7. a $y'' + 4y' = \begin{cases} 0 & 0 \leq t < 5 \\ \frac{t-5}{5} & 5 \leq t < 10 \\ 1 & t \geq 10 \end{cases}$ $y(0)=0$ , $y'(0)=0$.

The RHS:

let $g(t) = \begin{cases} 0 & 0 \leq t < 5 \\ \frac{t-5}{5} & 5 \leq t < 10 \\ 1 & t \geq 10 \end{cases}$

$\Rightarrow g(t) = 0(1 - u(t-5)) + \frac{t-5}{5}(u(t-5) - u(t-10)) + (1)\,u(t-10).$

$\Rightarrow g(t) = \frac{t-5}{5}(u(t-5) - u(t-10)) + u(t-10).$

$\Rightarrow g(t) = (1/5)(t-5)u(t-5) - (1/5)(t-5-5+5)u(t-10) + u(t-10)$

$\Rightarrow g(t) = (1/5)(t-5)u(t-5) - (1/5)(t-10)u(t-10) \cancel{-u(t-10)} \cancel{+u(t-10)}$

$\Rightarrow \underline{g(t) = (1/5)(t-5)u(t-5) - (1/5)(t-10)u(t-10)}.$

$\Rightarrow \mathcal{L}(g(t)) = (1/5)\mathcal{L}((t-5)u(t-5)) - (1/5)\mathcal{L}((t-10)u(t-10)).$

$$= (1/5)\frac{e^{-5s}}{s^2} - \frac{1}{5}\frac{e^{-10s}}{s^2} = \underline{\frac{(1/5)}{s^2}(e^{-5s} - e^{-10s})}.$$

the LHS.

$\mathcal{L}(y'' + 4y') = s^2\mathcal{L}(y) - s\,y(0) - y'(0) + 4(s\mathcal{L}(y) - y(0))$

$= \underline{(s^2 + 4s)\mathcal{L}(y)}$ since $y(0)=0$ & $y'(0)=0$.

since LHS = RHS.

$\Rightarrow (s^2+4s)\mathcal{L}(y) = \frac{(1/5)}{s^2}(e^{-5s} - e^{-10s}).$

$\Rightarrow \mathcal{L}(y) = \frac{(1/5)}{s^2(s^2+4s)}(e^{-5s} - e^{-10s}).$

$= \frac{(1/5)}{s^3(s+4)}(e^{-5s} - e^{-10s}).$

$= -A$

$= -1/64$

$\left(\frac{1}{s^3}e^{-5s}\right)$

$) + (1/4)\mathcal{L}^{-1}\left(\frac{1}{s^3}\right.$

$+ \frac{1}{4} * \frac{1}{2!}(t-5)^2$

$u(t-10) - \frac{1}{16}(t$

$p(t-10)u(t-10)$

$+ 1/4 * \frac{1}{2!}(t-10)$

partial fraction Decomposition:

$$\frac{1}{s^3(s+4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s+4}.$$

$$\Rightarrow \quad 1 \equiv As^2(s+4) + Bs(s+4) + C(s+4) + Ds^3.$$

$$\Rightarrow \quad 1 \equiv As^3 + 4As^2 + Bs^2 + 4Bs + Cs + 4C + Ds^3.$$

$\therefore$ $A + D = 0 \ldots (i).$

$4A + B = 0 \ldots (ii)$

$4B + C = 0 \ldots (iii)$

$4C = 1 \ldots (iv). \rightarrow C = 1/4$, $\overset{\text{from iii}}{B = -C/4} = -1/16$, $A = -B/4 = -\frac{1}{4}(-1/16) = \frac{1}{64}$, $D = -A = -\frac{1}{64}$

$$\therefore \frac{1}{s^3(s+4)} = \frac{(1/64)}{s} + \frac{(-1/16)}{s^2} + \frac{(1/4)}{s^3} + \frac{(-1/64)}{s+4}$$

thus $\mathcal{L}\{y\} = \left(\frac{1}{5}\right)\left(\frac{(1/64)}{s} + \frac{(-1/16)}{s^2} + \frac{(1/4)}{s^3} + \frac{(-1/64)}{s+4}\right)e^{-5s}$

$$- \left(\tfrac{1}{5}\right)\left(\frac{(1/64)}{s} + \frac{(-1/16)}{s^2} + \frac{(1/4)}{s^3} + \frac{(-1/64)}{s+4}\right)e^{-10s}.$$

