) \\ &= \cos(2\pi) + \sin\pi - 2(1)(-1) - \left(\cos\pi + \sin\left(\frac{\pi}{2}\right) - 1\left(\frac{1}{2}\right)(2)\right) \\ &= 4\end{aligned}$$

Notice that we also didn't need the gradient vector to actually do this line integral. However, for the practice of finding gradient vectors here it is,

$$\nabla f = \left\langle -\pi\sin(\pi x) - yz, \pi\cos(\pi y) - xz, -xy \right\rangle$$

The most important idea to get from this example is not how to do the integral as that's pretty simple, all we do is plug the final point and initial point into the function and subtract the two results. The important idea from this example (and hence about the Fundamental Theorem of Calculus) is that, for these kinds of line integrals, we didn't really need to know the path to get the answer. In other words, we could use any path we want and we'll always get the same results.

In the first section on line integrals (even though we weren't looking at vector fields) we saw that often when we change the path we will change the value of the line integral. We now have a type of line integral for which we know that changing the path will NOT change the value of the line integral.

Let's formalize this idea up a little. Here are some definitions. The first one we've already seen before, but it's been a while and it's important in this section so we'll give it again. The remaining definitions are new.

Definitions

First suppose that $\vec{F}$ is a continuous vector field in some domain *D*.

1. $\vec{F}$ is a **conservative** vector field if there is a function *f* such that $\vec{F} = \nabla f$. The function *f* is called a **potential function** for the vector field. We first saw this definition in the first section of this chapter.

2. $\int_C \vec{F} \bullet d\,\vec{r}$ is **independent of path** if $\int_{C_1} \vec{F} \bullet d\,\vec{r} = \int_{C_2} \vec{F} \bullet d\,\vec{r}$ for any two paths C_1 and C_2 in *D* with the same initial and final points.

3. A path *C* is called **closed** if its initial and final points are the same point. For example, a circle is a closed path.

4. A path *C* is **simple** if it doesn't cross itself. A circle is a simple curve while a figure 8 type curve is not simple.

5. A region *D* is **open** if it doesn't contain any of its boundary points.

6. A region *D* is **connected** if we can connect any two points in the region with a path that lies completely in *D*.

7. A region *D* is **simply-connected** if it is connected and it contains no holes. We won't need this one until the next section, but it fits in with all the other definitions given here so this was a natural place to put the definition.

With these definitions we can now give some nice facts.

Facts

1. $\int_C \nabla f \bullet d\,\vec{r}$ is independent of path.

   This is easy enough to prove since all we need to do is look at the theorem above. The theorem tells us that in order to evaluate this integral all we need are the initial and final points of the curve. This in turn tells us that the line integral must be independent of path.

2. If $\vec{F}$ is a conservative vector field then $\int_C \vec{F} \bullet d\,\vec{r}$ is independent of path.

   This fact is also easy enough to prove. If $\vec{F}$ is conservative then it has a potential function, *f*, and so the line integral becomes $\int_C \vec{F} \bullet d\,\vec{r} = \int_C \nabla f \bullet d\,\vec{r}$. Then using the first fact we know that this line integral must be independent of path.

3. If $\vec{F}$ is a continuous vector field on an open connected region *D* and if $\int_C \vec{F} \bullet d\,\vec{r}$ is independent of path (for any path in *D*) then $\vec{F}$ is a conservative vector field on *D*.

4. If $\int_C \vec{F} \bullet d\,\vec{r}$ is independent of path then $\int_C \vec{F} \bullet d\,\vec{r} = 0$ for every closed path *C*.

5. If $\int_C \vec{F} \bullet d\,\vec{r} = 0$ for every closed path *C* then $\int_C \vec{F} \bullet d\,\vec{r}$ is independent of path.

These are some nice facts to remember as we work with line integrals over vector fields. Also notice that 2 & 3 and 4 & 5 are converses of each other.

Section 5-6 : Conservative Vector Fields

In the previous section we saw that if we knew that the vector field $\vec{F}$ was conservative then $\int_C \vec{F} \bullet d\,\vec{r}$ was independent of path. This in turn means that we can easily evaluate this line integral provided we can find a potential function for $\vec{F}$.

In this section we want to look at two questions. First, given a vector field $\vec{F}$ is there any way of determining if it is a conservative vector field? Secondly, if we know that $\vec{F}$ is a conservative vector field how do we go about finding a potential function for the vector field?

The first question is easy to answer at this point if we have a two-dimensional vector field. For higher dimensional vector fields we'll need to wait until the final section in this chapter to answer this question. With that being said let's see how we do it for two-dimensional vector fields.

Theorem

Let $\vec{F} = P\,\vec{i} + Q\,\vec{j}$ be a vector field on an open and simply-connected region *D*. Then if *P* and *Q* have continuous first order partial derivatives in *D* and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

the vector field $\vec{F}$ is conservative.

Let's take a look at a couple of examples.

Example 1 Determine if the following vector fields are conservative or not.

(a) $\vec{F}(x,y) = \left(x^2 - yx\right)\vec{i} + \left(y^2 - xy\right)\vec{j}$

(b) $\vec{F}(x,y) = \left(2x\mathbf{e}^{xy} + x^2 y\mathbf{e}^{xy}\right)\vec{i} + \left(x^3\mathbf{e}^{xy} + 2y\right)\vec{j}$

Solution

Okay, there really isn't too much to these. All we do is identify *P* and *Q* then take a couple of derivatives and compare the results.

(a) $\vec{F}(x,y) = \left(x^2 - yx\right)\vec{i} + \left(y^2 - xy\right)\vec{j}$

In this case here is *P* and *Q* and the appropriate partial derivatives.

$$P = x^2 - yx \qquad \frac{\partial P}{\partial y} = -x$$

$$Q = y^2 - xy \qquad \frac{\partial Q}{\partial x} = -y$$

So, since the two partial derivatives are not the same this vector field is NOT conservative.

(b) $\vec{F}(x,y) = \left(2x\mathbf{e}^{xy} + x^2 y\mathbf{e}^{xy}\right)\vec{i} + \left(x^3\mathbf{e}^{xy} + 2y\right)\vec{j}$

Here is *P* and *Q* as well as the appropriate derivatives.

$$P = 2x\mathbf{e}^{xy} + x^2 y\mathbf{e}^{xy} \qquad \frac{\partial P}{\partial y} = 2x^2\mathbf{e}^{xy} + x^2\mathbf{e}^{xy} + x^3 y\mathbf{e}^{xy} = 3x^2\mathbf{e}^{xy} + x^3 y\mathbf{e}^{xy}$$

$$Q = x^3\mathbf{e}^{xy} + 2y \qquad \frac{\partial Q}{\partial x} = 3x^2\mathbf{e}^{xy} + x^3 y\mathbf{e}^{xy}$$

The two partial derivatives are equal and so this is a conservative vector field.

Now that we know how to identify if a two-dimensional vector field is conservative we need to address how to find a potential function for the vector field. This is actually a fairly simple process. First, let's assume that the vector field is conservative and so we know that a potential function, $f(x,y)$ exists. We can then say that,

$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} = P\vec{i} + Q\vec{j} = \vec{F}$$

Or by setting components equal we have,

$$\frac{\partial f}{\partial x} = P \qquad \text{and} \qquad \frac{\partial f}{\partial y} = Q$$

By integrating each of these with respect to the appropriate variable we can arrive at the following two equations.

$$f(x,y) = \int P(x,y)\,dx \qquad \text{or} \qquad f(x,y) = \int Q(x,y)\,dy$$

We saw this kind of integral briefly at the end of the section on iterated integrals in the previous chapter.

It is usually best to see how we use these two facts to find a potential function in an example or two.

Example 2 Determine if the following vector fields are conservative and find a potential function for the vector field if it is conservative.

(a) $\vec{F} = \left(2x^3y^4 + x\right)\vec{i} + \left(2x^4y^3 + y\right)\vec{j}$

(b) $\vec{F}(x,y) = \left(2x\mathbf{e}^{xy} + x^2 y\mathbf{e}^{xy}\right)\vec{i} + \left(x^3\mathbf{e}^{xy} + 2y\right)\vec{j}$

Solution

(a) $\vec{F} = \left(2x^3y^4 + x\right)\vec{i} + \left(2x^4y^3 + y\right)\vec{j}$

Let's first identify *P* and *Q* and then check that the vector field is conservative.

$$P = 2x^3y^4 + x \qquad \frac{\partial P}{\partial y} = 8x^3y^3$$

$$Q = 2x^4y^3 + y \qquad \frac{\partial Q}{\partial x} = 8x^3y^3$$

So, the vector field is conservative. Now let's find the potential function. From the first fact above we know that,

$$\frac{\partial f}{\partial x} = 2x^3y^4 + x \qquad \frac{\partial f}{\partial y} = 2x^4y^3 + y$$

From these we can see that

$$f(x,y) = \int 2x^3y^4 + x\,dx \qquad \text{or} \qquad f(x,y) = \int 2x^4y^3 + y\,dy$$

We can use either of these to get the process started. Recall that we are going to have to be careful with the "constant of integration" which ever integral we choose to use. For this example let's work with the first integral and so that means that we are asking what function did we differentiate with respect to *x* to get the integrand. This means that the "constant of integration" is going to have to be a function of *y* since any function consisting only of *y* and/or constants will differentiate to zero when taking the partial derivative with respect to *x*.

Here is the first integral.

$$\begin{aligned} f(x,y) &= \int 2x^3y^4 + x\,dx \\ &= \frac{1}{2}x^4y^4 + \frac{1}{2}x^2 + h(y) \end{aligned}$$

where $h(y)$ is the "constant of integration".

We now need to determine $h(y)$. This is easier than it might at first appear to be. To get to this point we've used the fact that we knew *P*, but we will also need to use the fact that we know *Q* to complete the problem. Recall that *Q* is really the derivative of *f* with respect to *y*. So, if we differentiate our function with respect to *y* we know what it should be.

So, let's differentiate *f* (including the $h(y)$) with respect to *y* and set it equal to *Q* since that is what the derivative is supposed to be.

$$\frac{\partial f}{\partial y} = 2x^4y^3 + h'(y) = 2x^4y^3 + y = Q$$

From this we can see that,

$$h'(y) = y$$

Notice that since $h'(y)$ is a function only of *y* so if there are any *x*'s in the equation at this point we will know that we've made a mistake. At this point finding $h(y)$ is simple.

$$h\left(y\right)=\int h'\left(y\right)dy=\int y\,dy=\frac{1}{2}y^{2}+c$$

So, putting this all together we can see that a potential function for the vector field is,

$$f\left(x,y\right)=\frac{1}{2}x^{4}y^{4}+\frac{1}{2}x^{2}+\frac{1}{2}y^{2}+c$$

Note that we can always check our work by verifying that $\nabla f=\vec{F}$. Also note that because the *c* can be anything there are an infinite number of possible potential functions, although they will only vary by an additive constant.

(b) $\vec{F}\left(x,y\right)=\left(2x\mathbf{e}^{xy}+x^{2}y\mathbf{e}^{xy}\right)\vec{i}+\left(x^{3}\mathbf{e}^{xy}+2y\right)\vec{j}$

Okay, this one will go a lot faster since we don't need to go through as much explanation. We've already verified that this vector field is conservative in the first set of examples so we won't bother redoing that.

Let's start with the following,

$$\frac{\partial f}{\partial x}=2x\mathbf{e}^{xy}+x^{2}y\mathbf{e}^{xy}\qquad\qquad\frac{\partial f}{\partial y}=x^{3}\mathbf{e}^{xy}+2y$$

This means that we can do either of the following integrals,

$$f\left(x,y\right)=\int 2x\mathbf{e}^{xy}+x^{2}y\mathbf{e}^{xy}\,dx\qquad\text{or}\qquad f\left(x,y\right)=\int x^{3}\mathbf{e}^{xy}+2y\,dy$$

While we can do either of these the first integral would be somewhat unpleasant as we would need to do integration by parts on each portion. On the other hand, the second integral is fairly simple since the second term only involves *y*'s and the first term can be done with the substitution $u=xy$. So, from the second integral we get,

$$f\left(x,y\right)=x^{2}\mathbf{e}^{xy}+y^{2}+h\left(x\right)$$

Notice that this time the "constant of integration" will be a function of *x*. If we differentiate this with respect to *x* and set equal to *P* we get,

$$\frac{\partial f}{\partial x}=2x\mathbf{e}^{xy}+x^{2}y\mathbf{e}^{xy}+h'\left(x\right)=2x\mathbf{e}^{xy}+x^{2}y\mathbf{e}^{xy}=P$$

So, in this case it looks like,

$$h'\left(x\right)=0\qquad\Rightarrow\qquad h\left(x\right)=c$$

So, in this case the "constant of integration" really was a constant. Sometimes this will happen and sometimes it won't.

Here is the potential function for this vector field.

$$f\left(x,y\right)=x^{2}\mathbf{e}^{xy}+y^{2}+c$$

Now, as noted above we don't have a way (yet) of determining if a three-dimensional vector field is conservative or not. However, if we are given that a three-dimensional vector field is conservative finding a potential function is similar to the above process, although the work will be a little more involved.

In this case we will use the fact that,

$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k} = P\,\vec{i} + Q\,\vec{j} + R\,\vec{k} = \vec{F}$$

Let's take a quick look at an example.

Example 3 Find a potential function for the vector field,

$$\vec{F} = 2xy^3z^4\,\vec{i} + 3x^2y^2z^4\,\vec{j} + 4x^2y^3z^3\,\vec{k}$$

Solution

Okay, we'll start off with the following equalities.

$$\frac{\partial f}{\partial x} = 2xy^3z^4 \qquad \frac{\partial f}{\partial y} = 3x^2y^2z^4 \qquad \frac{\partial f}{\partial z} = 4x^2y^3z^3$$

To get started we can integrate the first one with respect to *x*, the second one with respect to *y*, or the third one with respect to *z*. Let's integrate the first one with respect to *x*.

$$f(x,y,z) = \int 2xy^3z^4\,dx = x^2y^3z^4 + g(y,z)$$

Note that this time the "constant of integration" will be a function of both *y* and *z* since differentiating anything of that form with respect to *x* will differentiate to zero.

Now, we can differentiate this with respect to *y* and set it equal to *Q*. Doing this gives,

$$\frac{\partial f}{\partial y} = 3x^2y^2z^4 + g_y(y,z) = 3x^2y^2z^4 = Q$$

Of course we'll need to take the partial derivative of the constant of integration since it is a function of two variables. It looks like we've now got the following,

$$g_y(y,z) = 0 \qquad \Rightarrow \qquad g(y,z) = h(z)$$

Since differentiating $g(y,z)$ with respect to *y* gives zero then $g(y,z)$ could at most be a function of *z*. This means that we now know the potential function must be in the following form.

$$f(x,y,z) = x^2y^3z^4 + h(z)$$

To finish this out all we need to do is differentiate with respect to *z* and set the result equal to *R*.

$$\frac{\partial f}{\partial z} = 4x^2y^3z^3 + h'(z) = 4x^2y^3z^3 = R$$

So,

$$h'(z) = 0 \qquad \Rightarrow \qquad h(z) = c$$

The potential function for this vector field is then,

$$f(x,y,z) = x^2 y^3 z^4 + c$$

Note that to keep the work to a minimum we used a fairly simple potential function for this example. It might have been possible to guess what the potential function was based simply on the vector field. However, we should be careful to remember that this usually won't be the case and often this process is required.

