CALCULUS III

Practice Problems

Paul Dawkins

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Preface

Here are a set of practice problems for the Calculus III notes. If you are viewing the pdf version of this document (as opposed to viewing it on the web) this document contains only the problems themselves and no solutions are included in this document. Solutions can be found in a couple of places on the site.

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Note that some sections will have more problems than others and some will have more or less of a variety of problems. Most sections should have a range of difficulty levels in the problems although this will vary from section to section.

Outline

Here is a listing of sections for which practice problems have been written as well as a brief description of the material covered in the notes for that particular section.

cylindrical coordinates are really nothing more than a very natural extension of polar coordinates into a three dimensional setting.
Spherical Coordinates – In this section we will define the spherical coordinate system, yet another alternate coordinate system for the three dimensional coordinate system.

Partial Derivatives – In this chapter we'll take a brief look at limits of functions of more than one variable and then move into derivatives of functions of more than one variable. As we'll see if we can do derivatives of functions with one variable it isn't much more difficult to do derivatives of functions of more than one variable (with a very important subtlety). We will also discuss interpretations of partial derivatives, higher order partial derivatives and the chain rule as applied to functions of more than one variable. We will also define and discuss directional derivatives.

Limits – In the section we'll take a quick look at evaluating limits of functions of several variables. We will also see a fairly quick method that can be used, on occasion, for showing that some limits do not exist.
Partial Derivatives – In this section we will the idea of partial derivatives. We will give the formal definition of the partial derivative as well as the standard notations and how to compute them in practice (*i.e.* without the use of the definition). As you will see if you can do derivatives of functions of one variable you won't have much of an issue with partial derivatives. There is only one (very important) subtlety that you need to always keep in mind while computing partial derivatives.
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Directional Derivatives – In the section we introduce the concept of directional derivatives, including how to compute them and see a couple of nice facts pertaining to directional derivatives.

Applications of Partial Derivatives – In this chapter we will take a look at several applications of partial derivatives. We will find the equation of tangent planes to surfaces and we will revisit on of the more important applications of derivatives from earlier Calculus classes. We will spend a significant amount of time finding relative and absolute extrema of functions of multiple variables. We will also introduce Lagrange multipliers to find the absolute extrema of a function subject to one or more constraints.

Tangent Planes and Linear Approximations – In this section formally define just what a tangent plane to a surface is and how we use partial derivatives to find the equations of tangent planes to surfaces that can be written as $z = f\left(x, y\right)$. We will also see how tangent planes can be thought of as a linear approximation to the surface at a given point.
Gradient Vector, Tangent Planes and Normal Lines – In this section discuss how the gradient vector can be used to find tangent planes to a much more general function than in the previous section. We will also define the normal line and discuss how the gradient vector can be used to find the equation of the normal line.
Relative Minimums and Maximums – In this section we will define critical points for functions of two variables and discuss a method for determining if they are relative minimums, relative maximums or saddle points (*i.e.* neither a relative minimum or relative maximum).
Absolute Minimums and Maximums – In this section we will how to find the absolute extrema of a function of two variables when the independent variables are only allowed to come from a region that is bounded (*i.e.* no part of the region goes out to infinity) and closed (*i.e.* all of the points on the boundary are valid points that can be used in the process).
Lagrange Multipliers – In this section we'll see discuss how to use the method of Lagrange Multipliers to find the absolute minimums and maximums of functions of two or three variables in which the independent variables are subject to one or more constraints. We also give a brief justification for how/why the method works.

Multiple Integrals – In this chapter will be looking at double integrals, *i.e.* integrating functions of two variables in which the independent variables are from two dimensional regions, and triple integrals, *i.e.* integrating functions of three variables in which the independent variables are from three dimensional regions. Included will be double integrals in polar coordinates and triple integrals in cylindrical and spherical coordinates and more generally change in variables in double and triple integrals.
Double Integrals – In this section we will formally define the double integral as well as giving a quick interpretation of the double integral.
Iterated Integrals – In this section we will show how Fubini's Theorem can be used to evaluate double integrals where the region of integration is a rectangle.
Double Integrals over General Regions – In this section we will start evaluating double integrals over general regions, *i.e.* regions that aren't rectangles. We will illustrate how a double integral of a function can be interpreted as the net volume of the solid between the surface given by the function and the *xy*-plane.
Double Integrals in Polar Coordinates – In this section we will look at converting integrals (including *dA*) in Cartesian coordinates into Polar coordinates. The regions of integration in these cases will be all or portions of disks or rings and so we will also need to convert the original Cartesian limits for these regions into Polar coordinates.
Triple Integrals – In this section we will define the triple integral. We will also illustrate quite a few examples of setting up the limits of integration from the three dimensional region of integration. Getting the limits of integration is often the difficult part of these problems.
Triple Integrals in Cylindrical Coordinates – In this section we will look at converting integrals (including *dV*) in Cartesian coordinates into Cylindrical coordinates. We will also be converting the original Cartesian limits for these regions into Cylindrical coordinates.
Triple Integrals in Spherical Coordinates – In this section we will look at converting integrals (including *dV*) in Cartesian coordinates into Spherical coordinates. We will also be converting the original Cartesian limits for these regions into Spherical coordinates.