$\therefore y(t) = \left(\tfrac{1}{5}\right)\Big[\left(\tfrac{1}{64}\right)\mathcal{L}^{-1}\left(\frac{e^{-5s}}{s}\right) - \left(\tfrac{1}{16}\right)\mathcal{L}^{-1}\left(\frac{e^{-5s}}{s^2}\right) + \left(\tfrac{1}{4}\right)\mathcal{L}^{-1}\left(\frac{1}{s^3}e^{-5s}\right)$

$$- \left(\tfrac{1}{64}\right)\mathcal{L}^{-1}\left(\frac{e^{-5s}}{s+4}\right)\Big]$$

$$- \left(\tfrac{1}{5}\right)\Big[\left(\tfrac{1}{64}\right)\mathcal{L}^{-1}\left(\frac{e^{-10s}}{s}\right) - \left(\tfrac{1}{16}\right)\mathcal{L}^{-1}\left(\frac{e^{-10s}}{s^2}\right) + \left(\tfrac{1}{4}\right)\mathcal{L}^{-1}\left(\frac{1}{s^3}e^{-10s}\right)$$

$$- \left(\tfrac{1}{64}\right)\mathcal{L}^{-1}\left(\frac{e^{-10s}}{s+4}\right)\Big]$$

$$= \left(\tfrac{1}{5}\right)\Big[\tfrac{1}{64}u(t-5) - \tfrac{1}{16}(t-5)u(t-5) + \tfrac{1}{4} \times \tfrac{1}{2!}(t-5)^2 u(t-5)$$

$$- \tfrac{1}{64}e^{-4(t-5)}u(t-5)\Big] - \left(\tfrac{1}{5}\right)\Big[\tfrac{1}{64}u(t-10) - \tfrac{1}{16}(t-10)u(t-10)$$

$$+ \tfrac{1}{4} \times \tfrac{1}{2!}(t-10)^2 u(t-10) - \tfrac{1}{64}e^{-4(t-10)}u(t-10)\Big]$$

7.9

$$y(t) = \left[\left(\frac{1}{320}\right) - \left(\frac{1}{80}\right)(t-5) + \left(\frac{1}{40}\right)(t-5)^2 - \left(\frac{1}{320}\right)e^{-4(t-5)}\right]u(t-5)$$

$$+ \left[\left(\frac{1}{320}\right) - \left(\frac{1}{80}\right)(t-10) + \left(\frac{1}{40}\right)(t-10)^2 - \left(\frac{1}{320}\right)e^{-4(t-10)}\right]u(t-10).$$

$$\therefore\ y(t) = \begin{cases} 0 & 0 \le t < 5 \\ \frac{1}{320} - \frac{1}{80}(t-5) + \left(\frac{1}{40}\right)(t-5)^2 - \left(\frac{1}{320}\right)e^{-4(t-5)} & \to 5 \le t < 10. \\ \frac{1}{160} - \frac{1}{80}(t-15) + \left(\frac{1}{40}\right)(t-5)^2 + \frac{1}{40}(t-10)^2 - \frac{1}{320}e^{-4(t-5)} - \frac{1}{320}e^{-4(t-10)} & t \ge 10 \end{cases}$$

7.b

$$y'' + 9y' = \begin{cases} 8\sin t, & 0 \leq t < \pi \\ 0, & t \geq \pi \end{cases} \qquad y(0) = 0 \;.\; y'(0) = 4.$$

The RHS;

let

$$g(t) = \begin{cases} 8\sin t & 0 \leq t < \pi \\ 0 & t \geq \pi. \end{cases}$$

$$= 8\sin t\,(1 - u(t-\pi)) + 0\,u(t-\pi)$$
$$= 8\sin t - 8\sin t\, u(t-\pi)$$
$$= 8\sin t - 8\sin((t-\pi)+\pi)\, u(t-\pi)$$
$$= \underline{8\sin t + 8\sin(t-\pi)\, u(t-\pi)}.$$

and $\mathcal{L}(g(t)) = \mathcal{L}(8\sin t + 8\sin(t-\pi)u(t-\pi))$

$$= 8\mathcal{L}(\sin t) + 8\mathcal{L}(\sin(t-\pi)u(t-\pi))$$
$$= \underline{\frac{8}{s^2+1^2} + \frac{8e^{-\pi s}}{s^2+1^2}}$$

Thus. $y'' + 9y' = g(t)$.

