8. Evaluate the following double integrals

(a) $$\int_{-2}^{0}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x^2 \, dx\,dy$$

(b) $$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} \sqrt{x^2+y^2}\, dy\,dx$$

(c) $$\int_{0}^{8}\int_{y^{1/3}}^{2} \frac{y}{x^7+1}\, dx\,dy$$

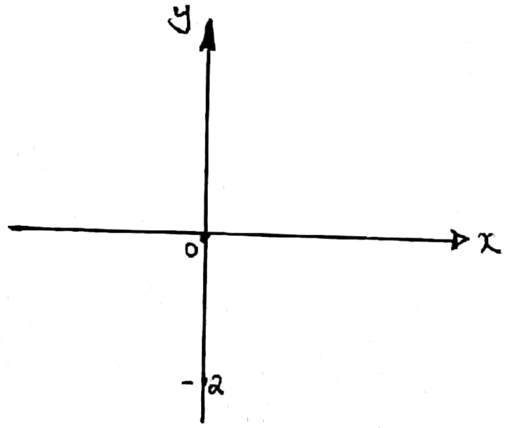
(d) $$\int_{-\varphi}^{0}\int_{\sqrt{-x}}^{2} x^{-2/3}\sqrt{y^{5/3}+1}\, dy\,dx$$

Solution

(a) $$\int_{-2}^{0}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x^2\, dx\,dy$$

$$D = \left\{(x,y) : -2 \leq y \leq 0,\ -\sqrt{1-y^2} \leq x \leq \sqrt{1-y^2}\right\}$$

Sketch of region D.

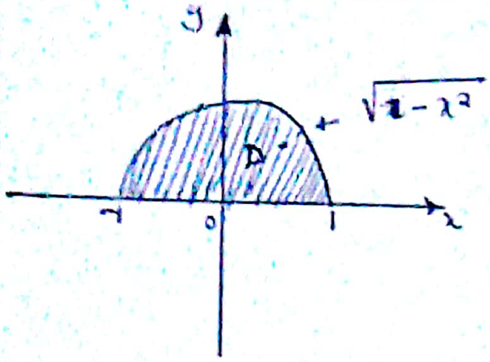


The set Builder notation above does no represent an actual region since there is no range of x values for all . y ; $-2 \leq y < -1$. Hence the region cannot be sketched thus integration is not possible.

(b) $$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} \sqrt{x^2+y^2}\, dy\, dx$$

$$D = \left\{ (x,y) : -1 \leq x \leq 1,\ 0 \leq y \leq \sqrt{1-x^2} \right\}$$

Sketch of the region



in polar Co-ordinates.

$$D = \left\{ (r,\theta) : 0 \leq r \leq 1,\ 0 \leq \theta \leq \pi \right\}.$$

$$\therefore \iint_D \sqrt{x^2+y^2}\, dA = \int_0^1\int_0^{\pi} \sqrt{(r\cos\theta)^2 + (r\sin\theta)^2}\, r\, d\theta\, dr$$

$$= \int_0^1\int_0^{\pi} \sqrt{r^2\cos^2\theta + r^2\sin^2\theta}\, r\, d\theta\, dr$$

$$= \int_0^1\int_0^{\pi} \sqrt{r^2(\cos^2\theta + \sin^2\theta)}\, r\, d\theta\, dr$$

$$= \int_0^1\int_0^{\pi} \sqrt{r^2}\, r\, d\theta\, dr$$

$$= \int_0^1\int_0^{\pi} r^2\, d\theta\, dr$$

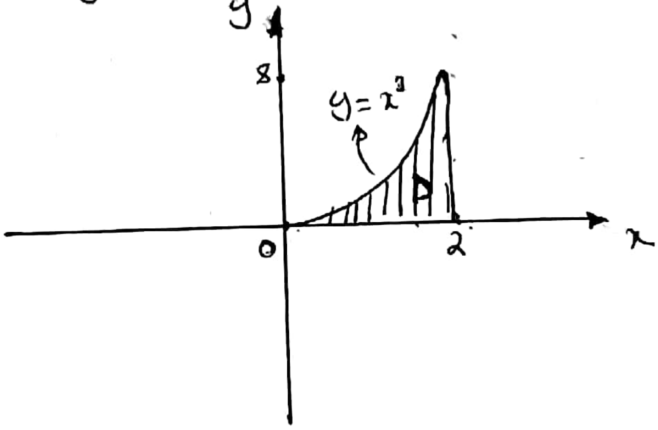
$$= \int_0^1 r^2 \left[\int_0^{\pi} d\theta \right] dr$$