Chapter 1 : 3-Dimensional Space

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The 3-D Coordinate System – In this section we will introduce the standard three dimensional coordinate system as well as some common notation and concepts needed to work in three dimensions.

Equations of Lines – In this section we will derive the vector form and parametric form for the equation of lines in three dimensional space. We will also give the symmetric equations of lines in three dimensional space. Note as well that while these forms can also be useful for lines in two dimensional space.

Equations of Planes – In this section we will derive the vector and scalar equation of a plane. We also show how to write the equation of a plane from three points that lie in the plane.

Quadric Surfaces – In this section we will be looking at some examples of quadric surfaces. Some examples of quadric surfaces are cones, cylinders, ellipsoids, and elliptic paraboloids.

Functions of Several Variables – In this section we will give a quick review of some important topics about functions of several variables. In particular we will discuss finding the domain of a function of several variables as well as level curves, level surfaces and traces.

Vector Functions – In this section we introduce the concept of vector functions concentrating primarily on curves in three dimensional space. We will however, touch briefly on surfaces as well. We will illustrate how to find the domain of a vector function and how to graph a vector function. We will also show a simple relationship between vector functions and parametric equations that will be very useful at times.

Calculus with Vector Functions – In this section here we discuss how to do basic calculus, *i.e.* limits, derivatives and integrals, with vector functions.

Tangent, Normal and Binormal Vectors – In this section we will define the tangent, normal and binormal vectors.

Arc Length with Vector Functions – In this section we will extend the arc length formula we used early in the material to include finding the arc length of a vector function. As we will see the new formula really is just an almost natural extension of one we've already seen.

Curvature – In this section we give two formulas for computing the curvature (*i.e.* how fast the function is changing at a given point) of a vector function.

Velocity and Acceleration – In this section we will revisit a standard application of derivatives, the velocity and acceleration of an object whose position function is given by a vector function. For the acceleration we give formulas for both the normal acceleration and the tangential acceleration.

Cylindrical Coordinates – In this section we will define the cylindrical coordinate system, an alternate coordinate system for the three dimensional coordinate system. As we will see cylindrical coordinates are really nothing more than a very natural extension of polar coordinates into a three dimensional setting.

Spherical Coordinates – In this section we will define the spherical coordinate system, yet another alternate coordinate system for the three dimensional coordinate system. This coordinates system is very useful for dealing with spherical objects. We will derive formulas to convert between cylindrical coordinates and spherical coordinates as well as between Cartesian and spherical coordinates (the more useful of the two).

Section 1-1 : The 3-D Coordinate System

1. Give the projection of $P = \left(3, -4, 6\right)$ onto the three coordinate planes.

2. Which of the points $P = \left(4, -2, 6\right)$ and $Q = \left(-6, -3, 2\right)$ is closest to the *yz*-plane?

3. Which of the points $P = \left(-1, 4, -7\right)$ and $Q = \left(6, -1, 5\right)$ is closest to the *z*-axis?

For problems 4 & 5 list all of the coordinates systems ($\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$) that the given equation will have a graph in. Do not sketch the graph.

4. $7x^2 - 9y^3 = 3x + 1$

5. $x^3 + \sqrt{y^2 + 1} - 6z = 2$

Section 1-2 : Equations of Lines

For problems 1 & 2 give the equation of the line in vector form, parametric form and symmetric form.

1. The line through the points $\left(2,-4,1\right)$ and $\left(0,4,-10\right)$.

2. The line through the point $\left(-7,2,4\right)$ and parallel to the line given by $x = 5 - 8t$, $y = 6 + t$, $z = -12t$.

3. Is the line through the points $\left(2,0,9\right)$ and $\left(-4,1,-5\right)$ parallel, orthogonal or neither to the line given by $\vec{r}\left(t\right) = \left\langle 5, 1-9t, -8-4t \right\rangle$?

For problems 4 & 5 determine the intersection point of the two lines or show that they do not intersect.

4. The line given by $x = 8 + t$, $y = 5 + 6t$, $z = 4 - 2t$ and the line given by $\vec{r}\left(t\right) = \left\langle -7+12t, 3-t, 14+8t \right\rangle$.

5. The line passing through the points $\left(1,-2,13\right)$ and $\left(2,0,-5\right)$ and the line given by $\vec{r}\left(t\right) = \left\langle 2+4t, -1-t, 3 \right\rangle$.

