Chapter 3

Quadratic curves, quadric surfaces

In this chapter we begin our study of curved surfaces. We focus on the quadric surfaces. To do this, we also need to look at quadratic curves, such as ellipses. We discuss:

- Equations and parametric descriptions of the plane quadratic curves: circles, ellipses, hyperbolas and parabolas.
- Equations and parametric descriptions of quadric surfaces, the 2–dimensional analogues of quadratic curves.

We also discuss aspects of *matrices*, since they are relevant for our discussion.

3.1 Plane quadratic curves

3.1.1 From linear to quadratic equations
Lines in the plane $\mathbf{R}^2$ are represented by linear equations and linear parametric descriptions. Degree 2 equations also correspond to curves you undoubtedly have come across before: circles, ellipses, hyperbolas and parabolas. This section is devoted to these curves. They will reoccur when we consider quadric surfaces, a class of fascinating shapes, since the intersection of a quadric surface with a plane consists of a quadratic curve. Lines differ from quadratic curves in various respects, one of which is that all lines look the same (only their position in the plane may differ), but that quadratic curves may truely differ in shape.

3.1.2 The general equation of a quadratic curve
The general equation of a line in $\mathbf{R}^2$ is $ax + by = c$. When we also allow terms of degree 2 in the variables x and y, i.e., x^2, xy and y^2, we obtain quadratic equations like

- $x^2 + y^2 = 1$, a circle.
- $x^2 + 2x + y = 3$, a parabola; probably you recognize it as such if it is rewritten in the form $y = 3 - x^2 - 2x$ or $y = -(x + 1)^2 + 4$.
- $x^2 - y^2 + 3x + 2y = 1$, a hyperbola.

Equations like $2x^3 - 5y^2 = 6$ or $\sin^2 x - y^2 = y$ are not quadratic.

The general equation of degree 2 in two variables x and y looks like

$$Ax^2 + Bxy + Cy^2 + Dx + Ey = F,$$

where A, B, etc., are the coefficients of the equation. From the examples just given, you can already conclude that quadratic curves tend to differ in appearance depending on the equations. This phenemenon implies that the equation of a given quadratic equation needs further investigation before you can tell the shape of the corresponding curve.

Below, we discuss the various types of quadratic curves. If such a curve is positioned nicely relative the coordinate system, its equation is relatively simple. In the list below we will use these so–called standard forms of the equations. First, we briefly discuss rotations and how to handle quadratic equations.

3.1.3 Intermezzo: Rotations around the origin in $\mathbf{R}^2$

Suppose we rotate the vector $\mathbf{x} = (x_1, x_2)$ counterclockwise over an angle of α (radians, say). Then what new vector do we get in terms of α, x_1 and x_2? To answer this question, we decompose $\mathbf{x}$ in its horizontal and vertical component using the standard basis vectors $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$ of Chapter 2 and rotate each of the two components. So we first write

$$\mathbf{x} = x_1\,\mathbf{e}_1 + x_2\,\mathbf{e}_2.$$

From Figure 3.1 we infer that $x_1\,\mathbf{e}_1$ transforms into $(x_1 \cos\alpha, x_1 \sin\alpha)$. Likewise, the vec-

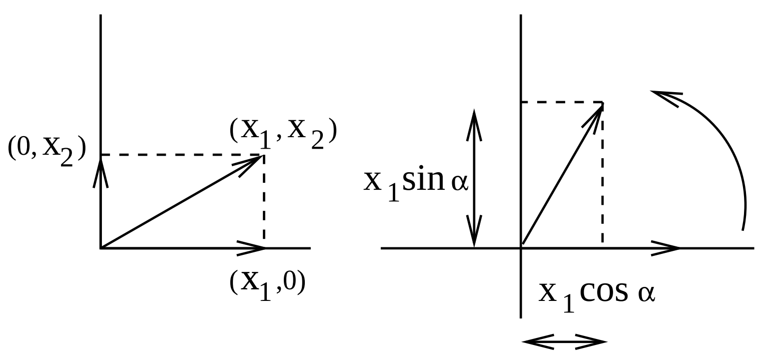

Figure 3.1: *To compute what happens to the coordinates of a vector* (x_1, x_2) *when we rotate it, we first decompose the vector in a horizontal and a vertical component (left). Then we rotate each of these components individually. This is illustrated for the component* $(x_1, 0) = x_1\mathbf{e}_1$. *This component rotates to* $(x_1 \cos\alpha, x_1 \sin\alpha)$ *(right). Finally, we add the results for the two components.*

tor $x_2\,\mathbf{e}_2$ is transformed into $(-x_2 \sin\alpha, x_2 \cos\alpha)$. Altogether this implies that (x_1, x_2) transforms into the sum of $(x_1 \cos\alpha, x_1 \sin\alpha)$ and $(-x_2 \sin\alpha, x_2 \cos\alpha)$:

$$(x_1 \cos\alpha - x_2 \sin\alpha, x_1 \sin\alpha + x_2 \cos\alpha).$$

It turns out to be convenient to rewrite this expression using matrices and the column form of vectors. Here is the rewritten expression:

$$\begin{pmatrix} x_1\cos\alpha - x_2\sin\alpha \\ x_1\sin\alpha + x_2\cos\alpha \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

with a matrix product on the right–hand side (see below). Here are the details:

- A *matrix* is a rectangular array of (possibly variable) numbers surrounded by a pair of brackets. It is a way to store (mathematical) information, which has proven its usefulness. In a way it is comparable to a table for collecting data. A *matrix* contains a number of rows and a number of columns. An r by s matrix is a matrix with r rows and s columns. Here is an example of a 2 by 3 matrix:

$$\begin{pmatrix} 2 & x^2 & -5\sin\alpha \\ 0 & -2 & t \end{pmatrix}$$

  The i, j–th element of the matrix is the element which is in the i–th row and in the j–th column. In the example, the 1, 3–th element is $-5\sin\alpha$.

- The left–hand side contains the vector $(x_1\cos\alpha - x_2\sin\alpha, x_1\sin\alpha + x_2\cos\alpha)$ written in column form. This is a special case of a matrix: a 2 by 1 matrix.

- The right–hand side contains the *product of two matrices*: a 2 by 2 matrix containing cosines and sines and a 2 by 1 matrix containing the column form of the vector (x_1, x_2). The result of the multiplication is the 2 by 1 matrix (vector in column form) on the left–hand side. The matrix multiplication works as follows. The product is a 2 by 1 matrix whose first entry is obtained by taking the (2d) inner product of the first row and the vector (x_1, x_2), i.e.,

$$(\cos\alpha, -\sin\alpha) \bullet (x_1, x_2) = x_1\cos\alpha - x_2\sin\alpha.$$

  The second entry is obtained similarly, by taking the inner product of the second row of the 2 by 2 matrix and the vector (x_1, x_2):

$$(\sin\alpha, \cos\alpha) \bullet (x_1, x_2) = x_1\sin\alpha + x_2\cos\alpha.$$

So the rotation over α (radians in our case) is encoded by the 2 by 2 matrix

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

in the sense that multiplying this matrix with the column form of the vector (x_1, x_2) produces the column form of the rotated vector. The way to remember this rotation matrix is the following:

- The first column contains the result of rotating $(1, 0)$ over α radians.
- The second column contains the result of rotating $(0, 1)$ over α radians.

Here are a few examples.

- The matrix for a rotation over $\pi/6$ or 30° is

$$\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

To rotate the vector $(4, 2)$, we multiply as follows

$$\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \cdot \frac{\sqrt{3}}{2} - 2 \cdot \frac{1}{2} \\ 4 \cdot \frac{1}{2} + 2 \cdot \frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} 2\sqrt{3} - 1 \\ 2 + \sqrt{3} \end{pmatrix}.$$

So $(4, 2)$ transforms into $(2\sqrt{3} - 1, 2 + \sqrt{3})$.

- The matrix describing a rotation over $-\alpha$ is $\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$.

We will have more to say on matrices and their use later.

3.1.4 A brief word on handling quadratic equations

We briefly address the question of how to handle a quadratic equation in order to get some idea of what shape the equation represents. There are two types of operations which are vital.

- **Splitting off squares**
Whenever an equation contains a square and a linear term in the same variable, like $2x^2 - 12x$, these two terms can be rewritten as follows:

$$2x^2 - 12x = 2(x^2 - 6x) = 2((x - 3)^2 - 9) = 2(x - 3)^2 - 18.$$

This technique is known as *spliting off a square.*. Using this technique, the equation

$$x^2 + 6x + y^2 + 4y = 23$$

can be rewritten as $(x + 3)^2 - 9 + (y + 2)^2 - 4 = 23$, and finally as

$$(x + 3)^2 + (y + 2)^2 = 36.$$

This represents a circle with center $(-3, -2)$ and radius 6 (see below for more on the circle). As you see, the technique of splitting off a square is related to translations.