$$= \int_0^1 r^2 \left(\theta \Big|_0^{\pi} \right) dr = \int_0^1 \pi r^2\, dr$$

$$= \pi\int_0^1 r^2 dr = \pi\left[\frac{r^3}{3}\right]_0^1$$
$$= \pi\left[\frac{1^3}{3} - \frac{0^3}{3}\right]$$
$$= \frac{\pi}{3}.$$

(c). $\displaystyle\int_0^8 \int_{y^{1/3}}^2 \frac{y}{x^7+1}\, dx\, dy$

$D = \{(x,y) : 0 \le y \le 8,\ y^{1/3} \le x \le 2\}$



since $y^{1/3} = x$
$\Rightarrow x^3 = y$ or $y = x^3$

$D = \{(x,y) : 0 \le x \le 2,\ 0 \le y \le x^3\}$

$$\therefore \int_0^8 \int_{y^{1/3}}^2 \frac{y}{x^7+1}\, dx\, dy = \int_0^2 \int_0^{x^3} \frac{y}{x^7+1}\, dy\, dx$$

$$= \int_0^2 \left[\int_0^{x^3} \frac{y}{x^7+1}\, dy\right] dx$$

$$= \int_0^2 \frac{1}{2}\left(\frac{y^2}{x^7+1}\right)\Bigg|_0^{x^3} dx$$

$$= \int_0^2 \frac{1}{2}\left[\frac{(x^3)^2}{x^7+1} - \frac{0^2}{x^7+1}\right] dx$$

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$$= \frac{1}{2}\int_0^2 \frac{x^6}{x^7+1}\,dx \qquad \text{let } u = x^7+1$$

$$\frac{du}{7} = \frac{\cancel{7}x^6\,dx}{\cancel{7}}$$

$$= \frac{1}{2}\int_0^2 \frac{x^6}{x^7+1}\,dx \qquad \Rightarrow x^6\,dx = \frac{1}{7}du$$

$$= \frac{1}{2}\int \frac{\left(\frac{1}{7}du\right)}{u} = \frac{1}{14}\int \frac{du}{u}$$

$$= \frac{1}{14}\ln(x^7+1)\Big|_0^2$$

$$= \frac{1}{14}\left(\ln(2^7+1) - \ln(0^7+1)\right)$$

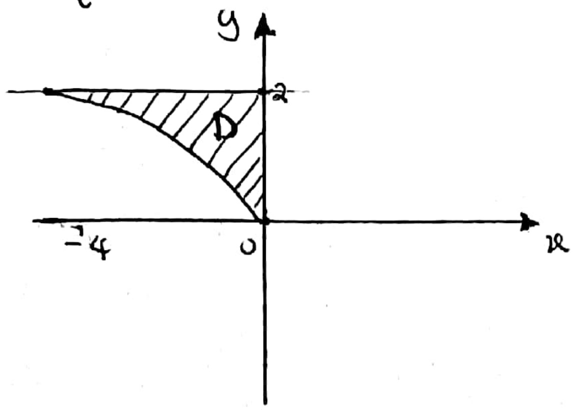
$$= \frac{1}{14}\left(\ln(129) - \ln(1)\right)$$

$$= \frac{\ln 129}{14}$$

(d). $$\int_{-4}^{0}\int_{\sqrt{-x}}^{2} x^{-2/3}\sqrt{y^{5/3}+1}\;dy\,dx$$

$$D = \{(x,y) : -4 \leq x \leq 0,\ \sqrt{-x} \leq y \leq 2\}$$

$$y = \sqrt{-x} \Rightarrow x = -y^2$$



$$D = \{(x,y) : 0 \leq y \leq 2,\ -y^2 \leq x \leq 0\}$$

$$\int_{-4}^{0}\int_{\sqrt{-x}}^{2} x^{-2/3}\sqrt{y^{5/3}+1}\;dy\,dx = \int_{0}^{2}\int_{-y^2}^{0} x^{-2/3}\sqrt{y^{5/3}+1}\;dx\,dy$$