$\mathcal{L}(y'' + 9y') = \mathcal{L}(g(t))$.

$$\Rightarrow s^2\mathcal{L}(y) - s\,y(0) - y'(0) + 9(s\mathcal{L}(y) - y(0)) = \frac{8}{s^2+1^2} + \frac{8e^{-\pi s}}{s^2+1^2}.$$

$$\Rightarrow (s^2+9s)\mathcal{L}(y) - 4 = \frac{8}{s^2+1^2} + \frac{8e^{-\pi s}}{s^2+1^2}.$$

$$\Rightarrow \mathcal{L}(y) = \frac{4}{s(s+9)} + \frac{8}{s(s+9)(s^2+1)} + \frac{8e^{-\pi s}}{s(s+9)(s^2+1)}.$$

partial fraction decompositions;

(i) $\dfrac{1}{s(s+9)} = \dfrac{A}{s} + \dfrac{B}{s+9}$

$\Rightarrow 1 = A(s+9) + Bs.$ when $s=0$

$\Rightarrow A = \underline{\underline{1/9}}$

when $s=-9 \Rightarrow B = -\frac{1}{9}$

$$\therefore \frac{1}{s(s+9)} = \frac{(1/9)}{s} + \frac{(-1/9)}{s+9}.$$

7.b

(ii)

$$\frac{1}{s(s+9)(s^2+1)} = \frac{A}{s} + \frac{B}{s+9} + \frac{Cs+D}{s^2+1}$$

$$\Rightarrow \quad 1 \equiv A(s+9)(s^2+1) + B\,s(s^2+1) + (Cs+D)s(s+9)$$

when $s=0$

$$\Rightarrow \quad 1 = A(9)(1) \Leftrightarrow A = \tfrac{1}{9}$$

when $s=-9$

$$\Rightarrow \quad 1 = B(-9)((-9)^2+1) \Leftrightarrow B = -\tfrac{1}{738}$$

$$1 \equiv As^3 + As + 9As^2 + 9A + Bs^3 + Bs + Cs^3 + 9Cs^2 + Ds^2 + 9Ds$$

$$1 \equiv (A+B+C)s^3 + (9A+9C+D)s^2 + (A+B+9D)s + 9A$$

$\therefore$

$A+B+C=0 \quad \dots (i)$

$9A+9C+D=0 \quad \dots (ii)$

$A+B+9D=0 \quad \dots (iii)$

$9A = 1 \quad \dots (iv)$

since $A = \tfrac{1}{9}$ & $B = -\tfrac{1}{738}$.

$\rightarrow$ $C = -A-B = -\tfrac{1}{9} + \tfrac{1}{738} = -\tfrac{9}{82}$, $\quad D = \tfrac{1}{9}(-A-B) = -\tfrac{1}{82}$

$$\Rightarrow \frac{1}{s(s+9)(s^2+1)} = \frac{(\tfrac{1}{9})}{s} + \frac{(-\tfrac{1}{738})}{s+9} + \frac{(-\tfrac{9}{82})s + (-\tfrac{1}{82})}{s^2+1}.$$

thus.

$$\mathcal{L}(y) = \frac{(\tfrac{4}{9})}{s} + \frac{(-\tfrac{4}{9})}{s+9} + \frac{(\tfrac{8}{9})}{s} + \frac{(-\tfrac{4}{369})}{s+9} + \frac{(-\tfrac{36}{41})s + (-\tfrac{4}{41})}{s^2+1}$$

$$+ \frac{(\tfrac{8}{9})}{s}e^{-\pi s} + \frac{(\tfrac{-4}{369})}{s+9}e^{-\pi s} + \left(\tfrac{-36}{41}\right)\frac{s}{s^2+1}e^{-\pi s} + \left(\tfrac{-4}{41}\right)\frac{1}{s^2+1}e^{-\pi s}$$

thus $y = \tfrac{4}{9}\mathcal{L}^{-1}\left(\tfrac{1}{s}\right) - \tfrac{4}{9}\mathcal{L}^{-1}\left(\tfrac{1}{s+9}\right) + \tfrac{8}{9}\mathcal{L}^{-1}\left(\tfrac{1}{s}\right) - \tfrac{4}{369}\mathcal{L}^{-1}\left(\tfrac{1}{s+9}\right)$

$- \tfrac{36}{41}\mathcal{L}^{-1}\left(\tfrac{s}{s^2+1}\right) - \tfrac{4}{41}\mathcal{L}^{-1}\left(\tfrac{1}{s^2+1}\right) + \tfrac{8}{9}\mathcal{L}^{-1}\left(\tfrac{e^{-\pi s}}{s}\right)$

$- \tfrac{4}{369}\mathcal{L}^{-1}\left(\tfrac{e^{-\pi s}}{s+9}\right) - \tfrac{36}{41}\mathcal{L}^{-1}\left(\tfrac{s e^{-\pi s}}{s^2+1}\right) - \tfrac{4}{41}\mathcal{L}^{-1}\left(\tfrac{1}{s^2+1}e^{-\pi s}\right)$