Also, there were several other paths that we could have taken to find the potential function. Each would have gotten us the same result.

Let's work one more slightly (and only slightly) more complicated example.

Example 4 Find a potential function for the vector field,

$$\vec{F} = \left(2x\cos(y) - 2z^3\right)\vec{i} + \left(3 + 2y\mathbf{e}^z - x^2\sin(y)\right)\vec{j} + \left(y^2\mathbf{e}^z - 6xz^2\right)\vec{k}$$

Solution

Here are the equalities for this vector field.

$$\frac{\partial f}{\partial x} = 2x\cos(y) - 2z^3 \qquad \frac{\partial f}{\partial y} = 3 + 2y\mathbf{e}^z - x^2\sin(y) \qquad \frac{\partial f}{\partial z} = y^2\mathbf{e}^z - 6xz^2$$

For this example let's integrate the third one with respect to *z*.

$$f(x,y,z) = \int y^2\mathbf{e}^z - 6xz^2\,dz = y^2\mathbf{e}^z - 2xz^3 + g(x,y)$$

The "constant of integration" for this integration will be a function of both *x* and *y*.

Now, we can differentiate this with respect to *x* and set it equal to *P*. Doing this gives,

$$\frac{\partial f}{\partial x} = -2z^3 + g_x(x,y) = 2x\cos(y) - 2z^3 = P$$

So, it looks like we've now got the following,

$$g_x(x,y) = 2x\cos(y) \qquad \Rightarrow \qquad g(x,y) = x^2\cos(y) + h(y)$$

The potential function for this problem is then,

$$f(x,y,z) = y^2\mathbf{e}^z - 2xz^3 + x^2\cos(y) + h(y)$$

To finish this out all we need to do is differentiate with respect to *y* and set the result equal to *Q*.

$$\frac{\partial f}{\partial y} = 2y\mathbf{e}^z - x^2\sin(y) + h'(y) = 3 + 2y\mathbf{e}^z - x^2\sin(y) = Q$$

So,

$$h'(y) = 3 \qquad \Rightarrow \qquad h(y) = 3y + c$$

The potential function for this vector field is then,

$$f(x,y,z) = y^2\mathbf{e}^z - 2xz^3 + x^2\cos(y) + 3y + c$$

So, a little more complicated than the others and there are again many different paths that we could have taken to get the answer.

We need to work one final example in this section.

Example 5 Evaluate $\int\limits_C \vec{F}\bullet d\,\vec{r}$ where $\vec{F} = \left(2x^3y^4 + x\right)\vec{i} + \left(2x^4y^3 + y\right)\vec{j}$ and *C* is given by $\vec{r}(t) = \left(t\cos(\pi t) - 1\right)\vec{i} + \sin\left(\frac{\pi t}{2}\right)\vec{j}$, $0 \le t \le 1$.

Solution

Now, we could use the techniques we discussed when we first looked at line integrals of vector fields however that would be particularly unpleasant solution.

Instead, let's take advantage of the fact that we know from Example 2a above this vector field is conservative and that a potential function for the vector field is,

$$f(x,y) = \frac{1}{2}x^4y^4 + \frac{1}{2}x^2 + \frac{1}{2}y^2 + c$$

Using this we know that integral must be independent of path and so all we need to do is use the theorem from the previous section to do the evaluation.

$$\int\limits_C \vec{F}\bullet d\,\vec{r} = \int\limits_C \nabla f\bullet d\,\vec{r} = f\left(\vec{r}(1)\right) - f\left(\vec{r}(0)\right)$$

where,

$$\vec{r}(1) = \langle -2,1\rangle \qquad\qquad \vec{r}(0) = \langle -1,0\rangle$$

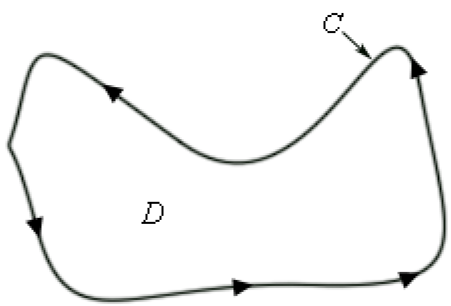
So, the integral is,

$$\begin{aligned}\int\limits_C \vec{F}\bullet d\,\vec{r} &= f(-2,1) - f(-1,0)\\ &= \left(\frac{21}{2} + c\right) - \left(\frac{1}{2} + c\right)\\ &= 10\end{aligned}$$

Section 5-7 : Green's Theorem

In this section we are going to investigate the relationship between certain kinds of line integrals (on closed paths) and double integrals.

Let's start off with a simple (recall that this means that it doesn't cross itself) closed curve *C* and let *D* be the region enclosed by the curve. Here is a sketch of such a curve and region.



First, notice that because the curve is simple and closed there are no holes in the region *D*. Also notice that a direction has been put on the curve. We will use the convention here that the curve *C* has a **positive orientation** if it is traced out in a counter-clockwise direction. Another way to think of a positive orientation (that will cover much more general curves as well see later) is that as we traverse the path following the positive orientation the region *D* must always be on the left.

Given curves/regions such as this we have the following theorem.

Green's Theorem

Let *C* be a positively oriented, piecewise smooth, simple, closed curve and let *D* be the region enclosed by the curve. If *P* and *Q* have continuous first order partial derivatives on *D* then,

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Before working some examples there are some alternate notations that we need to acknowledge. When working with a line integral in which the path satisfies the condition of Green's Theorem we will often denote the line integral as,

$$\oint_C P\,dx + Q\,dy \qquad \text{or} \qquad \circlearrowleft\!\!\!\!\int_C P\,dx + Q\,dy$$

Both of these notations do assume that *C* satisfies the conditions of Green's Theorem so be careful in using them.

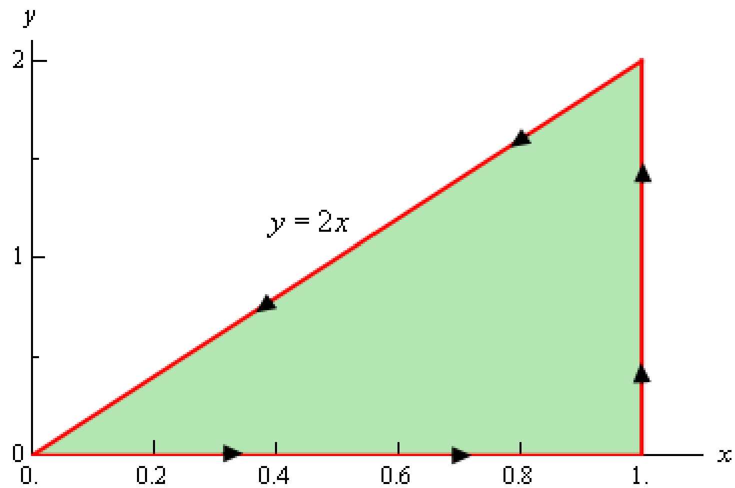
Also, sometimes the curve *C* is not thought of as a separate curve but instead as the boundary of some region *D* and in these cases you may see *C* denoted as ∂D.

Let's work a couple of examples.

Example 1 Use Green's Theorem to evaluate $\oint_C xy\,dx + x^2 y^3\,dy$ where *C* is the triangle with vertices $(0,0)$, $(1,0)$, $(1,2)$ with positive orientation.

Solution

Let's first sketch *C* and *D* for this case to make sure that the conditions of Green's Theorem are met for *C* and will need the sketch of *D* to evaluate the double integral.



So, the curve does satisfy the conditions of Green's Theorem and we can see that the following inequalities will define the region enclosed.

$$0 \le x \le 1 \qquad\qquad 0 \le y \le 2x$$

We can identify *P* and *Q* from the line integral. Here they are.

$$P = xy \qquad\qquad Q = x^2 y^3$$

So, using Green's Theorem the line integral becomes,

$$\oint_C xy\,dx + x^2y^3\,dy = \iint_D 2xy^3 - x\,dA$$
$$= \int_0^1\int_0^{2x} 2xy^3 - x\,dy\,dx$$
$$= \int_0^1 \left(\frac{1}{2}xy^4 - xy\right)\Big|_0^{2x} dx$$
$$= \int_0^1 8x^5 - 2x^2\,dx$$
$$= \left(\frac{4}{3}x^6 - \frac{2}{3}x^3\right)\Big|_0^1$$
$$= \frac{2}{3}$$

Example 2 Evaluate $\oint_C y^3\,dx - x^3\,dy$ where *C* is the positively oriented circle of radius 2 centered at the origin.

Solution

Okay, a circle will satisfy the conditions of Green's Theorem since it is closed and simple and so there really isn't a reason to sketch it.

Let's first identify *P* and *Q* from the line integral.

$$P = y^3 \qquad\qquad Q = -x^3$$

Be careful with the minus sign on *Q*!

Now, using Green's theorem on the line integral gives,

$$\oint_C y^3\,dx - x^3\,dy = \iint_D -3x^2 - 3y^2\,dA$$

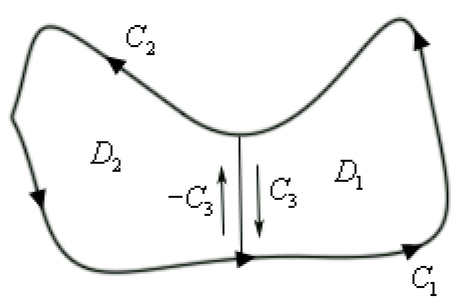
where *D* is a disk of radius 2 centered at the origin.

Since *D* is a disk it seems like the best way to do this integral is to use polar coordinates. Here is the evaluation of the integral.

$$
\begin{aligned}
\oint_C y^3\,dx - x^3\,dy &= -3\iint_D \left(x^2+y^2\right) dA \\
&= -3\int_0^{2\pi}\int_0^2 r^3\,dr\,d\theta \\
&= -3\int_0^{2\pi} \left.\frac{1}{4}r^4\right|_0^2 d\theta \\
&= -3\int_0^{2\pi} 4\,d\theta \\
&= -24\pi
\end{aligned}
$$

So, Green's theorem, as stated, will not work on regions that have holes in them. However, many regions do have holes in them. So, let's see how we can deal with those kinds of regions.

Let's start with the following region. Even though this region doesn't have any holes in it the arguments that we're going to go through will be similar to those that we'd need for regions with holes in them, except it will be a little easier to deal with and write down.



The region *D* will be $D_1 \cup D_2$ and recall that the symbol $\cup$ is called the union and means that *D* consists of both D_1 and D_2. The boundary of D_1 is $C_1 \cup C_3$ while the boundary of D_2 is $C_2 \cup \left(-C_3\right)$ and notice that both of these boundaries are positively oriented. As we traverse each boundary the corresponding region is always on the left. Finally, also note that we can think of the whole boundary, *C*, as,

$$C = \left(C_1 \cup C_3\right) \cup \left(C_2 \cup \left(-C_3\right)\right) = C_1 \cup C_2$$

since both C_3 and $-C_3$ will "cancel" each other out.

Now, let's start with the following double integral and use a basic property of double integrals to break it up.

$$\iint_D \left(Q_x - P_y\right) dA = \iint_{D_1 \cup D_2} \left(Q_x - P_y\right) dA = \iint_{D_1} \left(Q_x - P_y\right) dA + \iint_{D_2} \left(Q_x - P_y\right) dA$$

Next, use Green's theorem on each of these and again use the fact that we can break up line integrals into separate line integrals for each portion of the boundary.

$$
\begin{aligned}
\iint\limits_{D} \left(Q_x - P_y\right) dA &= \iint\limits_{D_1} \left(Q_x - P_y\right) dA + \iint\limits_{D_2} \left(Q_x - P_y\right) dA \\
&= \oint\limits_{C_1 \cup C_3} P\,dx + Q\,dy + \oint\limits_{C_2 \cup (-C_3)} P\,dx + Q\,dy \\
&= \oint\limits_{C_1} P\,dx + Q\,dy + \oint\limits_{C_3} P\,dx + Q\,dy + \oint\limits_{C_2} P\,dx + Q\,dy + \oint\limits_{-C_3} P\,dx + Q\,dy
\end{aligned}
$$

Next, we'll use the fact that,

$$\oint\limits_{-C_3} P\,dx + Q\,dy = -\oint\limits_{C_3} P\,dx + Q\,dy$$

Recall that changing the orientation of a curve with line integrals with respect to *x* and/or *y* will simply change the sign on the integral. Using this fact we get,

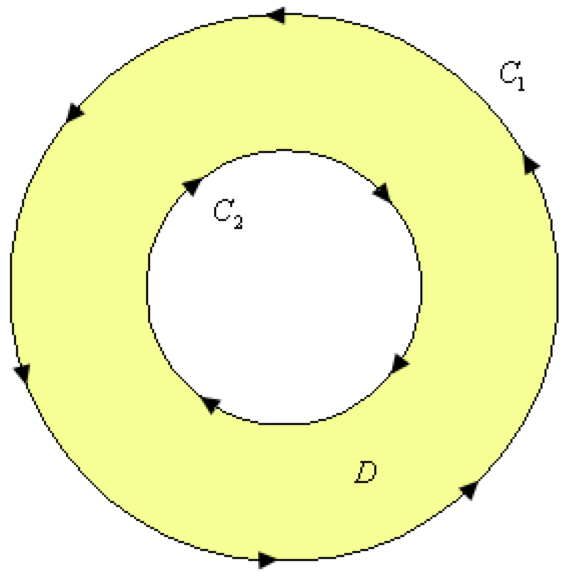
$$
\begin{aligned}
\iint\limits_{D} \left(Q_x - P_y\right) dA &= \oint\limits_{C_1} P\,dx + Q\,dy + \oint\limits_{C_3} P\,dx + Q\,dy + \oint\limits_{C_2} P\,dx + Q\,dy - \oint\limits_{C_3} P\,dx + Q\,dy \\
&= \oint\limits_{C_1} P\,dx + Q\,dy + \oint\limits_{C_2} P\,dx + Q\,dy
\end{aligned}
$$

Finally, put the line integrals back together and we get,

$$
\begin{aligned}
\iint\limits_{D} \left(Q_x - P_y\right) dA &= \oint\limits_{C_1} P\,dx + Q\,dy + \oint\limits_{C_2} P\,dx + Q\,dy \\
&= \oint\limits_{C_1 \cup C_2} P\,dx + Q\,dy \\
&= \oint\limits_{C} P\,dx + Q\,dy
\end{aligned}
$$

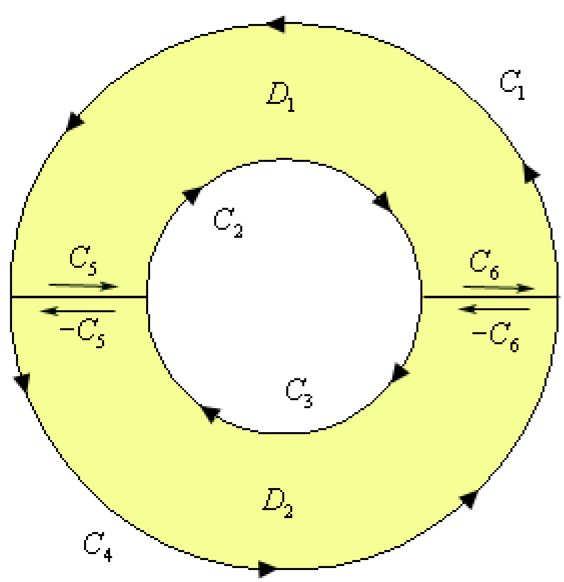
So, what did we learn from this? If you think about it this was just a lot of work and all we got out of it was the result from Green's Theorem which we already knew to be true. What this exercise has shown us is that if we break a region up as we did above then the portion of the line integral on the pieces of the curve that are in the middle of the region (each of which are in the opposite direction) will cancel out. This idea will help us in dealing with regions that have holes in them.

