Fourier Series: Examples

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1 Important Facts

1. Suppose $f(x)$ is a periodic function of period 2π which can be represented by a **TRIGONOMETRIC FOURIER SERIES**

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx.$$

(This means that the series above converges to $f(x)$.)

Then the **Fourier Coefficients** satisfy the **Euler Formulae**, namely:

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \quad \text{for } n = 1, 2, \ldots \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \quad \text{for } n = 1, 2, \ldots \end{aligned}$$

2. A function f is said to be **even** if

$$f(-x) = f(x) \quad \text{for all } x \in \mathbb{R}$$

and **odd** if

$$f(-x) = -f(x) \quad \text{for all } x \in \mathbb{R}$$

Recall the product of two even functions is even, the product of two odd functions is even and the product of an even and an odd function is odd. Compare the **multiplication** of even and odd **functions** to the **addition** of even and odd **integers**.

3. If f is an **odd** function then

$$\int_{-\pi}^{\pi} f(x)\,dx = 0,$$

while if f is an even function, then

$$\int_{-\pi}^{\pi} f(x)\,dx = 2\int_{0}^{\pi} f(x)\,dx$$

2 Exercises and Examples

Example 1. *Let f be a periodic function of period 2π such that*

$$f(x) = \pi^2 - x^2 \quad \textit{for } x \in (-\pi, \pi).$$

Supposing that f has a convergent trigonometric Fourier series, show that

$$\pi^2 - x^2 = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty} \frac{-4}{n^2}(-1)^n \cos nx. \tag{2.1}$$

SOLUTION: The solution can be effected in a number of separate steps:

- Check whether f is even or odd.
- If f is **odd**, all the Fourier coefficients $\mathbf{a_n}$ for $n = 0, 1, 2\ldots$ are **zero**; if f is **even**, all the Fourier coefficients $\mathbf{b_n}$ for $n = 1, 2\ldots$ are **zero**.
- Compute the remaining Fourier coefficients using the Euler Formulae. It is generally a good strategy to use **Integration by Parts**, successively **integrating** $\sin nx$ **and** $\cos nx$ and **differentiating** $f(x)$.
- Replace the expressions for the Fourier coefficients a_n, b_n in

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx.$$

STEP 1: $f(-x) = \pi^2 - (-x)^2 = \pi^2 - x^2 = f(x)$ so f is even.

STEP 2: Since $f(x)$ is even and $\sin nx$ is odd, $f(x)\sin nx$ is odd and hence

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0.$$

STEP 3: Since $f(x)$ is even and $\cos nx$ is even, $f(x)\cos nx$ is even, and so

$$\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 2\int_0^{\pi} f(x)\cos nx\,dx.$$

Therefore,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}2\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi}(\pi^2 - x^2)\cos nx \tag{2.2}$$

As suggested above, we calculate the integral in (2.2) by **Integration by Parts**. Recall the **Integration by Parts** formula:

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_{x=a}^{b} - \int_a^b f'(x)g(x)\,dx \tag{2.3}$$

Let

$$f(x) = \pi^2 - x^2 \quad \text{and} \quad g'(x) = \cos nx$$

so

$$f'(x) = -2x \quad \text{and} \quad g(x) = \int \cos nx\,dx = \frac{1}{n}\sin nx.$$

Using (2.3) and the above, we have

$$\begin{aligned}
&\int_0^{\pi} \underbrace{(\pi^2 - x^2)}_{f}\underbrace{\cos nx}_{g'}\,dx && (2.4)\\
&= \underbrace{(\pi^2 - x^2)}_{f}\underbrace{\frac{1}{n}\sin nx}_{g}\Big|_0^{\pi} - \int_0^{\pi}\underbrace{-2x}_{f'}\underbrace{\frac{1}{n}\sin nx}_{g}\,dx\\
&= (\pi^2 - \pi^2)\frac{1}{n}\sin n\pi - (\pi^2 - 0^2)\frac{1}{n}\sin 0 + \frac{2}{n}\int_0^{\pi} x\sin nx\,dx\\
&= \frac{2}{n}\int_0^{\pi} x\sin nx\,dx. && (2.5)
\end{aligned}$$

Now we calculate this last integral using integration by parts: let

$$f(x) = x \quad \text{and} \quad g'(x) = \sin nx,$$

so

$$f'(x) = 1 \quad \text{and} \quad g(x) = \int \sin nx\,dx = \frac{-\cos nx}{n}.$$

Using (2.3), and remembering that $\cos n\pi = (-1)^n$, $\sin n\pi = 0$ for n an integer, we have

$$\begin{aligned}
\int_0^{\pi}\underbrace{x}_{f}\underbrace{\sin nx}_{g'}\,dx &= \underbrace{x}_{f}\underbrace{\frac{-\cos nx}{n}}_{g}\Big|_0^{\pi} - \int_0^{\pi}\underbrace{1}_{f'}\underbrace{\frac{-\cos nx}{n}}_{g}\,dx\\
&= \pi\frac{-\cos n\pi}{n} - 0\frac{-\cos 0}{n} + \frac{1}{n}\int_0^{\pi}\cos nx\,dx\\
&= -\frac{1}{n}\pi(-1)^n + \frac{1}{n}\frac{\sin nx}{n}\Big|_0^{\pi} = -\frac{1}{n}\pi(-1)^n.
\end{aligned}$$