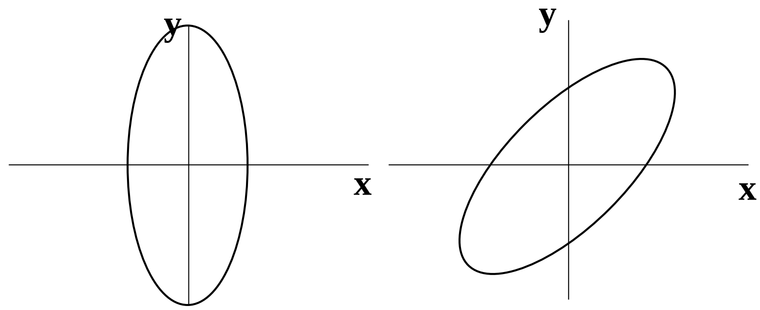

Figure 3.2: *The left–hand ellipse turns out to have a relatively simple equation without a mixed term xy. The right–hand side is an ellipse whose equation does contain a mixed term xy. By applying a suitable rotation around the origin, in this case over $45°$, the equation of the rotated figure simplifies and can then be recognized as an ellipse.*

- **Getting rid of a mixed term xy**
  Whenever a quadratic equation contains (a multiple of) the mixed product xy, a suitable rotation can be used to get rid of this term. For example, consider the equation $x^2 + xy + y^2 = 9$. Suppose we rotate the corresponding figure over an angle of $-\alpha$ radians. If (u, v) is on the rotated figure, then rotating it over α radians produces a point on $x^2+xy+y^2 = 9$. By the intermezzo, the rotation changes (u, v) into

  $$(u\cos\alpha - v\sin\alpha, u\sin\alpha + v\cos\alpha).$$

  The problem is to find α so that the equation satisfied by this point is 'simple'. So we proceed to substitute the coordinates in the equation:

  $$(u\cos\alpha - v\sin\alpha)^2 + (u\cos\alpha - v\sin\alpha)(u\sin\alpha + v\cos\alpha) + (u\sin\alpha + v\cos\alpha)^2 = 9.$$

  Expanding the expressions on the left–hand side and using the identity $\cos^2\alpha + \sin^2\alpha = 1$ yields the equation:

  $$(1 + \sin\alpha\cos\alpha)u^2 + (1 - \sin\alpha\cos\alpha)v^2 + uv(\cos^2\alpha - \sin^2\alpha) = 9.$$

  The term $uv(\cos^2\alpha - \sin^2\alpha)$ on the left–hand side is the crucial one: if we choose α in such a way that $\sin^2\alpha = \cos^2\alpha$, the whole term vanishes! This is, for example, the case if we choose $\alpha = \pi/4$ radians (or $45°$), since then $\cos\alpha = \sin\alpha = \dfrac{\sqrt{2}}{2}$. The equation reduces to

  $$\frac{3}{2}u^2 + \frac{1}{2}v^2 = 9 \text{ or } 3u^2 + v^2 = 18.$$

  Below you will learn that this equation describes an ellipse.

3.1.5 Listing all types of quadratic curves

Using techniques like the above, one can unravel the structure of any quadratic curve. These are roughly the steps.

a) First remove, by a suitable rotation, a mixed term cxy.

b) In the next step, the linear terms can be dealt with. Two cases need to be distinguished. If in the equation an expression occurs like $ax^2 + bx$ (with $a \neq 0$), then a square can be split off. If a variable, say x, occurs in a linear term, but x^2 does not occur in the equation, then we can not split off a square, and the linear term can not be removed by a suitable translation. For instance, the equation $x^2+2x-2y = 3$ can be rewritten as $2y = (x+1)^2 - 4$. The term $2y$ 'survives' so to speak. The equation represents a parabola.

In the end, one is left with an equation containing x^2, y^2 or both, containing at most one linear term, and a constant. For instance, $2x^2-y^2 = 1$ or $x^2-y = 0$. The list is then found by enumerating the possibilities: two squares with positive coefficients (like $2x^2+3y^2 = 1$), two squares precisely one of whose coefficients is positive (like $x^2 - 2y^2 = 2$), one square and a linear term (like $x^2 - 2y = 5$). Here are the details.

The circle

3.1.6 The circle

The circle with center $(0, 0)$ and radius r has the well–known equation $x^2 + y^2 = r^2$. If you translate the circle over the vector $\mathbf{t} = (a, b)$, then the new cirle will have equation $(x - a)^2 + (y - b)^2 = r^2$. This can be seen by using the method that we introduced in Section 2.6 of Chapter 2: if (u, v) is on the translated circle, then $(u - a, v - b)$ is on the circle $x^2 + y^2 = r^2$. So $(u - a, v - b)$ satisfies

$$(u - a)^2 + (v - b)^2 = r^2.$$

The circle is an extremely symmetric curve: rotating it over any angle around its center

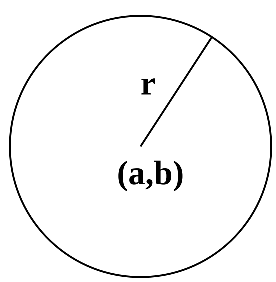

Figure 3.3: *Circle with center (a, b) and radius r.*

produces a copy that coincides with the original circle. Only individual points have actually moved to a new position.

In geometric terms, a circle consists of all points with a fixed distance to a given point. In vector terms, the circle with center $\mathbf{p}$ and radius r is described by $|\mathbf{x} - \mathbf{p}| = r$.

The ellipse

3.1.7 The standard equation of an ellipse

The circle is a special kind of ellipse. The standard form of the equation of an ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\,,$$

where a and b are non–zero constants. By convention, these are taken to be positive. If $a=b$, then the ellipse is a circle with radius a. You could view the ellipse as a kind of circle which is being stretched in the x– and y–direction. The numbers a and b determine the amount of stretching.

You may wonder why people prefer this standard form over, for instance, $a^2x^2+b^2y^2=1$. One reason is that with the first equation, the points of intersection with the x–axis and the y–axis look relatively simple: the points of intersection with the x–axis are $(a,0)$ and $(-a,0)$; the points of intersection with the y–axis are $(0,b)$ and $(0,-b)$. (If $a>b$, then the segment connecting $(-a,0)$ and $(a,0)$ is called the *long axis* of the ellipse, and the segment connecting $(-b,0)$ and $(b,0)$ the *short axis*). Anyway, the equation $a^2x^2+b^2y^2=1$ describes an ellipse just as well.

3.1.8 The geometric description of an ellipse

An ellipse can also be described by a geometric property, quite similar to the geometric description of a circle. However, the description is more involved. Here is this geometric

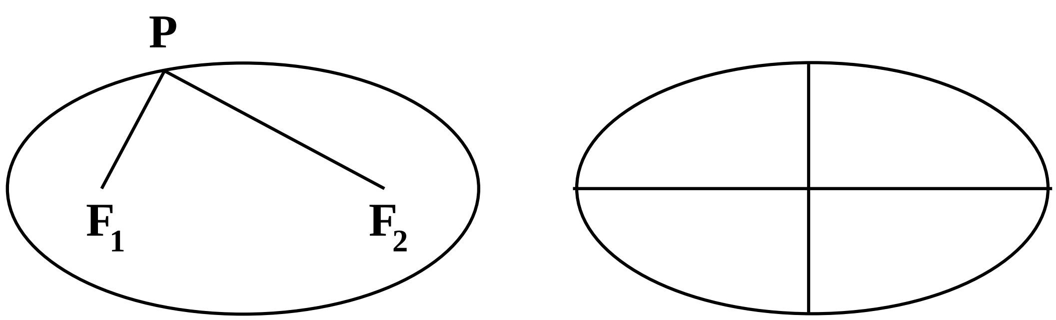

Figure 3.4: *Ellipse with foci F_1 en F_2 (left). The points on the ellipse, like P, satisfy $|PF_1|+|PF_2|=r$, for some constant $r>|F_1F_2|$. The figure on the right–hand side illustrates the long and short axes of the ellipse. Both axes are (if $a\neq b$ the only) axes of symmetry of the ellipse.*

description of an ellipse. Take two points in the plane, say F_1 and F_2. Fix a positive number r which is greater than the distance between F_1 and F_2. The set of points P satisfying

$$|PF_1|+|PF_2|=r$$

is called an *ellipse* with *foci* F_1 and F_2.

Here is how you get from this description to the (standard form) of the equation of an ellipse. Take the x–axis through the foci, and take the perpendicular bisector of F_1F_2 as y–axis. Suppose $F_1 = (-c, 0)$ and $F_2 = (c, 0)$. We write the defining property as

$$|PF_1| + |PF_2| = 2a,$$

where a is a positive constant with $a > c$ (the factor 2 is chosen so that we arrive at the standard form in the end; just wait and see). Using Pythagoras' Theorem in the triangles

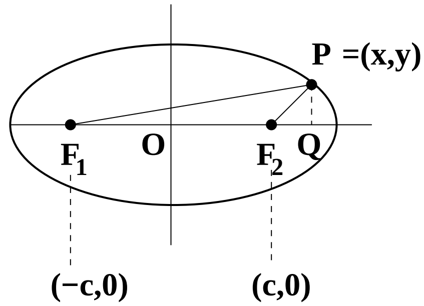

Figure 3.5: *Analysing the condition* $|PF_1|+|PF_2| = 2a$*. Use Pythagoras' theorem in the triangles* $\triangle PF_1Q$ *and* $\triangle PF_2Q$*. For instance,* $|PF_1| = \sqrt{|F_1Q|^2 + |PQ|^2} = \sqrt{(x+c)^2 + y^2}$.

$\triangle PF_1Q$ and $\triangle PF_2Q$ the equality $|PF_1| + |PF_2| = 2a$ becomes

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a.$$

Here are the steps that lead you to the standard equation.

- Bring one of the square roots to the right–hand side and square both sides of the resulting equation:

$$(x-c)^2 + y^2 = (2a - \sqrt{(x+c)^2 + y^2})^2.$$

- Expand the right–hand side and bring all terms except the term $-4a\sqrt{(x+c)^2 + y^2}$ to the left–hand side. Square both sides of the resulting equation.

- A bit of rewriting, which we leave out here, produces the equation

$$(a^2 - c^2)\,x^2 + a^2\,y^2 = a^2\,(a^2 - c^2).$$

  If we take $b = \sqrt{a^2 - c^2}$, then the equation becomes $b^2x^2 + a^2y^2 = a^2b^2$. Dividing both sides by a^2b^2 finally yields

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,.$$

3.1.9 Translating and rotating ellipses

The effect of a translation over $\mathbf{t} = (r, s)$ of the ellipse with equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is an ellipse whose equation is

$$\frac{(x-r)^2}{a^2} + \frac{(y-s)^2}{b^2} = 1\,,$$

as can be verified just like in the case of a circle.

The effect of a rotation is more involved: for most rotations, the result is an equation which is not easily recognized anymore as the equation of an ellipse. Here is one example. Rotate the ellipse over 90° in the positive direction, i.e., every point (x, y) is transformed into $(-y, x)$. If (u, v) is on the rotated ellipse, then rotate it back over 90°. The resulting point is $(v, -u)$ and lies on the ellipse we started with, so satisfies the equation

$$\frac{v^2}{a^2} + \frac{(-u)^2}{b^2} = 1.$$

So the rotated ellipse has equation

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1,$$

i.e., the roles of a and b have been interchanged.

The parabola

3.1.10 Parabolas

The standard form of the equation for a parabola is

$$y = ax^2,$$

where a is a non–zero constant. There is also a geometric description, which starts from a point F (the *focus* in the plane, and a line ℓ not through F. A parabola with focus F and *directrix* ℓ consists of the points P in the plane satisfying

$$|PF| = \text{distance of } P \text{ to } \ell.$$

Like in the case of an ellipse, the equation of a parabola can be derived from this geometric description.