To see this let's look at a ring.



Notice that both of the curves are oriented positively since the region *D* is on the left side as we traverse the curve in the indicated direction. Note as well that the curve C_2 seems to violate the original definition of positive orientation. We originally said that a curve had a positive orientation if it was traversed in a counter-clockwise direction. However, this was only for regions that do not have holes. For the boundary of the hole this definition won't work and we need to resort to the second definition that we gave above.

Now, since this region has a hole in it we will apparently not be able to use Green's Theorem on any line integral with the curve $C = C_1 \cup C_2$. However, if we cut the disk in half and rename all the various portions of the curves we get the following sketch.



The boundary of the upper portion (D_1)of the disk is $C_1 \cup C_2 \cup C_5 \cup C_6$ and the boundary on the lower portion (D_2)of the disk is $C_3 \cup C_4 \cup (-C_5) \cup (-C_6)$. Also notice that we can use Green's Theorem on each of these new regions since they don't have any holes in them. This means that we can do the following,

$$\begin{aligned}\iint_D (Q_x - P_y)\,dA &= \iint_{D_1} (Q_x - P_y)\,dA + \iint_{D_2} (Q_x - P_y)\,dA \\ &= \oint_{C_1 \cup C_2 \cup C_5 \cup C_6} P\,dx + Q\,dy + \oint_{C_3 \cup C_4 \cup (-C_5) \cup (-C_6)} P\,dx + Q\,dy\end{aligned}$$

Now, we can break up the line integrals into line integrals on each piece of the boundary. Also recall from the work above that boundaries that have the same curve, but opposite direction will cancel. Doing this gives,

$$\begin{aligned}\iint_D (Q_x - P_y)\,dA &= \iint_{D_1} (Q_x - P_y)\,dA + \iint_{D_2} (Q_x - P_y)\,dA \\ &= \oint_{C_1} P\,dx + Q\,dy + \oint_{C_2} P\,dx + Q\,dy + \oint_{C_3} P\,dx + Q\,dy + \oint_{C_4} P\,dx + Q\,dy\end{aligned}$$

But at this point we can add the line integrals back up as follows,

$$\begin{aligned}\iint_D (Q_x - P_y)\,dA &= \oint_{C_1 \cup C_2 \cup C_3 \cup C_4} P\,dx + Q\,dy \\ &= \oint_C P\,dx + Q\,dy\end{aligned}$$

The end result of all of this is that we could have just used Green's Theorem on the disk from the start even though there is a hole in it. This will be true in general for regions that have holes in them.

Let's take a look at an example.

Example 3 Evaluate $\oint_C y^3\,dx - x^3\,dy$ where *C* are the two circles of radius 2 and radius 1 centered at the origin with positive orientation.

Solution

Notice that this is the same line integral as we looked at in the second example and only the curve has changed. In this case the region *D* will now be the region between these two circles and that will only change the limits in the double integral so we'll not put in some of the details here.

Here is the work for this integral.

$$\begin{aligned}\oint_C y^3\,dx - x^3\,dy &= -3\iint_D \left(x^2+y^2\right)dA \\ &= -3\int_0^{2\pi}\int_1^2 r^3\,dr\,d\theta \\ &= -3\int_0^{2\pi} \left.\frac{1}{4}r^4\right|_1^2 d\theta \\ &= -3\int_0^{2\pi}\frac{15}{4}\,d\theta \\ &= -\frac{45\pi}{2}\end{aligned}$$

We will close out this section with an interesting application of Green's Theorem. Recall that we can determine the area of a region *D* with the following double integral.

$$A = \iint_D dA$$

Let's think of this double integral as the result of using Green's Theorem. In other words, let's assume that

$$Q_x - P_y = 1$$

and see if we can get some functions *P* and *Q* that will satisfy this.

There are many functions that will satisfy this. Here are some of the more common functions.

$$\begin{array}{ccc} P = 0 & \qquad P = -y & \qquad P = -\dfrac{y}{2} \\ Q = x & \qquad Q = 0 & \qquad Q = \dfrac{x}{2} \end{array}$$

Then, if we use Green's Theorem in reverse we see that the area of the region *D* can also be computed by evaluating any of the following line integrals.

$$A = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C x\,dy - y\,dx$$

where *C* is the boundary of the region *D*.

Let's take a quick look at an example of this.

Example 4 Use Green's Theorem to find the area of a disk of radius *a*.

Solution

We can use either of the integrals above, but the third one is probably the easiest. So,

$$A = \frac{1}{2}\oint_C x\,dy - y\,dx$$

where *C* is the circle of radius *a*. So, to do this we'll need a parameterization of *C*. This is,

$$x = a\cos t \qquad y = a\sin t \qquad 0 \le t \le 2\pi$$

The area is then,

$$\begin{aligned} A &= \frac{1}{2}\oint_C x\,dy - y\,dx \\ &= \frac{1}{2}\left(\int_0^{2\pi} a\cos t\left(a\cos t\right)dt - \int_0^{2\pi} a\sin t\left(-a\sin t\right)dt\right) \\ &= \frac{1}{2}\int_0^{2\pi} a^2\cos^2 t + a^2\sin^2 t\,dt \\ &= \frac{1}{2}\int_0^{2\pi} a^2\,dt \\ &= \pi a^2 \end{aligned}$$

Chapter 6 : Surface Integrals

In the previous chapter we looked at evaluating integrals of functions or vector fields where the points came from a curve in two- or three-dimensional space. We now want to extend this idea and integrate functions and vector fields where the points come from a surface in three-dimensional space. These integrals are called surface integrals.

Here is a list of the topics covered in this chapter.

Curl and Divergence – In this section we will introduce the concepts of the curl and the divergence of a vector field. We will also give two vector forms of Green's Theorem and show how the curl can be used to identify if a three dimensional vector field is conservative field or not.

Parametric Surfaces – In this section we will take a look at the basics of representing a surface with parametric equations. We will also see how the parameterization of a surface can be used to find a normal vector for the surface (which will be very useful in a couple of sections) and how the parameterization can be used to find the surface area of a surface.

Surface Integrals – In this section we introduce the idea of a surface integral. With surface integrals we will be integrating over the surface of a solid. In other words, the variables will always be on the surface of the solid and will never come from inside the solid itself. Also, in this section we will be working with the first kind of surface integrals we'll be looking at in this chapter : surface integrals of functions.

Surface Integrals of Vector Fields – In this section we will introduce the concept of an oriented surface and look at the second kind of surface integral we'll be looking at : surface integrals of vector fields.

Stokes' Theorem – In this section we will discuss Stokes' Theorem.

Divergence Theorem – In this section we will discuss the Divergence Theorem.

Section 6-1 : Curl and Divergence

Before we can get into surface integrals we need to get some introductory material out of the way. That is the purpose of the first two sections of this chapter.

In this section we are going to introduce the concepts of the curl and the divergence of a vector.

Let's start with the curl. Given the vector field $\vec{F} = P\,\vec{i} + Q\,\vec{j} + R\,\vec{k}$ the curl is defined to be,

$$\operatorname{curl}\vec{F} = \left(R_y - Q_z\right)\vec{i} + \left(P_z - R_x\right)\vec{j} + \left(Q_x - P_y\right)\vec{k}$$

There is another (potentially) easier definition of the curl of a vector field. To use it we will first need to define the ∇ *operator*. This is defined to be,

$$\nabla = \frac{\partial}{\partial x}\,\vec{i} + \frac{\partial}{\partial y}\,\vec{j} + \frac{\partial}{\partial z}\,\vec{k}$$

We use this as if it's a function in the following manner.

$$\nabla f = \frac{\partial f}{\partial x}\,\vec{i} + \frac{\partial f}{\partial y}\,\vec{j} + \frac{\partial f}{\partial z}\,\vec{k}$$

So, whatever function is listed after the ∇ is substituted into the partial derivatives. Note as well that when we look at it in this light we simply get the gradient vector.

Using the ∇ we can define the curl as the following cross product,

$$\operatorname{curl}\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

We have a couple of nice facts that use the curl of a vector field.

Facts

1. If $f\left(x,y,z\right)$ has continuous second order partial derivatives then $\operatorname{curl}\left(\nabla f\right)=\vec{0}$. This is easy enough to check by plugging into the definition of the derivative so we'll leave it to you to check.

2. If $\vec{F}$ is a conservative vector field then $\operatorname{curl}\vec{F}=\vec{0}$. This is a direct result of what it means to be a conservative vector field and the previous fact.

3. If $\vec{F}$ is defined on all of $\mathbb{R}^3$ whose components have continuous first order partial derivative and $\operatorname{curl}\vec{F}=\vec{0}$ then $\vec{F}$ is a conservative vector field. This is not so easy to verify and so we won't try.

Example 1 Determine if $\vec{F}=x^2y\,\vec{i}+xyz\,\vec{j}-x^2y^2\,\vec{k}$ is a conservative vector field.

Solution
So, all that we need to do is compute the curl and see if we get the zero vector or not.

$$\begin{aligned}\operatorname{curl}\vec{F}&=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ x^2y & xyz & -x^2y^2\end{vmatrix}\\ &=-2x^2y\,\vec{i}+yz\,\vec{k}-\left(-2xy^2\,\vec{j}\right)-xy\,\vec{i}-x^2\vec{k}\\ &=-\left(2x^2y+xy\right)\vec{i}+2xy^2\,\vec{j}+\left(yz-x^2\right)\vec{k}\\ &\neq\vec{0}\end{aligned}$$

So, the curl isn't the zero vector and so this vector field is not conservative.

Next, we should talk about a physical interpretation of the curl. Suppose that $\vec{F}$ is the velocity field of a flowing fluid. Then $\operatorname{curl}\vec{F}$ represents the tendency of particles at the point $\left(x,y,z\right)$ to rotate about the axis that points in the direction of $\operatorname{curl}\vec{F}$. If $\operatorname{curl}\vec{F}=\vec{0}$ then the fluid is called irrotational.

Let's now talk about the second new concept in this section. Given the vector field $\vec{F}=P\,\vec{i}+Q\,\vec{j}+R\,\vec{k}$ the divergence is defined to be,

$$\operatorname{div}\vec{F}=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}$$

There is also a definition of the divergence in terms of the ∇ operator. The divergence can be defined in terms of the following dot product.

$$\operatorname{div}\vec{F}=\nabla\bullet\vec{F}$$

Example 2 Compute $\text{div}\,\vec{F}$ for $\vec{F} = x^2 y\,\vec{i} + xyz\,\vec{j} - x^2 y^2\,\vec{k}$

Solution
There really isn't much to do here other than compute the divergence.

$$\text{div}\,\vec{F} = \frac{\partial}{\partial x}\left(x^2 y\right) + \frac{\partial}{\partial y}\left(xyz\right) + \frac{\partial}{\partial z}\left(-x^2 y^2\right) = 2xy + xz$$

We also have the following fact about the relationship between the curl and the divergence.

$$\text{div}\left(\text{curl}\,\vec{F}\right) = 0$$

Example 3 Verify the above fact for the vector field $\vec{F} = yz^2\,\vec{i} + xy\,\vec{j} + yz\,\vec{k}$.

Solution
Let's first compute the curl.

$$\begin{aligned} \text{curl}\,\vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz^2 & xy & yz \end{vmatrix} \\ &= z\,\vec{i} + 2yz\,\vec{j} + y\,\vec{k} - z^2\vec{k} \\ &= z\vec{i} + 2yz\,\vec{j} + \left(y - z^2\right)\vec{k} \end{aligned}$$

Now compute the divergence of this.

$$\text{div}\left(\text{curl}\,\vec{F}\right) = \frac{\partial}{\partial x}\left(z\right) + \frac{\partial}{\partial y}\left(2yz\right) + \frac{\partial}{\partial z}\left(y - z^2\right) = 2z - 2z = 0$$

We also have a physical interpretation of the divergence. If we again think of $\vec{F}$ as the velocity field of a flowing fluid then $\text{div}\,\vec{F}$ represents the net rate of change of the mass of the fluid flowing from the point $\left(x, y, z\right)$ per unit volume. This can also be thought of as the tendency of a fluid to diverge from a point. If $\text{div}\,\vec{F} = 0$ then the $\vec{F}$ is called incompressible.

The next topic that we want to briefly mention is the **Laplace** operator. Let's first take a look at,

$$\text{div}\left(\nabla f\right) = \nabla \bullet \nabla f = f_{xx} + f_{yy} + f_{zz}$$

The Laplace operator is then defined as,

$$\nabla^2 = \nabla \bullet \nabla$$

The Laplace operator arises naturally in many fields including heat transfer and fluid flow.

The final topic in this section is to give two vector forms of Green's Theorem. The first form uses the curl of the vector field and is,

$$\oint_C \vec{F} \cdot d\,\vec{r} = \iint_D \left(\text{curl}\,\vec{F}\right) \cdot \vec{k}\, dA$$

where $\vec{k}$ is the standard unit vector in the positive *z* direction.

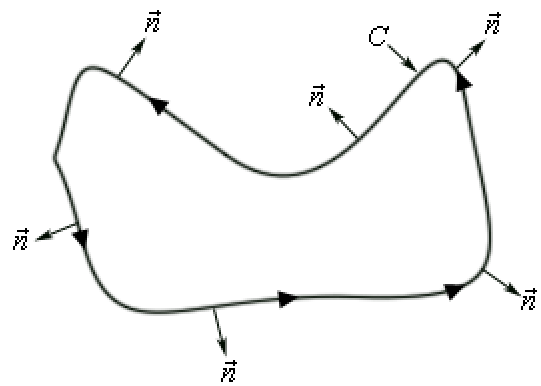
The second form uses the divergence. In this case we also need the outward unit normal to the curve *C*. If the curve is parameterized by

$$\vec{r}\left(t\right) = x\left(t\right)\vec{i} + y\left(t\right)\vec{j}$$

then the outward unit normal is given by,

$$\vec{n} = \frac{y'\left(t\right)}{\left\|\vec{r}'\left(t\right)\right\|}\vec{i} - \frac{x'\left(t\right)}{\left\|\vec{r}'\left(t\right)\right\|}\vec{j}$$

Here is a sketch illustrating the outward unit normal for some curve *C* at various points.



The vector form of Green's Theorem that uses the divergence is given by,

$$\oint_C \vec{F} \cdot \vec{n}\, ds = \iint_D \text{div}\,\vec{F}\, dA$$

Section 6-2 : Parametric Surfaces

The final topic that we need to discuss before getting into surface integrals is how to parameterize a surface. When we parameterized a curve we took values of *t* from some interval $\left[a,b\right]$ and plugged them into

$$\vec r\left(t\right) = x\left(t\right)\vec i + y\left(t\right)\vec j + z\left(t\right)\vec k$$

and the resulting set of vectors will be the position vectors for the points on the curve.