7.b

$$y(t) = 4/9 - 4/9\, e^{-9t} + 8/9 - \frac{4}{369} e^{-9t} - 36/41 \cos t - \frac{4}{41} \sin t$$
$$+ 8/9\, u(t-\pi) - \frac{4}{369} e^{-9(t-\pi)} u(t-\pi) - \frac{36}{41} \cos(t-\pi) u(t-\pi)$$
$$- 4/41 \sin(t-\pi) u(t-\pi).$$

$$y(t) = 4/3 - \frac{56}{123} e^{-9t} - 36/41 \cos t - 4/41 \sin t$$
$$+ \left( 8/9 - 4/369\, e^{-9(t-\pi)} - 36/41 \cos(t-\pi) - 4/41 \sin(t-\pi) \right) u(t-\pi)$$

$$y(t) = \begin{cases} 4/3 - 56/123\, e^{-9t} - 36/41 \cos t - 4/41 \sin t & 0 \le t < \pi, \\ 20/9 - 56/123\, e^{-9t} - 4/369\, e^{-9(t-\pi)} - 36/41 \cos t \\ - 36/41 \cos(t-\pi) - 4/41 \sin t - 4/41 \sin(t-\pi) & t \ge \pi. \end{cases}$$

$\quad\quad 8 \quad\quad t \geq 1$

let $g(t) = \begin{cases} 4t & 0 \leq t < 1 \\ 8 & t \geq 1 \end{cases}$ , $y(0) = 0$, $y'(0) = 0$

$\Rightarrow g(t) = 4t(1 - u(t-1)) + 8u(t-1)$.

$= 4t - 4t\,u(t-1) + 8u(t-1)$.

$= 4t - (4(t-1+1) - 8)u(t-1)$

$= \underline{4t - 4(t-1)u(t-1) + 4u(t-1)}$.

thus $\mathcal{L}(g(t)) = 4\mathcal{L}(t) - 4\mathcal{L}((t-1)u(t-1)) + 4\mathcal{L}(u(t-1))$

$= \underline{4/s^2 - \frac{4}{s^2}e^{-s} + \frac{4e^{-s}}{s}}$

$\Rightarrow y'' + 3y' + 2y = g(t)$

$\Rightarrow \mathcal{L}(y'' + 3y' + 2y) = \mathcal{L}(g(t))$

$\Rightarrow (s^2\mathcal{L}(y) - s\,y(0) - y'(0)) + 3(s\mathcal{L}(y) - y(0)) + 2\mathcal{L}(y) = \mathcal{L}(g(t))$

$\Rightarrow (s^2 + 3s + 2)\mathcal{L}(y) = \mathcal{L}(g(t))$

$\Rightarrow (s+1)(s+2)\mathcal{L}(y) = \frac{4}{s^2} - \frac{4}{s^2}e^{-s} + \frac{4}{s}e^{-s}$.

$\Rightarrow \mathcal{L}(y) = \frac{4}{s^2(s+1)(s+2)} - \frac{4\,e^{-s}}{s^2(s+1)(s+2)} + \frac{4}{s(s+1)(s+2)}e^{-s}$.

<u>Partial fraction decompositions:</u>

(i) $\frac{1}{s^2(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{s+2}$.

$\Rightarrow 1 \equiv As(s+1)(s+2) + B(s+1)(s+2) + C\,s^2(s+2) + D\,s^2(s+1)$

when $s = -1$

$\Rightarrow 1 = C(-1)^2(-1+2) = C \Leftrightarrow \underline{C = 1}$

when $s = -2$

$\Rightarrow 1 = D(-2)^2(-2+1) = -4D \Leftrightarrow D = \underline{-1/4}$

when $s = 0$

$\Rightarrow 1 \equiv B(2) \Leftrightarrow B = 1/2$

when $s = 1$

$\Rightarrow 1 = A(1)(2)(3) + \frac{1}{2}(2)(3) + (1)^2(3) - \frac{1}{4}1^2(2)$

$\Rightarrow 1 = 6A + 3 + 3 - 1/2$

$\Rightarrow \underline{A = -3/4}$

7.c $\therefore \dfrac{1}{s^2(s+1)(s+2)} = \dfrac{(-3/4)}{s} + \dfrac{(1/2)}{s^2} + \dfrac{1}{s+1} + \dfrac{(-1/4)}{s+2}.$

(ii) $\dfrac{1}{s(s+1)(s+2)} = \dfrac{A}{s} + \dfrac{B}{s+1} + \dfrac{C}{s+2}$

$\Rightarrow \quad 1 = A(s+1)(s+2) + B\,s(s+2) + C\,s(s+1)$

when $s=-1$

$\Rightarrow \quad 1 = B(-1)(-1+2) \Leftrightarrow B = -1$

when $s=-2$

$\Rightarrow \quad 1 = C(-2)(-2+1) \Leftrightarrow C = \frac{1}{2}$

when $s=0$

$\Rightarrow \quad 1 = 2A \Leftrightarrow A = \frac{1}{2}$

$\therefore \dfrac{1}{s(s+1)(s+2)} = \dfrac{(\frac{1}{2})}{(s)} + \dfrac{(-1)}{(s+1)} + \dfrac{(\frac{1}{2})}{(s+2)}$

thus.