If a mirror is designed in the shape of a parabola, then light rays are reflected through the focus.

The hyperbola

3.1.11 Hyperbolas

The standard equation of a hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\,,$$

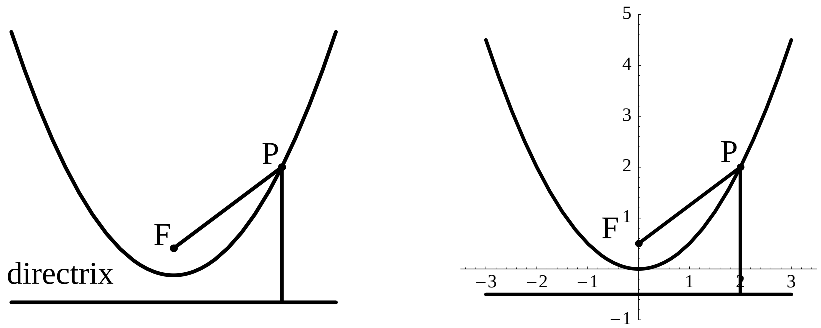

Figure 3.6: *The defining property of a parabola (left): the distance of a point P on the parabola to F is equal to its distance to the directrix. For focus $F = (0, 1/2)$ and direction line $y = -1/2$, the points P on the parabola have equal distance to F and to the line. This translates into $\sqrt{x^2 + (y - 1/2)^2} = |y + 1/2|$. Simplifying yields the equation $y = x^2/2$ (right).*

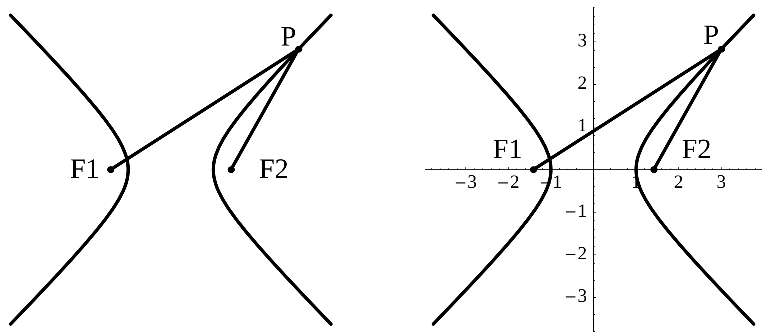

Figure 3.7: *The defining property of a hyperbola (left): the absolute value of the difference of the distances of a point P on the curve to the foci F_1 and F_2 is constant. In the case $F_1 = (-\sqrt{2}, 0)$, $F_2 = (\sqrt{2}, 0)$, the equation $||PF_1| - |PF_2|| = 2$ translates into $x^2 - y^2 = 1$ (right).*

where a and b are, by convention, positive constants. The hyperbola is special in the sense that it consists of two separate branches.

Again there is a geometric definition of a hyperbola, this time involving two foci F_1 and F_2. Given these points and a positive constant d, the set of points P such that the absolute value of the difference of the distances $|PF_1|$ and $|PF_2|$ is equal to d, i.e.,

$$||PF_1| - |PF_2|| = d$$

is by definition a *hyperbola*. Just as in the case of an ellipse or a parabola, an equation of a hyperbola can be derived from this definition. In fact, if the foci are $(-\sqrt{a^2 + b^2}, 0)$ and $(\sqrt{a^2 + b^2}, 0)$, and the constant is $2a$ (with a and b positive), then it can be shown that the equation of the corresponding hyperbola is the standard equation $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

Remaining types: degenerate quadratic curves

3.1.12 'Degenerate' quadratic curves
Strictly speaking there are two more types of quadratic curves:

a) A pair of distinct lines. This occurs when the quadratic expression $q(x, y)$ in the equation $q(x, y) = 0$ can be written as a product of two linear factors: $q(x, y) = l(x, y)m(x, y)$. For example, $x^2 - y^2 = 0$ is such an equation. It can be rewritten as

$$(x + y)(x - y) = 0.$$

So (x, y) satisfies this equation precisely if it satisfies $x + y = 0$ (a line) or $x - y = 0$ (another line). Again, it may not always be clear from a given quadratic equation, whether it can be written as a product of two linear factors.

b) The second case is even more special: suppose $q(x, y)$ in the equation $q(x, y) = 0$ is actually a square of a linear factor, say $q(x, y) = l^2(x, y)$. Then $q(x, y) = 0$ is equivalent with $l(x, y) = 0$. So the equation represents a line. (Mathematicians say 'a line with multiplicity 2'.)

3.2 Parametrizing quadratic curves

3.2.1 Parametric descriptions versus equations
In Chapter 2 we described planes in 3–space in two ways: with equations and with parametric descriptions. The first way defines the points on a plane in an implicit way (you have to solve the equation to get the actual coordinates of points), the second way provides you with explicit coordinates of the points, once you substitute actual values for the parameters. Both approaches have their advantages. In this section we discuss parametric descriptions of quadratic curves. Although the various types of curves admit more parametric descriptions, the best known parametric descriptions are derived from the two identities $\cos^2 u + \sin^2 u = 1$ and $\cosh^2 u - \sinh^2 u = 1$.

3.2.2 Parametrizing circles and ellipses
A useful parametric description of the circle $x_1^2 + x_2^2 = r^2$ (with $r > 0$) is based on the identity $\cos^2 \phi + \sin^2 \phi = 1$. Here it is:

$$\begin{aligned} x_1 &= r\cos\phi \\ x_2 &= r\sin\phi. \end{aligned}$$

The parameter ϕ has a clear geometric meaning: it is the angle between the vectors $(1, 0)$ and $\mathbf{x} = (x_1, x_2)$, and runs, say, from 0 to 2π. For instance, the circle $x_1^2 + x_2^2 = 16$ has parametric description $(4\cos\phi, 4\sin\phi)$. For $\phi = 30°$ we get the point

$$(4 \cdot \frac{\sqrt{3}}{2}, 4 \cdot \frac{1}{2}) = (2\sqrt{3}, 2).$$

To deal with the ellipse with equation $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$, we note that a small change of the parametric description of the circle works:

$$\begin{aligned} x_1 &= a\,\cos\phi \\ x_2 &= b\,\sin\phi, \end{aligned}$$

with ϕ ranging from, say, 0 to 2π. Here is how you check that this parametric description really satisfies the equation:

$$\frac{(a\cos\phi)^2}{a^2} + \frac{(b\sin\phi)^2}{b^2} = \frac{a^2\cos^2\phi}{a^2} + \frac{b^2\sin^2\phi}{b^2} = \cos^2\phi + \sin^2\phi = 1.$$

In the parametric description $x_1 = a\cos\phi$, $x_2 = b\sin\phi$, the parameter ϕ does not have such a clear geometric meaning as in the case of a circle.

Some other ways to parametrize circles and ellipses are discussed in the exercises.

3.2.3 Parametrizing hyperbolas

Let us first look at the equation $x_1^2 - x_2^2 = 1$, representing a relatively simple hyperbola. This equation contains a difference of two squares. In parametrizing a sum of two squares we used the cosine and the sine, because the sum of their squares is 1. Now maybe you recall from previous mathematics courses that there also exist two functions such that the difference of their squares equals 1, the hyperbolic cosine and the hyperbolic sine: $\cosh^2 u - \sinh^2 u = 1$. This leads to the following parametric description for the hyperbola $x_1^2 - x_2^2 = 1$:

$$\begin{aligned} x_1 &= \cosh u \\ x_2 &= \sinh u, \end{aligned}$$

where u runs through the real numbers. This parametric description is easily adapted to one for the standard equation $\frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} = 1$:

$$\begin{aligned} x_1 &= a\,\cosh u \\ x_2 &= b\,\sinh u \end{aligned}$$

Unfortunately, these parametric descriptions only catch one half of the hyperbola: since $\cosh u > 0$ for all u, no points of the left–hand branch of the hyperbola are parametrized. For the hyperbola, we need a separate parametric description for the left–hand branch, for instance:

$$\begin{aligned} x_1 &= -\cosh u \\ x_2 &= \sinh u, \end{aligned}$$

where u run through the real numbers.

3.2.4 Parametrizing parabolas

The standard parabola $x_2 = x_1^2$ is of the form $y = f(x)$, where $f : \mathbf{R} \to \mathbf{R}$ is the function with $f(x) = x^2$. So, it is easy to parametrize the parabola as follows:

$$\begin{aligned} x_1 &= u \\ x_2 &= u^2. \end{aligned}$$

3.3 Quadric surfaces

3.3.1 Quadratic equations and quadric surfaces

Planes are examples of flat objects, and they are represented by linear equations in a cartesian coordinate system. In such a coordinate system, curved surfaces are represented by nonlinear equations. The simplest among them are the *quadric surfaces* or quadratic surfaces, whose equations have so–called degree 2: they involve non–linear terms like x^2 or xz, but each term contains at most the product of two variables. A few examples of such equations are:

- $x^2 + y^2 + z^2 = 1$, which you probably recognize as a sphere of radius 1,
- $z^2 = x^2 + y^2$,
- $xz - y^2 + y = 0$.

The general degree 2 equation in three variables is

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0,$$

where A, B, $C, \ldots, J$ are (ten) constants. These surfaces generalize the plane quadratic curves, like the circle, ellipse, parabola and the hyperbola. Quadratic equations are more complicated than linear equations, but they represent a great variety of interesting shapes, which occur in the design of various structures, ranging from towers to antennas for astronomical purposes. One way to see that quadric surfaces are a higher dimensional analogue of quadratic curves, is to intersect a quadric surface with a plane, like $z = 3$, to keep things simple. The resulting curve of intersection is clearly a quadratic curve: just substitute 3 for z in the equation and a quadratic equation in two variables appears. (A minor detail: in some cases the resulting curve is just a line. Do you see why this possibility may occur?)

