Quadric Surfaces

Introduction

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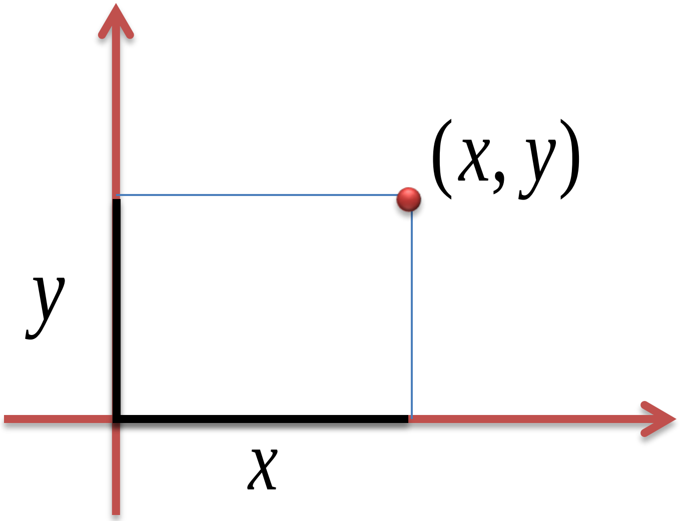
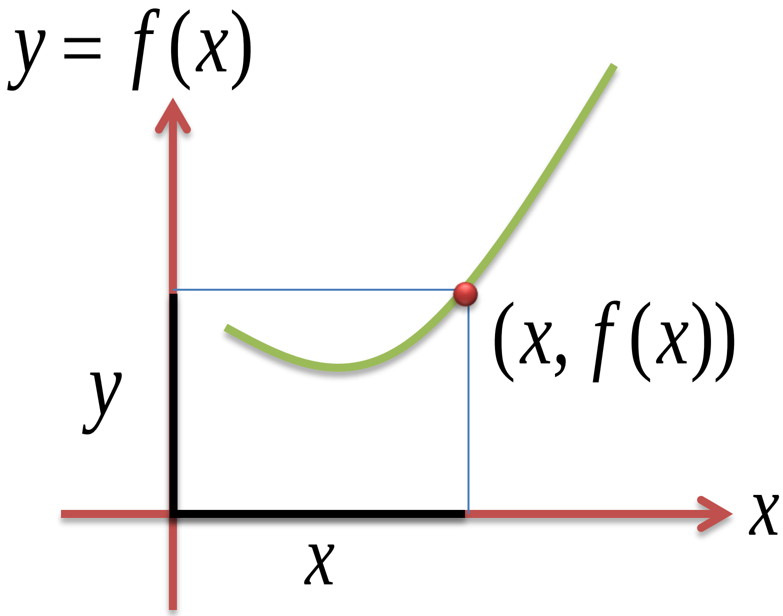
He made an astonishing observation of his time that the position of a tiny object on a wall can be located by drawing two perpendiculars from the object on the respective axes defining the wall.



Introduction

Later his idea was used to study the shape of curves in *xy*-plane defined by algebraic functions. The resultant curve is a graph of the function $y = f(x)$. A simple understanding is given below.

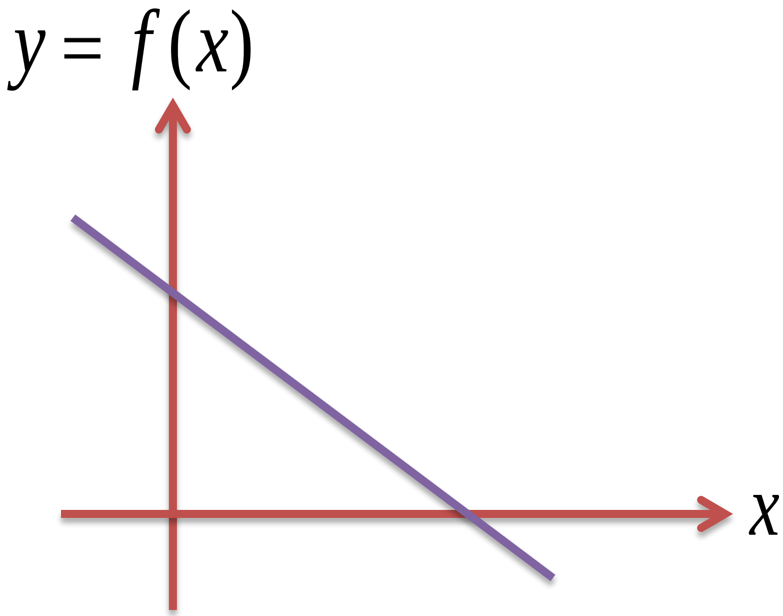


Examples:

1. Straight Line: A straight line has a graph

$$y = f(x) = m\,x + c$$

where m and c denote slope and intercept of straight line.



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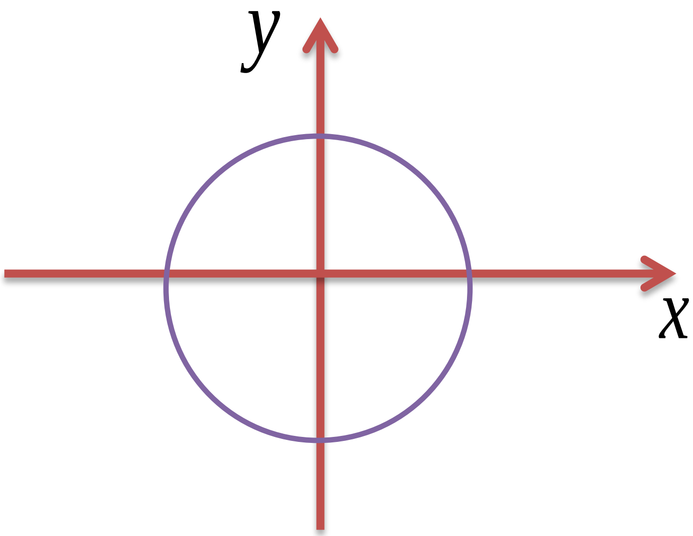
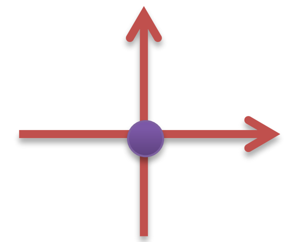
which is an implicit function in which x and y can not be separated such that they define a one-one mapping.

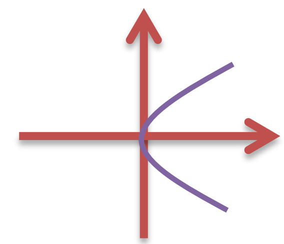


Examples:

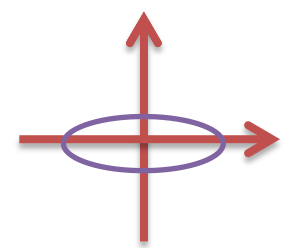
3. Origin: $$x^2 + y^2 = 0$$



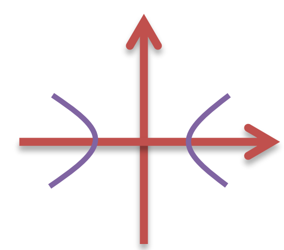
4. Parabola: $$x = \pm a y^2$$



5. Ellipse: $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = c^2$$



6. Hyperbola: $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = c^2$$



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Now ask yourself the following questions to construct more examples.

1. What is the contribution of linear terms in the above equation?
2. What geometrical difference does it bring to the known graphs?

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$$a\,x^2 + b\,y^2 + c\,x + d\,y + e = 0$$

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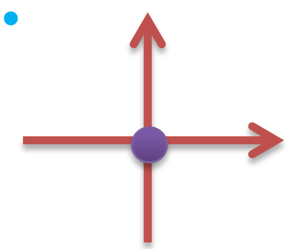
Justification:

$$x^2 + y^2 = 0$$



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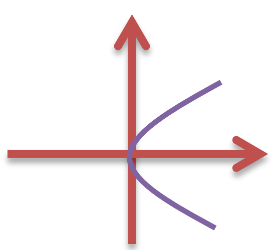
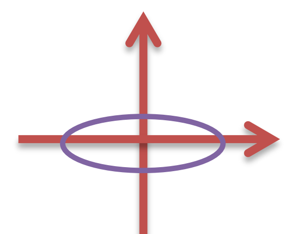
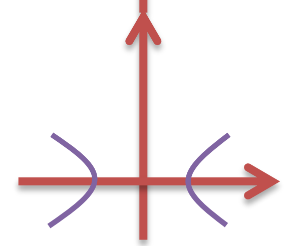