$$L(y) = \frac{-3}{s} + \frac{2}{s^2} + \frac{4}{s+1} - \frac{1}{s+2}$$
$$- \left(-\frac{3}{s}e^{-s} + \frac{2}{s^2}e^{-s} + \frac{4}{s+1}e^{-s} - \frac{1}{s+2}e^{-s}\right)$$
$$+ \left(\frac{2}{s}e^{-s} - \frac{4}{s+1}e^{-s} + \frac{2}{s+2}e^{-s}\right)$$
$$= -\frac{3}{s} + \frac{2}{s^2} + \frac{4}{s+1} - \frac{1}{s+2} + \frac{5}{s}e^{-s} + \frac{2}{s^2}e^{-s} - \frac{8}{s+1}e^{-s} + \frac{3}{s+2}e^{-s}.$$

$$\therefore y = -3L^{-1}\left(\frac{1}{s}\right) + 2L^{-1}\left(\frac{1}{s^2}\right) + 4L^{-1}\left(\frac{1}{s+1}\right) - L\left(\frac{1}{s+2}\right)$$
$$+5L^{-1}\left(\frac{e^{-s}}{s}\right) + 2L^{-1}\left(\frac{1}{s^2}e^{-s}\right) - 8L^{-1}\left(\frac{1}{s+1}e^{-s}\right)$$
$$+3L^{-1}\left(\frac{1}{s+2}e^{-s}\right)$$

$$\Rightarrow y(t) = (-3) + 2t + 4e^{-t} - e^{-2t} + 5u(t-1) + 2(t-1)u(t-1)$$
$$-8e^{-(t-1)}u(t-1)) + 3e^{-2(t-1)}u(t-1)$$
$$= (-3 + 2t + 4e^{-t} - e^{-2t}) + (5 + 2(t-1) - 8e^{-(t-1)} + 3e^{-2(t-1)})u(t-1)$$

7. C

$$y(t) = \begin{cases} -3 + 2t + 4e^{-t} - e^{-2t} & 0 \le t < 1 \\ -3 + 2t + 4e^{-t} - e^{-2t} + 5 + (2t-1) - 8e^{-(t-1)} + 3e^{-2(t-1)} & t \geq 1 \end{cases}$$

$$y'' + \ldots + 5y = \begin{cases} 10 \sin t & 0 \leq t < 2\pi \\ 0 & t \geq 2\pi \end{cases} \qquad y(\pi) = 1, \quad y'(\pi) = 2e^{-\pi} - 2.$$

let
$$g(t) = \begin{cases} 10 \sin t & 0 \leq t < 2\pi \\ 0 & t \geq 2\pi. \end{cases}$$

$$\Rightarrow g(t) = 10 \sin t \,(1 - u(t-2\pi)) + 0\, u(t-2\pi).$$

$$\Rightarrow g(t) = 10 \sin t - 10 \sin t \, u(t-2\pi).$$

$$= 10 \sin t - 10 \sin((t-2\pi)+2\pi)\, u(t-2\pi).$$

$$* \left\{ \begin{aligned} &\sin((t-2\pi)+2\pi) \\ &= \sin(t-2\pi)\cos 2\pi + \sin 2\pi \cos(t-2\pi) \\ &= \sin(t-2\pi). \end{aligned} \right\}$$

$$\Rightarrow g(t) = 10 \sin t - 10 \sin(t-2\pi)\, u(t-2\pi).$$

Thus $y'' + 2y' + 5y = g(t)$.

also let $\tilde{t} = t - \pi$. thus $\tilde{y}(0) = 1$, $\tilde{y}'(0) = 2e^{-\pi} - 2$.

then $\tilde{y}'' + 2\tilde{y}' + 5\tilde{y} = \tilde{g}$

where $\tilde{g}(\tilde{t}) = 10 \sin(\tilde{t} + \pi) - 10 \sin(\tilde{t} - \pi)\, u(\tilde{t} - \pi)$.