6. Does the line given by $x = 9 + 21t$, $y = -7$, $z = 12 - 11t$ intersect the *xy*-plane? If so, give the point.

7. Does the line given by $x = 9 + 21t$, $y = -7$, $z = 12 - 11t$ intersect the *xz*-plane? If so, give the point.

Section 1-3 : Equations of Planes

For problems 1 – 3 write down the equation of the plane.

1. The plane containing the points $\left(4,-3,1\right)$, $\left(-3,-1,1\right)$ and $\left(4,-2,8\right)$.

2. The plane containing the point $\left(3,0,-4\right)$ and orthogonal to the line given by $\vec{r}\left(t\right)=\left\langle 12-t,1+8t,4+6t\right\rangle$.

3. The plane containing the point $\left(-8,3,7\right)$ and parallel to the plane given by $4x+8y-2z=45$.

For problems 4 & 5 determine if the two planes are parallel, orthogonal or neither.

4. The plane given by $4x-9y-z=2$ and the plane given by $x+2y-14z=-6$.

5. The plane given by $-3x+2y+7z=9$ and the plane containing the points $\left(-2,6,1\right)$, $\left(-2,5,0\right)$ and $\left(-1,4,-3\right)$.

For problems 6 & 7 determine where the line intersects the plane or show that it does not intersect the plane.

6. The line given by $\vec{r}\left(t\right)=\left\langle -2t,2+7t,-1-4t\right\rangle$ and the plane given by $4x+9y-2z=-8$.

7. The line given by $\vec{r}\left(t\right)=\left\langle 4+t,-1+8t,3+2t\right\rangle$ and the plane given by $2x-y+3z=15$.

8. Find the line of intersection of the plane given by $3x+6y-5z=-3$ and the plane given by $-2x+7y-z=24$.

9. Determine if the line given by $x=8-15t$, $y=9t$, $z=5+18t$ and the plane given by $10x-6y-12z=7$ are parallel, orthogonal or neither.

Section 1-4 : Quadric Surfaces

Sketch each of the following quadric surfaces.

1. $\dfrac{y^2}{9} + z^2 = 1$

2. $\dfrac{x^2}{4} + \dfrac{y^2}{9} + \dfrac{z^2}{6} = 1$

3. $z = \dfrac{x^2}{4} + \dfrac{y^2}{4} - 6$

4. $y^2 = 4x^2 + 16z^2$

5. $x = 4 - 5y^2 - 9z^2$

Section 1-5 : Functions of Several Variables

For problems 1 – 4 find the domain of the given function.

1. $f\left(x,y\right)=\sqrt{x^2-2y}$

2. $f\left(x,y\right)=\ln\left(2x-3y+1\right)$

3. $f\left(x,y,z\right)=\dfrac{1}{x^2+y^2+4z}$

4. $f\left(x,y\right)=\dfrac{1}{x}+\sqrt{y+4}-\sqrt{x+1}$

For problems 5 – 7 identify and sketch the level curves (or contours) for the given function.

5. $2x-3y+z^2=1$

6. $4z+2y^2-x=0$

7. $y^2=2x^2+z$

For problems 8 & 9 identify and sketch the traces for the given curves.

8. $2x-3y+z^2=1$

9. $4z+2y^2-x=0$

Section 1-6 : Vector Functions

For problems 1 & 2 find the domain of the given vector function.

1. $\vec r\left(t \right) = \left\langle {{t^2} + 1,\frac{1}{{t + 2}},\sqrt {t + 4} } \right\rangle$

2. $\vec r\left(t \right) = \left\langle {\ln \left({4 - {t^2}} \right),\sqrt {t + 1} } \right\rangle$

For problems 3 – 5 sketch the graph of the given vector function.

3. $\vec r\left(t \right) = \left\langle {4t,10 - 2t} \right\rangle$

4. $\vec r\left(t \right) = \left\langle {t + 1,\frac{1}{4}{t^2} + 3} \right\rangle$

5. $\vec r\left(t \right) = \left\langle {4\sin \left(t \right),8\cos \left(t \right)} \right\rangle$

For problems 6 & 7 identify the graph of the vector function without sketching the graph.

6. $\vec r\left(t \right) = \left\langle {3\cos \left({6t} \right), - 4,\sin \left({6t} \right)} \right\rangle$

7. $\vec r\left(t \right) = \left\langle {2 - t,4 + 7t, - 1 - 3t} \right\rangle$

For problems 8 & 9 write down the equation of the line segment between the two points.

8. The line segment starting at $\left({1,3} \right)$ and ending at $\left({ - 4,6} \right)$.

9. The line segment starting at $\left({0,2, - 1} \right)$ and ending at $\left({7, - 9,2} \right)$.

Section 1-7 : Calculus with Vector Functions

For problems 1 – 3 evaluate the given limit.