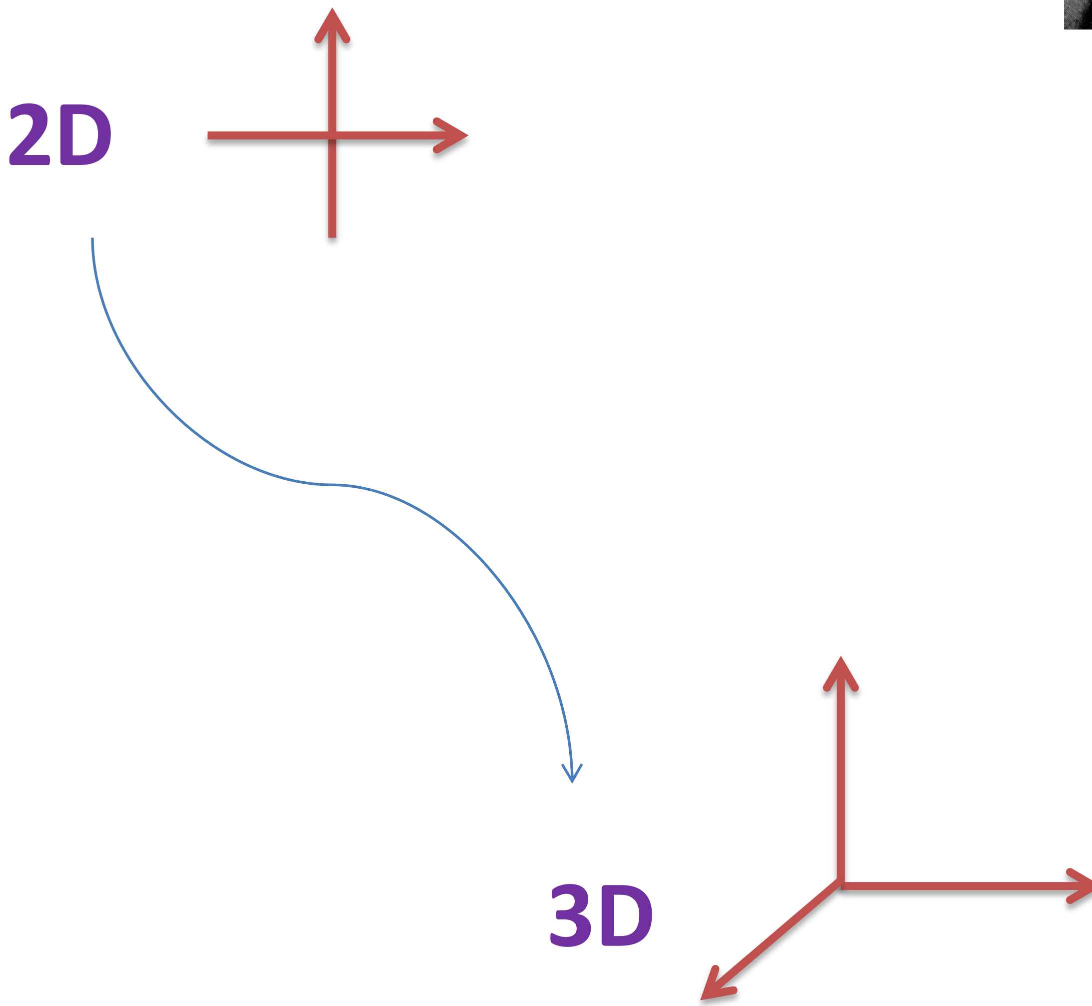
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = c^2$$



$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = c^2$$



Quadric Surfaces



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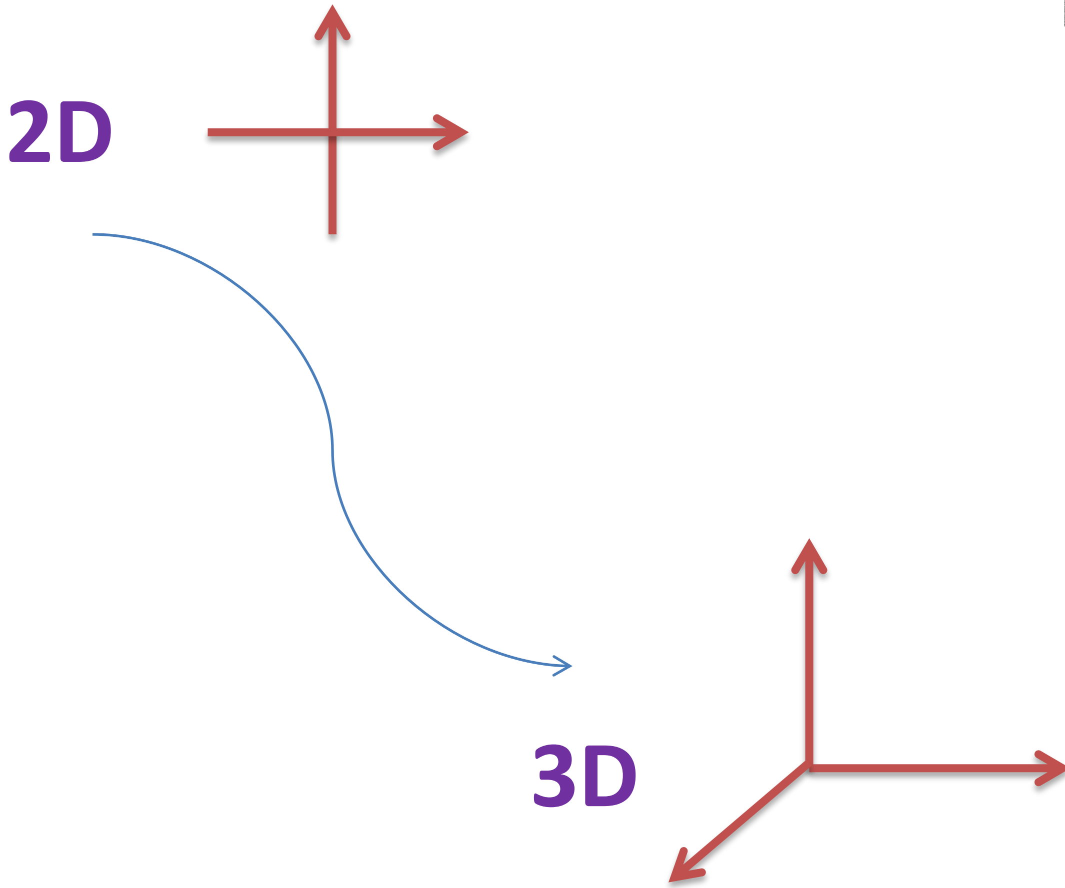
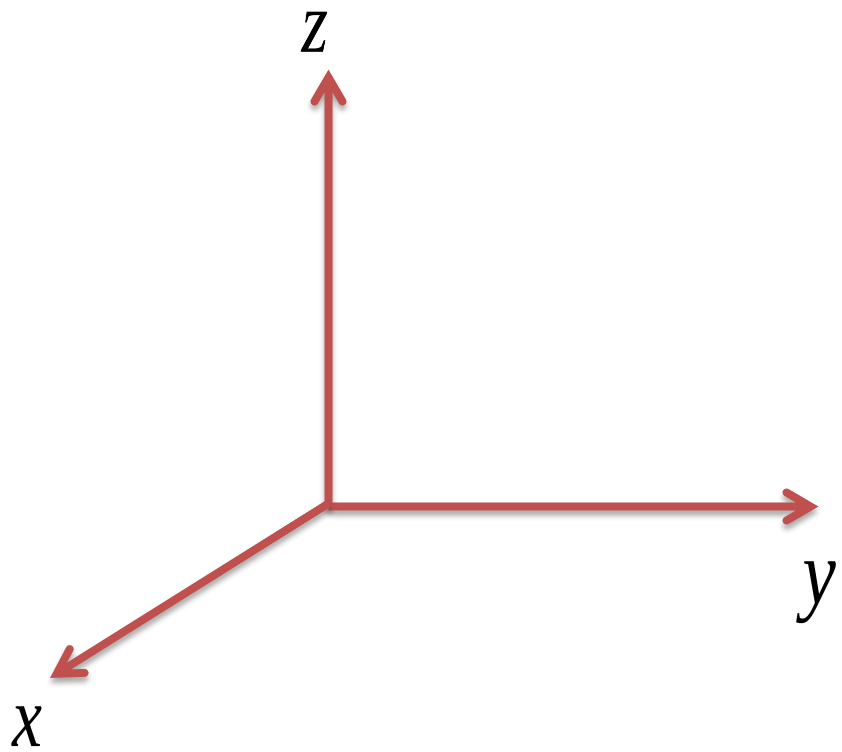