With surfaces we'll do something similar. We will take points, $\left(u,v\right)$, out of some two-dimensional space *D* and plug them into

$$\vec r\left(u,v\right) = x\left(u,v\right)\vec i + y\left(u,v\right)\vec j + z\left(u,v\right)\vec k$$

and the resulting set of vectors will be the position vectors for the points on the surface *S* that we are trying to parameterize. This is often called the **parametric representation** of the **parametric surface** *S*.

We will sometimes need to write the **parametric equations** for a surface. There are really nothing more than the components of the parametric representation explicitly written down.

$$x = x\left(u,v\right) \qquad\qquad y = y\left(u,v\right) \qquad\qquad z = z\left(u,v\right)$$

Example 1 Determine the surface given by the parametric representation

$$\vec r\left(u,v\right) = u\,\vec i + u\cos v\,\vec j + u\sin v\,\vec k$$

Solution

Let's first write down the parametric equations.

$$x = u \qquad\qquad y = u\cos v \qquad\qquad z = u\sin v$$

Now if we square *y* and *z* and then add them together we get,

$$y^2 + z^2 = u^2\cos^2 v + u^2\sin^2 v = u^2\left(\cos^2 v + \sin^2 v\right) = u^2 = x^2$$

So, we were able to eliminate the parameters and the equation in *x*, *y*, and *z* is given by,

$$x^2 = y^2 + z^2$$

From the Quadric Surfaces section notes we can see that this is a cone that opens along the *x*-axis.

We are much more likely to need to be able to write down the parametric equations of a surface than identify the surface from the parametric representation so let's take a look at some examples of this.

Example 2 Give parametric representations for each of the following surfaces.

(a) The elliptic paraboloid $x = 5y^2 + 2z^2 - 10$.

(b) The elliptic paraboloid $x = 5y^2 + 2z^2 - 10$ that is in front of the *yz*-plane.

(c) The sphere $x^2 + y^2 + z^2 = 30$.

(d) The cylinder $y^2 + z^2 = 25$.

Solution

(a) The elliptic paraboloid $x = 5y^2 + 2z^2 - 10$.

This one is probably the easiest one of the four to see how to do. Since the surface is in the form $x = f(y,z)$ we can quickly write down a set of parametric equations as follows,

$$x = 5y^2 + 2z^2 - 10 \qquad y = y \qquad z = z$$

The last two equations are just there to acknowledge that we can choose *y* and *z* to be anything we want them to be. The parametric representation is then,

$$\vec{r}(y,z) = \left(5y^2 + 2z^2 - 10\right)\vec{i} + y\,\vec{j} + z\,\vec{k}$$

(b) The elliptic paraboloid $x = 5y^2 + 2z^2 - 10$ that is in front of the *yz*-plane.

This is really a restriction on the previous parametric representation. The parametric representation stays the same.

$$\vec{r}(y,z) = \left(5y^2 + 2z^2 - 10\right)\vec{i} + y\,\vec{j} + z\,\vec{k}$$

However, since we only want the surface that lies in front of the *yz*-plane we also need to require that $x \ge 0$. This is equivalent to requiring,

$$5y^2 + 2z^2 - 10 \ge 0 \qquad \text{or} \qquad 5y^2 + 2z^2 \ge 10$$

(c) The sphere $x^2 + y^2 + z^2 = 30$.

This one can be a little tricky until you see how to do it. In spherical coordinates we know that the equation of a sphere of radius *a* is given by,

$$\rho = a$$

and so the equation of this sphere (in spherical coordinates) is $\rho = \sqrt{30}$. Now, we also have the following conversion formulas for converting Cartesian coordinates into spherical coordinates.

$$x = \rho\sin\varphi\cos\theta \qquad y = \rho\sin\varphi\sin\theta \qquad z = \rho\cos\varphi$$

However, we know what ρ is for our sphere and so if we plug this into these conversion formulas we will arrive at a parametric representation for the sphere. Therefore, the parametric representation is,

$$\vec{r}(\theta,\varphi) = \sqrt{30}\sin\varphi\cos\theta\,\vec{i} + \sqrt{30}\sin\varphi\sin\theta\,\vec{j} + \sqrt{30}\cos\varphi\,\vec{k}$$

All we need to do now is come up with some restriction on the variables. First, we know that we have the following restriction.

$$0 \le \varphi \le \pi$$

This is enforced upon us by choosing to use spherical coordinates. Also, to make sure that we only trace out the sphere once we will also have the following restriction.

$$0 \le \theta \le 2\pi$$

(d) The cylinder $y^2 + z^2 = 25$.

As with the last one this can be tricky until you see how to do it. In this case it makes some sense to use cylindrical coordinates since they can be easily used to write down the equation of a cylinder.

In cylindrical coordinates the equation of a cylinder of radius *a* is given by

$$r = a$$

and so the equation of the cylinder in this problem is $r = 5$.

Next, we have the following conversion formulas.

$$x = x \qquad\qquad y = r\sin\theta \qquad\qquad z = r\cos\theta$$

Notice that they are slightly different from those that we are used to seeing. We needed to change them up here since the cylinder was centered upon the *x*-axis.

Finally, we know what *r* is so we can easily write down a parametric representation for this cylinder.

$$\vec r\left(x,\theta\right) = x\,\vec i + 5\sin\theta\,\vec j + 5\cos\theta\,\vec k$$

We will also need the restriction $0 \le \theta \le 2\pi$ to make sure that we don't retrace any portion of the cylinder. Since we haven't put any restrictions on the "height" of the cylinder there won't be any restriction on *x*.

In the first part of this example we used the fact that the function was in the form $x = f\left(y,z\right)$ to quickly write down a parametric representation. This can always be done for functions that are in this basic form.

$$z = f\left(x,y\right) \quad \Rightarrow \quad \vec r\left(x,y\right) = x\,\vec i + y\,\vec j + f\left(x,y\right)\vec k$$

$$x = f\left(y,z\right) \quad \Rightarrow \quad \vec r\left(y,z\right) = f\left(y,z\right)\vec i + y\,\vec j + z\,\vec k$$

$$y = f\left(x,z\right) \quad \Rightarrow \quad \vec r\left(x,z\right) = x\,\vec i + f\left(x,z\right)\vec j + z\,\vec k$$

Okay, now that we have practice writing down some parametric representations for some surfaces let's take a quick look at a couple of applications.

Let's take a look at finding the tangent plane to the parametric surface *S* given by,

$$\vec r\left(u,v\right) = x\left(u,v\right)\vec i + y\left(u,v\right)\vec j + z\left(u,v\right)\vec k$$

First, define

$$\vec{r}_u(u,v) = \frac{\partial x}{\partial u}(u,v)\,\vec{i} + \frac{\partial y}{\partial u}(u,v)\,\vec{j} + \frac{\partial z}{\partial u}(u,v)\,\vec{k}$$
$$\vec{r}_v(u,v) = \frac{\partial x}{\partial v}(u,v)\,\vec{i} + \frac{\partial y}{\partial v}(u,v)\,\vec{j} + \frac{\partial z}{\partial v}(u,v)\,\vec{k}$$

If we hold $v = v_0$ fixed then $\vec{r}_u(u, v_0)$ will be tangent to the curve given by $\vec{r}(u, v_0)$ (and yes this is a curve given that only one of the variables, *u*, is changing….) provided $\vec{r}_u(u, v_0) \neq \vec{0}$. Similarly, if we hold $u = u_0$ fixed then $\vec{r}_v(u_0 v)$ will be tangent to the curve given by $\vec{r}(u_0, v)$ (again, because only *v* is changing this is a curve) provided $\vec{r}_v(u_0, v) \neq \vec{0}$.

Therefore, both $\vec{r}_u(u_0, v_0)$ and $\vec{r}_v(u_0, v_0)$, provided neither one is the zero vector) will be tangent to the surface, *S*, given by $\vec{r}(u,v)$ at (u_0, v_0) and the tangent plane to the surface at (u_0, v_0) will be the plane containing both $\vec{r}_u(u_0, v_0)$ and $\vec{r}_v(u_0, v_0)$.

To help make things a little clearer we did the work at a particular point, but this fact is true at any point for which neither $\vec{r}_u$ or $\vec{r}_v$ are the zero vector.

This, in turn, means that provided $\vec{r}_u \times \vec{r}_v \neq \vec{0}$ the vector $\vec{r}_u \times \vec{r}_v$ will be orthogonal to the surface *S* and so it can be used for the normal vector that we need in order to write down the equation of a tangent plane. This is an important idea that will be used many times throughout the next couple of sections.

Let's take a look at an example.

Example 3 Find the equation of the tangent plane to the surface given by

$$\vec{r}(u,v) = u\,\vec{i} + 2v^2\,\vec{j} + \left(u^2 + v\right)\vec{k}$$

at the point $(2,2,3)$.

Solution

Let's first compute $\vec{r}_u \times \vec{r}_v$. Here are the two individual vectors.

$$\vec{r}_u(u,v) = \vec{i} + 2u\,\vec{k} \qquad\qquad \vec{r}_v(u,v) = 4v\,\vec{j} + \vec{k}$$

Now the cross product (which will give us the normal vector $\vec{n}$) is,

$$\vec{n} = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2u \\ 0 & 4v & 1 \end{vmatrix} = -8uv\,\vec{i} - \vec{j} + 4v\,\vec{k}$$

Now, this is all fine, but in order to use it we will need to determine the value of *u* and *v* that will give us the point in question. We can easily do this by setting the individual components of the parametric representation equal to the coordinates of the point in question. Doing this gives,

$$\begin{aligned} 2 &= u & \Rightarrow \quad & u = 2 \\ 2 &= 2v^2 & \Rightarrow \quad & v = \pm 1 \\ 3 &= u^2 + v & & \end{aligned}$$

Now, as shown, we have the value of *u*, but there are two possible values of *v*. To determine the correct value of *v* let's plug *u* into the third equation and solve for *v*. This should tell us what the correct value is.

$$3 = 4 + v \qquad \Rightarrow \qquad v = -1$$

Okay so we now know that we'll be at the point in question when $u = 2$ and $v = -1$. At this point the normal vector is,

$$\vec{n} = 16\vec{i} - \vec{j} - 4\vec{k}$$

The tangent plane is then,

$$\begin{aligned} 16(x-2) - (y-2) - 4(z-3) &= 0 \\ 16x - y - 4z &= 18 \end{aligned}$$

You do remember how to write down the equation of a plane, right?

The second application that we want to take a quick look at is the surface area of the parametric surface *S* given by,

$$\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$$

and as we will see it again comes down to needing the vector $\vec{r}_u \times \vec{r}_v$.

So, provided *S* is traced out exactly once as (u,v) ranges over the points in *D* the surface area of *S* is given by,

$$A = \iint_D \left\| \vec{r}_u \times \vec{r}_v \right\| dA$$

Let's take a look at an example.

Example 4 Find the surface area of the portion of the sphere of radius 4 that lies inside the cylinder $x^2 + y^2 = 12$ and above the *xy*-plane.

Solution

Okay we've got a couple of things to do here. First, we need the parameterization of the sphere. We parameterized a sphere earlier in this section so there isn't too much to do at this point. Here is the parameterization.

$$\vec{r}(\theta,\varphi) = 4\sin\varphi\cos\theta\,\vec{i} + 4\sin\varphi\sin\theta\,\vec{j} + 4\cos\varphi\,\vec{k}$$

Next, we need to determine *D*. Since we are not restricting how far around the *z*-axis we are rotating with the sphere we can take the following range for θ.

$$0 \le \theta \le 2\pi$$

Now, we need to determine a range for φ. This will take a little work, although it's not too bad. First, let's start with the equation of the sphere.

$$x^2 + y^2 + z^2 = 16$$

Now, if we substitute the equation for the cylinder into this equation we can find the value of *z* where the sphere and the cylinder intersect.

$$\begin{aligned} x^2 + y^2 + z^2 &= 16 \\ 12 + z^2 &= 16 \\ z^2 &= 4 \qquad \Rightarrow \qquad z = \pm 2 \end{aligned}$$

Now, since we also specified that we only want the portion of the sphere that lies above the *xy*-plane we know that we need $z = 2$. We also know that $\rho = 4$. Plugging this into the following conversion formula we get,

$$\begin{aligned} z &= \rho \cos\varphi \\ 2 &= 4\cos\varphi \\ \cos\varphi &= \frac{1}{2} \qquad \Rightarrow \qquad \varphi = \frac{\pi}{3} \end{aligned}$$

So, it looks like the range of φ will be,

$$0 \le \varphi \le \frac{\pi}{3}$$

Finally, we need to determine $\vec{r}_\theta \times \vec{r}_\varphi$. Here are the two individual vectors.

$$\begin{aligned} \vec{r}_\theta\left(\theta,\varphi\right) &= -4\sin\varphi\sin\theta\,\vec{i} + 4\sin\varphi\cos\theta\,\vec{j} \\ \vec{r}_\varphi\left(\theta,\varphi\right) &= 4\cos\varphi\cos\theta\,\vec{i} + 4\cos\varphi\sin\theta\,\vec{j} - 4\sin\varphi\,\vec{k} \end{aligned}$$

Now let's take the cross product.

$$\begin{aligned} \vec{r}_\theta \times \vec{r}_\varphi &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -4\sin\varphi\sin\theta & 4\sin\varphi\cos\theta & 0 \\ 4\cos\varphi\cos\theta & 4\cos\varphi\sin\theta & -4\sin\varphi \end{vmatrix} \\ &= -16\sin^2\varphi\cos\theta\,\vec{i} - 16\sin\varphi\cos\varphi\sin^2\theta\,\vec{k} - 16\sin^2\varphi\sin\theta\,\vec{j} - 16\sin\varphi\cos\varphi\cos^2\theta\,\vec{k} \\ &= -16\sin^2\varphi\cos\theta\,\vec{i} - 16\sin^2\varphi\sin\theta\,\vec{j} - 16\sin\varphi\cos\varphi\left(\sin^2\theta + \cos^2\theta\right)\vec{k} \\ &= -16\sin^2\varphi\cos\theta\,\vec{i} - 16\sin^2\varphi\sin\theta\,\vec{j} - 16\sin\varphi\cos\varphi\,\vec{k} \end{aligned}$$

We now need the magnitude of this,

$$\begin{aligned}\left\|\vec{r}_\theta \times \vec{r}_\varphi\right\| &= \sqrt{256\sin^4\varphi\cos^2\theta + 256\sin^4\varphi\sin^2\theta + 256\sin^2\varphi\cos^2\varphi} \\ &= \sqrt{256\sin^4\varphi\left(\cos^2\theta + \sin^2\theta\right) + 256\sin^2\varphi\cos^2\varphi} \\ &= \sqrt{256\sin^2\varphi\left(\sin^2\varphi + \cos^2\varphi\right)} \\ &= 16\sqrt{\sin^2\varphi} \\ &= 16\left|\sin\varphi\right| \\ &= 16\sin\varphi\end{aligned}$$

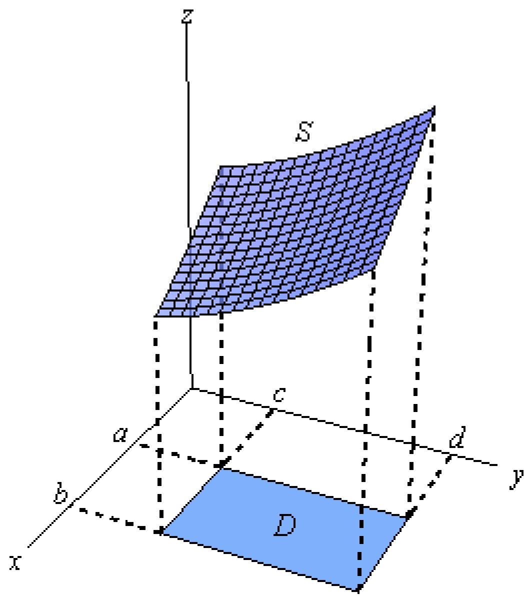
We can drop the absolute value bars in the sine because sine is positive in the range of φ that we are working with.