3.3.2 Example. The equation $z = x^2 + y^2$ defines a quadric surface. One way to obtain some

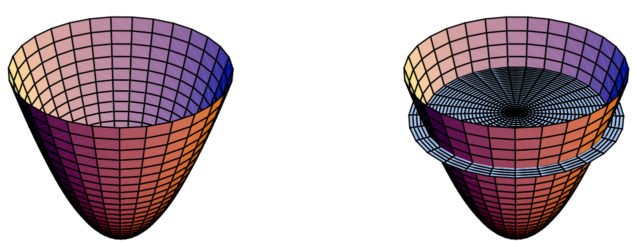

Figure 3.8: *Part of the quadric $z = x^2 + y^2$ (left). On the right the intersection with a horizontal plane is sketched.*

feeling for the shape of the surface is to intersect it with planes. Here are a few examples:

- Intersecting with the plane $y = 0$ produces $z = x^2$, $y = 0$, a parabola in the plane $y = 0$.
- Intersecting with the plane $z = 1$ produces a circle with radius 1:
$$x^2 + y^2 = 1, \quad z = 1.$$
- Intersecting with planes like $x = z$ produces curves which are more difficult to analyse. In the case at hand: $x = x^2 + y^2$ and $x = z$ doesn't help much in understanding what the curve looks like.

3.3.3 Example. The quadric surface $z^2 = x^2 + y^2$ has a sharp peak at the origin. Here are a

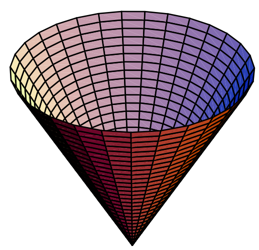

Figure 3.9: *Part of the quadric $z^2 = 4x^2 + 4y^2$, where $z \geq 0$, $x^2 + y^2 \leq 4$.*

few relatively simple intersections:

- The intersection with the plane $y = 0$ produces the equations $4z^2 = x^2$ and $y = 0$. Since $4z^2 - x^2 = (2z + x)(2z - x)$, the intersection consists of two lines, both in the plane $y = 0$, one with equation $2z + x = 0$ and one with equation $2z - x = 0$.
- The intersection with the plane $z = 1$ is a circle with radius 2.

3.3.4 Types of quadrics

By translating and rotating, any plane can be moved in 3–space to any other plane. In particular, this means that, geometrically, all planes are essentially the same, only their position in space may differ. For quadrics the situation is different: their are various types of quadrics, which really differ in shape; even quadrics which roughly look the same may differ on a more detailed level. The examples above already gave an indication of the variety of shapes among quadrics. Below we discuss most of these types, though not in every detail. The algebraic derivation of all these types requires techniques which are beyond the scope of these lectures. Also note that we describe the various types by relatively simple equations ('equations in standard form'): depending on the position of the quadric relative to the coordinate system, the complexity of the equation changes. Before we list the various types, we briefly discuss rotations around the coordinate axes in $\mathbf{R}^3$.

3.3.5 Intermezzo: Rotations around coordinate axes in $\mathbf{R}^3$

A rotation of $\mathbf{x} = (x_1, x_2, x_3)$ around the z–axis over α radians does not affect the third coordinate, x_3. But the change in x_1 and x_2 is like the change in rotating (x_1, x_2) in the x_1, x_2–plane around the origin: so (x_1, x_2, x_3) is rotated to $(x_1 \cos\alpha - x_2 \sin\alpha, x_1 \sin\alpha + x_2 \cos\alpha)$. As in the planar case, it turns out to be more convenient to use matrices. So we write the vectors in column form (a 3 by 1 matrix) and use a matrix product to describe the rotation:

$$\begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \cos\alpha - x_2 \sin\alpha \\ x_1 \sin\alpha + x_2 \cos\alpha \\ x_3 \end{pmatrix}$$

The left–hand side contains a 3 by 3 matrix containing all information regarding the rotation. Similar to the 2d–case, the first column contains the result of rotating $\mathbf{e}_1 = (1, 0, 0)$, the second column the result of rotating $\mathbf{e}_2 = (0, 1, 0)$ and the third column the result of rotating $\mathbf{e}_3 = (0, 0, 1)$. Here is how the matrix product works:

- The product of a 3 by 3 matrix and a 3 by 1 matrix (like a vector in column form) is a 3 by 1 matrix (which appears on the right–hand side of the =–sign).

- The first entry of the matrix product is obtained by taking the inner product of the first row of the 3 by 3 matrix and the vector. In our case:

  $$(\cos\alpha, -\sin\alpha, 0) \bullet (x_1, x_2, x_3) = x_1 \cos\alpha - x_2 \sin\alpha.$$

  Likewise, the second entry is obtained by taking the inner product of the second row of the 3 by 3 matrix and the vector:

  $$(\sin\alpha, \cos\alpha, 0) \bullet (x_1, x_2, x_3) = x_1 \sin\alpha + x_2 \cos\alpha.$$

  Finally, the inner product of the third row and the vector yields the third entry: $(0, 0, 1) \bullet (x_1, x_2, x_3) = x_3$.

Rotations around the other coordinate axes can be dealt with in a similar manner. For instance, a rotation around the x_1–axis over α radians leaves the first coordinate of $\mathbf{x} = (x_1, x_2, x_3)$ fixed but changes the second and third coordinates: $\mathbf{x} \mapsto (x_1, x_2 \cos\alpha - x_3 \sin\alpha, x_2 \sin\alpha + x_3 \cos\alpha)$. In matrix form, this becomes:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \cos\alpha - x_3 \sin\alpha \\ x_2 \sin\alpha + x_3 \cos\alpha \end{pmatrix}$$

For instance, the second entry on the right–hand side is obtained as the inner product $(0, \cos\alpha, -\sin\alpha) \bullet (x_1, x_2, x_3) = x_2 \cos\alpha - x_3 \sin\alpha$.

3.3.6 Spheres

Probably the most familiar quadric surface is the *sphere*, the collection of points or vectors at a fixed distance from a fixed point or vector (the center). In cartesian coordinates, a sphere with center at $(0,0,0)$ and radius r has equation

$$x^2 + y^2 + z^2 = r^2.$$

The sphere is extremely symmetric: no matter how you rotate the sphere around its center, the result will always coincide with the original sphere. But if you move the sphere so that its center is at (a,b,c), then the equation does change of course. If (u,v,w) is on the new sphere, then $(u-a, v-b, w-c)$ is on $x^2+y^2+z^2=r^2$, so that (u,v,w) satisfies

$$(u-a)^2 + (v-b)^2 + (w-c)^2 = r^2.$$

In conclusion, the equation of the sphere with center at (a,b,c) and radius r is $(x-a)^2 +$

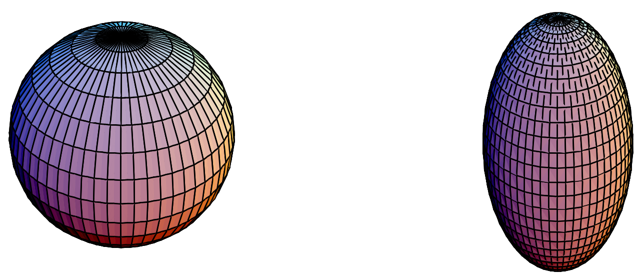

Figure 3.10: *A sphere and an ellipsoid.*

$(y-b)^2 + (z-c)^2 = r^2$ (note the minus signs!).

3.3.7 Ellipsoids

A sphere is a special kind of ellipsoid, more or less like a circle is a special kind of ellipse. The equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

(by convention a, b and c are all taken to be positive: $a > 0$, $b > 0$, $c > 0$) represents an *ellipsoid* with *semi–axes* (of length) a, b, and c. In the special case $a = b = c$, the ellipsoid simplifies to a sphere with radius a.

The presence of the parameters a, b, and c, linked to the three coordinate variables x, y, and z, sort of breaks the symmetry present in the sphere. Some of the symmetry is restored if, for instance, $a = b$. If you intersect the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} = 1$$

with the plane $z = c/2$, a glimpse of this symmetry becomes apparent: Within the plane $z = c/2$, the equation of the resulting curve is

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = \frac{3}{4}\,,$$

representing a circle with radius $\dfrac{a\sqrt{3}}{2}$.

Translating the ellipsoid over $\mathbf{t} = (t_1, t_2, t_3)$ produces an ellipsoid with equation

$$\frac{(x-t_1)^2}{a^2} + \frac{(y-t_2)^2}{b^2} + \frac{(z-t_3)^2}{c^2} = 1.$$

Since our ellipsoid lacks rotational symmetry in general, rotating it will produce a surface with a different equation. For instance, rotating around the z–axis over 90° according to $(x, y, z) \mapsto (-y, x, z)$ (how does this follow from our discussion on rotations?) will produce the following change. if (u, v, w) is on the rotated ellipsoid, then rotating it back yields $(v, -u, w)$, which must be on our ellipsoid. So $(v, -u, w)$ must satisfy

$$\frac{v^2}{a^2} + \frac{(-u)^2}{b^2} + \frac{w^2}{c^2} = 1\,,$$

i.e., the equation of the rotated ellipsoid is

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} = 1\,.$$

Note that the effect of the rotation on the equation is that the coefficients a and b have been interchanged.

Other rotations may change the equation so drastically, that from the new equation it is much harder to recognize the ellipsoid.

3.3.8 Cylinders

The equation

$$x^2 + y^2 = r^2$$

represents a (right–circular) *cylinder* of radius r, whose so–called *axis* is the z–axis. Note that this equation is independent of z. This means that, given any point on the cylinder, the whole vertical line through that point is also on the cylinder.

Translating the cylinder over $\mathbf{t} = (t_1, t_2, t_3)$ changes the equation in the following, by now straightforward, way:

$$(x - t_1)^2 + (y - t_2)^2 = r^2.$$

(Note that t_3 does not occur. Can you explain this geometrically?)