$$= \int_{0}^{2}\sqrt{y^{5/3}+1}\left[\int_{-y^2}^{0} x^{-2/3}\,dx\right]dy$$

$$= \int_{0}^{2}\sqrt{y^{5/3}+1}\left[3x^{1/3}\right]_{-y^2}^{0}dy$$

$$= \int_{0}^{2} 3\sqrt{y^{5/3}+1}\left(0-(-y^2)^{1/3}\right)dy$$

$$= 3\int_{0}^{2} y^{2/3}\sqrt{y^{5/3}+1}\;dy$$

Let $u = y^{5/3}+1 \Rightarrow du = \frac{5}{3}y^{2/3}\,dy$

$\Rightarrow \underline{y^{2/3}\,dy} = \frac{3}{5}\,du$

$$3\int_{0}^{2} y^{2/3}\sqrt{y^{5/3}+1}\;dy = 3\int\sqrt{u}\left(\frac{3}{5}\,du\right)$$

$$= \frac{9}{5}\int\sqrt{u}\,du$$

$$= \frac{\cancel{9}^{3}}{5}\,u^{3/2}\times\frac{2}{\cancel{3}}$$

$$= \frac{6}{5}\,u^{3/2}$$

$$= \frac{6}{5}\left(y^{5/3}+1\right)^{3/2}\Big|_{0}^{2}$$

$$= \frac{6}{5}\left[\left(2^{5/3}+1\right)^{3/2}-\left(0^{5/3}+1\right)^{3/2}\right]$$

$$= \frac{6}{5}\left[\left(2^{5/3}+1\right)^{3/2}-1\right]$$

$$= \frac{6}{5}\left(2^{5/3}+1\right)^{3/2}-\frac{6}{5}$$

9 Use the double integrals to determine the area of the following regions in the xy-plane.

(a) The region bounded by $y = x^2+1$ and $y = \frac{1}{2}x^2+3$

(b) The region bounded by $x = -y^2$ and $x = y-6$

Solution

To find intercept solve simultaneously.

$x^2+1 = \frac{1}{2}x^2+3$

$\Rightarrow x^2 - \frac{1}{2}x^2 = 3-1$

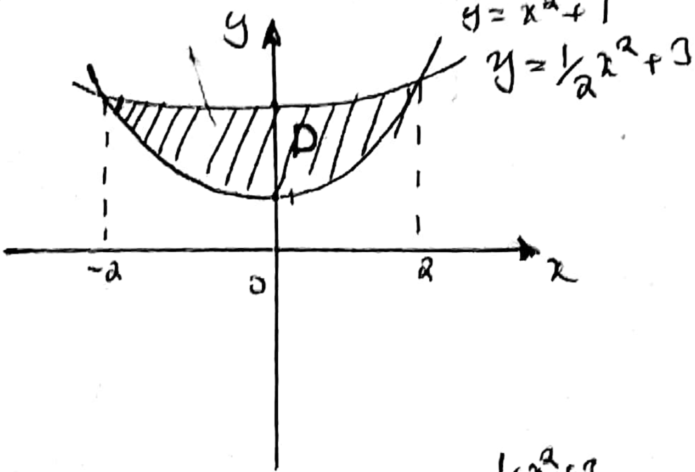
$\Rightarrow \frac{1}{2}x^2 = 2 \Rightarrow x^2 = 4$

$\Rightarrow x^2-4 = 0 \Rightarrow (x-2)(x+2) = 0$

$\Rightarrow x = 2$ or $x = -2$.

$y = (2)^2+1 = 5$ for $x = 2$

$y = (-2)^2+1 = 5$ for $x = -2$



$\therefore$ Area $= \iint_D dA = \int_{-2}^{2}\int_{x^2+1}^{\frac{1}{2}x^2+3} dy\,dx$