1. $\lim\limits_{t\to 1}\left\langle \mathbf{e}^{t-1}, 4t, \dfrac{t-1}{t^2-1}\right\rangle$

2. $\lim\limits_{t\to -2}\left(\dfrac{1-\mathbf{e}^{t+2}}{t^2+t-2}\vec{i}+\vec{j}+\left(t^2+6t\right)\vec{k}\right)$

3. $\lim\limits_{t\to \infty}\left\langle \dfrac{1}{t^2}, \dfrac{2t^2}{1-t-t^2}, \mathbf{e}^{-t}\right\rangle$

For problems 4 – 6 compute the derivative of the given vector function.

4. $\vec{r}\left(t\right)=\left(t^3-1\right)\vec{i}+\mathbf{e}^{2t}\,\vec{j}+\cos\left(t\right)\vec{k}$

5. $\vec{r}\left(t\right)=\left\langle \ln\left(t^2+1\right), t\mathbf{e}^{-t}, 4\right\rangle$

6. $\vec{r}\left(t\right)=\left\langle \dfrac{t+1}{t-1}, \tan\left(4t\right), \sin^2\left(t\right)\right\rangle$

For problems 7 – 9 evaluate the given integral.

7. $\int \vec{r}\left(t\right)dt$, where $\vec{r}\left(t\right)=t^3\,\vec{i}-\dfrac{2t}{t^2+1}\vec{j}+\cos^2\left(3t\right)\vec{k}$

8. $\int_{-1}^{2}\vec{r}\left(t\right)dt$ where $\vec{r}\left(t\right)=\left\langle 6, 6t^2-4t, t\mathbf{e}^{2t}\right\rangle$

9. $\int \vec{r}\left(t\right)dt$, where $\vec{r}\left(t\right)=\left\langle \left(1-t\right)\cos\left(t^2-2t\right), \cos\left(t\right)\sin\left(t\right), \sec^2\left(4t\right)\right\rangle$

Section 1-8 : Tangent, Normal and Binormal Vectors

For problems 1 & 2 find the unit tangent vector for the given vector function.

1. $\vec{r}\left(t\right)=\left\langle t^2+1,3-t,t^3\right\rangle$

2. $\vec{r}\left(t\right)=t\mathbf{e}^{2t}\,\vec{i}+\left(2-t^2\right)\vec{j}-\mathbf{e}^{2t}\vec{k}$

For problems 3 & 4 find the tangent line to the vector function at the given point.

3. $\vec{r}\left(t\right)=\cos\left(4t\right)\vec{i}+3\sin\left(4t\right)\vec{j}+t^3\vec{k}$ at $t=\pi$.

4. $\vec{r}\left(t\right)=\left\langle 7\mathbf{e}^{2-t},\dfrac{16}{t^3},5-t\right\rangle$ at $t=2$.

5. Find the unit normal and the binormal vectors for the following vector function.

$$\vec{r}\left(t\right)=\left\langle\cos\left(2t\right),\sin\left(2t\right),3\right\rangle$$

Section 1-9 : Arc Length with Vector Functions

For problems 1 & 2 determine the length of the vector function on the given interval.

1. $\vec{r}\left(t\right)=\left(3-4t\right)\vec{i}+6t\,\vec{j}-\left(9+2t\right)\vec{k}$ from $-6\le t\le 8$.

2. $\vec{r}\left(t\right)=\left\langle \frac{1}{3}t^3,4t,\sqrt{2}t^2\right\rangle$ from $0\le t\le 2$.

For problems 3 & 4 find the arc length function for the given vector function.

3. $\vec{r}\left(t\right)=\left\langle t^2,2t^3,1-t^3\right\rangle$

4. $\vec{r}\left(t\right)=\left\langle 4t,-2t,\sqrt{5}\ t^2\right\rangle$

5. Determine where on the curve given by $\vec{r}\left(t\right)=\left\langle t^2,2t^3,1-t^3\right\rangle$ we are after traveling a distance of 20.

Section 1-10 : Curvature

Find the curvature for each the following vector functions.

1. $\vec r\left(t \right) = \left\langle {\cos \left({2t} \right), - \sin \left({2t} \right),4t} \right\rangle$

2. $\vec r\left(t \right) = \left\langle {4t, - {t^2},2{t^3}} \right\rangle$

Section 1-11 : Velocity and Acceleration

1. An objects acceleration is given by $\vec{a} = 3t\,\vec{i} - 4\mathbf{e}^{-t}\,\vec{j} + 12t^2\vec{k}$. The objects initial velocity is $\vec{v}\left(0\right) = \vec{j} - 3\vec{k}$ and the objects initial position is $\vec{r}\left(0\right) = -5\vec{i} + 2\vec{j} - 3\vec{k}$. Determine the objects velocity and position functions.