Of course, the cylinder $x^2 + y^2 = r^2$ has 'circular' symmetry with respect to the z–axis: if you turn the cylinder around the z–axis over any angle, the result will coincide with the original cylinder (although individual points have moved of course to another place on the cylinder). Here is the algebraic verification. By the intermezzo, a rotation over an angle α is described by the transformation

$$(x, y, z) \mapsto (x \cos\alpha - y \sin\alpha, x \sin\alpha + y \cos\alpha, z).$$

Rotating backwards, i.e., rotating over an angle $-\alpha$, is then described by $(x, y, z) \mapsto (x \cos\alpha + y \sin\alpha, -x \sin\alpha + y \cos\alpha, z)$ (where we have used $\cos(-\alpha) = \cos\alpha$ and $\sin(-\alpha) = -\sin(\alpha)$). If (u, v, w) is on the rotated cylinder, then turning it back produces $(u \cos\alpha + v \sin\alpha, -u \sin\alpha + v \cos\alpha, z)$. These coordinates should satisfy the equation $x^2 + y^2 = r^2$:

$$(u \cos\alpha + v \sin\alpha)^2 + (-u \sin\alpha + v \cos\alpha)^2 = r^2.$$

Now this looks like an awful expression, but see what happens to the left–hand side, mainly thanks to the fact that $\cos^2\alpha + \sin^2\alpha = 1$:

$$\begin{aligned}
&(u \cos\alpha + v \sin\alpha)^2 + (-u \sin\alpha + v \cos\alpha)^2 \\
&= u^2 \cos^2\alpha + 2u \cos\alpha \cdot v \sin\alpha + v^2 \sin^2\alpha + (-u)^2 \sin^2\alpha - 2u \sin\alpha \cdot v \cos\alpha + v^2 \cos^2\alpha \\
&= u^2(cos^2\alpha + \sin^2\alpha) + v^2(\cos^2\alpha + \sin^2\alpha) \\
&= u^2 + v^2.
\end{aligned}$$

Conclusion: the rotated cylinder has the same equation, $x^2 + y^2 = r^2$. Probably you agree that it is quite remarkable, that such an obvious geometric fact involves such a lengthy computation.

The equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

(with, by convention, $a > 0$ and $b > 0$) represents a cylinder whose horizontal cross section is an ellipse. If $a = b$, then this equation reduces to $x^2 + y^2 = a^2$, i.e., a circular cylinder of radius a. You can also view the cylinder as being constructed in the following way. Start

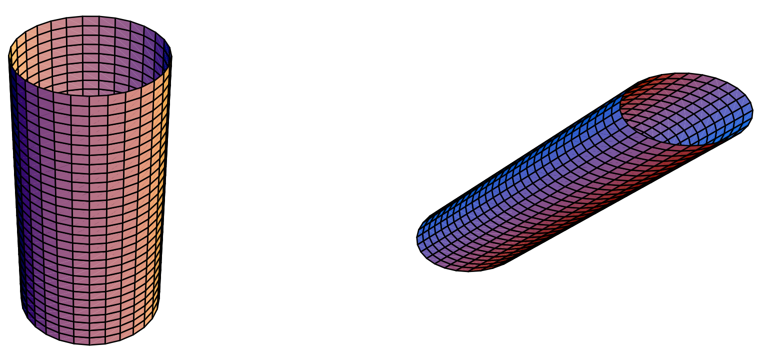

Figure 3.11: *On the left, the cylinder* $x^2+y^2 = 1$. *On the right the cylinder* $(x-z)^2+y^2 = 1$.

with a circle, say $x^2 + y^2 = 1$ in the plane $z = 0$. Now the vertical lines passing through the circle together form the cylinder. Given this construction you can wonder what happens if instead of vertical lines, you take lines with a fixed direction which is not vertical, as illustrated in the figure. We will discuss such an example when we deal with parametric descriptions of quadrics.

Instead of considering cylinders 'over circles', you could just as well consider cylinders over other quadratic curves, like a parabola. For example, the equation $y = x^2$ describes a *right–parabolic cylinder.*

3.3.9 Cones

The equation

$$x^2 + y^2 = z^2$$

describes a *right–circular cone* with the z–axis as vertical axis. A horizontal cross–section at level $z = r$ produces the circle $x^2 + y^2 = r^2$ (and just a point for $r = 0$). A vertical

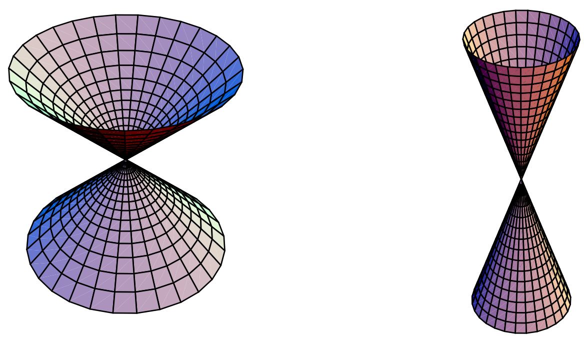

Figure 3.12: *Part of the cone $x^2+y^2-z^2 = 0$ (left) and the cone $x^2+y^2-z^2/4 = 0$ (right). The coefficient of z^2 controls the steepness.*

cross–section, say with the plane $y = 0$, produces a pair of lines in the plane $y = 0$: $x^2 = z^2$ (and $y = 0$), which reduces to

$$x - z = 0 \text{ and } y = 0, \text{ or } x + z = 0 \text{ and } y = 0.$$

In particular, a cone contains lines. The vertical cross–section with the plane $y = r$ for $r \neq 0$ is a hyperbola: $x^2 - z^2 = -r^2$.

By introducing parameters in the equation, the shape can be adapted. For example, the parameter a in $x^2 + y^2 = a^2z^2$ increases or decreases the steepness of the cone. The standard equation of the general cone is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}\,.$$

It has less symmetry than our right–circular cone.

The cone also arises as follows: take a circle at level $z = 1$, say $x^2 + y^2 = 1$. Now connect each point of this circle by a line with the origin. These lines then run through a cone. To see this, take an arbitrary point on the circle, say $(\cos u, \sin u, 1)$ for some real number u. The line through the origin and this point is given by the parametric description $\lambda\,(\cos u, \sin u, 1)$. So any vector on the surface is described by some vector of the form $\mathbf{x}(u,\lambda) = \lambda\,(\cos u, \sin u, 1)$. These vectors satisfy the equation of the cone:

$$(\lambda \cos u)^2 + (\lambda \sin u)^2 - \lambda^2 = \lambda^2(\cos^2 u + \sin^2 u) - \lambda^2 = \lambda^2 - \lambda^2 = 0.$$

3.3.10 Paraboloids

This type is related to the parabola. We distinguish two types:

- The *elliptic paraboloid.* The equation (in standard form)

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

  describes such a surface. The intersection of this quadric with a vertical plane, like the plane $y = 0$ (or $y = c$, any constant) yields a parabola, hence the name paraboloid. By intersecting with a horizontal plane, like $z = 1$, we get an ellipse. This explains the adjective elliptic.

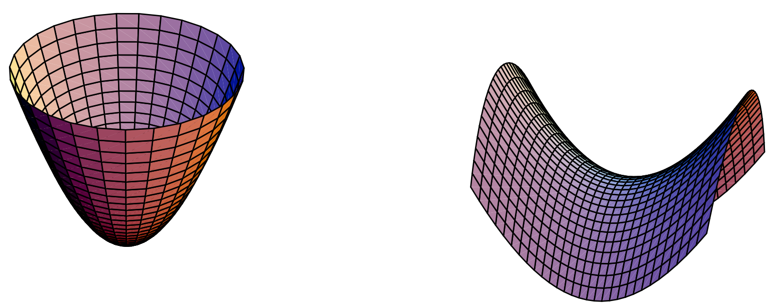

Figure 3.13: *An elliptic paraboloid (left) and a hyperbolic paraboloid (right).*

- The *hyperbolic paraboloid.* Its equation in standard form is

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

  (note the minus sign). The intersection with a horizontal plane consists of a hyperbola. For instance, for $z = 1$, we get the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ (in the plane $z = 1$!). The intersection with a vertical plane produces a parabola.

3.3.11 Hyperboloids

The hyperboloids come in two types, a one–sheeted and a two–sheeted version.

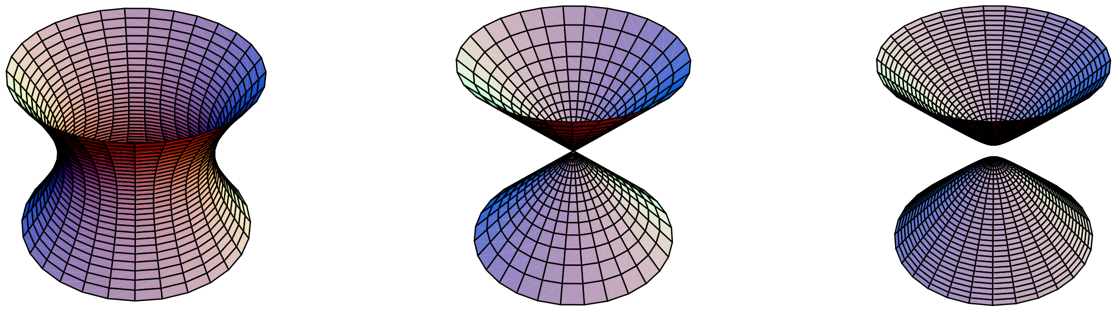

Figure 3.14: *Part of the hyperboloid* $x^2 + y^2 - z^2 = 1$ *(left). The right–hand side illustrates what happens when the equation changes from* $x^2 + y^2 - z^2 = 1$ *(hyperboloid of one sheet) to* $x^2 + y^2 - z^2 = 0$ *(cone), and finally to* $x^2 + y^2 - z^2 = -1$ *(hyperboloid of two sheets).*

a) The standard equation of a *hyperboloid of one sheet* is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,$$

where a, b and c are positive constants. We illustrate the fact that this type of hyperboloid consists of one piece (one sheet) by taking a closer look at the hyperboloid with $a = b = c = 1$, i.e., $x^2+y^2-z^2 = 1$. If you rewrite the equation as $x^2+y^2 = 1+z^2$, then you clearly see that the intersection with any horizontal plane z = constant produces a circle. For instance, for $z = 0$, we get the circle $x^2 + y^2 = 1$ in the plane $z = 0$. By varying the position of these horizontal planes, you find a varying family of circles. The intersection with a vertical plane usually consists of a hyperbola. For instance, the plane $x = 0$ meets $x^2 + y^2 - z^2 = 1$ in the hyperbola with equation $y^2 - z^2 = 1$ (in the plane $x = 0$).

b) The standard equation of the *hyperboloid of two sheets* is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1,$$

with positive constants a, b and c. Note that the only difference with the previous equation is the minus sign on the right–hand side. To explain the two–sheetedness, we take a closer look at $x^2 + y^2 - z^2 = -1$. For $|z| < 1$, this equation has no solutions, i.e., between the horizontal planes $z = -1$ and $z = 1$, there is no point of the hyperboloid. But the intersection of the figure with the plane $z = 3$, for example is again a circle. In general, the cross section with a horizontal plane $z = c$ with $c \geq 1$ produces a circle. All these cross sections together make up one sheet. The cross sections with the planes $z = c$ with $c \leq -1$ make up the other sheet.