$= \int_{-2}^{2} y \Big|_{x^2+1}^{\frac{1}{2}x^2+3} dx$

$= \int_{-2}^{2}\left(\left[\frac{1}{2}x^2+3\right] - \left[x^2+1\right]\right) dx$

$$= \int_{-2}^{2} \left[-\tfrac{1}{2}x^2 + 2\right] dx$$

$$= \left[-\frac{x^3}{6} + 2x\right]_{-2}^{2}$$

$$= \left[-\frac{(2)^3}{6} + 2(2)\right] - \left[-\frac{(-2)^3}{6} + 2(-2)\right]$$

$$= \left[-\frac{8}{6} + 4\right] - \left[\frac{8}{6} - 4\right]$$

$$= \left[-\frac{16}{6} + 8\right]$$

$$= \left[-\frac{8}{3} + 8\right] = \frac{16}{3} \text{ units}^2$$

(b) $x = -y^2$, $x = y - 6$

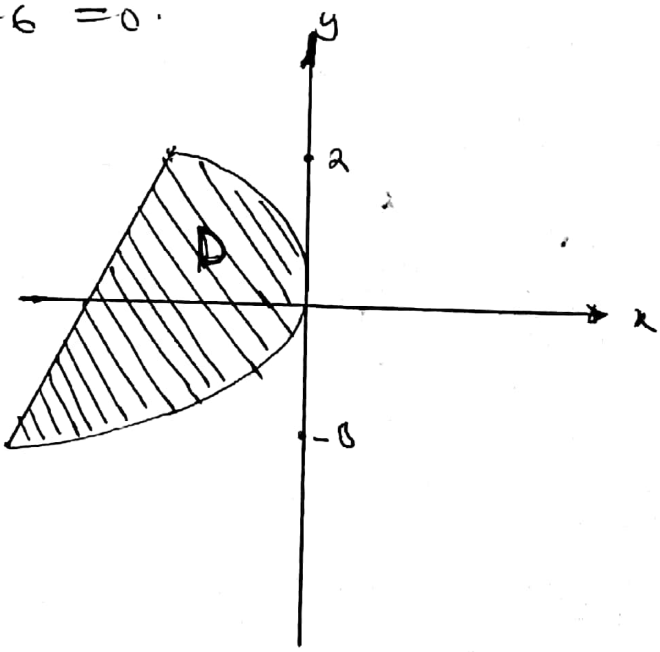
$-y^2 = y - 6 \Rightarrow y^2 + y - 6 = 0$.

$\Rightarrow y^2 + 3y - 2y - 6 = 0$.

$\Rightarrow y(y+3) - 2(y+3) = 0$.

$\Rightarrow (y-2)(y+3) = 0$

$\Rightarrow y = 2$ or $y = -3$.



$x = 2 - 6$
$= -4$, $y = 2$

$x = -3 - 6 = -9$
$y = -3$.

$$\iint_D dA = \int_{-3}^{2} \int_{y-6}^{-y^2} dx\, dy$$

$$= \int_{-3}^{2} x \Big|_{y-6}^{-y^2} dy$$

$$= \int_{-3}^{2} \left(-y^2 - (y-6)\right) dy$$

$$= \int_{-3}^{2} \left(-y^2 - y + 6\right) dy$$

$$= \left[-\frac{y^3}{3} - \frac{y^2}{2} + 6y \right]_{-3}^{2}$$

$$= \left[-\frac{(2)^3}{3} - \frac{(2)^2}{2} + 6(2) \right] - \left[-\frac{(-3)^3}{3} - \frac{(-3)^2}{2} + 6(-3) \right]$$

$$= \left[-\frac{8}{3} - \frac{4}{2} + 12 \right] - \left[+\frac{27}{3} - \frac{9}{2} - 18 \right]$$

$$= \left[-\frac{8}{3} + 10 \right] - \left[-9 - \frac{9}{2} \right]$$

$$= \left[\frac{22}{3} \right] + \left[\frac{27}{2} \right]$$

$$= \left[\frac{44}{6} \right] + \left[\frac{81}{6} \right]$$

$$= \frac{125}{6}$$

10. Use a double integral to determine the volume of the following solids.

(a) The solid bounded by the planes $z = 4 - 2x - 2y$, $y = 2x$, $x = 0$, and $z = 0$.

(b) The solid ~~bounded by the~~ that is inside both the Cylinder $x^2 + y^2 = 9$ and $x^2 + y^2 + z^2 = 16$

(c) The solid that is bounded by $z = 12 - 3x^2 - 3y^2$ and $z = x^2 + y^2 - 8$

Solution

find intersection of planes.

$x = 0$, $z = 4 - 2x - 2y \Rightarrow z = 4 - 2y$.