2. Determine the tangential and normal components of acceleration for the object whose position is given by $\vec{r}\left(t\right) = \left\langle \cos\left(2t\right), -\sin\left(2t\right), 4t \right\rangle$.

Section 1-12 : Cylindrical Coordinates

For problems 1 & 2 convert the Cartesian coordinates for the point into Cylindrical coordinates.

1. $\left(4,-5,2\right)$

2. $\left(-4,-1,8\right)$

3. Convert the following equation written in Cartesian coordinates into an equation in Cylindrical coordinates.

$$x^3 + 2x^2 - 6z = 4 - 2y^2$$

For problems 4 & 5 convert the equation written in Cylindrical coordinates into an equation in Cartesian coordinates.

4. $z\,r = 2 - r^2$

5. $4\sin\left(\theta\right) - 2\cos\left(\theta\right) = \dfrac{r}{z}$

For problems 6 & 7 identify the surface generated by the given equation.

6. $r^2 - 4r\cos\left(\theta\right) = 14$

7. $z = 7 - 4r^2$

Section 1-13 : Spherical Coordinates

For problems 1 & 2 convert the Cartesian coordinates for the point into Spherical coordinates.

1. $\left(3,-4,1\right)$

2. $\left(-2,-1,-7\right)$

3. Convert the Cylindrical coordinates for the point $\left(2,0.345,-3\right)$ into Spherical coordinates.

4. Convert the following equation written in Cartesian coordinates into an equation in Spherical coordinates.

$$x^2+y^2=4x+z-2$$

For problems 5 & 6 convert the equation written in Spherical coordinates into an equation in Cartesian coordinates.

5. $\rho^2=3-\cos\varphi$

6. $\csc\varphi=2\cos\theta+4\sin\theta$

For problems 7 & 8 identify the surface generated by the given equation.

7. $\varphi=\dfrac{4\pi}{5}$

8. $\rho=-2\sin\varphi\cos\theta$

Chapter 2 : Partial Derivatives

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Higher Order Partial Derivatives – In the section we will take a look at higher order partial derivatives. Unlike Calculus I however, we will have multiple second order derivatives, multiple third order derivatives, *etc*. because we are now working with functions of multiple variables. We will also discuss Clairaut's Theorem to help with some of the work in finding higher order derivatives.

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Directional Derivatives – In the section we introduce the concept of directional derivatives. With directional derivatives we can now ask how a function is changing if we allow all the independent variables to change rather than holding all but one constant as we had to do with partial derivatives. In addition, we will define the gradient vector to help with some of the notation and work here. The gradient vector will be very useful in some later sections as well. We will also give a nice fact that will allow us to determine the direction in which a given function is changing the fastest.

Section 2-1 : Limits

Evaluate each of the following limits.

1. $\displaystyle \lim_{(x,y,z)\to(-1,0,4)} \frac{x^3 - z\mathbf{e}^{2y}}{6x + 2y - 3z}$

2. $\displaystyle \lim_{(x,y)\to(2,1)} \frac{x^2 - 2xy}{x^2 - 4y^2}$

3. $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x - 4y}{6y + 7x}$

4. $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x^2 - y^6}{xy^3}$

Section 2-2 : Partial Derivatives

For problems 1 – 8 find all the 1st order partial derivatives.

1. $f\left(x,y,z\right)=4x^3y^2-\mathbf{e}^zy^4+\dfrac{z^3}{x^2}+4y-x^{16}$

2. $w=\cos\left(x^2+2y\right)-\mathbf{e}^{4x-z^4y}+y^3$

3. $f\left(u,v,p,t\right)=8u^2t^3p-\sqrt{v}\,p^2t^{-5}+2u^2t+3p^4-v$

4. $f\left(u,v\right)=u^2\sin\left(u+v^3\right)-\sec\left(4u\right)\tan^{-1}\left(2v\right)$

5. $f\left(x,z\right)=\mathbf{e}^{-x}\sqrt{z^4+x^2}-\dfrac{2x+3z}{4z-7x}$

6. $g\left(s,t,v\right)=t^2\ln\left(s+2t\right)-\ln\left(3v\right)\left(s^3+t^2-4v\right)$

7. $R\left(x,y\right)=\dfrac{x^2}{y^2+1}-\dfrac{y^2}{x^2+y}$

8. $z=\dfrac{p^2\left(r+1\right)}{t^3}+pr\,\mathbf{e}^{2p+3r+4t}$

9. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ for the following function.

$$x^2\sin\left(y^3\right)+x\mathbf{e}^{3z}-\cos\left(z^2\right)=3y-6z+8$$

Section 2-3 : Interpretations of Partial Derivatives

1. Determine if $f\left(x,y\right)=x\ln\left(4y\right)+\sqrt{x+y}$ is increasing or decreasing at $\left(-3,6\right)$ if

   (a) we allow *x* to vary and hold *y* fixed.
   (b) we allow *y* to vary and hold *x* fixed.