3.3.12 Degenerate quadrics

As with quadratic curves, some quadric surfaces are more or less 'degenerate'. Here are typical examples.

a) The equation $x^2 - z^2 = 0$ describes a union of two planes. This is easily seen once you rewrite the equation as $(x+z)(x-z) = 0$.

b) The equation $x^2 = 1$ describes two parallel planes, which is evident if you rewrite the equation as $(x+1)(x-1) = 0$.

c) The equation $z^2 = 0$ is just the plane $z = 0$ ('with multiplicity two').

3.4 Parametrizing quadrics

3.4.1 (Co)sines and hyperbolic (co)sines

The standard equations of quadrics contain sums and differences of squares. This makes it possible to use (co)sines and hyperbolic (co)sines in an appropriate way to parametrize quadrics. Parametric descriptions of quadrics not only produce explicit points on the quadric, but also provide tilings of the surfaces.

3.4.2 Parametrizing cylinders

We start with the equation $x_1^2 + x_2^2 = r^2$ in $\mathbf{R}^3$ of a right–circular cylinder, since its structure reminds one immediately of a circle. We use one parameter, say ϕ, to go around the cylinder (in a horizontal plane), and one parameter, say u, to indicate the z–coordinate. This results in:

$$x_1 = r\cos\phi, \quad x_2 = r\sin\phi, \quad x_3 = u.$$

Here, $0 \le \phi \le 2\pi$ and u runs through all the real numbers.

To parametrize the cylinder with equation $\dfrac{x_1^2}{a^2} + \dfrac{x_2^2}{b^2} = 1$, the same simple change as for a circle can be used:

$$x_1 = ar\cos\phi, \quad x_2 = br\sin\phi, \quad x_3 = u.$$

When we introduced cylinders, we mentioned the construction of other cylinders. For instance, start with the circle $x^2+y^2 = 1$ in the plane $z = 0$, and take all lines through points of the circle with direction $(1,0,1)$. To deal with this situation, we turn the description into a parametric description. A point on the circle is given by $(\cos u, \sin u, 0)$. Now use this as support vector for a line with direction $(1,0,1)$: $(\cos u, \sin u, 0) + v(1,0,1)$. So

$$\begin{aligned} x_1 &= \cos u + v \\ x_2 &= \sin u \\ x_3 &= v \end{aligned}$$

From this parametric description, it is fairly easy to obtain an equation. By subtracting x_3 from x_1 we get rid of v. Then we use $\cos^2 u + \sin^2 u = 1$. This leads to: $(x_1 - x_3)^2 + x_2^2 = \cos^2 u + \sin^2 u = 1$. So an equation is $(x_1 - x_3)^2 + x_2^2 = 1$.

3.4.3 Parametrizing spheres and ellipsoids

We start with the sphere $x_1^2 + x_2^2 + x_3^2 = 1$. The left–hand side contains a sum of three squares instead of two squares. To relate this situation to a sum of two squares, we rewrite the equation as

$$\left(\sqrt{x_1^2 + x_2^2}\right)^2 + x_3^2 = 1.$$

Now, this is a sum of two squares, where the square root also contains a sum of two squares. We then proceed in two steps:

a) For a given point (x_1, x_2, x_3) satisfying the equation, choose u between 0 and π so that $\sqrt{x_1^2 + x_2^2} = \sin u$ and $x_3 = \cos u$. (Why is this possible? Think of the parametric description of a circle.)

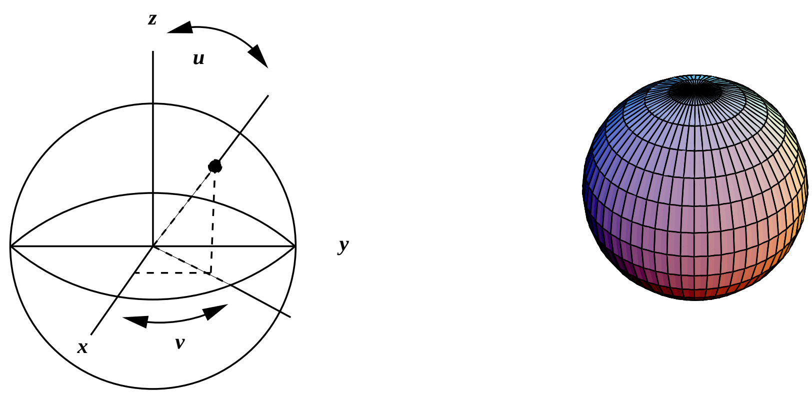

Figure 3.15: *Spherical coordinates* $(r\cos v \sin u, r \sin v \sin u, r \cos u)$ *with* $0 \leq u \leq \pi$ *and* $0 \leq v \leq 2\pi$ *are a widely used way of parametrizing a sphere with center* $(0, 0, 0)$ *and radius* r*. The parameters have a clear geometric meaning. If* $\mathbf{x} = (r\cos v \sin u, r \sin v \sin u, r \cos u)$*, then* u *is the angle between* $\mathbf{x}$ *and the positive* z*–axis, and* v *is the angle between the positive* x*–axis and the projection* $(x_1, x_2, 0)$ *of* $\mathbf{x}$ *on the* x, y*–plane. The horizontal and vertical curves in the picture on the right–hand side are curves where either* u *or* v *is kept constant.*

b) Now parametrize, for fixed u, the equation $x_1^2 + x_2^2 = \sin^2 u$ in the standard way:

$$x_1 = \cos v \sin u, \quad x_2 = \sin v \sin u.$$

(With v between 0 and 2π.) Collecting the two steps leads to the following parametrization of the sphere $x_1^2 + x_2^2 + x_3^2 = 1$:

$$x_1 = \cos v \sin u, \quad x_2 = \sin v \sin u, \quad x_3 = \cos u.$$

This parametric description also goes by the name *parametric description in spherical coordinates*. Both u and v have a clear geometric meaning: u is the angle between $\mathbf{e}_3 = (0, 0, 1)$ and $\mathbf{x}$, and v is the angle between $\mathbf{e}_1$ and the vector $(x_1, x_2, 0)$.

From here on, it is fairly easy to parametrize the ellipsoid $\dfrac{x_1^2}{a^2} + \dfrac{x_2^2}{b^2} + \dfrac{x_3^2}{c^2} = 1$:

$$x_1 = a\cos v \sin u, \quad x_2 = b \sin v \sin u, \quad x_3 = c \cos u.$$

3.4.4 Parametrizing hyperboloids

To parametrize the hyperboloid of one sheet with equation $x_1^2 + x_2^2 - x_3^2 = 1$, we will need both (co)sines and hyperbolic (co)sines to deal with the plus and minus signs. First, we rewrite our equation as a difference of squares:

$$\left(\sqrt{x_1^2 + x_2^2}\right)^2 - x_3^2 = 1,$$

as in the case of a sphere. Then we find a parametrization in the following two steps:

a) First we handle the difference of squares with hyperbolic (co)sines:

$$\sqrt{x_1^2 + x_2^2} = \cosh u, \quad x_3 = \sinh u.$$

b) For fixed u, we then parametrize the equation $x_1^2 + x_2^2 = \cosh^2 u$ in the usual way: $x_1 = \cosh u \cos v$, $x_2 = \cosh u \sin v$.

The resulting parametric description is

$$x_1 = \cosh u \cos v, \quad x_2 = \cosh u \sin v, \quad x_3 = \sinh u.$$

To parametrize (one sheet of) a hyperboloid of two sheets, say with equation $x_1^2+x_2^2-x_3^2 = -1$, the first option is to rewrite the equation in the following way as a difference of two squares:

$$x_3^2 - \left(\sqrt{x_1^2 + x_2^2}\right)^2 = 1.$$

Next, we handle the difference of two squares with $x_3 = \cosh u$, $\sqrt{x_1^2 + x_2^2} = \sinh u$. For fixed u we parametrize the circle $x_1^2 + x_2^2 = \sinh^2 u$ with $x_1 = \sinh u \cos v$ and $x_2 = \sinh u \sin v$. The resulting parametrization

$$x_1 = \sinh u \cos v, \quad x_2 = \sinh u \sin v, \quad x_3 = \cosh u$$

describes only the part of the hyperboloid above the x_1, x_2–plane. Do you see why?

3.4.5 Parametrizations and curves on quadrics

With a parametric description of a quadric come two families of curves on the quadric. These families of curves can be visualized as special patterns on the quadric.

Consider, for example, the cylinder $x^2 + y^2 = 1$ parametrized by $x = \cos u$, $y = \sin u$, $z = v$ (with $0 \leq u \leq 2\pi$ and $-\infty < v < \infty$). If we keep u fixed, say $u = \pi$, and let v run through the real numbers, the result is a vertical line on the cylinder. If, on the other hand, we keep v fixed and let u run through the interval $[0, 2\pi]$, we obtain a circle at height v on the cylinder.

A different parametrization usually results in a different pattern on the cylinder as is illustrated in the figure. If we keep u constant in the parametrization $x = \cos(u + v/2)$,

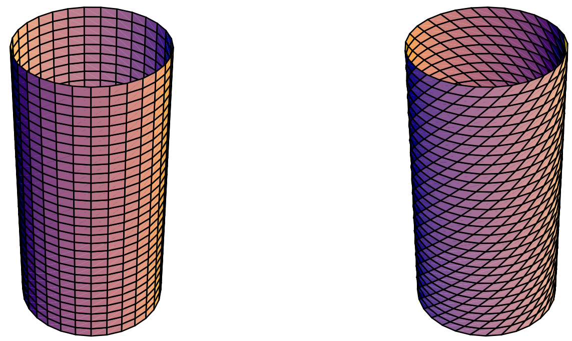

Figure 3.16: *On the left, the cylinder* $x^2 + y^2 = 1$ *parametrized by* $x = \cos u$, $y = \sin u$, $z = v$. *This parametrization results in the pattern illustrated in the figure. The vertical lines on the cylinder are curves where* u *is constant. The horizontal circles are curves where* v *is constant. On the right the parametrization* $x = \cos(u + v/2)$, $y = \sin(u + v/2)$, $z = 2v$ *is used, resulting in a different pattern on the cylinder.*

$y = \sin(u + v/2)$, $z = 2v$, then the resulting curve is no longer a vertical line, but a curve which spirals upwards around the cylinder.