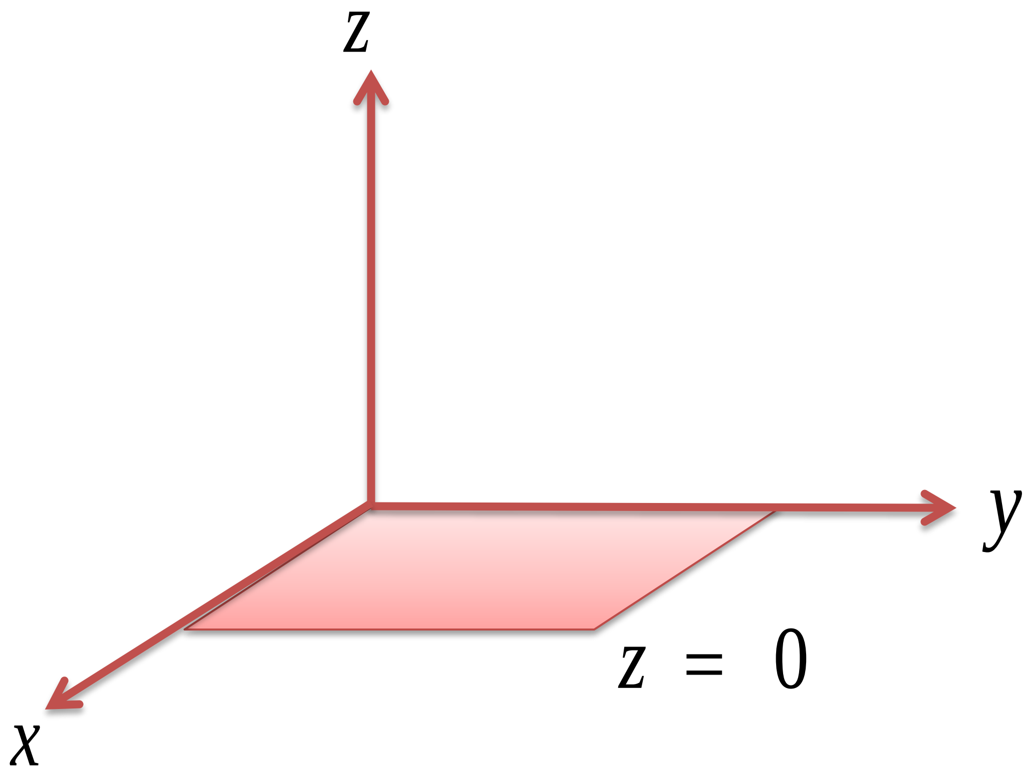
The complete story about construction of a function in 3D will be unveiled in the study of multivariable functions. We first start from the question of solutions of an equation of second degree in three variables.

Quadric Surfaces



3D

Quadric Surfaces



The equation of *xy*-plane is $z = 0$.

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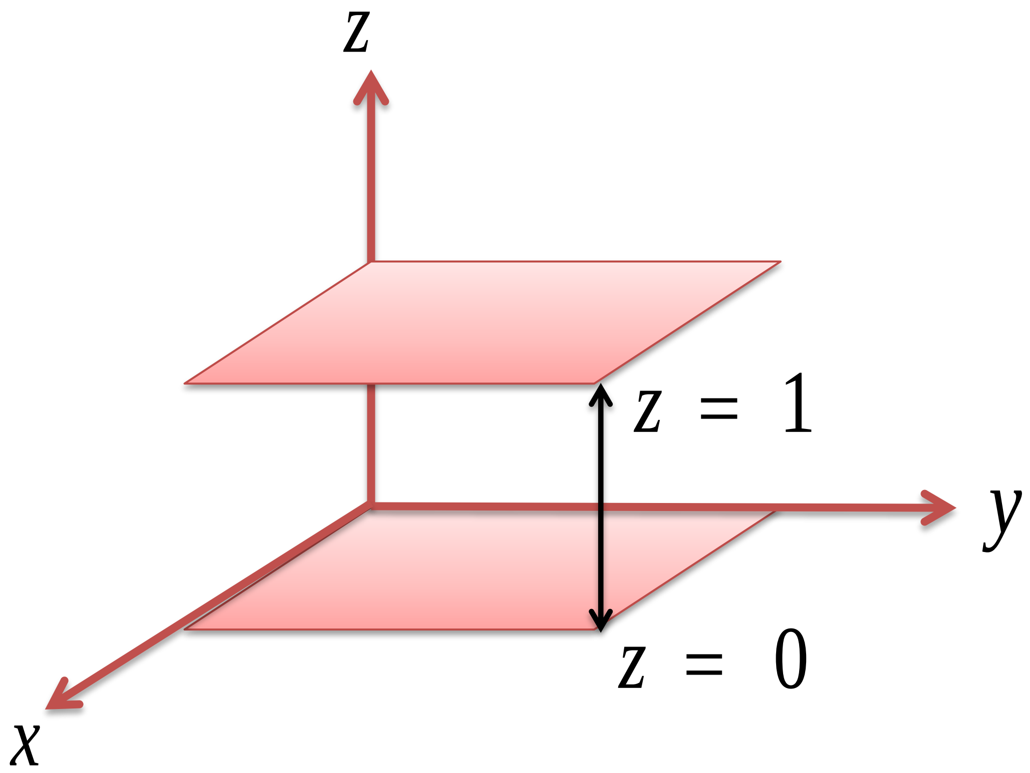

The equation of a plane parallel to *xy*-plane and one unit above is $z = 1$.

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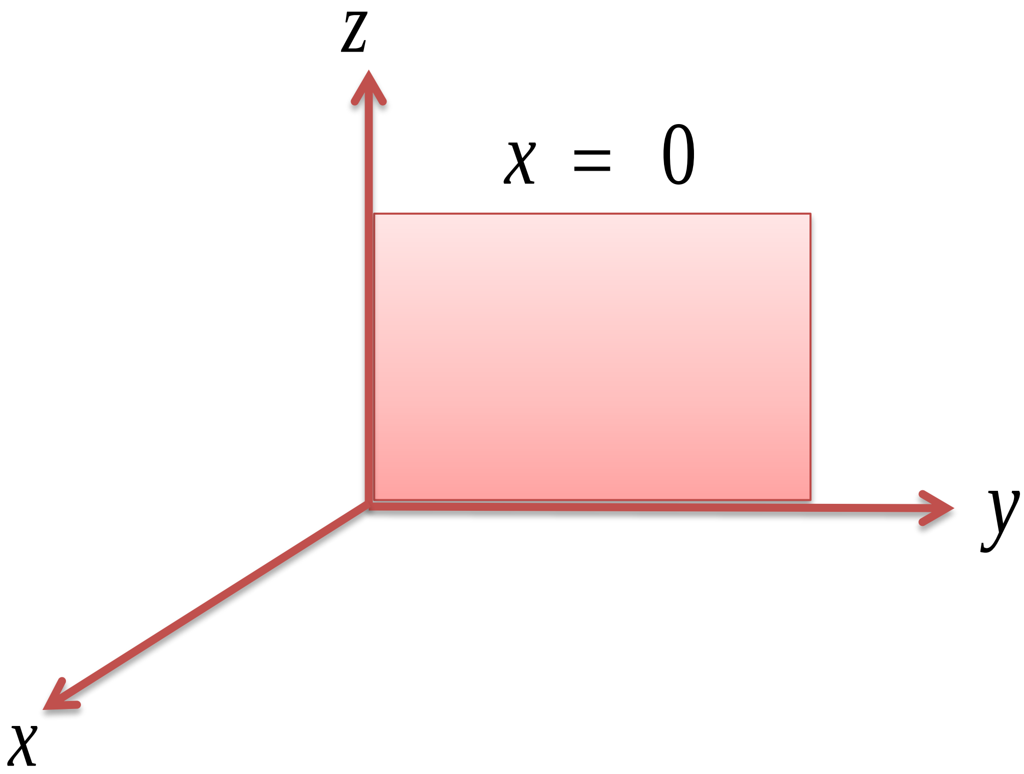
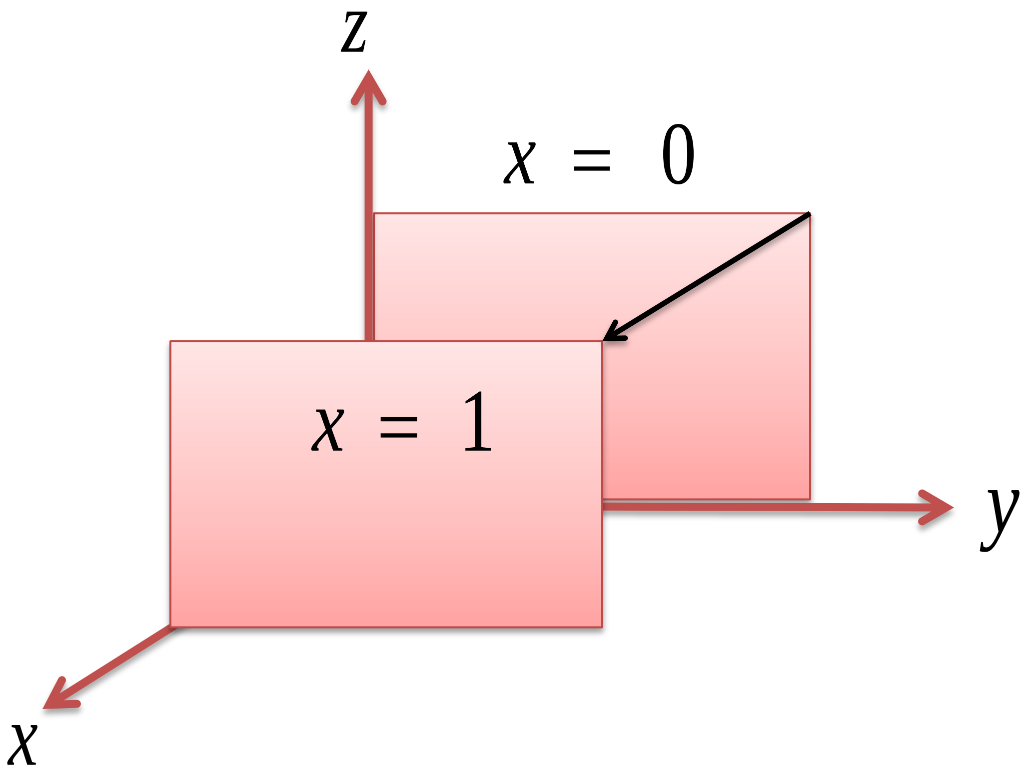