2. Determine if $f\left(x,y\right)=x^2\sin\left(\frac{\pi}{y}\right)$ is increasing or decreasing at $\left(-2,\frac{3}{4}\right)$ if

   (a) we allow *x* to vary and hold *y* fixed.
   (b) we allow *y* to vary and hold *x* fixed.

3. Write down the vector equations of the tangent lines to the traces for $f\left(x,y\right)=x\,\mathbf{e}^{2x-y^2}$ at $\left(2,0\right)$.

Section 2-4 : Higher Order Partial Derivatives

For problems 1 & 2 verify Clairaut's Theorem for the given function.

1. $f\left(x,y\right)=x^3y^2-\dfrac{4y^6}{x^3}$

2. $A\left(x,y\right)=\cos\left(\dfrac{x}{y}\right)-x^7y^4+y^{10}$

For problems 3 – 6 find all 2nd order derivatives for the given function.

3. $g\left(u,v\right)=u^3v^4-2u\sqrt{v^3}+u^6-\sin\left(3v\right)$

4. $f\left(s,t\right)=s^2t+\ln\left(t^2-s\right)$

5. $h\left(x,y\right)=\mathbf{e}^{x^4y^6}-\dfrac{y^3}{x}$

6. $f\left(x,y,z\right)=\dfrac{x^2y^6}{z^3}-2x^6z+8y^{-3}x^4+4z^2$

7. Given $f\left(x,y\right)=x^4y^3z^6$ find $\dfrac{\partial^6 f}{\partial y\partial z^2\partial y\partial x^2}$.

8. Given $w=u^2\mathbf{e}^{-6v}+\cos\left(u^6-4u+1\right)$ find w_{vuuvv}.

9. Given $G\left(x,y\right)=y^4\sin\left(2x\right)+x^2\left(y^{10}-\cos\left(y^2\right)\right)^7$ find $G_{yyyxxxy}$.

Section 2-5 : Differentials

Compute the differential of each of the following functions.

1. $z = x^2 \sin(6y)$

2. $f(x,y,z) = \ln\left(\dfrac{xy^2}{z^3}\right)$

Section 2-6 : Chain Rule

1. Given the following information use the Chain Rule to determine $\dfrac{dz}{dt}$.

$$z = \cos\left(y\,x^2\right) \qquad x = t^4 - 2t, \quad y = 1 - t^6$$

2. Given the following information use the Chain Rule to determine $\dfrac{dw}{dt}$.

$$w = \frac{x^2 - z}{y^4} \qquad x = t^3 + 7, \quad y = \cos\left(2t\right), \quad z = 4t$$

3. Given the following information use the Chain Rule to determine $\dfrac{dz}{dx}$.

$$z = x^2 y^4 - 2y \qquad y = \sin\left(x^2\right)$$

4. Given the following information use the Chain Rule to determine $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$.

$$z = x^{-2} y^6 - 4x \qquad x = u^2 v, \quad y = v - 3u$$

5. Given the following information use the Chain Rule to determine z_t and z_p .

$$z = 4y\sin\left(2x\right) \qquad x = 3u - p, \quad y = p^2 u, \quad u = t^2 + 1$$

6. Given the following information use the Chain Rule to determine $\dfrac{\partial w}{\partial t}$ and $\dfrac{\partial w}{\partial s}$.

$$w = \sqrt{x^2 + y^2} + \frac{6z}{y} \qquad x = \sin\left(p\right), \quad y = p + 3t - 4s, \quad z = \frac{t^3}{s^2}, \quad p = 1 - 2t$$

7. Determine formulas for $\dfrac{\partial w}{\partial t}$ and $\dfrac{\partial w}{\partial v}$ for the following situation.

$$w = w\left(x, y\right) \qquad x = x\left(p, q, s\right), \quad y = y\left(p, u, v\right), \quad s = s\left(u, v\right), \quad p = p\left(t\right)$$

8. Determine formulas for $\dfrac{\partial w}{\partial t}$ and $\dfrac{\partial w}{\partial u}$ for the following situation.

$$w = w\left(x, y, z\right) \qquad x = x\left(t\right), \quad y = y\left(u, v, p\right), \quad z = z\left(v, p\right), \quad v = v\left(r, u\right), \quad p = p\left(t, u\right)$$

9. Compute $\dfrac{dy}{dx}$ for the following equation.

$$x^2 y^4 - 3 = \sin\left(xy\right)$$

10. Compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for the following equation.

$$\mathbf{e}^{zy} + xz^2 = 6xy^4z^3$$

11. Determine f_{uu} for the following situation.

$$f = f\left(x, y\right) \qquad x = u^2 + 3v, \quad y = uv$$

Section 2-7 : Directional Derivatives

For problems 1 & 2 determine the gradient of the given function.