In general, in a parametric description $(x(u, v), y(u, v), z(u, v))$ of a quadric, we get two families of curves: one by keeping u fixed at an arbitrary value, and one by keeping v fixed at an arbitrary value.

3.5 Geometric ins and outs on quadrics

3.5.1 Intersecting planes with quadrics produces quadratic curves

Classically, quadrics were called *conic sections*. This is related to the fact that if you intersect a quadric and a plane, the resulting figure is usually a quadratic curve. In the case the plane is given by $z = 0$ this is obvious: just substitute 0 for z in the equation of the quadric and you are usually left with a quadratic equation in x and y. The reason we say 'usually' is that sometimes the intersection is empty or consists of a line. But in general a quadratic curve arises. A nice example of this phenomenon can be observed in the theatre: the light of a spot light creates a spot on stage with the shape of, for instance, an ellipse.

This section discusses some aspects of the intersections of planes and quadrics.

3.5.2 Intersecting a sphere with a plane

In Chapter 1 we discussed such intersections, so a brief discussion suffices here. The intersection of a sphere and a plane is a circle (strictly speaking, the intersection can be empty, a single point or a circle), but why a circle? Here is an analytic–geometric approach to seeing this. We rescale so that our sphere has equation $x^2 + y^2 + z^2 = 1$. Our strategy is to rotate the plane so that its equation becomes simple. Of course, rotating the sphere does not change its position. In the following we only need to consider the case of a plane at distance at most 1 from the origin. If the distance to the plane is greater than 1, then the plane does not meet the sphere, since all points of the sphere are at distance 1 from the origin.

a) Suppose we have a plane at distance r from the origin with $0 \leq r \leq 1$. Rotate the plane around the origin, so that the rotated plane is horizontal (parallel with the x, y–plane) and above the x, y–plane (or coincides with it, for $r = 0$). Its equation will be $z = r$.

b) Now the intersection of $z = r$ and $x^2 + y^2 + z^2 = 1$ is a circle in the plane $z = r$ with equation $x^2 + y^2 = 1 - r^2$. If $r = 1$, then we get precisely one point (the point $(0, 0, 1)$, the 'north pole'). If $r = 0$, we get a circle with radius 1 in the x, y–plane: $x^2 + y^2 = 1$ and $z = 0$.

3.5.3 Intersecting a right–circular cylinder with a plane: geometric approach

If you cut the cylinder $x^2 + y^2 = 1$ with a horizontal plane $z = r$, then the resulting curve of intersection is a circle with radius 1. But, if the plane is not quite horizontal, then the intersection is not a circle anymore, but looks more like an ellipse. But is it? (If the plane is vertical and meets the cylinder, the intersection consists of one or two lines; in the first case, the plane is tangent to the cylinder.) To decide on this matter, we first use a purely geometric approach, which uses the geometric definition of an ellipse: by a famous trick due to the French–Belgian G.P. Dandelin (1794–1847), we can locate two points which turn out to be the foci of the ellipse. Take two spheres of radius 1. Insert one in the cylinder from above, and the other from below. Move them down (resp. up) so that they touch the plane. Each sphere touches the plane in exactly one point, so we get two points, say F_1 and F_2. The spheres meet the cylinder in horizontal circles of radius 1.

To show that the intersection of the cylinder and the plane is an ellipse, we take an arbitrary point P on the intersection and show that $|PF_1| + |PF_2|$ is constant. Now, $|PF_1|$ equals the distance $|PB|$ of P to the lower circle, and $|PF_2|$ equals the distance $|PA|$ of P to the upper circle. But then $|PF_1| + |PF_2|$ equals the distance between the two circles, which is obviously constant.

In a similar way, all other types of intersections can be analysed. The right–hand picture in Figure (3.17) illustrates the 'Dandelin–approach' to the intersection of a plane parallel to the line OC with a cone. The resulting intersection curve is a parabola. In the exercises we will discuss this case analytically.

3.5.4 Intersecting a right–circular cylinder with a plane: analytic approach

In terms of equations, things become complicated quite soon. To simplify matters, we rotate the whole situation so that the plane is given by the equation $z = 0$. For the sake of concreteness, let us assume that the cylinder has been rotated around the y–axis over an angle of α radians. So if we rotate a point (u, v, w) on the rotated cylinder back over α radians the resulting coordinates should satisfy the equation $x^2 + y^2 = 1$. This means that

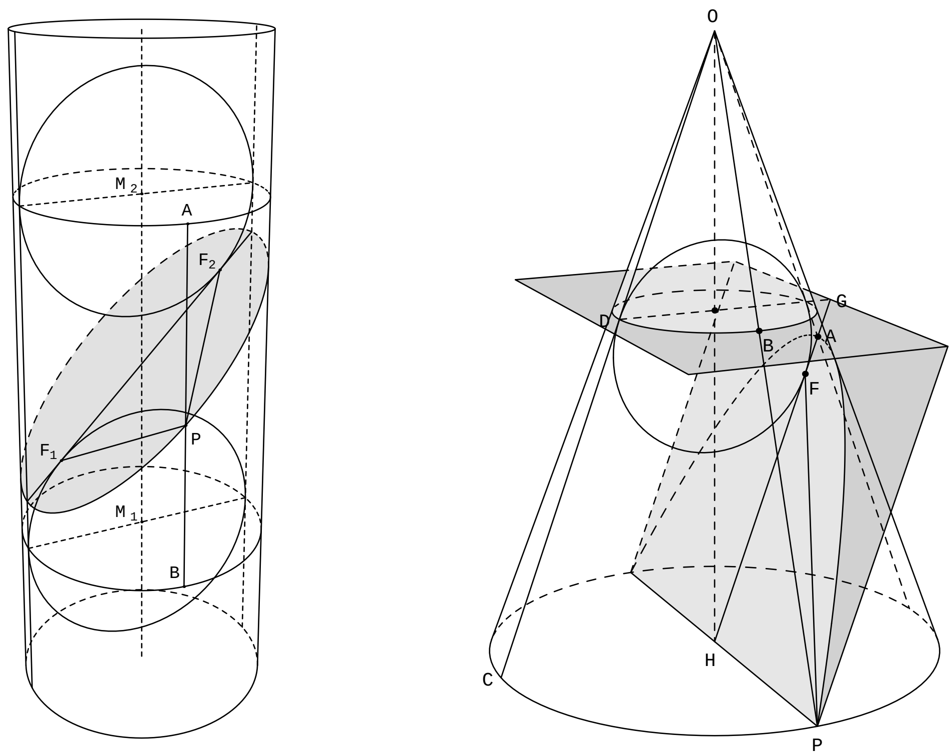

Figure 3.17: *On the left, Dandelin's approach to proving that the intersection of a plane with the right–circular cylinder is an ellipse if the plane is not vertical. On the right, a similar illustration for a plane intersecting a cone along a parabola. This case is not explained in the text.*

$(u\cos\alpha - w\sin\alpha, v, u\sin\alpha + w\cos\alpha)$ satisfies $x^2 + y^2 = 1$:

$$(u\cos\alpha - w\sin\alpha)^2 + v^2 = 1.$$

The interesection with the plane $z = 0$ is obtained by setting $w = 0$:

$$u^2\cos^2\alpha + v^2 = 1 \text{ and } w = 0,$$

evidently the equation of an ellipse.

3.5.5 Lines on hyperboloids

Usually, the intersection of a plane with a quadric yields a truly curved curve. But sometimes, surprises occur. Let us illustrate this in the case of the hyperboloid of one sheet $x^2 + y^2 - z^2 = 1$. Although this is a pretty curved object, it contains straight lines. For instance, the line $(1, 0, 0) + \lambda(0, 1, 1)$ is on the hyperboloid. Just substitute $(1, \lambda, \lambda)$ to verify this:

$$1^2 + \lambda^2 - \lambda^2 = 1.$$

This idea of lines on a hyperboloid was used in the design of the Vortex, which will possibly be built in England. Visually, the presence of lines on the hyperboloid may be clear, mathematically, it is harder to get them in your hands. So here is how these lines show up algebraically. Rewrite the equation as

$$x^2 - z^2 = 1^2 - y^2$$

(a difference of two squares on both sides of the equality). Now use the identities $x^2 - z^2 = (x+z)(x-z)$ and $1 - y^2 = (1+y)(1-y)$. Our equation becomes

$$(x+z)(x-z) = (1+y)(1-y).$$

So what? This does not look like simplifying things at all. But from this version of the equation lots of lines on the hyperboloid can be extracted. Choose a real number, say 2. The intersection of the planes

$$x + z = 2(1+y), \quad 2(x-z) = 1 - y$$

is on the hyperboloid: if (x, y, z) satisfies both $x + z = 2(1+y)$ and $2(x-z) = 1-y$, then (x, y, z) also satisfies $(x+z)(x-z) = (1+y)(1-y)$ (multiply the left–hand sides, and multiply the right–hand sides):

$$(x+z) \cdot 2 \cdot (x-z) = 2 \cdot (1+y) \cdot (1-y),$$

which is our equation up to a factor 2. In general, for any fixed number λ, the line of intersection of the two planes

$$x + z = \lambda(1+y), \quad \lambda(x-z) = 1 - y$$

is on the hyperboloid. Even more lines can be found: for any μ, the intersection of the

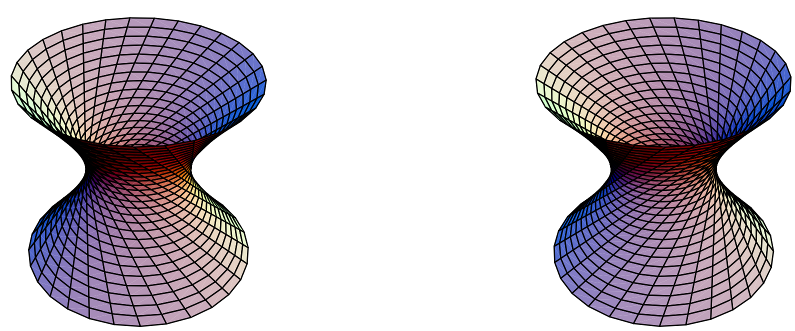

Figure 3.18: *Two systems of straight lines on a hyperboloid. We have visualized both systems by using the two parametric descriptions:* $(\cos u + v\sin u, \sin u - v\cos u, v)$ *and* $(\cos u - v\sin u, \sin u + v\cos u, v)$.

planes

$$x + z = \mu(1-y), \quad \mu(x-z) = 1 + y$$

produces a line on the hyperboloid. We say that the hyperboloid $x^2 + y^2 - z^2 = 1$ contains two *systems of lines*.