<u>the laplace of the RHS:</u>

$$\Rightarrow \mathcal{L}(\tilde{y}'' + 2\tilde{y}' + 5\tilde{y}) = \mathcal{L}(\tilde{g}).$$

$$* \mathcal{L}(\tilde{g}) = \mathcal{L}\left(10 \sin(\tilde{t} + \pi) - 10 \sin(\tilde{t} - \pi)\, u(\tilde{t} - \pi)\right).$$

$$\sin(\tilde{t} + \pi) = \sin \tilde{t} \cos \pi + \sin \pi \cos \tilde{t}$$
$$= \underline{-\sin \tilde{t}}.$$

$$\Rightarrow \mathcal{L}(\tilde{g}) = \mathcal{L}(-10 \sin \tilde{t}) - 10\, \mathcal{L}(\sin(\tilde{t} - \pi)\, u(\tilde{t} - \pi))$$

$$= \frac{-10}{s^2 + 1^2} - \frac{10\, e^{-\pi s}}{s^2 + 1^2}$$

$* \mathcal{L}(f(t-a)u(t-a)) = e^{-as} F(s)$ ; $F(s) = \mathcal{L}(f)$

<u>the laplace of the LHS;</u>

$$\mathcal{L}(\tilde{y}'' + 2\tilde{y}' + 5\tilde{y}) = s^2 \mathcal{L}(\tilde{y}) - s\tilde{y}(0) - \tilde{y}'(0)$$
$$+ 2(s\mathcal{L}(\tilde{y}) - \tilde{y}(0)) + 5\mathcal{L}(\tilde{y})$$
$$= (s^2 + 2s + 5)\mathcal{L}(\tilde{y}) - s(1) - (2e^{-\pi} - 2) - 2(1)$$
$$= (s^2 + 2s + 5)\mathcal{L}(\tilde{y}) - s - 2e^{-\pi}.$$

7. d thus.

$$(s^2+2s+5)\mathcal{L}(\tilde{y}) - s - 2e^{-\pi} = \frac{-10}{s^2+1^2} - \frac{10e^{-\pi s}}{s^2+1^2}.$$

$$\Rightarrow \mathcal{L}(\tilde{y}) = \frac{-10}{(s^2+1^2)(s^2+2s+5)} - \frac{10e^{-\pi s}}{(s^2+1^2)(s^2+2s+5)} + \frac{s}{(s^2+2s+5)} + \frac{2e^{-\pi}}{(s^2+2s+5)}.$$

Partial fraction decomposition:

$$\frac{1}{(s^2+1^2)(s^2+2s+5)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2s+5}$$

$$\Rightarrow 1 \equiv (As+B)(s^2+2s+5) + (Cs+D)(s^2+1)$$

$$\Rightarrow 1 \equiv As^3 + 2As^2 + 5As + Bs^2 + 2Bs + 5B + Cs^3 + Cs + Ds^2 + D$$

$$\Rightarrow 1 = (A+C)s^3 + (2A+B+D)s^2 + (5A+2B+C)s + (5B+D).$$

$\therefore$ $A+C=0$ (i)

$2A+B+D=0$ (ii) $\longrightarrow$ $C=-A$ from Eqn i

$5A+2B+C=0$ (iii)  $D=1-5B$ from Eqn (iv)

$5B+D=1$ (iv)

Eqn ii

$2A+B+(1-5B)=0$

$\Rightarrow 2A-4B=-1$ ...(v)

Eqn iii

$5A+2B-A=0$

$\Rightarrow 4A+2B=0$

$\Rightarrow B=-2A$ ...(vi)

thus. $2A-4(-2A)=-1$

$10A=-1 \Rightarrow A=-\frac{1}{10}$, $B=-2\left(-\frac{1}{10}\right)=\frac{1}{5}$, $C=-A=\frac{1}{10}$, $D=0$

thus.

$$\frac{1}{(s^2+1^2)(s^2+2s+5)} = \frac{(-\frac{1}{10})s+\frac{1}{5}}{s^2+1} + \frac{(\frac{1}{10})s}{s^2+2s+5}.$$

since $s^2+2s+5 = (s+1)^2+5-1$
$= (s+1)^2+2^2.$

thus.

$$\mathcal{L}(\tilde{y}) = \frac{+s+(-2)}{s^2+1} - \frac{s}{(s+1)^2+2^2} + \frac{(s-2)e^{-\pi s}}{s^2+1} - \frac{se^{-\pi s}}{(s+1)^2+2^2}$$
$$+ \frac{s}{(s+1)^2+2^2} + \frac{2e^{-\pi}}{(s+1)^2+2^2}$$