We can finally get the surface area.

$$\begin{aligned}A &= \iint\limits_D 16\sin\varphi\, dA \\ &= \int_0^{2\pi}\int_0^{\frac{\pi}{3}} 16\sin\varphi\, d\varphi\, d\theta \\ &= \int_0^{2\pi} -16\cos\varphi\Big|_0^{\pi/3}\, d\theta \\ &= \int_0^{2\pi} 8\, d\theta \\ &= 16\pi\end{aligned}$$

Section 6-3 : Surface Integrals

It is now time to think about integrating functions over some surface, *S*, in three-dimensional space. Let's start off with a sketch of the surface *S* since the notation can get a little confusing once we get into it. Here is a sketch of some surface *S*.



The region *S* will lie above (in this case) some region *D* that lies in the *xy*-plane. We used a rectangle here, but it doesn't have to be of course. Also note that we could just as easily looked at a surface *S* that was in front of some region *D* in the *yz*-plane or the *xz*-plane. Do not get so locked into the *xy*-plane that you can't do problems that have regions in the other two planes.

Now, how we evaluate the surface integral will depend upon how the surface is given to us. There are essentially two separate methods here, although as we will see they are really the same.

First, let's look at the surface integral in which the surface *S* is given by $z = g(x,y)$. In this case the surface integral is,

$$\iint_S f(x,y,z)\,dS = \iint_D f\left(x,y,g(x,y)\right)\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}\,dA$$

Now, we need to be careful here as both of these look like standard double integrals. In fact the integral on the right is a standard double integral. The integral on the left however is a surface integral. The way to tell them apart is by looking at the differentials. The surface integral will have a *dS* while the standard double integral will have a *dA*.

In order to evaluate a surface integral we will substitute the equation of the surface in for *z* in the integrand and then add on the often messy square root. After that the integral is a standard double integral and by this point we should be able to deal with that.

Note as well that there are similar formulas for surfaces given by $y = g\left(x, z\right)$ (with *D* in the *xz*-plane) and $x = g\left(y, z\right)$ (with *D* in the *yz*-plane). We will see one of these formulas in the examples and we'll leave the other to you to write down.

The second method for evaluating a surface integral is for those surfaces that are given by the parameterization,

$$\vec{r}\left(u,v\right) = x\left(u,v\right)\vec{i} + y\left(u,v\right)\vec{j} + z\left(u,v\right)\vec{k}$$

In these cases the surface integral is,

$$\iint\limits_S f\left(x,y,z\right)dS = \iint\limits_D f\left(\vec{r}\left(u,v\right)\right)\left\|\vec{r}_u \times \vec{r}_v\right\|dA$$

where *D* is the range of the parameters that trace out the surface *S*.

Before we work some examples let's notice that since we can parameterize a surface given by $z = g\left(x, y\right)$ as,

$$\vec{r}\left(x,y\right) = x\vec{i} + y\vec{j} + g\left(x,y\right)\vec{k}$$

we can always use this form for these kinds of surfaces as well. In fact, it can be shown that,

$$\left\|\vec{r}_x \times \vec{r}_y\right\| = \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}$$

for these kinds of surfaces. You might want to verify this for the practice of computing these cross products.

Let's work some examples.

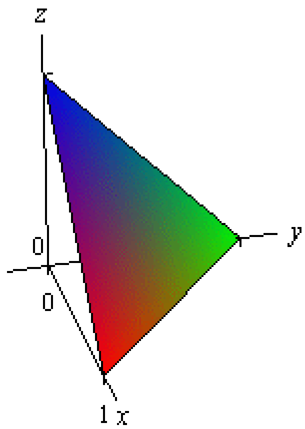
Example 1 Evaluate $\iint\limits_S 6xy\,dS$ where *S* is the portion of the plane $x + y + z = 1$ that lies in the 1st octant and is in front of the *yz*-plane.

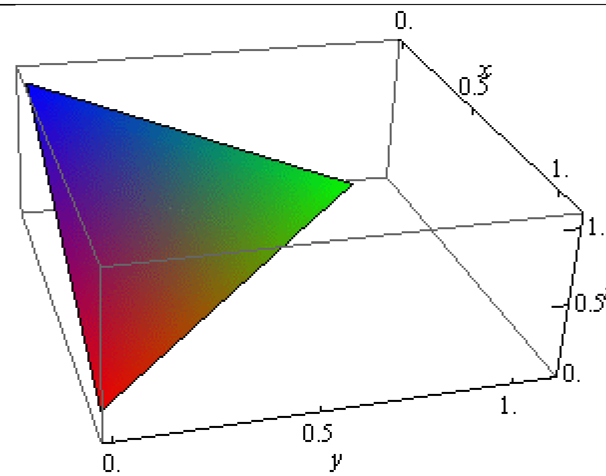
Solution

Okay, since we are looking for the portion of the plane that lies in front of the *yz*-plane we are going to need to write the equation of the surface in the form $x = g\left(y,z\right)$. This is easy enough to do.

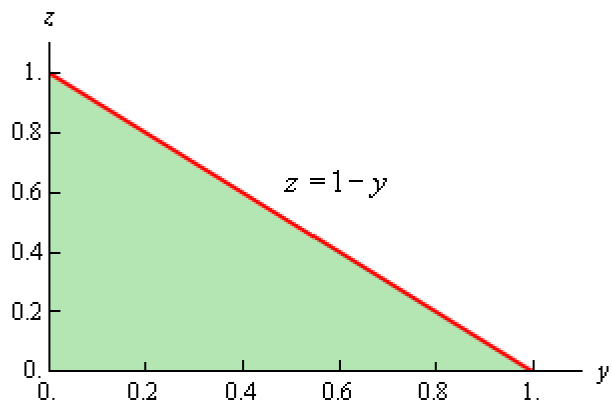
$$x = 1 - y - z$$

Next, we need to determine just what *D* is. Here is a sketch of the surface *S*.





Here is a sketch of the region *D*.



Notice that the axes are labeled differently than we are used to seeing in the sketch of *D*. This was to keep the sketch consistent with the sketch of the surface. We arrived at the equation of the hypotenuse by setting *x* equal to zero in the equation of the plane and solving for *z*. Here are the ranges for *y* and *z*.

$$0 \le y \le 1 \qquad 0 \le z \le 1 - y$$

Now, because the surface is not in the form $z = g\left(x, y\right)$ we can't use the formula above. However, as noted above we can modify this formula to get one that will work for us. Here it is,

$$\iint\limits_S f\left(x,y,z\right)dS = \iint\limits_D f\left(g\left(y,z\right),y,z\right)\sqrt{1+\left(\frac{\partial g}{\partial y}\right)^2+\left(\frac{\partial g}{\partial z}\right)^2}\,dA$$

The changes made to the formula should be the somewhat obvious changes. So, let's do the integral.

$$\iint\limits_S 6xy\,dS = \iint\limits_D 6\left(1-y-z\right)y\sqrt{1+\left(-1\right)^2+\left(-1\right)^2}\,dA$$

Notice that we plugged in the equation of the plane for the *x* in the integrand. At this point we've got a fairly simple double integral to do. Here is that work.

$$\iint_S 6xy\,dS = \sqrt{3}\iint_D 6\left(y - y^2 - zy\right)dA$$
$$= 6\sqrt{3}\int_0^1\int_0^{1-y} y - y^2 - zy\,dz\,dy$$
$$= 6\sqrt{3}\int_0^1 \left(yz - zy^2 - \frac{1}{2}z^2 y\right)\Big|_0^{1-y} dy$$
$$= 6\sqrt{3}\int_0^1 \frac{1}{2}y - y^2 + \frac{1}{2}y^3\,dy$$
$$= 6\sqrt{3}\left(\frac{1}{4}y^2 - \frac{1}{3}y^3 + \frac{1}{8}y^4\right)\Big|_0^1 = \frac{\sqrt{3}}{4}$$

Example 2 Evaluate $\iint\limits_S z\,dS$ where *S* is the upper half of a sphere of radius 2.

Solution
We gave the parameterization of a sphere in the previous section. Here is the parameterization for this sphere.

$$\vec r\left(\theta,\varphi\right) = 2\sin\varphi\cos\theta\,\vec i + 2\sin\varphi\sin\theta\,\vec j + 2\cos\varphi\,\vec k$$

Since we are working on the upper half of the sphere here are the limits on the parameters.

$$0 \le \theta \le 2\pi \qquad\qquad 0 \le \varphi \le \frac{\pi}{2}$$

Next, we need to determine $\vec r_\theta \times \vec r_\varphi$. Here are the two individual vectors.

$$\vec r_\theta\left(\theta,\varphi\right) = -2\sin\varphi\sin\theta\,\vec i + 2\sin\varphi\cos\theta\,\vec j$$
$$\vec r_\varphi\left(\theta,\varphi\right) = 2\cos\varphi\cos\theta\,\vec i + 2\cos\varphi\sin\theta\,\vec j - 2\sin\varphi\,\vec k$$

Now let’s take the cross product.

$$\vec r_\theta \times \vec r_\varphi = \begin{vmatrix} \vec i & \vec j & \vec k \\ -2\sin\varphi\sin\theta & 2\sin\varphi\cos\theta & 0 \\ 2\cos\varphi\cos\theta & 2\cos\varphi\sin\theta & -2\sin\varphi \end{vmatrix}$$
$$= -4\sin^2\varphi\cos\theta\,\vec i - 4\sin\varphi\cos\varphi\sin^2\theta\,\vec k - 4\sin^2\varphi\sin\theta\,\vec j - 4\sin\varphi\cos\varphi\cos^2\theta\,\vec k$$
$$= -4\sin^2\varphi\cos\theta\,\vec i - 4\sin^2\varphi\sin\theta\,\vec j - 4\sin\varphi\cos\varphi\left(\sin^2\theta + \cos^2\theta\right)\vec k$$
$$= -4\sin^2\varphi\cos\theta\,\vec i - 4\sin^2\varphi\sin\theta\,\vec j - 4\sin\varphi\cos\varphi\,\vec k$$

Finally, we need the magnitude of this,

$$\begin{aligned}\left\|\vec{r}_\theta \times \vec{r}_\varphi\right\| &= \sqrt{16\sin^4\varphi\cos^2\theta + 16\sin^4\varphi\sin^2\theta + 16\sin^2\varphi\cos^2\varphi} \\ &= \sqrt{16\sin^4\varphi\left(\cos^2\theta + \sin^2\theta\right) + 16\sin^2\varphi\cos^2\varphi} \\ &= \sqrt{16\sin^2\varphi\left(\sin^2\varphi + \cos^2\varphi\right)} \\ &= 4\sqrt{\sin^2\varphi} \\ &= 4\left|\sin\varphi\right| \\ &= 4\sin\varphi\end{aligned}$$

We can drop the absolute value bars in the sine because sine is positive in the range of φ that we are working with. The surface integral is then,

$$\iint\limits_S z\,dS = \iint\limits_D 2\cos\varphi\left(4\sin\varphi\right)dA$$

Don't forget that we need to plug in for *x*, *y* and/or *z* in these as well, although in this case we just needed to plug in *z*. Here is the evaluation for the double integral.

$$\begin{aligned}\iint\limits_S z\,dS &= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} 4\sin\left(2\varphi\right)d\varphi\,d\theta \\ &= \int_0^{2\pi}\left(-2\cos\left(2\varphi\right)\right)\Big|_0^{\frac{\pi}{2}}\,d\theta \\ &= \int_0^{2\pi} 4\,d\theta \\ &= 8\pi\end{aligned}$$

Example 3 Evaluate $\iint\limits_S y\,dS$ where *S* is the portion of the cylinder $x^2 + y^2 = 3$ that lies between $z = 0$ and $z = 6$.

Solution

We parameterized up a cylinder in the previous section. Here is the parameterization of this cylinder.

$$\vec{r}\left(z,\theta\right) = \sqrt{3}\cos\theta\,\vec{i} + \sqrt{3}\sin\theta\,\vec{j} + z\,\vec{k}$$

The ranges of the parameters are,

$$0 \le z \le 6 \qquad 0 \le \theta \le 2\pi$$

Now we need $\vec{r}_z \times \vec{r}_\theta$. Here are the two vectors.

$$\begin{aligned}\vec{r}_z\left(z,\theta\right) &= \vec{k} \\ \vec{r}_\theta\left(z,\theta\right) &= -\sqrt{3}\sin\theta\,\vec{i} + \sqrt{3}\cos\theta\,\vec{j}\end{aligned}$$

Here is the cross product.

$$\vec{r}_z \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & 1 \\ -\sqrt{3}\sin\theta & \sqrt{3}\cos\theta & 0 \end{vmatrix}$$
$$= -\sqrt{3}\cos\theta\,\vec{i} - \sqrt{3}\sin\theta\,\vec{j}$$

The magnitude of this vector is,

$$\left\| \vec{r}_z \times \vec{r}_\theta \right\| = \sqrt{3\cos^2\theta + 3\sin^2\theta} = \sqrt{3}$$

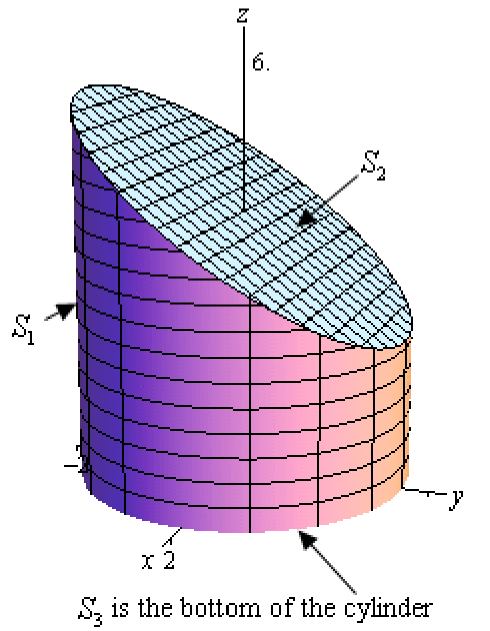
The surface integral is then,

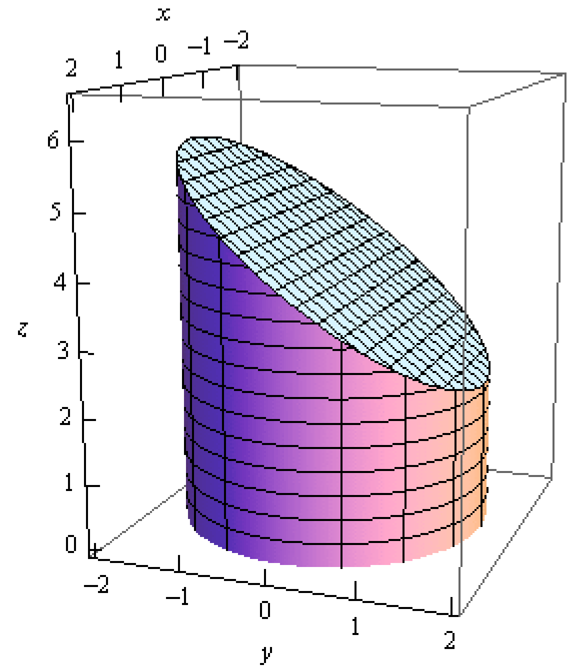
$$\iint\limits_S y\,dS = \iint\limits_D \sqrt{3}\sin\theta\left(\sqrt{3}\right)dA$$
$$= 3\int_0^{2\pi}\int_0^6 \sin\theta\,dz\,d\theta$$
$$= 3\int_0^{2\pi} 6\sin\theta\,d\theta$$
$$= \left(-18\cos\theta\right)\Big|_0^{2\pi}$$
$$= 0$$

Example 4 Evaluate $\iint\limits_S y + z\,dS$ where *S* is the surface whose side is the cylinder $x^2 + y^2 = 3$, whose bottom is the disk $x^2 + y^2 \le 3$ in the *xy*-plane and whose top is the plane $z = 4 - y$.