The equation of *yz*-plane is $x = 0$.

Quadric Surfaces



The equation of a plane parallel to *yz*-plane which is one unit upfront is $x = 1$.

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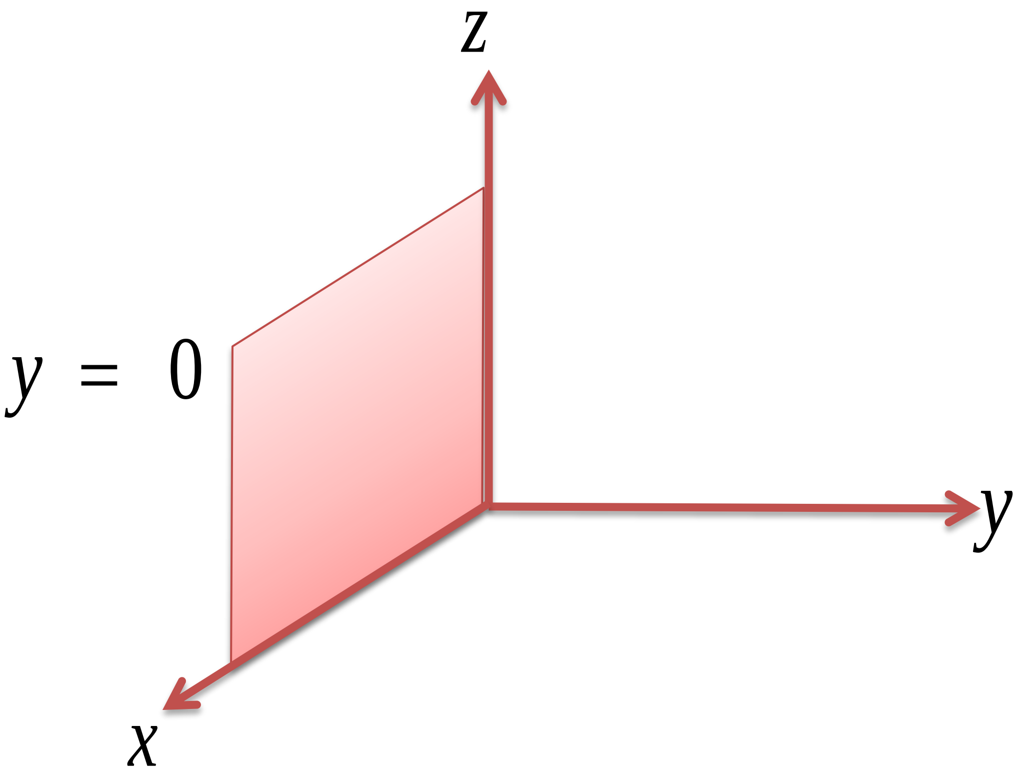
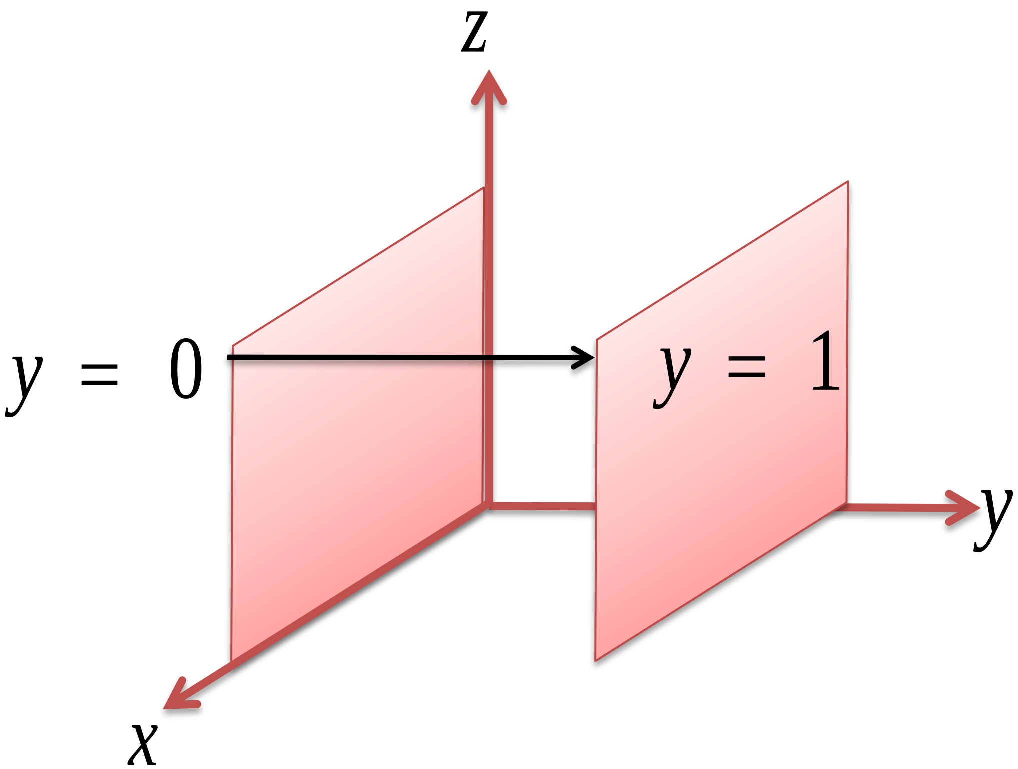


The equation of xz-plane is y = 0.

Quadric Surfaces



The equation of a plane *xz*-plane parallel to *xz*-plane and one unit to right is $y = 1$.

Quadric Surface

A quadric surface is a graph in space of a second degree equation in *x*,*y* and *z*. The general form of the equation is

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fzx + Gx + Hy + Jz + K = 0$$

where *A*,…, *K* are all constants.

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What are the solutions of above equation?

The answer can be found for different possible combinations of constants. Interestingly it leads to the classification of several basic surfaces in 3D. Now we see how to sketch few basic surfaces.

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Sketching:

We study these surfaces with a simple observation that a surface is generated by infinite family of curves that lie on it. Normally a surface can be drawn by tracing out a few curves and joining them together to form it. To do this we need to formally define a procedure which is known as sketching the traces.

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The answer is a circle.

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Trace:

A trace of a surface *S* is a geometrical curve which one obtain on a plane that intersects the surface.

Basic traces are found on the fundamental planes:

- *xy*- plane **z = 0**
- plane parallel to *xy*- plane **z = 1**
- *yz*- plane **x = 0**
- *xz*- plane **y = 0**

Example:

We use basic functions from calculus – 1 and extend them in 3D.