7. d

$$\mathcal{L}(\tilde{y}) = \frac{s}{s^2+1} + (-2)\frac{(1)}{s^2+1^2} - \frac{(\cancel{s+1})}{(\cancel{s+1})^2+2^2} + \left(\frac{1}{2}\right)\frac{\cancel{2}}{\cancel{(s+1)^2+2^2}}$$

$$+\frac{s}{s^2+1^2}e^{-\pi s} + (-2)\frac{1}{s^2+1^2}e^{-\pi s} - \frac{(s+1)e^{-\pi s}}{(s+1)^2+2^2}$$

$$+\left(\frac{1}{2}\right)\frac{2}{(s+1)^2+2^2}e^{-\pi s} + \frac{(\cancel{s+1})}{\cancel{(s+1)^2+2^2}} - \left(\frac{1}{2}\right)\frac{\cancel{2}}{\cancel{(s+1)^2+2^2}} + (e^{-\pi})\frac{2}{(s+1)^2+2^2}$$

$$\therefore \tilde{y} = \mathcal{L}\left(\frac{s}{s^2+1}\right) + (-2)\mathcal{L}\left(\frac{1}{s^2+1^2}\right) + \mathcal{L}\left(\frac{s}{s^2+1^2}e^{-\pi s}\right)$$

$$+(-2)\mathcal{L}\left(\frac{1}{s^2+1^2}e^{-\pi s}\right) - \mathcal{L}\left(\frac{(s+1)}{(s+1)^2+2^2}e^{-\pi s}\right)$$

$$+\left(\frac{1}{2}\right)\mathcal{L}\left(\frac{2}{(s+1)^2+2^2}e^{-\pi s}\right) + (e^{-\pi})\mathcal{L}\left(\frac{2}{(s+1)^2+2^2}\right)$$

$$\Rightarrow \tilde{y} = \cos\tilde{t} - 2\sin\tilde{t} + \cos(\tilde{t}-\pi)u(\tilde{t}-\pi) - 2\sin(\tilde{t}-\pi)u(\tilde{t}-\pi)$$

$$- e^{-(\tilde{t}-\pi)}\cos 2(\tilde{t}-\pi)u(\tilde{t}-\pi) + \frac{1}{2}e^{-(\tilde{t}-\pi)}\sin 2(\tilde{t}-\pi)u(\tilde{t}-\pi)$$

$$+ e^{-\pi}e^{-\tilde{t}}\sin(2\tilde{t})$$

thus since $\tilde{t} = t-\pi.$

$$\Rightarrow y(t) = \cos(t-\pi) - 2\sin(t-\pi) + \cos(t-2\pi)u(t-2\pi)$$

$$-2\sin(t-2\pi)u(t-2\pi) - e^{-(t-2\pi)}\cos 2(t-2\pi)u(t-2\pi)$$

$$+\frac{1}{2}e^{-(t-2\pi)}\sin 2(t-2\pi)u(t-2\pi) + e^{-\pi}e^{-(t-\pi)}\sin(2t-2\pi)$$

$$\Rightarrow y(t) = \cos(t-\pi) - 2\sin(t-\pi) + e^{-t}\sin 2(t-\pi)$$

$$+\Big(\cos(t-2\pi) - 2\sin(t-2\pi) - e^{-(t-2\pi)}\cos 2(t-2\pi)$$

$$+\frac{1}{2}e^{-(t-2\pi)}\sin 2(t-2\pi)\Big)u(t-2\pi).$$

$$y(t) = \begin{cases} \cos(t-\pi) - 2\sin(t-\pi) + e^{-t}\sin 2(t-\pi), & 0 < t < 2\pi. \\ \cos(t-\pi) - 2\sin(t-\pi) + e^{-t}\sin 2(t-\pi) \\ + \cos(t-2\pi) - 2\sin(t-2\pi) - e^{-(t-2\pi)}\cos 2(t-2\pi) \\ + \frac{1}{2} e^{-(t-2\pi)}\sin 2(t-2\pi), & t \geq 2\pi. \end{cases}$$

$$\frac{20}{(s^2+1)(s^2+25)} = (4)\,\frac{1}{s^2+1^2} \times \frac{5}{s^2+5^2}$$

from the convolution theorem:

$$f*g = \int_0^t f(\tau)\,g(t-\tau)\,d\tau$$

such that $\mathcal{L}(f*g) = F(s)G(s) \Leftrightarrow \mathcal{L}^{-1}(F(s)G(s)) = f*g$

& $F(s) = \mathcal{L}(f)$, $G(s) = \mathcal{L}(g)$.

let $F(s) = (4)\,\dfrac{1}{s^2+1^2} \Leftrightarrow f(t) = 4\sin t$.