1. $f\left(x,y\right)=x^2\sec\left(3x\right)-\frac{x^2}{y^3}$

2. $f\left(x,y,z\right)=x\cos\left(xy\right)+z^2y^4-7xz$

For problems 3 & 4 determine $D_{\vec{u}}f$ for the given function in the indicated direction.

3. $f\left(x,y\right)=\cos\left(\frac{x}{y}\right)$ in the direction of $\vec{v}=\left\langle 3,-4\right\rangle$

4. $f\left(x,y,z\right)=x^2y^3-4xz$ in the direction of $\vec{v}=\left\langle -1,2,0\right\rangle$

5. Determine $D_{\vec{u}}f\left(3,-1,0\right)$ for $f\left(x,y,z\right)=4x-y^2\mathbf{e}^{3xz}$ direction of $\vec{v}=\left\langle -1,4,2\right\rangle$.

For problems 6 & 7 find the maximum rate of change of the function at the indicated point and the direction in which this maximum rate of change occurs.

6. $f\left(x,y\right)=\sqrt{x^2+y^4}$ at $\left(-2,3\right)$

7. $f\left(x,y,z\right)=\mathbf{e}^{2x}\cos\left(y-2z\right)$ at $\left(4,-2,0\right)$

Chapter 3 : Applications of Partial Derivatives

Here are a set of practice problems for the Applications of Partial Derivatives chapter of the Calculus II notes. If you are viewing the pdf version of this document (as opposed to viewing it on the web) this document contains only the problems themselves and no solutions are included in this document. Solutions can be found in a couple of places on the site.

7. If you'd like a pdf document containing the solutions the download tab on the website contains links to pdf's containing the solutions for the full book, chapter and section. At this time, I do not offer pdf's for solutions to individual problems.

8. If you'd like to view the solutions on the web go to the problem set web page, click the solution link for any problem and it will take you to the solution to that problem.

Note that some sections will have more problems than others and some will have more or less of a variety of problems. Most sections should have a range of difficulty levels in the problems although this will vary from section to section.

Here is a list of all the sections for which practice problems have been written as well as a brief description of the material covered in the notes for that particular section.

Tangent Planes and Linear Approximations – In this section formally define just what a tangent plane to a surface is and how we use partial derivatives to find the equations of tangent planes to surfaces that can be written as $z = f\left(x, y\right)$. We will also see how tangent planes can be thought of as a linear approximation to the surface at a given point.

Gradient Vector, Tangent Planes and Normal Lines – In this section discuss how the gradient vector can be used to find tangent planes to a much more general function than in the previous section. We will also define the normal line and discuss how the gradient vector can be used to find the equation of the normal line.

Relative Minimums and Maximums – In this section we will define critical points for functions of two variables and discuss a method for determining if they are relative minimums, relative maximums or saddle points (*i.e.* neither a relative minimum or relative maximum).

Absolute Minimums and Maximums – In this section we will how to find the absolute extrema of a function of two variables when the independent variables are only allowed to come from a region that is bounded (*i.e.* no part of the region goes out to infinity) and closed (*i.e.* all of the points on the boundary are valid points that can be used in the process).

Lagrange Multipliers – In this section we'll see discuss how to use the method of Lagrange Multipliers to find the absolute minimums and maximums of functions of two or three variables in which the independent variables are subject to one or more constraints. We also give a brief justification for how/why the method works.

Section 3-1 : Tangent Planes and Linear Approximations

1. Find the equation of the tangent plane to $z = x^2 \cos(\pi y) - \dfrac{6}{xy^2}$ at $(2, -1)$.

2. Find the equation of the tangent plane to $z = x\sqrt{x^2 + y^2} + y^3$ at $(-4, 3)$.

3. Find the linear approximation to $z = 4x^2 - y\mathbf{e}^{2x+y}$ at $(-2, 4)$.

Section 3-2 : Gradient Vector, Tangent Planes and Normal Lines

1. Find the tangent plane and normal line to $x^2 y = 4z\mathbf{e}^{x+y} - 35$ at $\left(3, -3, 2\right)$.

2. Find the tangent plane and normal line to $\ln\left(\frac{x}{2y}\right) = z^2\left(x - 2y\right) + 3z + 3$ at $\left(4, 2, -1\right)$.

Section 3-3 : Relative Minimums and Maximums

Find and classify all the critical points of the following functions.

1. $f\left(x,y\right)=\left(y-2\right)x^2-y^2$

2. $f\left(x,y\right)=7x-8y+2xy-x^2+y^3$

3. $f\left(x,y\right)=\left(3x+4x^3\right)\left(y^2+2y\right)$

4. $f\left(x,y\right)=3y^3-x^2y^2+8y^2+4x^2-20y$

Section 3-4 : Absolute Minimums and Maximums

1. Find the absolute minimum and absolute maximum of $f\left(x,y\right)=192x^3+y^2-4xy^2$ on the triangle with vertices $\left(0,0\right)$, $\left(4,2\right)$ and $\left(-2,2\right)$.