Suppose we have a simple parabola that is to be sketched in 3D

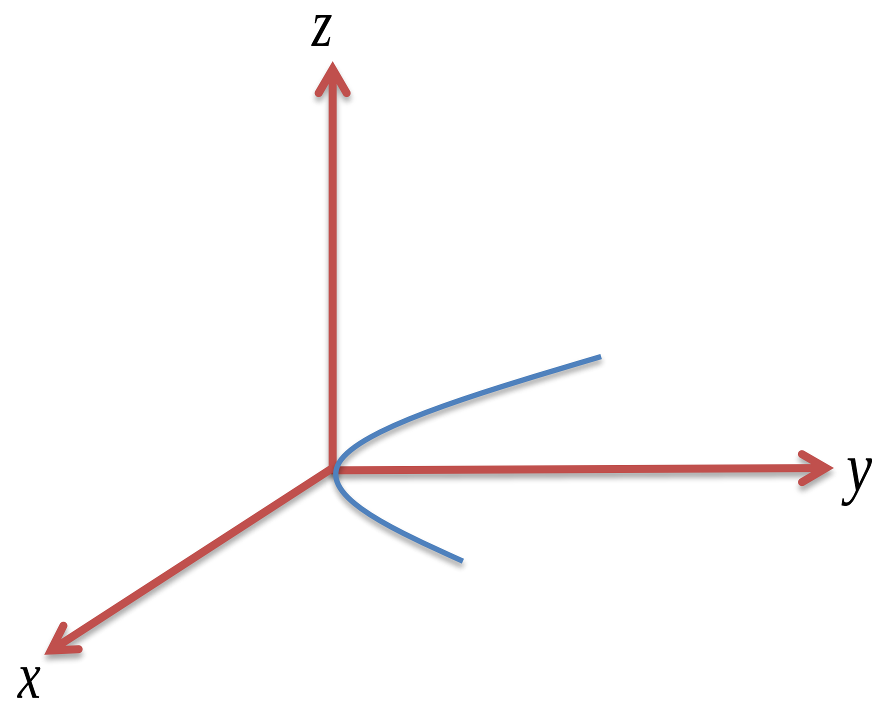
$$y = x^2, \quad 0 < z < 1$$

Step – 1: Sketch the parabola on xy-plane

Step – 2: Sketch a similar parabola on a plane parallel to xy-plane

Step – 3: Join all edges uniformly

Step – 4: Identify the surface



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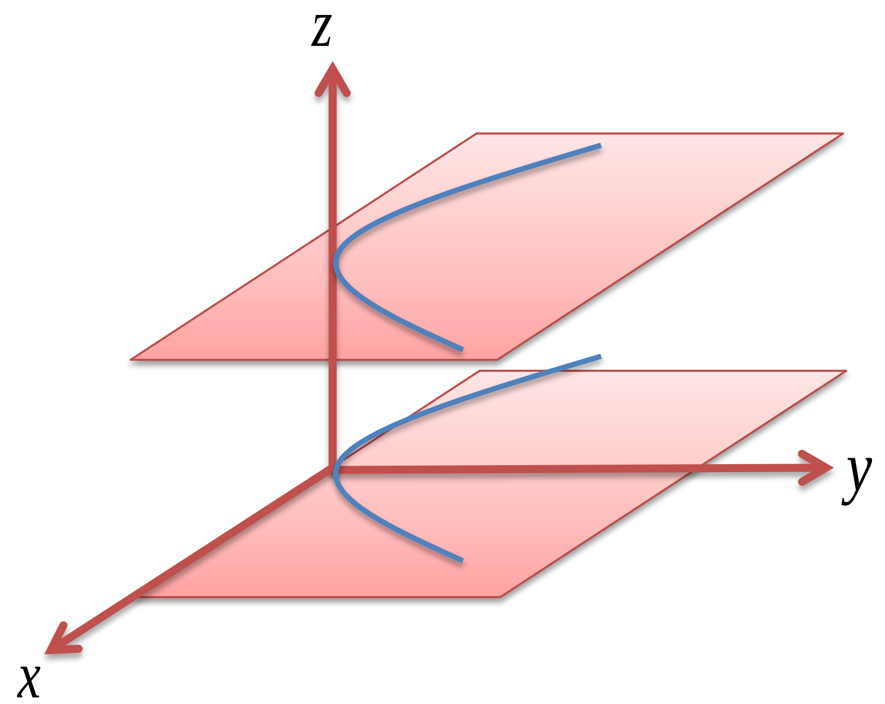

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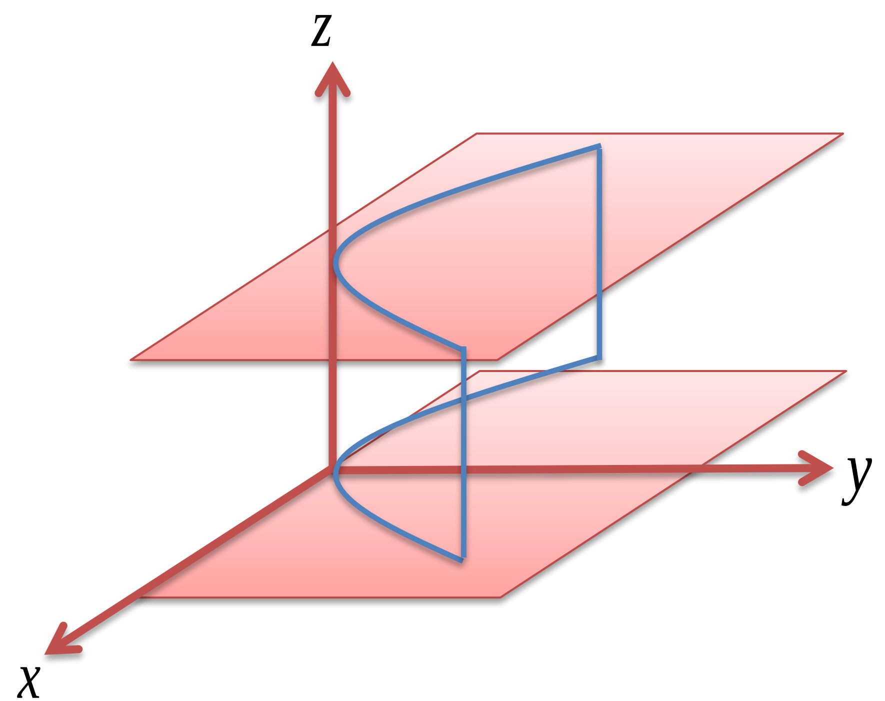

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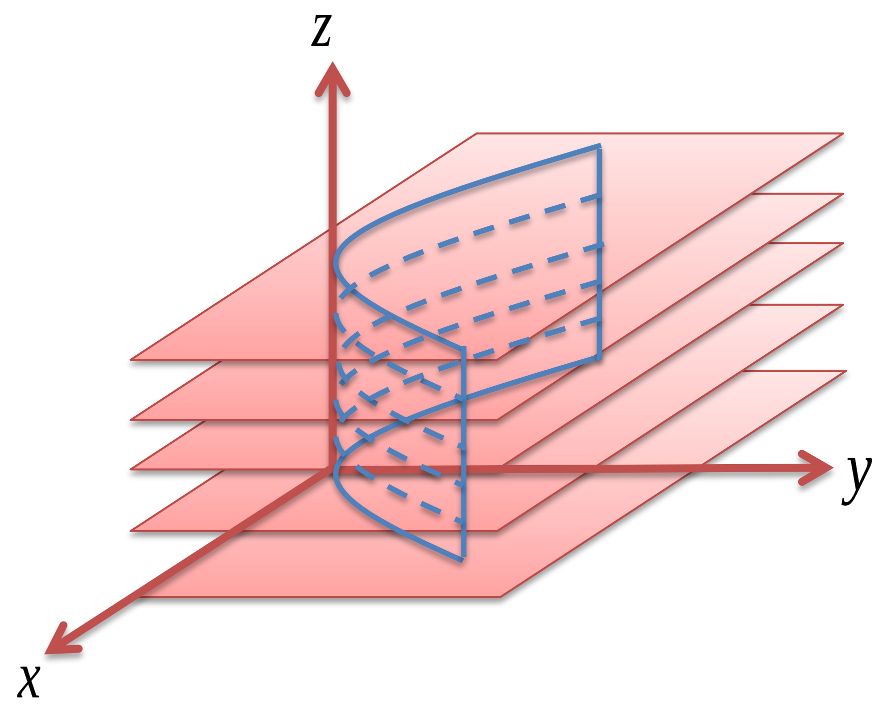

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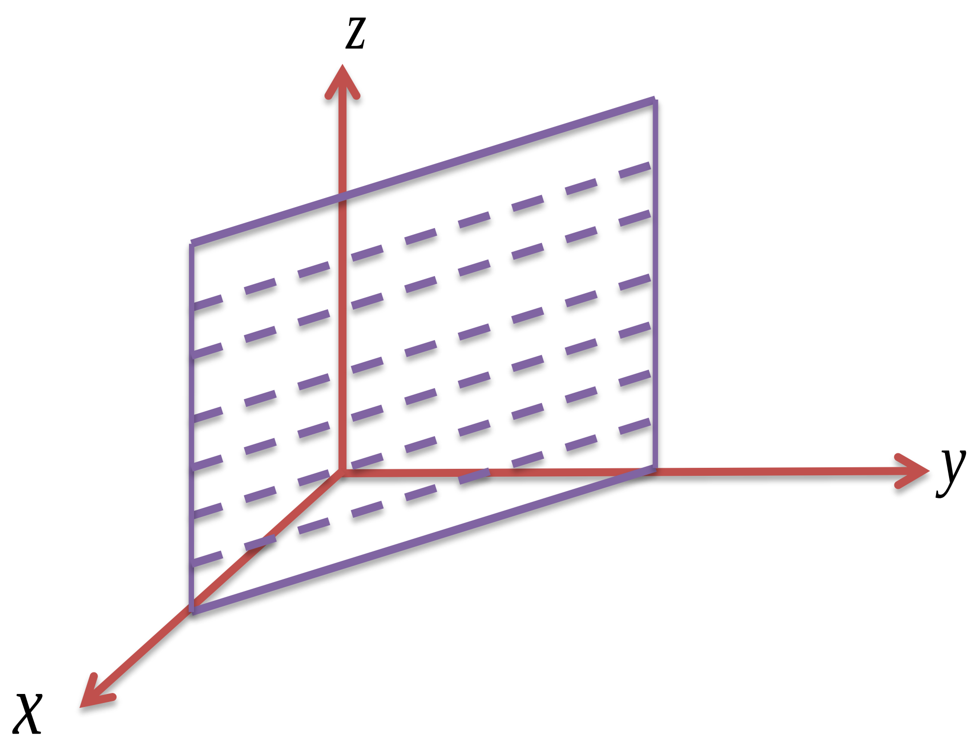