$G(s) = \dfrac{5}{s^2+5^2} \Leftrightarrow g(t) = \sin 5t$.

thus $f*g = (4\sin t)*(\sin 5t) = \displaystyle\int_0^t 4\sin\tau\,\sin 5(t-\tau)\,d\tau$.

the integrand; $4\sin\tau\,\sin 5(t-\tau)$

$* \sin u \sin v = \frac{1}{2}(\cos(u-v) - \cos(u+v))$

$$= 4\left(\frac{1}{2}\left(\cos(\tau-5t+5\tau) - \cos(\tau+5t-5\tau)\right)\right)$$

$$= 2\left[\cos(6\tau-5t) - \cos(-4\tau+5t)\right]$$

$$= 2\left[\cos(6\tau-5t) - \cos(4\tau-5t)\right]$$

$$\therefore\ f*g = \int_0^t \left(2\cos(6\tau-5t) - 2\cos(4\tau-5t)\right)d\tau$$

$$= 2\int_0^t \cos(6\tau-5t)\,d\tau - 2\int_0^t \cos(4\tau-5t)\,d\tau$$

$$= 2\left(\frac{1}{6}\sin(6\tau-5t)\Big|_0^t\right) - 2\left(\frac{1}{4}\sin(4\tau-5t)\Big|_0^t\right)$$

$$= 2\left[\frac{1}{6}\sin t - \frac{1}{6}\sin(-5t)\right] - 2\left[\frac{1}{4}\sin(-t) - \frac{1}{4}\sin(-5t)\right]$$

$$= \frac{1}{3}\sin t + \frac{1}{3}\sin 5t + \frac{1}{2}\sin t - \frac{1}{2}\sin 5t$$

$$\frac{\quad}{(s^2+1)(s^2+25)} = (4)\,\frac{1}{s^2+1^2} \times \frac{5}{s^2+5^2}$$

from ...

8.9 $$f*g = \frac{5}{6}\sin t - \frac{1}{6}\sin 5t.$$

thus. $$\mathcal{L}^{-1}\left(\frac{20}{(s^2+1)(s^2+5^2)}\right) = \frac{5}{6}\sin t - \frac{1}{6}\sin 5t$$

8.6 $\dfrac{\omega}{s^2(s^2+\omega^2)} = \dfrac{1}{s^2} * \dfrac{\omega}{s^2+\omega^2}$.

using the convolution theorem.

$\mathcal{L}(f*g) = F(s)G(s) \Leftrightarrow \mathcal{L}^{-1}(F(s)G(s)) = f*g$.

where $f*g = \int_0^t f(\tau)g(t-\tau)\,d\tau$. & $F(s) = \mathcal{L}(f)$, $G(s) = \mathcal{L}(g)$.

let $F(s) = \dfrac{\omega}{s^2+\omega^2} \Leftrightarrow f(t) = \mathcal{L}^{-1}(F(s))$

$= \underline{\sin\omega t}$.

, $G(s) = \dfrac{1}{s^2} \Leftrightarrow g(t) = \underline{t}$.

$\therefore f*g = (\sin\omega t)*(t) = \int_0^t \sin\omega\tau\,(t-\tau)\,d\tau$

$= \int_0^t (t\sin\omega\tau\,d\tau - \tau\sin\omega\tau)\,d\tau$

$= \int_0^t t\sin\omega\tau\,d\tau - \underbrace{\int_0^t \tau\sin\omega\tau\,d\tau}$.

$*$ $\int u\,dv = uv - \int v\,du = \int_0 \tau\sin\omega\tau\,d\tau$.

$u = \tau$, $dv = \sin\omega\tau\,d\tau$

$\Rightarrow du = d\tau$, $v = \int \sin\omega\tau\,d\tau$

$= -\frac{1}{\omega}\cos\omega\tau$.

$\therefore \int \tau\sin\omega\tau\,d\tau = -\frac{\tau}{\omega}\cos\omega\tau + \frac{1}{\omega^2}\sin\omega\tau + C$.

$\Rightarrow f*g = t\left(-\frac{1}{\omega}\cos\omega\tau\Big|_0^t\right) - \left(\left(-\frac{\tau}{\omega}\cos\omega\tau + \frac{1}{\omega^2}\sin\omega\tau\right)\Big|_0^t\right)$

$= -\frac{t}{\omega}\cos\omega t + \frac{t}{\omega} - \left(-\frac{t}{\omega}\cos\omega t + \frac{1}{\omega^2}\sin\omega t + 0\right)$.

$= -\cancel{\frac{t}{\omega}\cos\omega t} + \frac{t}{\omega} + \cancel{\frac{t}{\omega}\cos\omega t} - \frac{1}{\omega^2}\sin\omega t$.

$= \frac{t}{\omega} - \frac{1}{\omega^2}\sin\omega t$.

$\therefore \mathcal{L}^{-1}\left(\dfrac{\omega}{s^2(s^2+\omega^2)}\right) = \underline{\underline{\frac{t}{\omega} - \frac{1}{\omega^2}\sin\omega t}}$ ∎