$z = 0$, $z = 4 - 2x - 2y \Rightarrow 0 = 4 - 2x - 2y$

$\Rightarrow \frac{2x + 2y}{2} = \frac{4}{2} \Rightarrow x + y = 2.$

for $z = 4 - 2y$, when $y = 0$ $z = 4$.

when $z = 0$ $4 - 2y = 0 \Rightarrow 2y = 4$

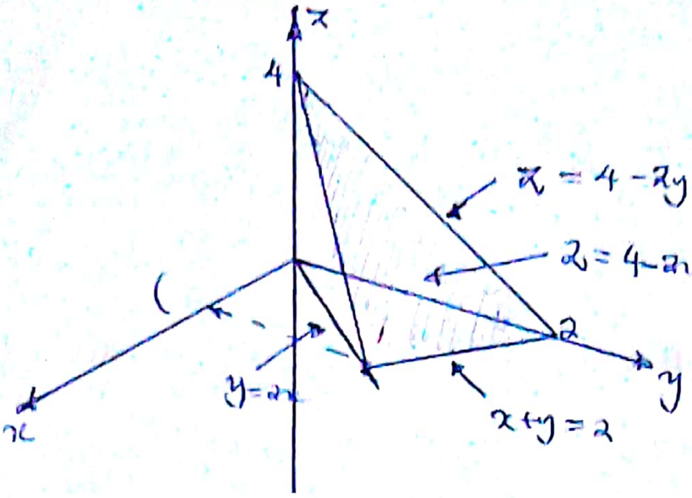
$\Rightarrow y = 2.$

also for $x + y = 2$, and $y = 2x$

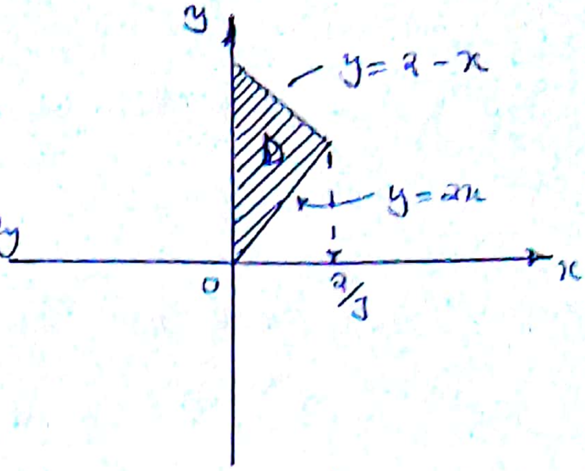
$\Rightarrow 2x + x = 2 \Rightarrow \frac{3x}{3} = \frac{2}{3}$

$x = 2/3$, $y = 2(2/3) = 4/3$

Sketch 3D



Sketch of region D



$$\therefore V = \iint_D f(x,y)\, dA = \int_0^{2/3} \int_{2x}^{2-x} (4 - 2x - 2y)\, dy\, dx$$

$$= \int_0^{2/3} \left[4y - 2xy - 2y^2/2 \right]_{2x}^{2-x} dx$$

$$= \int_0^{2/3} \left[4y - 2xy - y^2 \right]_{2x}^{2-x} dx$$

$$= \int_0^{2/3} \Big(\big(4(2-x) - 2x(2-x) - (2-x)^2\big) - \big(4(2x) - 2x(2x) - (2x)^2\big) \Big) dx$$

$$= \int_0^{2/3} \Big[\big(8 - 4x - 4x + 2x^2 - (4 - 4x + x^2)\big) - \big(8x - 4x^2 - 4x^2\big) \Big] dx$$

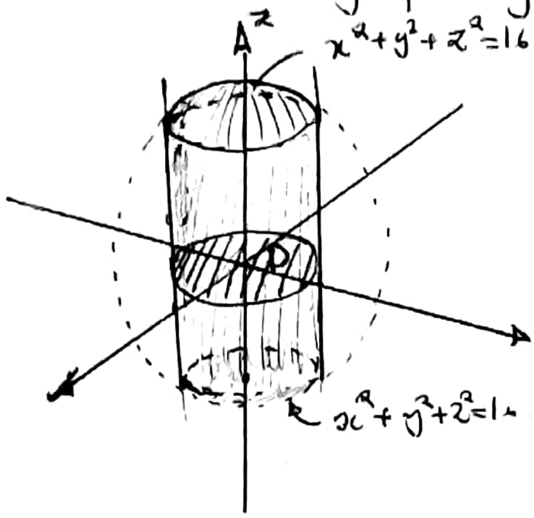
$$= \int_0^{2/3} \Big[\big(4 - 4x + x^2\big) - \big(8x - 4x^2 - 4x^2\big) \Big] dx$$