Using (2.2), (2.5) and the above, we have

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^\pi (\pi^2 - x^2)\cos nx\, dx = \frac{2}{\pi}\frac{2}{n}\int_0^\pi x\cos nx\, dx \\ &= \frac{2}{\pi}\frac{2}{n} \cdot \frac{1}{n}\pi(-1)^n = \frac{-4}{n^2}(-1)^n. \end{aligned}$$

It remains to calculate a_0, which is given by

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi}2\int_0^\pi \pi^2 - x^2\, dx \\ &= \frac{2}{\pi}\left(\pi^2 x - \frac{x^3}{3}\right)\Big|_0^\pi = \frac{2}{\pi}\left(\pi^3 - \frac{\pi^3}{3}\right) = \frac{4\pi^3}{3} \end{aligned}$$

where we use the fact that $f(x) = \pi^2 - x^2$ is even.

STEP 4: Using the formulae obtained above for the Fourier coefficients, we have

$$\pi^2 - x^2 = \frac{2\pi^3}{3} + \sum_{n=1}^{\infty}\frac{-4}{n^2}(-1)^n\cos nx + 0.\sin nx = \frac{2\pi^3}{3} + \sum_{n=1}^{\infty}\frac{-4}{n^2}(-1)^n\cos nx$$

Example 2. *Show that the trigonometric Fourier series of $f(x) = 3x$ for $x \in (-\pi, \pi)$ is given by*

$$\sum_{n=1}^{\infty}\frac{-6}{n}(-1)^n\sin nx.$$

SOLUTION:

STEP 1: $f(-x) = 3.-x = -3x = -f(x)$, so f is an odd function.

STEP 2: Since $f(x)$ is odd and $\cos nx$ is even, it follows that $f(x)\cos nx$ is odd, so

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{1}{\pi}.0 = 0.$$

Moreover, since f is odd

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi}.0 = 0.$$

STEP 3: We need to calculate the Fourier coefficients using the Euler Formulae. However, noting that $f(x)$ and $\sin nx$ are odd, and therefore that $f(x)\sin nx$ is even we have

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{1}{\pi}2\int_0^\pi f(x)\sin nx\, dx = \frac{6}{\pi}\int_0^\pi x\cos nx\, dx. \quad (2.6)$$

The latter integral is calculated using integration by parts.

Exercise 2.1. Show that

$$\int_0^\pi x \sin nx \, dx = x \frac{-\cos nx}{n}\Big|_0^\pi - \int_0^\pi \frac{-\cos nx}{n} \, dx = \frac{-\pi}{n}(-1)^n.$$

By virtue of Exercise 2.1, we have, from (2.6)

$$b_n = \frac{6}{\pi}\frac{-\pi}{n}(-1)^n = -\frac{6}{n}(-1)^n.$$

STEP 4: The Fourier series of $f(x) = 3x$ is given by

$$\begin{aligned} \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx &= 0 + \sum_{n=1}^{\infty} 0 \cos nx + -\frac{6}{n}(-1)^n \sin nx \\ &= \sum_{n=1}^{\infty} -\frac{6}{n}(-1)^n \sin nx. \end{aligned}$$

Now try the following
Exercise 2.2.

(i) Show that $x^3 \cos nx$ is an odd function and $x^3 \sin nx$ is an even function. Hence give the value of

$$\int_{-\pi}^{\pi} x^3 \cos nx \, dx$$

and write down another expression equal to

$$\int_{-\pi}^{\pi} x^3 \sin nx \, dx.$$

(ii) By integrating by parts, show that

$$\int_0^\pi x^3 \sin nx \, dx = -\frac{(-1)^n \pi^3}{n} + \frac{3}{n}\int_0^\pi x^2 \cos nx \, dx.$$

Hint: Recall for integer values of n that $\cos n\pi = (-1)^n$.

(iii) Given that

$$\int_0^\pi x^2 \cos nx \, dx = -\frac{2}{n}\int_0^\pi x \sin nx \, dx$$

and

$$\int_0^\pi x \sin nx \, dx = -\frac{\pi}{n}(-1)^n,$$

use part (ii) to prove that

$$\int_0^\pi x^3 \sin nx \, dx = \frac{6\pi}{n^3}(-1)^n - \frac{\pi^3}{n}(-1)^n.$$

(iv) Using parts (i) and (iii), and supposing that the Fourier series converges, show for all $x \in (-\pi, \pi)$ that

$$x^3 = \sum_{n=1}^{\infty} 2(-1)^n \left(\frac{6}{n^3} - \frac{\pi^2}{n}\right) \sin nx.$$