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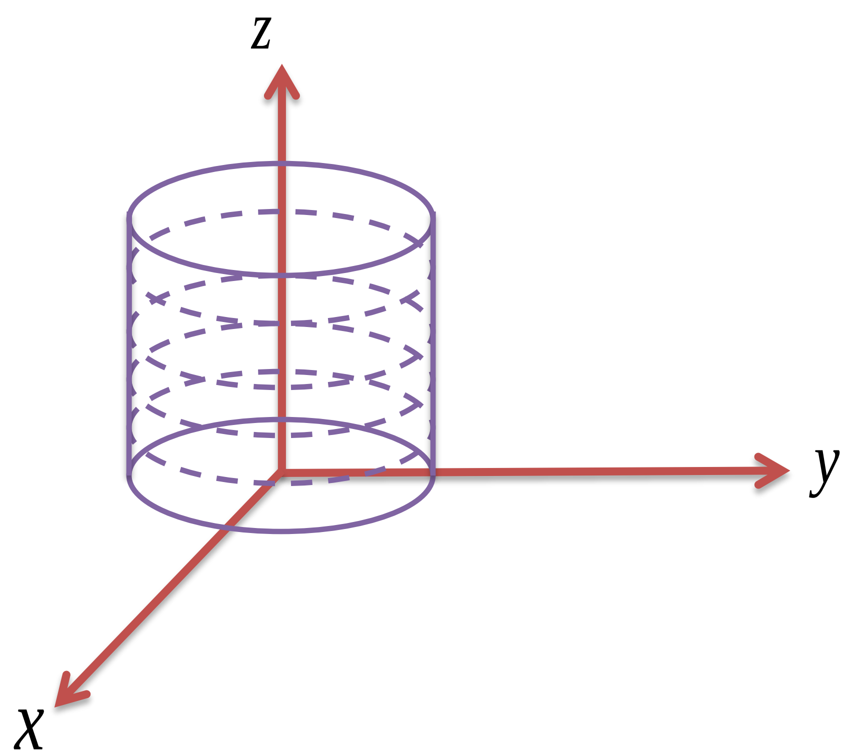

Examples:

Generate 3D surfaces for the following functions

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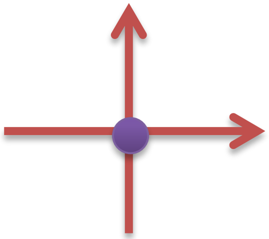

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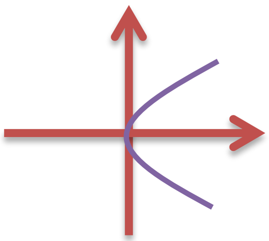

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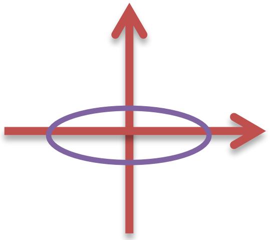

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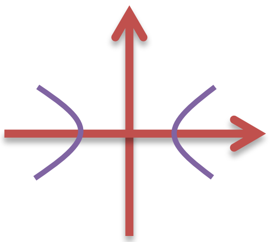

Example:

Follow the same lines and sketch

$$z = x^2, \ -1 < y < 1$$

Sketch

$$y = z^2, \ 0 < x < 1$$

Ellipsoid

Problem: Identify the surface by sketching out traces on different coordinate planes.

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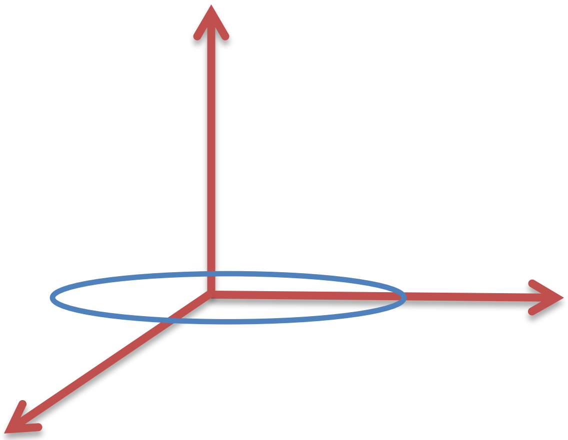

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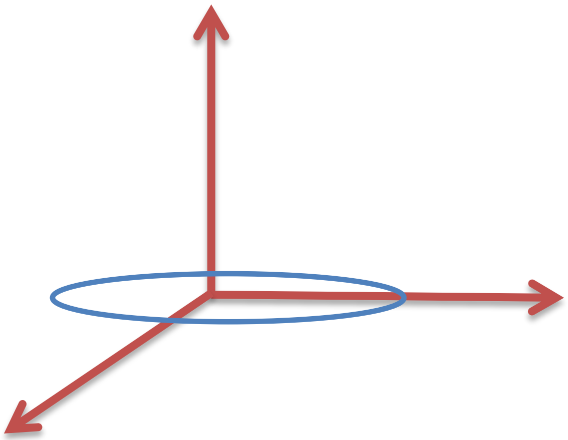

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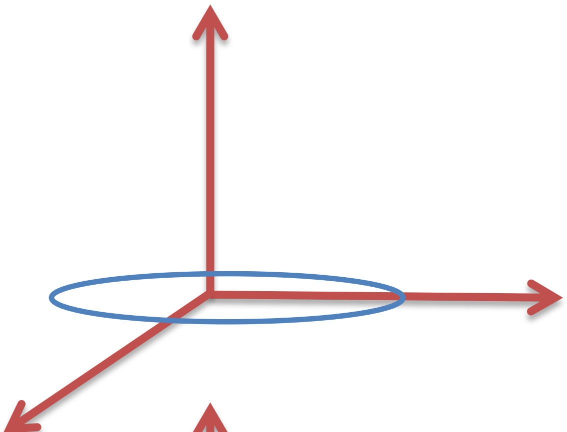
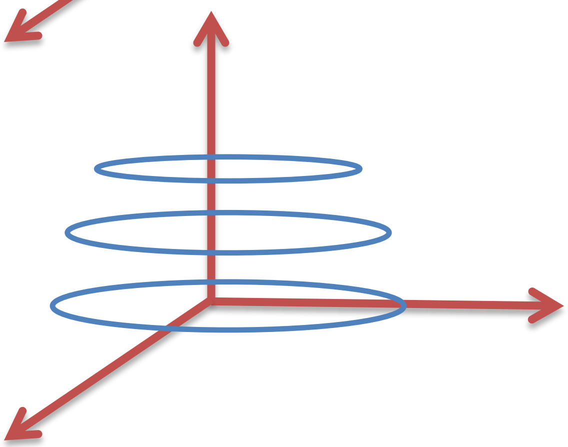
which is an ellipse of shorter width, respectively for z = 2, … Even if we take negative values of z we get the same patter below xy-plane.