Solution

There is a lot of information that we need to keep track of here. First, we are using pretty much the same surface (the integrand is different however) as the previous example. However, unlike the previous example we are putting a top and bottom on the surface this time. Let's first start out with a sketch of the surface.





We need to be careful here. There is more to this sketch than the actual surface itself. We're going to let S_1 be the portion of the cylinder that goes from the *xy*-plane to the plane. In other words, the top of the cylinder will be at an angle. We'll call the portion of the plane that lies inside (*i.e.* the cap on the cylinder) S_2. Finally, the bottom of the cylinder (not shown here) is the disk of radius $\sqrt{3}$ in the *xy*-plane and is denoted by S_3.

In order to do this integral we'll need to note that just like the standard double integral, if the surface is split up into pieces we can also split up the surface integral. So, for our example we will have,

$$\iint_S y + z\,dS = \iint_{S_1} y + z\,dS + \iint_{S_2} y + z\,dS + \iint_{S_3} y + z\,dS$$

We're going to need to do three integrals here. However, we've done most of the work for the first one in the previous example so let's start with that.

S_1 : <u>The Cylinder</u>

The parameterization of the cylinder and $\left\| \vec r_z \times \vec r_\theta \right\|$ is,

$$\vec r\left(z,\theta \right) = \sqrt{3}\cos\theta\,\vec i + \sqrt{3}\sin\theta\,\vec j + z\,\vec k \qquad\qquad \left\| \vec r_z \times \vec r_\theta \right\| = \sqrt{3}$$

The difference between this problem and the previous one is the limits on the parameters. Here they are.

$$0 \le \theta \le 2\pi$$
$$0 \le z \le 4 - y = 4 - \sqrt{3}\sin\theta$$

The upper limit for the *z*'s is the plane so we can just plug that in. However, since we are on the cylinder we know what *y* is from the parameterization so we will also need to plug that in.

Here is the integral for the cylinder.

$$\begin{aligned}\iint\limits_{S_1} y+z\,dS &= \iint\limits_{D}\left(\sqrt{3}\sin\theta+z\right)\left(\sqrt{3}\right)dA \\ &= \sqrt{3}\int_0^{2\pi}\int_0^{4-\sqrt{3}\sin\theta}\sqrt{3}\sin\theta+z\,dz\,d\theta \\ &= \sqrt{3}\int_0^{2\pi}\sqrt{3}\sin\theta\left(4-\sqrt{3}\sin\theta\right)+\frac{1}{2}\left(4-\sqrt{3}\sin\theta\right)^2 d\theta \\ &= \sqrt{3}\int_0^{2\pi} 8-\frac{3}{2}\sin^2\theta\,d\theta \\ &= \sqrt{3}\int_0^{2\pi} 8-\frac{3}{4}\left(1-\cos\left(2\theta\right)\right)d\theta \\ &= \sqrt{3}\left(\frac{29}{4}\theta+\frac{3}{8}\sin\left(2\theta\right)\right)\Bigg|_0^{2\pi} \\ &= \frac{29\sqrt{3}\,\pi}{2}\end{aligned}$$

S_2 : Plane on Top of the Cylinder

In this case we don't need to do any parameterization since it is set up to use the formula that we gave at the start of this section. Remember that the plane is given by $z = 4 - y$. Also note that, for this surface, *D* is the disk of radius $\sqrt{3}$ centered at the origin.

Here is the integral for the plane.

$$\begin{aligned}\iint\limits_{S_2} y+z\,dS &= \iint\limits_{D}\left(y+4-y\right)\sqrt{\left(0\right)^2+\left(-1\right)^2+1}\,dA \\ &= \sqrt{2}\iint\limits_{D} 4\,dA\end{aligned}$$

Don't forget that we need to plug in for *z*! Now at this point we can proceed in one of two ways. Either we can proceed with the integral or we can recall that $\iint\limits_{D} dA$ is nothing more than the area of *D* and we know that *D* is the disk of radius $\sqrt{3}$ and so there is no reason to do the integral.

Here is the remainder of the work for this problem.

$$\iint_{S_2} y + z\, dS = 4\sqrt{2}\iint_D dA$$
$$= 4\sqrt{2}\left(\pi\left(\sqrt{3}\right)^2\right)$$
$$= 12\sqrt{2}\,\pi$$

S_3 : Bottom of the Cylinder

Again, this is set up to use the initial formula we gave in this section once we realize that the equation for the bottom is given by $g\left(x, y\right) = 0$ and *D* is the disk of radius $\sqrt{3}$ centered at the origin. Also, don't forget to plug in for *z*.

Here is the work for this integral.

$$\iint_{S_3} y + z\, dS = \iint_D \left(y + 0\right)\sqrt{\left(0\right)^2 + \left(0\right)^2 + 1}\, dA$$
$$= \iint_D y\, dA$$
$$= \int_0^{2\pi}\int_0^{\sqrt{3}} r^2 \sin\theta\, dr\, d\theta$$
$$= \int_0^{2\pi} \left(\frac{1}{3} r^3 \sin\theta\right)\Bigg|_0^{\sqrt{3}} d\theta$$
$$= \int_0^{2\pi} \sqrt{3}\sin\theta\, d\theta$$
$$= -\sqrt{3}\cos\theta\Big|_0^{2\pi}$$
$$= 0$$

We can now get the value of the integral that we are after.

$$\iint_S y + z\, dS = \iint_{S_1} y + z\, dS + \iint_{S_2} y + z\, dS + \iint_{S_3} y + z\, dS$$
$$= \frac{29\sqrt{3}\,\pi}{2} + 12\sqrt{2}\,\pi + 0$$
$$= \frac{\pi}{2}\left(29\sqrt{3} + 24\sqrt{2}\right)$$

Section 6-4 : Surface Integrals of Vector Fields

Just as we did with line integrals we now need to move on to surface integrals of vector fields. Recall that in line integrals the orientation of the curve we were integrating along could change the answer. The same thing will hold true with surface integrals. So, before we really get into doing surface integrals of vector fields we first need to introduce the idea of an **oriented surface**.

Let's start off with a surface that has two sides (while this may seem strange, recall that the Mobius Strip is a surface that only has one side!) that has a tangent plane at every point (except possibly along the boundary). Making this assumption means that every point will have two unit normal vectors, $\vec{n}_1$ and $\vec{n}_2 = -\vec{n}_1$. This means that every surface will have two sets of normal vectors. The set that we choose will give the surface an orientation.

There is one convention that we will make in regard to certain kinds of oriented surfaces. First, we need to define a **closed surface**. A surface *S* is closed if it is the boundary of some solid region *E*. A good example of a closed surface is the surface of a sphere. We say that the closed surface *S* has a **positive** orientation if we choose the set of unit normal vectors that point outward from the region *E* while the **negative** orientation will be the set of unit normal vectors that point in towards the region *E*.

Note that this convention is only used for closed surfaces.

In order to work with surface integrals of vector fields we will need to be able to write down a formula for the unit normal vector corresponding to the orientation that we've chosen to work with. We have two ways of doing this depending on how the surface has been given to us.

First, let's suppose that the function is given by $z = g\left(x,y\right)$. In this case we first define a new function,

$$f\left(x,y,z\right) = z - g\left(x,y\right)$$

In terms of our new function the surface is then given by the equation $f\left(x,y,z\right) = 0$. Now, recall that ∇f will be orthogonal (or normal) to the surface given by $f\left(x,y,z\right) = 0$. This means that we have a normal vector to the surface. The only potential problem is that it might not be a unit normal vector. That isn't a problem since we also know that we can turn any vector into a unit vector by dividing the vector by its length. In our case this is,

$$\vec{n} = \frac{\nabla f}{\left\|\nabla f\right\|}$$

In this case it will be convenient to actually compute the gradient vector and plug this into the formula for the normal vector. Doing this gives,

$$\vec{n} = \frac{\nabla f}{\left\|\nabla f\right\|} = \frac{-g_x\,\vec{i} - g_y\,\vec{j} + \vec{k}}{\sqrt{\left(g_x\right)^2 + \left(g_y\right)^2 + 1}}$$

Now, from a notational standpoint this might not have been so convenient, but it does allow us to make a couple of additional comments.

First, notice that the component of the normal vector in the *z*-direction (identified by the $\vec{k}$ in the normal vector) is always positive and so this normal vector will generally point upwards. It may not point directly up, but it will have an upwards component to it.

This will be important when we are working with a closed surface and we want the positive orientation. If we know that we can then look at the normal vector and determine if the "positive" orientation should point upwards or downwards. Remember that the "positive" orientation must point out of the region and this may mean downwards in places. Of course, if it turns out that we need the downward orientation we can always take the negative of this unit vector and we'll get the one that we need. Again, remember that we always have that option when choosing the unit normal vector.

Before we move onto the second method of giving the surface we should point out that we only did this for surfaces in the form $z = g\left(x, y\right)$. We could just as easily done the above work for surfaces in the form $y = g\left(x, z\right)$ (so $f\left(x, y, z\right) = y - g\left(x, z\right)$) or for surfaces in the form $x = g\left(y, z\right)$ (so $f\left(x, y, z\right) = x - g\left(y, z\right)$).

Now, we need to discuss how to find the unit normal vector if the surface is given parametrically as,

$$\vec{r}\left(u, v\right) = x\left(u, v\right)\vec{i} + y\left(u, v\right)\vec{j} + z\left(u, v\right)\vec{k}$$

In this case recall that the vector $\vec{r}_u \times \vec{r}_v$ will be normal to the tangent plane at a particular point. But if the vector is normal to the tangent plane at a point then it will also be normal to the surface at that point. So, this is a normal vector. In order to guarantee that it is a unit normal vector we will also need to divide it by its magnitude.

So, in the case of parametric surfaces one of the unit normal vectors will be,

$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{\left\|\vec{r}_u \times \vec{r}_v\right\|}$$

As with the first case we will need to look at this once it's computed and determine if it points in the correct direction or not. If it doesn't then we can always take the negative of this vector and that will point in the correct direction.

Finally, remember that we can always parameterize any surface given by $z = g\left(x, y\right)$ (or $y = g\left(x, z\right)$ or $x = g\left(y, z\right)$) easily enough and so if we want to we can always use the parameterization formula to find the unit normal vector.

Okay, now that we've looked at oriented surfaces and their associated unit normal vectors we can actually give a formula for evaluating surface integrals of vector fields.

Given a vector field $\vec{F}$ with unit normal vector $\vec{n}$ then the surface integral of $\vec{F}$ over the surface *S* is given by,

$$\iint_S \vec{F} \bullet d\vec{S} = \iint_S \vec{F} \bullet \vec{n}\, dS$$

where the right hand integral is a standard surface integral. This is sometimes called the **flux of $\vec{F}$ across *S***.

Before we work any examples let's notice that we can substitute in for the unit normal vector to get a *somewhat* easier formula to use. We will need to be careful with each of the following formulas however as each will assume a certain orientation and we may have to change the normal vector to match the given orientation.

Let's first start by assuming that the surface is given by $z = g(x, y)$. In this case let's also assume that the vector field is given by $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ and that the orientation that we are after is the "upwards" orientation. Under all of these assumptions the surface integral of $\vec{F}$ over *S* is,

$$\begin{aligned}\iint_S \vec{F} \bullet d\vec{S} &= \iint_S \vec{F} \bullet \vec{n}\, dS \\ &= \iint_D \left(P\vec{i} + Q\vec{j} + R\vec{k}\right) \bullet \left(\frac{-g_x\,\vec{i} - g_y\,\vec{j} + \vec{k}}{\sqrt{(g_x)^2 + (g_y)^2 + 1}}\right) \sqrt{(g_x)^2 + (g_y)^2 + 1}\, dA \\ &= \iint_D \left(P\vec{i} + Q\vec{j} + R\vec{k}\right) \bullet \left(-g_x\,\vec{i} - g_y\,\vec{j} + \vec{k}\right) dA \\ &= \iint_D -Pg_x - Qg_y + R\, dA\end{aligned}$$

Now, remember that this assumed the "upward" orientation. If we'd needed the "downward" orientation, then we would need to change the signs on the normal vector. This would in turn change the signs on the integrand as well. So, we really need to be careful here when using this formula. In general, it is best to rederive this formula as you need it.

When we've been given a surface that is not in parametric form there are in fact 6 possible integrals here. Two for each form of the surface $z = g(x, y)$, $y = g(x, z)$ and $x = g(y, z)$. Given each form of the surface there will be two possible unit normal vectors and we'll need to choose the correct one to match the given orientation of the surface. However, the derivation of each formula is similar to that given here and so shouldn't be too bad to do as you need to.

Notice as well that because we are using the unit normal vector the messy square root will always drop out. This means that when we do need to derive the formula we won't really need to put this in. All we'll need to work with is the numerator of the unit vector. We will see at least one more of these derived in the examples below. It should also be noted that the square root is nothing more than,

$$\sqrt{(g_x)^2 + (g_y)^2 + 1} = \|\nabla f\|$$

so in the following work we will probably just use this notation in place of the square root when we can to make things a little simpler.

Let's now take a quick look at the formula for the surface integral when the surface is given parametrically by $\vec{r}\left(u,v\right)$. In this case the surface integral is,

$$\begin{aligned}\iint_S \vec{F}\bullet d\vec{S} &= \iint_S \vec{F}\bullet\vec{n}\,dS \\ &= \iint_D \vec{F}\bullet\left(\frac{\vec{r}_u\times\vec{r}_v}{\left\|\vec{r}_u\times\vec{r}_v\right\|}\right)\left\|\vec{r}_u\times\vec{r}_v\right\|dA \\ &= \iint_D \vec{F}\bullet\left(\vec{r}_u\times\vec{r}_v\right)dA\end{aligned}$$

Again, note that we may have to change the sign on $\vec{r}_u\times\vec{r}_v$ to match the orientation of the surface and so there is once again really two formulas here. Also note that again the magnitude cancels in this case and so we won't need to worry that in these problems either.

Note as well that there are even times when we will use the definition, $\iint_S \vec{F}\bullet d\vec{S} = \iint_S \vec{F}\bullet\vec{n}\,dS$, directly. We will see an example of this below.