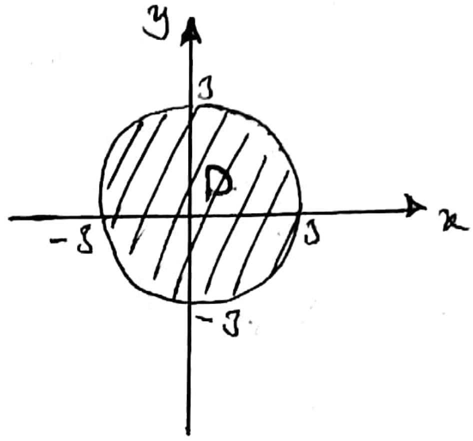
$$= \int_0^{2/3} \left(4 - 12x + 9x^2 \right) dx$$

$$= \left[4x - 6x^2 + 3x^3 \right]_0^{2/3}$$

$$= \left[4\left(\tfrac{2}{3}\right) - 6\left(\tfrac{2}{3}\right)^2 + 3\left(\tfrac{2}{3}\right)^3 \right] - \left[4(0) - 6(0)^2 + 3(0)^3 \right] = \underline{\underline{\tfrac{8}{9} \text{ units}^3}}$$

(b). The region inside $x^2+y^2=9$ and $x^2+y^2+z^2=16$ can be shown graphically.





$$D=\{(r,\theta):0\le r\le 3,\ 0\le\theta\le 2\pi\}$$

But the bottom part is:

$z=-\sqrt{16-x^2-y^2}$

while the top part is:

$z=\sqrt{16-x^2-y^2}$

since $x^2+y^2=9$

$\Rightarrow x^2+y^2=r^2$

hence the integral will be from

$$V=\int_0^3\int_0^{2\pi}\sqrt{16-(r\cos\theta)^2-(r\sin\theta)^2}-\left(-\sqrt{16-(r\cos\theta)^2-(r\sin\theta)^2}\right)r\,d\theta\,dr$$

$$\Rightarrow V=\int_0^3\int_0^{2\pi}r\left(\sqrt{16-r^2}+\sqrt{16-r^2}\right)d\theta\,dr$$

$$V=\int_0^3\int_0^{2\pi}2r\sqrt{16-r^2}\,d\theta\,dr$$

$$V=\int_0^3 2r\sqrt{16-r^2}\int_0^{2\pi}d\theta\,dr$$

$$=\int_0^3(2r)(\sqrt{16-r^2})\left[2\pi-0\right]dr$$

$$=\int_0^3(2r)(\sqrt{16-r^2})(2\pi)\,dr=4\pi\int_0^3 r\sqrt{16-r^2}\,dr$$

$$V = 4\pi \int_0^3 r\sqrt{4^2 - r^2}\, dr$$

let $u = 4^2 - r^2 \Rightarrow du = -2r\,dr$

$\Rightarrow r\,dr = -\frac{1}{2}du$

$$V = 4\pi \int \sqrt{u}\left(-\tfrac{1}{2}du\right)$$

$$= -2\pi \int \sqrt{u}\, du$$

$$= -2\pi\, u^{3/2} \cdot \tfrac{2}{3}$$

$$= -\tfrac{4\pi}{3} \left(16 - r^2\right)^{3/2}\Big|_0^3$$

$$= -\tfrac{4\pi}{3}\left[\left(16 - 3^2\right)^{3/2} - \left(16 - 0\right)^{3/2}\right]$$