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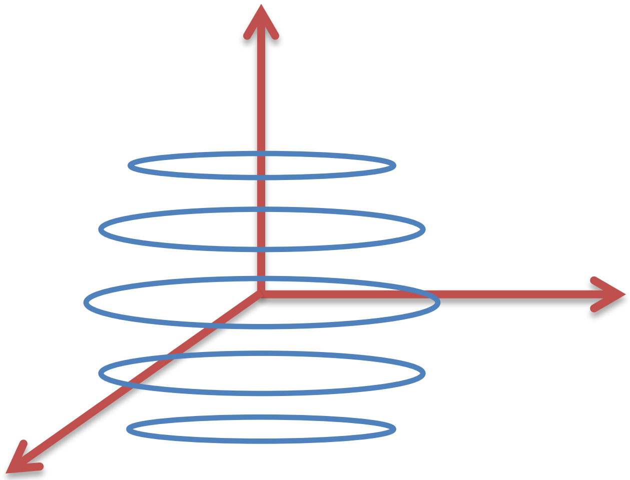

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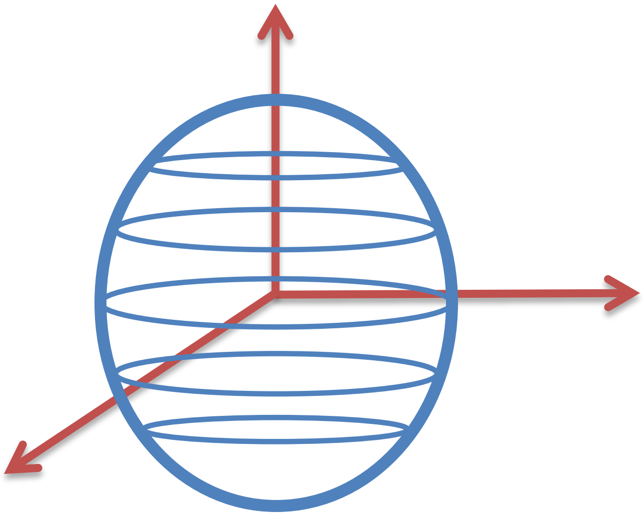
Solution:

1. If we join the edges we get the required surface just using sequence of one type of traces.



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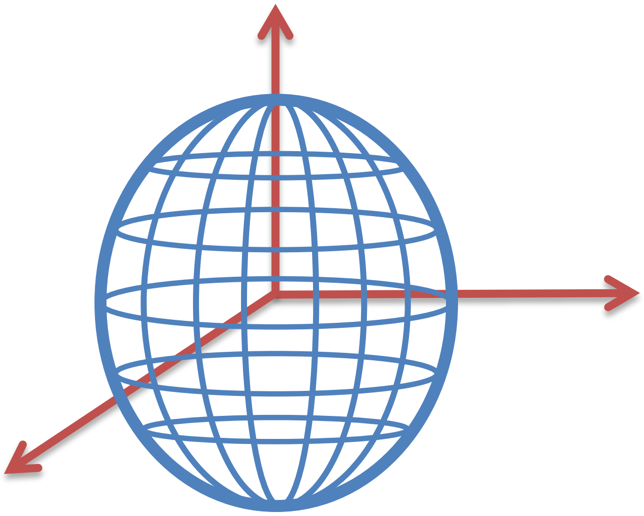
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \qquad b>c>a$$

Solution:

1. If we join the edges we get the required surface just using sequence of one type of traces. However if we take other traces we can easily determine that each trace result into an ellipse.



yz- Trace, x=0; $\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ Ellipse

zx- Trace, y=0; $\frac{x^2}{a^2}+\frac{z^2}{c^2}=1$ Ellipse

Trace on plane parallel to xy plane, z = ± k such that k <c

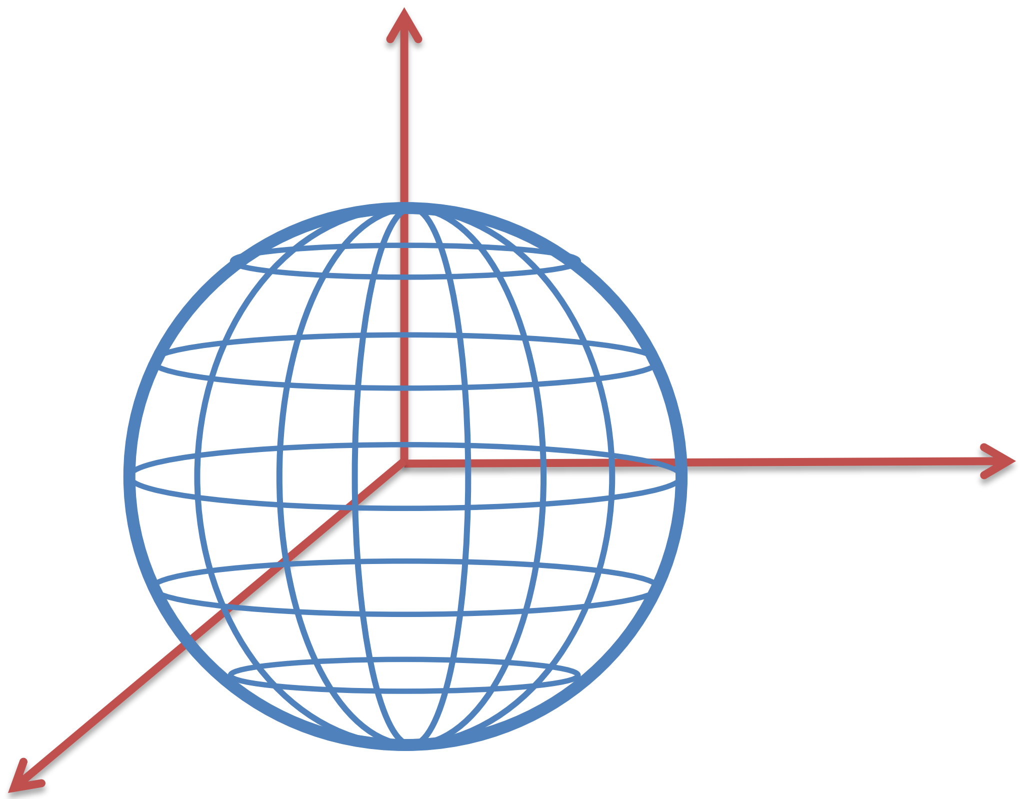
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1-\frac{k^2}{c^2}$$

Sphere

If a = b = c, the surface is a sphere

$$x^2 + y^2 + z^2 = a^2$$

whose traces are circles in different planes.



Elliptic Cone

Problem: Identify the surface by sketching out traces on different coordinate planes.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0 \qquad b > c > a$$

Elliptic Cone

Problem: Identify the surface by sketching out traces on different coordinate planes.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0 \qquad b > c > a$$

Solution: The coordinate with negative coefficient, z axis, is the axis of cone.

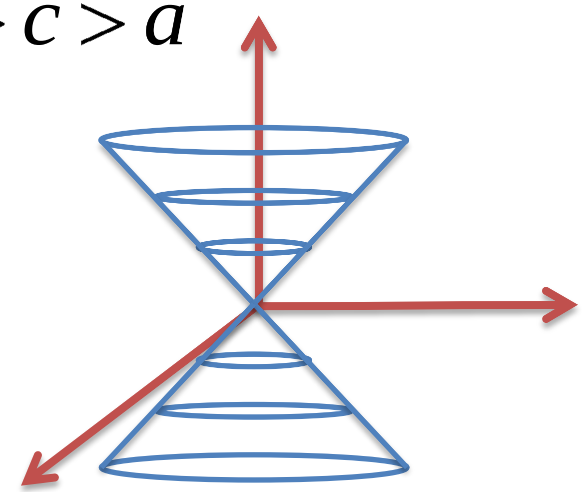
xy- Trace, z=0; $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$ origin

Elliptic Cone

Problem: Identify the surface by sketching out traces on different coordinate planes.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$$

$b > c > a$



Solution: The coordinate with negative coefficient, z axis, is the axis of cone.

xy- Trace, z=0; $\frac{x^2}{a^2}+\frac{y^2}{b^2}=0$ origin

yz- Trace, x=0; $z=\pm(c/b)y$ St. lines intersecting at origin

zx- Trace, y=0; $z=\pm(c/b)x$ St. lines intersecting at origin

Trace on plane parallel to xy plane, z = ± c

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Ellipse

Practice Questions:

Practice-1: Identify the surface by sketching out traces on different coordinate planes.

$$x^2 + y^2 = a^2, -1 < z < 1$$

Practice-2: Identify the surface by sketching out traces on different coordinate planes.