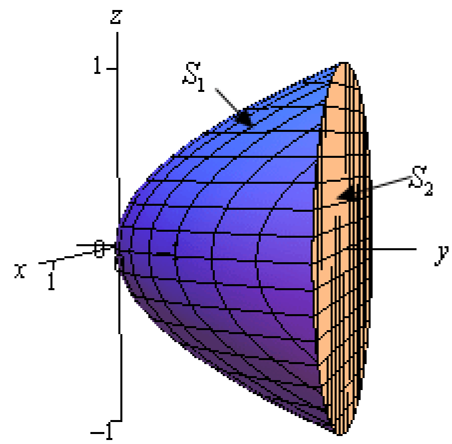
Let's now work a couple of examples.

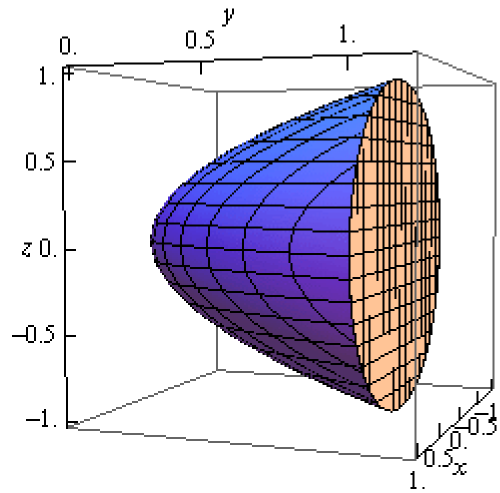
Example 1 Evaluate $\iint_S \vec{F}\bullet d\vec{S}$ where $\vec{F} = y\,\vec{j} - z\,\vec{k}$ and *S* is the surface given by the paraboloid $y = x^2 + z^2$, $0 \le y \le 1$ and the disk $x^2 + z^2 \le 1$ at $y = 1$. Assume that *S* has positive orientation.

Solution

Okay, first let's notice that the disk is really nothing more than the cap on the paraboloid. This means that we have a closed surface. This is important because we've been told that the surface has a positive orientation and by convention this means that all the unit normal vectors will need to point outwards from the region enclosed by *S*.

Let's first get a sketch of *S* so we can get a feel for what is going on and in which direction we will need to unit normal vectors to point.





As noted in the sketch we will denote the paraboloid by S_1 and the disk by S_2. Also note that in order for unit normal vectors on the paraboloid to point away from the region they will all need to point generally in the negative *y* direction. On the other hand, unit normal vectors on the disk will need to point in the positive *y* direction in order to point away from the region.

Since *S* is composed of the two surfaces we'll need to do the surface integral on each and then add the results to get the overall surface integral. Let's start with the paraboloid. In this case we have the surface in the form $y = g(x,z)$ so we will need to derive the correct formula since the one given initially wasn't for this kind of function. This is easy enough to do however. First define,

$$f(x,y,z) = y - g(x,z) = y - x^2 - z^2$$

We will next need the gradient vector of this function.

$$\nabla f = \langle -2x, 1, -2z \rangle$$

Now, the *y* component of the gradient is positive and so this vector will generally point in the positive *y* direction. However, as noted above we need the normal vector point in the negative *y* direction to make sure that it will be pointing away from the enclosed region. This means that we will need to use

$$\vec{n} = \frac{-\nabla f}{\|-\nabla f\|} = \frac{\langle 2x, -1, 2z \rangle}{\|\nabla f\|}$$

Let's note a couple of things here before we proceed. We don't really need to divide this by the magnitude of the gradient since this will just cancel out once we actually do the integral. So, because of this we didn't bother computing it. Also, the dropping of the minus sign is not a typo. When we compute the magnitude we are going to square each of the components and so the minus sign will drop out.

S_1 : The Paraboloid

Okay, here is the surface integral in this case.

$$\iint_{S_1} \vec{F}\bullet d\vec{S} = \iint_D \left(y\,\vec{j} - z\,\vec{k}\right)\bullet\left(\frac{\langle 2x,-1,2z\rangle}{\|\nabla f\|}\right)\|\nabla f\|\,dA$$
$$= \iint_D -y - 2z^2\,dA$$
$$= \iint_D -\left(x^2+z^2\right) - 2z^2\,dA$$
$$= -\iint_D x^2 + 3z^2\,dA$$

Don't forget that we need to plug in the equation of the surface for *y* before we actually compute the integral. In this case *D* is the disk of radius 1 in the *xz*-plane and so it makes sense to use polar coordinates to complete this integral. Here are polar coordinates for this region.

$$x = r\cos\theta \qquad\qquad z = r\sin\theta$$
$$0 \le \theta \le 2\pi \qquad\qquad 0 \le r \le 1$$

Note that we kept the *x* conversion formula the same as the one we are used to using for *x* and let *z* be the formula that used the sine. We could have done it any order, however in this way we are at least working with one of them as we are used to working with.

Here is the evaluation of this integral.

$$\iint_{S_1} \vec{F}\bullet d\vec{S} = -\iint_D x^2 + 3z^2\,dA$$
$$= -\int_0^{2\pi}\int_0^1 \left(r^2\cos^2\theta + 3r^2\sin^2\theta\right) r\,dr\,d\theta$$
$$= -\int_0^{2\pi}\int_0^1 \left(\cos^2\theta + 3\sin^2\theta\right) r^3\,dr\,d\theta$$
$$= -\int_0^{2\pi} \left(\frac{1}{2}\left(1+\cos\left(2\theta\right)\right) + \frac{3}{2}\left(1-\cos\left(2\theta\right)\right)\right)\left(\frac{1}{4}r^4\right)\Bigg|_0^1\,d\theta$$
$$= -\frac{1}{8}\int_0^{2\pi} 4 - 2\cos\left(2\theta\right)d\theta$$
$$= -\frac{1}{8}\left(4\theta - \sin\left(2\theta\right)\right)\Bigg|_0^{2\pi}$$
$$= -\pi$$

S_2 : <u>The Cap of the Paraboloid</u>

We can now do the surface integral on the disk (cap on the paraboloid). This one is actually fairly easy to do and in fact we can use the definition of the surface integral directly. First let's notice that the disk is really just the portion of the plane $y = 1$ that is in front of the disk of radius 1 in the *xz*-plane.

Now we want the unit normal vector to point away from the enclosed region and since it must also be orthogonal to the plane $y = 1$ then it must point in a direction that is parallel to the *y*-axis, but we already have a unit vector that does this. Namely,

$$\vec{n} = \vec{j}$$

the standard unit basis vector. It also points in the correct direction for us to use. Because we have the vector field and the normal vector we can plug directly into the definition of the surface integral to get,

$$\iint\limits_{S_2} \vec{F} \bullet d\vec{S} = \iint\limits_{S_2} \left(y\,\vec{j} - z\,\vec{k} \right) \bullet \left(\vec{j} \right) dS = \iint\limits_{S_2} y\, dS$$

At this point we need to plug in for *y* (since S_2 is a portion of the plane $y = 1$ we do know what it is) and we'll also need the square root this time when we convert the surface integral over to a double integral. In this case since we are using the definition directly we won't get the canceling of the square root that we saw with the first portion. To get the square root well need to acknowledge that

$$y = 1 = g\left(x, z\right)$$

and so the square root is,

$$\sqrt{\left(g_x\right)^2 + 1 + \left(g_z\right)^2}$$

The surface integral is then,

$$\begin{aligned} \iint\limits_{S_2} \vec{F} \bullet d\vec{S} &= \iint\limits_{S_2} y\, dS \\ &= \iint\limits_{D} 1\sqrt{0 + 1 + 0}\, dA = \iint\limits_{D} dA \end{aligned}$$

At this point we can acknowledge that *D* is a disk of radius 1 and this double integral is nothing more than the double integral that will give the area of the region *D* so there is no reason to compute the integral. Here is the value of the surface integral.

$$\iint\limits_{S_2} \vec{F} \bullet d\vec{S} = \pi$$

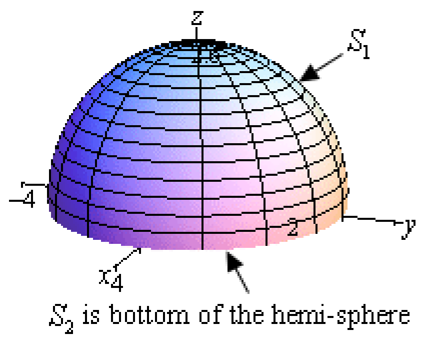
Finally, to finish this off we just need to add the two parts up. Here is the surface integral that we were actually asked to compute.

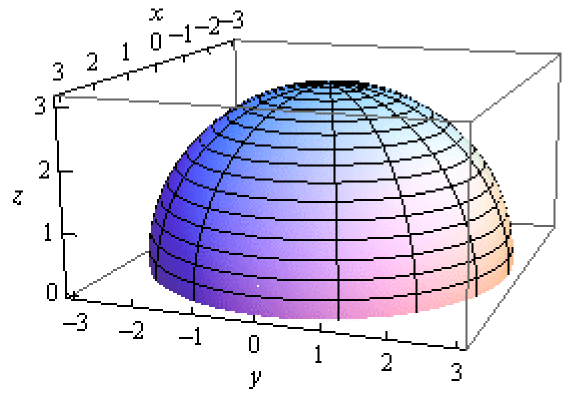
$$\iint\limits_{S} \vec{F} \bullet d\vec{S} = \iint\limits_{S_1} \vec{F} \bullet d\vec{S} + \iint\limits_{S_2} \vec{F} \bullet d\vec{S} = -\pi + \pi = 0$$

Example 2 Evaluate $\iint\limits_{S} \vec{F} \bullet d\vec{S}$ where $\vec{F} = x\,\vec{i} + y\,\vec{j} + z^4\,\vec{k}$ and *S* is the upper half the sphere $x^2 + y^2 + z^2 = 9$ and the disk $x^2 + y^2 \le 9$ in the plane $z = 0$. Assume that *S* has the positive orientation.

Solution

So, as with the previous problem we have a closed surface and since we are also told that the surface has a positive orientation all the unit normal vectors must point away from the enclosed region. To help us visualize this here is a sketch of the surface.





We will call S_1 the hemisphere and S_2 will be the bottom of the hemisphere (which isn't shown on the sketch). Now, in order for the unit normal vectors on the sphere to point away from enclosed region they will all need to have a positive *z* component. Remember that the vector must be normal to the surface and if there is a positive *z* component and the vector is normal it will have to be pointing away from the enclosed region.

On the other hand, the unit normal on the bottom of the disk must point in the negative *z* direction in order to point away from the enclosed region.

S_1 : <u>The Sphere</u>

Let's do the surface integral on S_1 first. In this case since the surface is a sphere we will need to use the parametric representation of the surface. This is,

$$\vec{r}(\theta,\varphi) = 3\sin\varphi\cos\theta\,\vec{i} + 3\sin\varphi\sin\theta\,\vec{j} + 3\cos\varphi\,\vec{k}$$

Since we are working on the hemisphere here are the limits on the parameters that we'll need to use.

$$0 \le \theta \le 2\pi \qquad\qquad 0 \le \varphi \le \frac{\pi}{2}$$

Next, we need to determine $\vec{r}_\theta \times \vec{r}_\varphi$. Here are the two individual vectors and the cross product.

$$\vec{r}_\theta(\theta,\varphi) = -3\sin\varphi\sin\theta\,\vec{i} + 3\sin\varphi\cos\theta\,\vec{j}$$

$$\vec{r}_\varphi(\theta,\varphi) = 3\cos\varphi\cos\theta\,\vec{i} + 3\cos\varphi\sin\theta\,\vec{j} - 3\sin\varphi\,\vec{k}$$

$$\begin{aligned}\vec{r}_\theta \times \vec{r}_\varphi &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3\sin\varphi\sin\theta & 3\sin\varphi\cos\theta & 0 \\ 3\cos\varphi\cos\theta & 3\cos\varphi\sin\theta & -3\sin\varphi \end{vmatrix} \\ &= -9\sin^2\varphi\cos\theta\,\vec{i} - 9\sin\varphi\cos\varphi\sin^2\theta\,\vec{k} - 9\sin^2\varphi\sin\theta\,\vec{j} - 9\sin\varphi\cos\varphi\cos^2\theta\,\vec{k} \\ &= -9\sin^2\varphi\cos\theta\,\vec{i} - 9\sin^2\varphi\sin\theta\,\vec{j} - 9\sin\varphi\cos\varphi\left(\sin^2\theta + \cos^2\theta\right)\vec{k} \\ &= -9\sin^2\varphi\cos\theta\,\vec{i} - 9\sin^2\varphi\sin\theta\,\vec{j} - 9\sin\varphi\cos\varphi\,\vec{k}\end{aligned}$$

Note that we won't need the magnitude of the cross product since that will cancel out once we start doing the integral.

Notice that for the range of φ that we've got both sine and cosine are positive and so this vector will have a negative *z* component and as we noted above in order for this to point away from the enclosed area we will need the *z* component to be positive. Therefore, we will need to use the following vector for the unit normal vector.

$$\vec{n} = -\frac{\vec{r}_\theta \times \vec{r}_\varphi}{\left\|\vec{r}_\theta \times \vec{r}_\varphi\right\|} = \frac{9\sin^2\varphi\cos\theta\,\vec{i} + 9\sin^2\varphi\sin\theta\,\vec{j} + 9\sin\varphi\cos\varphi\,\vec{k}}{\left\|\vec{r}_\theta \times \vec{r}_\varphi\right\|}$$

Again, we will drop the magnitude once we get to actually doing the integral since it will just cancel in the integral.

Okay, next we'll need

$$\vec{F}\left(\vec{r}\left(\theta,\varphi\right)\right) = 3\sin\varphi\cos\theta\,\vec{i} + 3\sin\varphi\sin\theta\,\vec{j} + 81\cos^4\varphi\,\vec{k}$$

Remember that in this evaluation we are just plugging in the *x* component of $\vec{r}\left(\theta,\varphi\right)$ into the vector field *etc*.

We also may as well get the dot product out of the way that we know we are going to need.

$$\begin{aligned}\vec{F}\left(\vec{r}\left(\theta,\varphi\right)\right)\bullet\left(-\vec{r}_\theta \times \vec{r}_\varphi\right) &= 27\sin^3\varphi\cos^2\theta + 27\sin^3\varphi\sin^2\theta + 729\sin\varphi\cos^5\varphi \\ &= 27\sin^3\varphi + 729\sin\varphi\cos^5\varphi\end{aligned}$$

Now we can do the integral.

$$\begin{aligned}\iint\limits_{S_1} \vec{F}\bullet d\vec{S} &= \iint\limits_{D} \vec{F}\bullet\left(\frac{\vec{r}_\theta \times \vec{r}_\varphi}{\left\|\vec{r}_\theta \times \vec{r}_\varphi\right\|}\right)\left\|\vec{r}_\theta \times \vec{r}_\varphi\right\|\,dA \\ &= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} 27\sin^3\varphi + 729\sin\varphi\cos^5\varphi\,d\varphi\,d\theta \\ &= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} 27\sin\varphi\left(1-\cos^2\varphi\right) + 729\sin\varphi\cos^5\varphi\,d\varphi\,d\theta \\ &= -\int_0^{2\pi}\left(27\left(\cos\varphi - \frac{1}{3}\cos^3\varphi\right) + \frac{729}{6}\cos^6\varphi\right)\Bigg|_0^{\frac{\pi}{2}}\,d\theta \\ &= \int_0^{2\pi}\frac{279}{2}\,d\theta \\ &= 279\pi\end{aligned}$$

S_2 : <u>The Bottom of the Hemi-Sphere</u>

Now, we need to do the integral over the bottom of the hemisphere. In this case we are looking at the disk $x^2 + y^2 \le 9$ that lies in the plane $z = 0$ and so the equation of this surface is actually $z = 0$.