$$= -\tfrac{4\pi}{3}\left[\left(16 - 9\right)^{3/2} - (16)^{3/2}\right]$$

$$= -\tfrac{4\pi}{3}\left[(7)^{3/2} - 64\right]$$

$$= \tfrac{4\pi}{3}\left[64 - 7^{3/2}\right] \text{ unit}^3$$

(c) $z = 12 - 3x^2 - 3y^2$, and $z = x^2 + y^2 - 8$

intersection

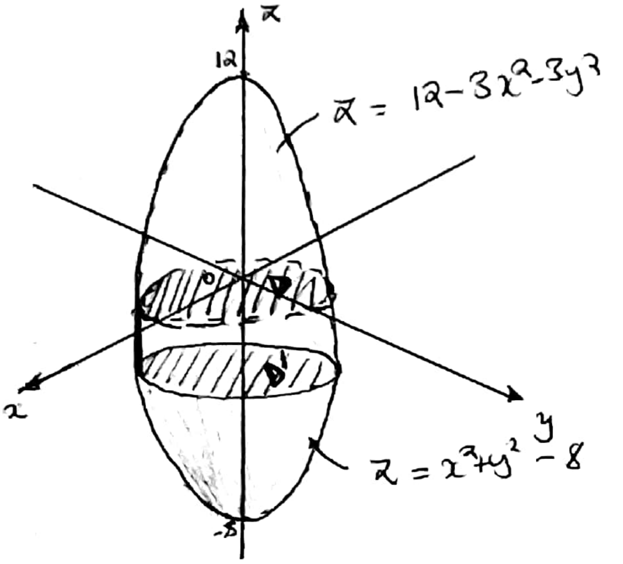
$\Rightarrow \quad 12 - 3x^2 - 3y^2 = x^2 + y^2 - 8.$

$\Rightarrow \quad 12 + 8 = x^2 + 3x^2 + y^2 + 3y^2$

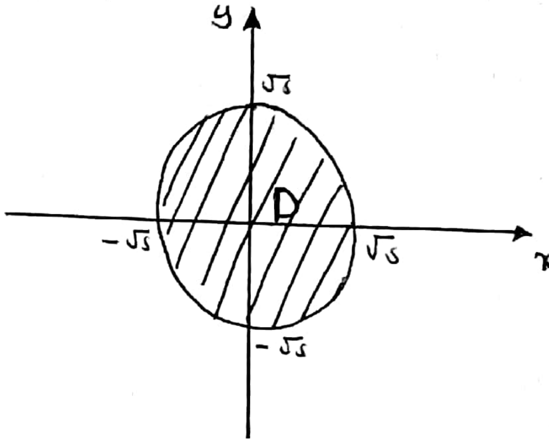
$\Rightarrow \quad \frac{20}{4} = \frac{\cancel{4}x^2}{\cancel{4}} + \frac{\cancel{4}y^2}{\cancel{4}}$

$x^2 + y^2 = 5 = (\sqrt{5})^2$

$z = 5 - 8 = -3.$



Projecting the region onto the xy-plane we have.



$\therefore D = \{(r,\theta) : 0 \le r \le \sqrt{5} , 0 \le \theta \le 2\pi\}$

$$\therefore V = \iint_D z_2(x,y) - z_1(x,y) \, dA$$

$$= \int_0^{\sqrt{5}} \int_0^{2\pi} (12 - 3x^2 - 3y^2) - (x^2 + y^2 - 8) \, r \, d\theta \, dr$$

$$= \int_0^{\sqrt{5}} \int_0^{2\pi} (20 - 4(r\cos\theta)^2 - 4(r\sin\theta)^2) \, r \, d\theta \, dr.$$

$$= \int_0^{\sqrt{5}} \int_0^{2\pi} \left(20 - 4r^2(\cos^2\theta + \sin^2\theta)\right) r \, d\theta \, dr$$

$$= \int_0^{\sqrt{5}} \int_0^{2\pi} (20 - 4r^2) \, r \, d\theta \, dr$$

$$V = \left[\int_0^{\sqrt{5}} (20r - 4r^3)\,dr\right]\left[\int_0^{2\pi} d\theta\right]$$

$$V = \left(\left[10r^2 - r^4\right]_0^{\sqrt{5}}\right)\left(\left[\theta\right]_0^{2\pi}\right)$$

$$= \left(\left[10(\sqrt{5})^2 - (\sqrt{5})^4\right] - \left[10(0)^2 - (0)^4\right]\right)\left(2\pi - 0\right)$$

$$= \left(50 - 25\right)\left(2\pi\right)$$

$$= (25)(2\pi) = \underline{50\pi}$$

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