3.6 Exercises

1 Parametrizing the circle
In each of the following parametric descriptions, let u run from 0 to 2π and sketch what happens to the corresponding point on the circle $x_1^2 + x_2^2 = 1$.

a) The familiar parametric description $(\cos u, \sin u)$.

b) The parametric description $(\sin u, \cos u)$.

c) The parametric description $(\sin(u+\pi), \cos(u+\pi))$.

d) What happens if you take $(\cos 2u, \sin 2u)$ and let u run from 0 to 2π?

2 Parametric descriptions of the circle
Here are some other ways to parametrize (parts of) the circle $x_1^2 + x_2^2 = r^2$.

a) Suppose you want to parametrize the upperhalf of the circle, i.e., the part with $x_2 \geq 0$, by taking $x_1 = u$. What is the resulting parametric description for the circle? Visualize this parametric description by intersecting a vertical line with the circle. Relate this description to the graph of a function and identify the function.

b) Find a similar parametrization for the lower half of the circle, i.e., the part with $x_2 \leq 0$.

c) Find similar parametric descriptions for the right half (i.e., the part with $x_1 \geq 0$) and the left half (i.e., the part with $x_1 \leq 0$) of the circle.

3 Parametrizing translates
Parametrize each of the following curves.

a) $(x_1 - 2)^2 + (x_2 + 3)^2 = 25$.

b) $\dfrac{(x_1 - 2)^2}{3} + \dfrac{(x_2 + 3)^2}{5} = 25$.

c) $\dfrac{(x_1 - 2)^2}{3} - \dfrac{(x_2 + 3)^2}{5} = 1$.

4 The equation of a parabola

a) Find the equation of the parabola with focus $F = (2, 0)$ and directrix ℓ: $x = -2$.

b) Find, either directly or by using a), the equation of the parabola with focus $F' = (6, 2)$ and directrix ℓ': $x = 2$.

5 The equation of a hyperbola

a) Find the equation of the hyperbola with foci $(-\sqrt{2}, 0)$ and $(\sqrt{2}, 0)$ and constant 2.

b) If you rotate the hyperbola over $90°$ around the origin, how does the equation change?

6 Parametrizing non–standard equations
In (3.1.4) on page 60 we saw that if you rotate $3x^2 + y^2 = 18$ over $45°$ you get the ellipse $x^2 + xy + y^2 = 9$. Use this rotation to find a parametric description of $x^2 + xy + y^2 = 9$ from one for $3x^2 + y^2 = 18$.

7 Rotating quadratic curves

a) Start with the ellipse $2x^2 + 3y^2 = 10$ in the x, y–plane. Determine the equation of the surface you get by rotating this ellipse around the x–axis in 3–space. What kind of surface is it?

b) Similar question: rotate the parabola $y = x^2$ in the x, y–plane around the y–axis in 3–space. What kind of quadric do you get?

8 Quadrics and products of planes
Take two equations of planes, say $2x - y + 3z + 5 = 0$ and $x + y - 2z - 2 = 0$.

a) Show that by multiplying these equations you get a quadratic equation.

b) Argue why the product of any two linear equations $a_1x + b_1y + c_1z - d_1 = 0$ and $a_2x + b_2y + c_2z - d_2 = 0$ will produce a quadric.

c) Go back to item b): each linear equation contains 3 degrees of freedom. Here is why. To begin with, there are 4 coefficients, but if you multiply all coefficients with the same (non–zero) scalar, the resulting new equation represents the same plane. Similarly, each quadratic equation in three variables contains 9 degrees of freedom $(= 10 - 1)$. Use this to make plausible that by taking products of linear equations it is impossible to get all quadric surfaces.

9 Rotations in $\mathbf{R}^3$

a) In the text we have described rotations around the x_1–axis and the x_3–axis. Now describe rotations around the x_2–axis by 3 by 3 matrices, i.e., determine the matrix of a rotation over α around the x_2–axis.

b) Take the cylinder $x^2 + y^2 = 2$ in 3–space. Rotate the cylinder over $45°$ around the y–axis and determine the equation of the resulting surface.

10 Parametrizing spheres

a) Illustrate the geometric meaning of u and v in the parametric description $x_1 = \cos v \sin u$, $x_2 = \sin v \sin u$ and $x_3 = \cos u$ of the sphere $x_1^2 + x_2^2 + x_3^2 = 1$.

b) Show that $x_1 = \cos v \cos u$, $x_2 = \sin v \cos u$, $x_3 = \sin u$ is also a parametrization of the sphere. Suggest a range for u and v. Give a geometric interpretation of u and v.

11 Parametrizing paraboloids

a) Construct a parametric description of (the standard equation of) an elliptic paraboloid.

b) Similar question for a hyperbolic paraboloid.

c) Rotate $z = x^2 + y^2$ and $z = x^2 - y^2$ around the z–axis over 45°. How do the two equations change?

12 Lines on quadrics

a) Rotate the surface with equation $z = xy$ over a suitable angle to get rid of the term xy. Determine the type of the surface.

b) Determine two systems of lines on the surface $z = xy$. [Hint: write $z = xy$ as $1 \cdot z = x \cdot y$ and use the technique of (3.5.5) on page 83.]

13 Parabolic cylinders

a) We construct a parabolic cylinder as follows. Start with the parabola $y = x^2$ in the plane $z = 0$. Through each point of the parabola take a line with direction vector $(1, 1, 1)$. Determine a parametric description of the resulting surface and an equation of the surface. Conclude that the surface is a quadric.

b) Similar question, now for the hyperbola $x^2 - y^2 = 1$.

14 Cross–sections of quadrics
The cylinder with equation $x^2 + 2y^2 = 1$ has elliptical cross–sections in horizontal planes. Find a vector **a** so that planes perpendicular to **a** have circular cross–sections with the cylinder. [Hint: intersect a line through the origin and perpendicular to the x–axis with the cylinder and think of long and short axes.]

15 Intersecting with planes

a) Compute the intersection of the hyperboloid of one sheet $x^2 + y^2 - z^2 = 1$ with the plane $x = 1$. Show that this intersection is not a hyperbola. There is one other plane of the form $x =$constant so that the intersection with the hyperboloid is not a hyperbola. Which plane is this?

b) By using a) and an appropriate rotation, argue that the intersection of the hyperboloid $x^2 + y^2 - z^2 = 1$ with any vertical plane at distance 1 from the origin consists of two lines.

c) Determine what the intersection of the hyperboloid of two sheets $x^2 + y^2 - z^2 = -1$ with any vertical plane looks like.

16 Changing shapes

a) For every real number a the equation $x^2 + y^2 = az^2 + (1 - a)$ is quadric surface. Describe what happens to the surface if a changes.

b) Parametric descriptions of quadrics yield a kind of tiling of the surface. What is the effect of changing the standard parametric description

$$\mathbf{x}(u, v) = (\cosh u \cos v, \cosh u \sin v, \sinh u)$$

into $\mathbf{x}(u, v) = (\cosh u \cos(v + u/2), \cosh u \sin(v + u/2), \sinh u)$?

17 Intersecting a cone with a plane

If you intersect a cone with a plane, the intersection can be any type of quadratic curve. In this exercise we illustrate some aspects of this. We take the cone $C : x^2 + y^2 = z^2$.

a) If you intersect the cone with a vertical plane through $(0, 0, 0)$, you get a pair of lines. Show this by intersecting with the plane $x = 0$.

b) The cone C contains the whole line ℓ: $\lambda(1, 0, 1)$. Why?

c) Next, we investigate the intersection of the cone with a plane parallel to ℓ. To do this first rotate the cone around the y–axis over $-45°$. What is the new equation? Now intersect with the plane $z = 1/2$. Determine the type of the intersection.

d) There is also a plane containing ℓ which meets C in precisely ℓ. Determine this plane and verify that the intersection with C is indeed precisely ℓ.

18 Intersecting spheres with spheres

If you intersect a sphere with a plane, then the intersection is either empty, consists of one point, or is a circle. In this exercise we take a closer look at the intersection of two spheres.

a) Suppose we have two spheres, one with center $\mathbf{a}$ and radius r, the other with center $\mathbf{b}$ and radius s. Formulate a condition in terms of $\mathbf{a}$, $\mathbf{b}$, r and s that expresses when the intersection of the two spheres is non–empty.

b) Suppose $\mathbf{a} = (0, 0, 0)$, so that the equation of the sphere is $x^2 + y^2 + z^2 = r^2$. Now take the equation of any other sphere and show by manipulating these two equations one step that the intersection of the two spheres is also the intersection of the first sphere and a plane. So what possibilities arise if you intersect two spheres?

c) Is it possible to draw an ellipse on a sphere? Explain!

d) The intersection of two spheres lies within a plane if it is non–empty. Take a right–circular cylinder and a sphere. Suppose the radius of the cylinder (the radius of a circular cross–section) is less than the radius of the sphere. If their intersection is not empty, then it usually consists of two pieces. Can these pieces be flat?

19 A cone touching a sphere

Let S be the sphere with equation $x^2 + y^2 + z^2 = 1$.

a) For each value of c the equation $x^2 + y^2 = c(z - 2)^2$ is quadratic. Describe for each value of c the solutions to the equation. We call this T_c.

b) Suppose c is positive and consider the intersection of the sphere with T_c. Show that z satisfies a quadratic equation (in which c still appears). For which value of c does this quadratic equation have only one solution? [Hint: discriminant of a quadratic equation $ax^2 + bx + c = 0$ is $b^2 - 4ac$.] What does this mean for the relative position of the sphere and the corresponding cone? What is the intersection in that case?