The disk is really the region *D* that tells us how much of the surface we are going to use. This also means that we can use the definition of the surface integral here with

$$\vec{n} = -\vec{k}$$

We need the negative since it must point away from the enclosed region.

The surface integral in this case is,

$$\begin{aligned}\iint\limits_{S_2} \vec{F}\bullet d\vec{S} &= \iint\limits_{S_2} \left(x\,\vec{i} + y\,\vec{j} + z^4\,\vec{k}\right)\bullet\left(-\vec{k}\right)dS \\ &= \iint\limits_{S_2} -z^4\,dS\end{aligned}$$

Remember, however, that we are in the plane given by $z = 0$ and so the surface integral becomes,

$$\iint\limits_{S_2} \vec{F}\bullet d\vec{S} = \iint\limits_{S_2} -z^4\,dS = \iint\limits_{S_2} 0\,dS = 0$$

The last step is to then add the two pieces up. Here is surface integral that we were asked to look at.

$$\iint\limits_{S} \vec{F}\bullet d\vec{S} = \iint\limits_{S_1} \vec{F}\bullet d\vec{S} + \iint\limits_{S_2} \vec{F}\bullet d\vec{S} = 279\pi + 0 = 279\pi$$

We will leave this section with a quick interpretation of a surface integral over a vector field. If $\vec{v}$ is the velocity field of a fluid then the surface integral

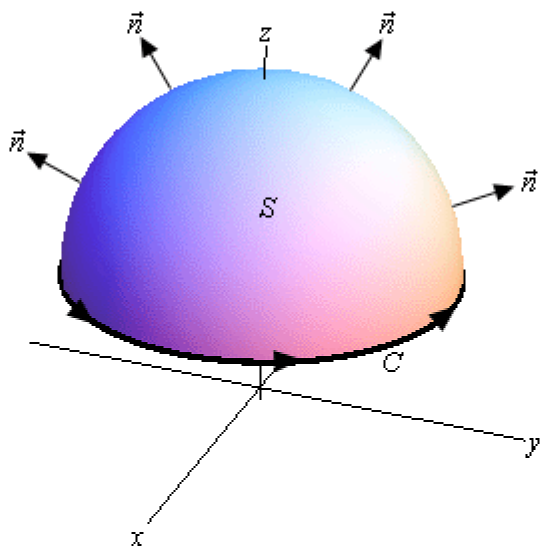
$$\iint\limits_{S} \vec{v}\bullet d\vec{S}$$

represents the volume of fluid flowing through *S* per time unit (*i.e.* per second, per minute, or whatever time unit you are using).

Section 6-5 : Stokes' Theorem

In this section we are going to take a look at a theorem that is a higher dimensional version of Green's Theorem. In Green's Theorem we related a line integral to a double integral over some region. In this section we are going to relate a line integral to a surface integral. However, before we give the theorem we first need to define the curve that we're going to use in the line integral.

Let's start off with the following surface with the indicated orientation.



Around the edge of this surface we have a curve *C*. This curve is called the **boundary curve**. The orientation of the surface *S* will induce the **positive orientation of *C***. To get the positive orientation of *C* think of yourself as walking along the curve. While you are walking along the curve if your head is pointing in the same direction as the unit normal vectors while the surface is on the left then you are walking in the positive direction on *C*.

Now that we have this curve definition out of the way we can give Stokes' Theorem.

Stokes' Theorem

Let *S* be an oriented smooth surface that is bounded by a simple, closed, smooth boundary curve *C* with positive orientation. Also let $\vec{F}$ be a vector field then,

$$\int_C \vec{F} \bullet d\,\vec{r} = \iint_S \operatorname{curl} \vec{F} \bullet d\vec{S}$$

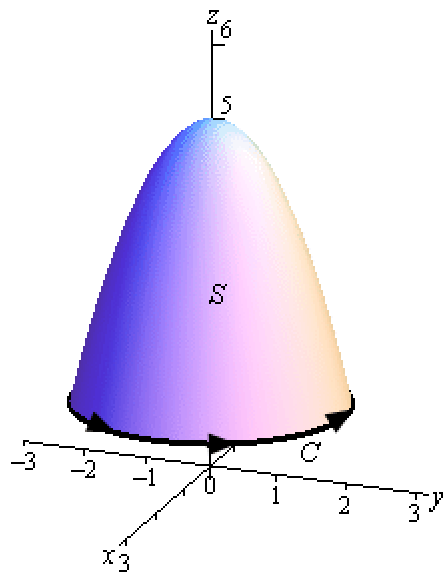
In this theorem note that the surface *S* can actually be any surface so long as its boundary curve is given by *C*. This is something that can be used to our advantage to simplify the surface integral on occasion.

Let's take a look at a couple of examples.

Example 1 Use Stokes' Theorem to evaluate $\iint\limits_S \operatorname{curl}\vec F \bullet d\vec S$ where $\vec F = z^2\,\vec i - 3xy\,\vec j + x^3y^3\,\vec k$ and *S* is the part of $z = 5 - x^2 - y^2$ above the plane $z = 1$. Assume that *S* is oriented upwards.

Solution

Let's start this off with a sketch of the surface.



In this case the boundary curve *C* will be where the surface intersects the plane $z = 1$ and so will be the curve

$$1 = 5 - x^2 - y^2$$
$$x^2 + y^2 = 4 \qquad\qquad \text{at } z = 1$$

So, the boundary curve will be the circle of radius 2 that is in the plane $z = 1$. The parameterization of this curve is,

$$\vec r\left(t\right) = 2\cos t\,\vec i + 2\sin t\,\vec j + \vec k, \quad 0 \le t \le 2\pi$$

The first two components give the circle and the third component makes sure that it is in the plane $z = 1$.

Using Stokes' Theorem we can write the surface integral as the following line integral.

$$\iint\limits_S \operatorname{curl}\vec F \bullet d\vec S = \int\limits_C \vec F \bullet d\,\vec r = \int_0^{2\pi} \vec F\left(\vec r\left(t\right)\right)\bullet \vec r\,'\left(t\right)dt$$

So, it looks like we need a couple of quantities before we do this integral. Let's first get the vector field evaluated on the curve. Remember that this is simply plugging the components of the parameterization into the vector field.

$$\vec{F}\left(\vec{r}\left(t\right)\right) = \left(1\right)^2 \vec{i} - 3\left(2\cos t\right)\left(2\sin t\right)\vec{j} + \left(2\cos t\right)^3\left(2\sin t\right)^3\vec{k}$$
$$= \vec{i} - 12\cos t\sin t\,\vec{j} + 64\cos^3 t\sin^3 t\,\vec{k}$$

Next, we need the derivative of the parameterization and the dot product of this and the vector field.

$$\vec{r}'\left(t\right) = -2\sin t\,\vec{i} + 2\cos t\,\vec{j}$$
$$\vec{F}\left(\vec{r}\left(t\right)\right) \bullet \vec{r}'\left(t\right) = -2\sin t - 24\sin t\cos^2 t$$

We can now do the integral.

$$\iint\limits_S \text{curl}\,\vec{F} \bullet d\vec{S} = \int_0^{2\pi} -2\sin t - 24\sin t\cos^2 t\,dt$$
$$= \left(2\cos t + 8\cos^3 t\right)\Big|_0^{2\pi}$$
$$= 0$$

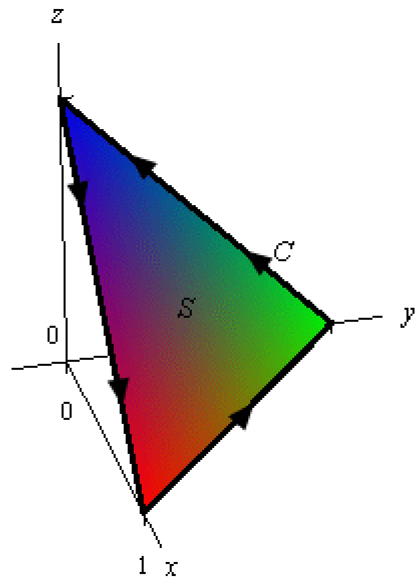
Example 2 Use Stokes' Theorem to evaluate $\int\limits_C \vec{F} \bullet d\,\vec{r}$ where $\vec{F} = z^2\,\vec{i} + y^2\,\vec{j} + x\,\vec{k}$ and *C* is the triangle with vertices $\left(1,0,0\right)$, $\left(0,1,0\right)$ and $\left(0,0,1\right)$ with counter-clockwise rotation.

Solution

We are going to need the curl of the vector field eventually so let's get that out of the way first.

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z^2 & y^2 & x \end{vmatrix} = 2z\,\vec{j} - \vec{j} = \left(2z - 1\right)\vec{j}$$

Now, all we have is the boundary curve for the surface that we'll need to use in the surface integral. However, as noted above all we need is any surface that has this as its boundary curve. So, let's use the following plane with upwards orientation for the surface.



Since the plane is oriented upwards this induces the positive direction on *C* as shown. The equation of this plane is,

$$x + y + z = 1 \qquad \Rightarrow \qquad z = g\left(x, y\right) = 1 - x - y$$

Now, let's use Stokes' Theorem and get the surface integral set up.

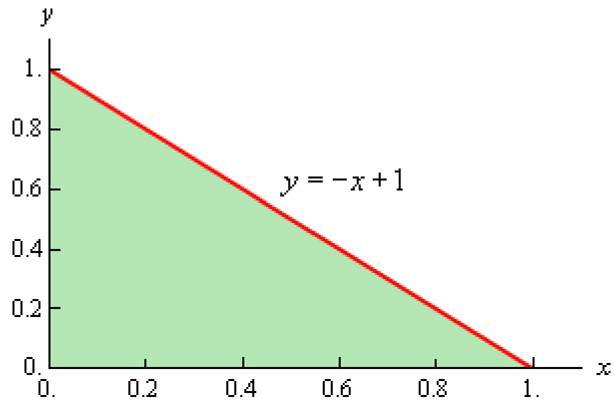
$$\begin{aligned} \int_C \vec{F} \bullet d\,\vec{r} &= \iint_S \text{curl}\,\vec{F} \bullet d\vec{S} \\ &= \iint_S \left(2z - 1\right)\vec{j} \bullet d\vec{S} \\ &= \iint_D \left(2z - 1\right)\vec{j} \bullet \frac{\nabla f}{\left\| \nabla f \right\|} \left\| \nabla f \right\| dA \end{aligned}$$

Okay, we now need to find a couple of quantities. First let's get the gradient. Recall that this comes from the function of the surface.

$$f\left(x, y, z\right) = z - g\left(x, y\right) = z - 1 + x + y$$
$$\nabla f = \vec{i} + \vec{j} + \vec{k}$$

Note as well that this also points upwards and so we have the correct direction.

Now, *D* is the region in the *xy*-plane shown below,



We get the equation of the line by plugging in $z = 0$ into the equation of the plane. So based on this the ranges that define *D* are,

$$0 \le x \le 1 \qquad 0 \le y \le -x+1$$

The integral is then,

$$\int_C \vec{F} \bullet d\,\vec{r} = \iint_D \left(2z-1\right)\vec{j} \bullet \left(\vec{i}+\vec{j}+\vec{k}\right) dA$$
$$= \int_0^1 \int_0^{-x+1} 2\left(1-x-y\right) - 1\, dy\, dx$$

Don't forget to plug in for *z* since we are doing the surface integral on the plane. Finishing this out gives,

$$\int_C \vec{F} \bullet d\,\vec{r} = \int_0^1 \int_0^{-x+1} 1-2x-2y\, dy\, dx$$
$$= \int_0^1 \left(y - 2xy - y^2\right)\Big|_0^{-x+1} dx$$
$$= \int_0^1 x^2 - x\, dx$$
$$= \left(\frac{1}{3}x^3 - \frac{1}{2}x^2\right)\Big|_0^1$$
$$= -\frac{1}{6}$$

In both of these examples we were able to take an integral that would have been somewhat unpleasant to deal with and by the use of Stokes' Theorem we were able to convert it into an integral that wasn't too bad.

Section 6-6 : Divergence Theorem

In this section we are going to relate surface integrals to triple integrals. We will do this with the Divergence Theorem.

Divergence Theorem

Let E be a simple solid region and S is the boundary surface of E with positive orientation. Let $\vec{F}$ be a vector field whose components have continuous first order partial derivatives. Then,

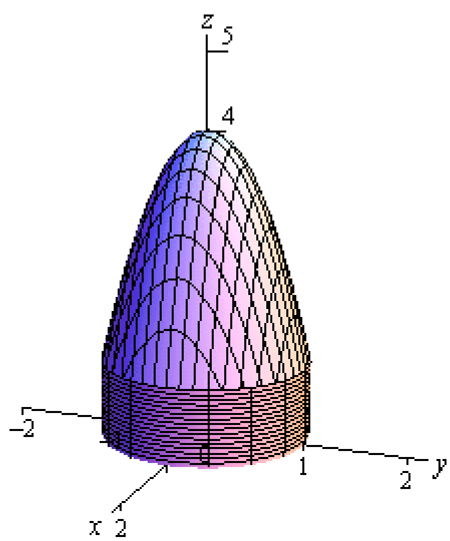
$$\iint\limits_S \vec{F} \bullet d\vec{S} = \iiint\limits_E \operatorname{div} \vec{F} \, dV$$

Let's see an example of how to use this theorem.

Example 1 Use the divergence theorem to evaluate $\iint\limits_S \vec{F} \bullet d\vec{S}$ where $\vec{F} = xy\,\vec{i} - \frac{1}{2}y^2\,\vec{j} + z\,\vec{k}$ and the surface consists of the three surfaces, $z = 4 - 3x^2 - 3y^2$, $1 \le z \le 4$ on the top, $x^2 + y^2 = 1$, $0 \le z \le 1$ on the sides and $z = 0$ on the bottom.

Solution

Let's start this off with a sketch of the surface.



The region E for the triple integral is then the region enclosed by these surfaces. Note that cylindrical coordinates would be a perfect coordinate system for this region. If we do that here are the limits for the ranges.

$$0 \le z \le 4 - 3r^2$$
$$0 \le r \le 1$$
$$0 \le \theta \le 2\pi$$

We'll also need the divergence of the vector field so let's get that.

$$\operatorname{div} \vec{F} = y - y + 1 = 1$$

The integral is then,

$$\begin{aligned}
\iint\limits_S \vec{F} \cdot d\vec{S} &= \iiint\limits_E \operatorname{div} \vec{F}\, dV \\
&= \int_0^{2\pi} \int_0^1 \int_0^{4-3r^2} r\, dz\, dr\, d\theta \\
&= \int_0^{2\pi} \int_0^1 4r - 3r^3\, dr\, d\theta \\
&= \int_0^{2\pi} \left.\left(2r^2 - \frac{3}{4} r^4\right)\right|_0^1 d\theta \\
&= \int_0^{2\pi} \frac{5}{4} d\theta \\
&= \frac{5}{2}\pi
\end{aligned}$$