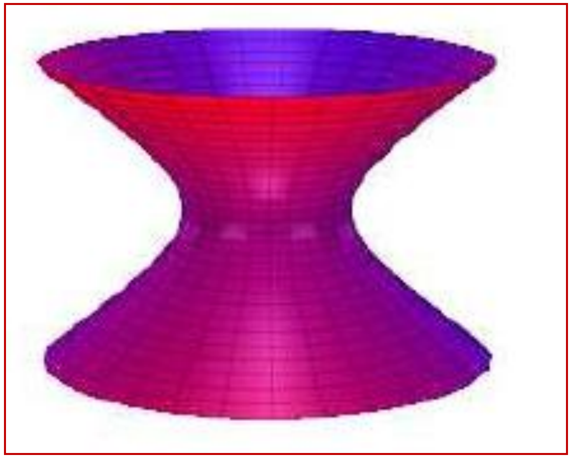
$$-\frac{x^2}{4} + y^2 + z^2 = 0$$

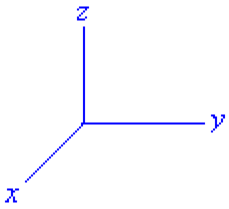
Practice-3: Identify the surface by sketching out traces on different coordinate planes.

$$x^2 - y^2 + z^2 = 1$$

Hyperboloid of one sheet

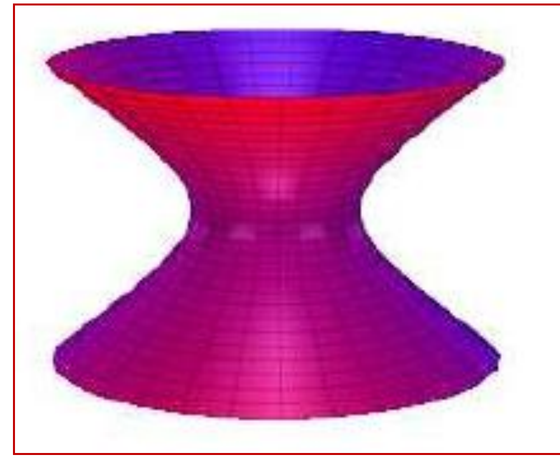
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$



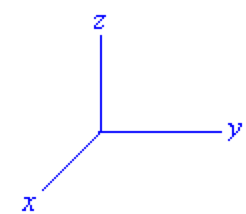


Hyperboloid of one sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$



The coordinate with negative coefficient, Z axis, is the axis of hyperboloid.



xy- Trace, z=0; $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$ Ellipse

yz- Trace, x=0; $$\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$ Hyperbola

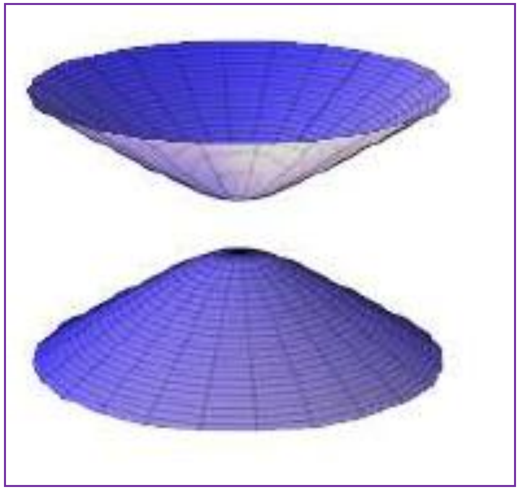
zx- Trace, y=0; $$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1$$ Hyperbola

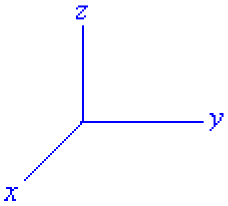
Trace on plane parallel to xy plane, z = ± c

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$$ Ellipse

Hyperboloid of two sheet

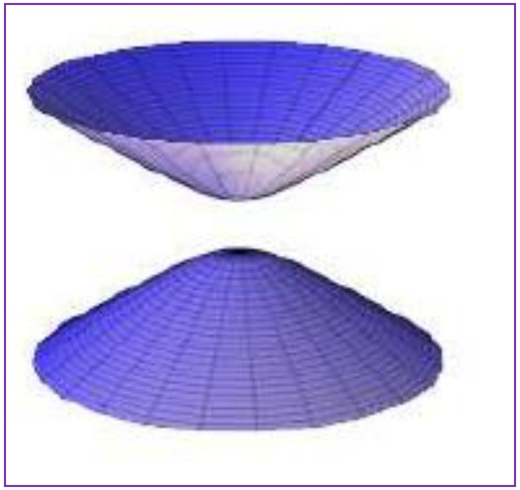
$$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

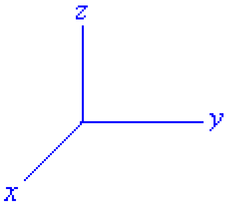




Hyperboloid of two sheet

$$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$





The coordinate with positive coefficient,
Z axis, is the axis of hyperboloid.

xy- Trace, z=0; $-\frac{x^2}{a^2}-\frac{y^2}{a^2}=1$ None

yz- Trace, x=0; $-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ Hyperbola

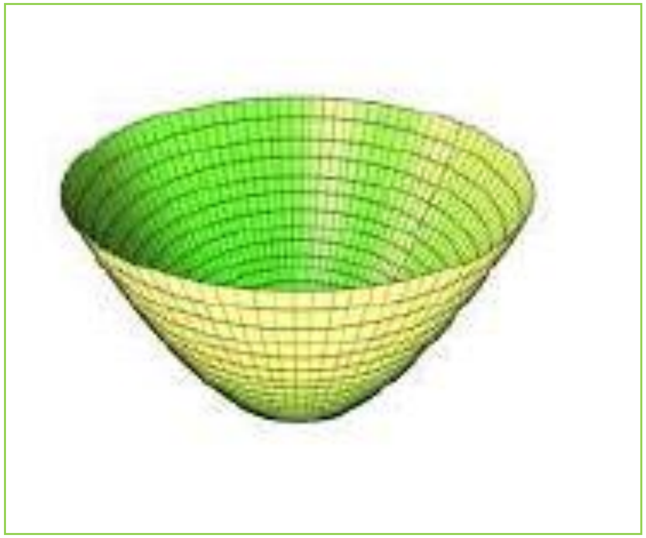
zx- Trace, y=0; $-\frac{x^2}{a^2}+\frac{z^2}{c^2}=1$ Hyperbola

Trace on plane parallel to xy plane, z = ± k such that k>c

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=-1+\frac{k^2}{c^2}$$

Elliptic Paraboloid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$$

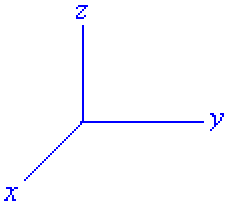


The coordinate with power 1, Z axis, is the axis of elliptic Paraboloid. Paraboloid will lie above xy-plane if c is +ve and below xy-plane if c is –ve.

xy- Trace, z=0; $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$ origin

yz- Trace, x=0; $z = \frac{c}{b^2} y^2$ Parabola

zx- Trace, y=0; $z = \frac{c}{a^2} x^2$ Parabola



Trace on plane parallel to xy plane, z = c

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$ Ellipse

Elliptic Paraboloid

Paraboloid with axis along x-axis

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{x}{a}$$

Paraboloid with axis along y-axis

$$\frac{x^2}{a^2} + \frac{z^2}{c^2} = \frac{y}{b}$$

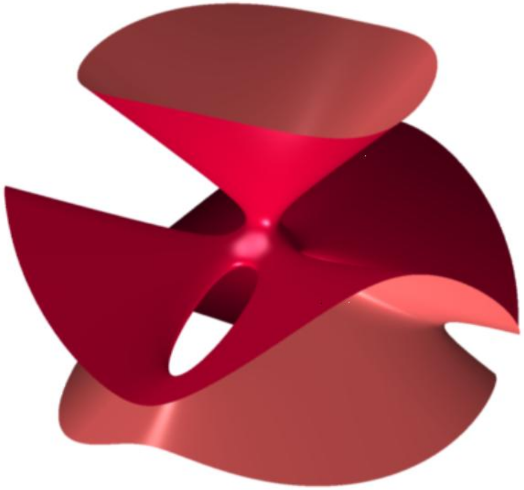
Mathematician World

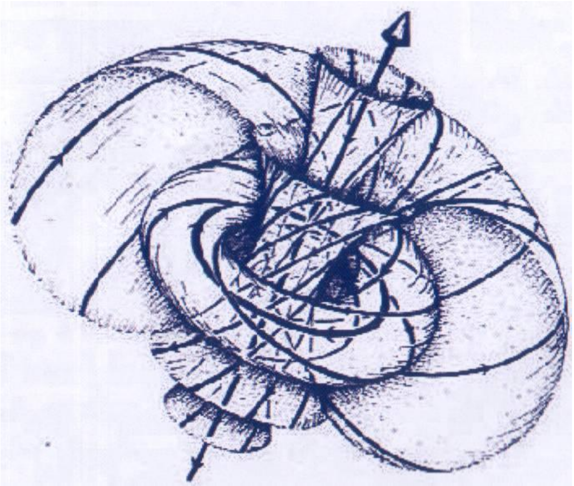
These examples are not part of your course. However it is important to see how the same concept of traces is used to work out surfaces in higher dimensions.

1. Clebsch Surface (1871):

$$x_0 + x_1 + x_2 + x_3 = 0$$

$$x_0^{\ 3} + x_1^{\ 3} + x_2^{\ 3} + x_3^{\ 3} = 0$$





2. Penrose Twistor (1969):

In these sketches they have used the same trick that a surface in higher dimensions can be realized by looking at their intersections with traces (which are 3D hyperplanes).

Mathematician World

These examples are not part of your course. However it is important to see how the same concept of traces is used to work out surfaces in higher dimensions.

3. Klein Bottle (1869):

$$(x^2+y^2+z^2+2y-1)\left[(x^2+y^2+z^2-2y-1)^2-8z^2\right]$$
$$+16xz(x^2+y^2+z^2-2y-1)=0$$