2. Find the absolute minimum and absolute maximum of $f\left(x,y\right)=\left(9x^2-1\right)\left(1+4y\right)$ on the rectangle given by $-2\le x\le 3$, $-1\le y\le 4$.

Section 3-5 : Lagrange Multipliers

1. Find the maximum and minimum values of $f\left(x,y\right)=81x^2+y^2$ subject to the constraint $4x^2+y^2=9$.

2. Find the maximum and minimum values of $f\left(x,y\right)=8x^2-2y$ subject to the constraint $x^2+y^2=1$.

3. Find the maximum and minimum values of $f\left(x,y,z\right)=y^2-10z$ subject to the constraint $x^2+y^2+z^2=36$.

4. Find the maximum and minimum values of $f\left(x,y,z\right)=xyz$ subject to the constraint $x+9y^2+z^2=4$. Assume that $x\geq 0$ for this problem. Why is this assumption needed?

5. Find the maximum and minimum values of $f\left(x,y,z\right)=3x^2+y$ subject to the constraints $4x-3y=9$ and $x^2+z^2=9$.

Chapter 4 : Multiple Integrals

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Here is a list of all the sections for which practice problems have been written as well as a brief description of the material covered in the notes for that particular section.

Double Integrals – In this section we will define the double integral.

Iterated Integrals – In this section we will show how Fubini's Theorem can be used to evaluate double integrals where the region of integration is a rectangle.

Double Integrals over General Regions – In this section we will start evaluating double integrals over general regions, *i.e.* regions that aren't rectangles. We will illustrate how a double integral of a function can be interpreted as the net volume of the solid between the surface given by the function and the *xy*-plane.

Double Integrals in Polar Coordinates – In this section we will look at converting integrals (including *dA*) in Cartesian coordinates into Polar coordinates. The regions of integration in these cases will be all or portions of disks or rings and so we will also need to convert the original Cartesian limits for these regions into Polar coordinates.

Triple Integrals – In this section we will define the triple integral. We will also illustrate quite a few examples of setting up the limits of integration from the three dimensional region of integration. Getting the limits of integration is often the difficult part of these problems.

Triple Integrals in Cylindrical Coordinates – In this section we will look at converting integrals (including *dV*) in Cartesian coordinates into Cylindrical coordinates. We will also be converting the original Cartesian limits for these regions into Cylindrical coordinates.

Triple Integrals in Spherical Coordinates – In this section we will look at converting integrals (including *dV*) in Cartesian coordinates into Spherical coordinates. We will also be converting the original Cartesian limits for these regions into Spherical coordinates.

Change of Variables – In previous sections we've converted Cartesian coordinates in Polar, Cylindrical and Spherical coordinates. In this section we will generalize this idea and discuss how we convert integrals in Cartesian coordinates into alternate coordinate systems. Included will be a derivation of the *dV* conversion formula when converting to Spherical coordinates.

Surface Area – In this section we will show how a double integral can be used to determine the surface area of the portion of a surface that is over a region in two dimensional space.

Area and Volume Revisited – In this section we summarize the various area and volume formulas from this chapter.

Section 4-1 : Double Integrals

1. Use the Midpoint Rule to estimate the volume under $f\left(x,y\right)=x^2+y$ and above the rectangle given by $-1\le x\le 3$, $0\le y\le 4$ in the *xy*-plane. Use 4 subdivisions in the *x* direction and 2 subdivisions in the *y* direction.

Section 4-2 : Iterated Integrals

1. Compute the following double integral over the indicated rectangle **(a)** by integrating with respect to *x* first and **(b)** by integrating with respect to *y* first.

$$\iint\limits_R 12x - 18y\,dA \qquad R = [-1,4]\times[2,3]$$

For problems 2 – 8 compute the given double integral over the indicated rectangle.

2. $\displaystyle\iint\limits_R 6y\sqrt{x} - 2y^3\,dA \qquad R = [1,4]\times[0,3]$

3. $\displaystyle\iint\limits_R \frac{\mathbf{e}^x}{2y} - \frac{4x-1}{y^2}\,dA \qquad R = [-1,0]\times[1,2]$

4. $\displaystyle\iint\limits_R \sin(2x) - \frac{1}{1+6y}\,dA \qquad R = \left[\tfrac{\pi}{4},\tfrac{\pi}{2}\right]\times[0,1]$

5. $\displaystyle\iint\limits_R y\mathbf{e}^{y^2-4x}\,dA \qquad R = [0,2]\times\left[0,\sqrt{8}\right]$

6. $\displaystyle\iint\limits_R xy^2\sqrt{x^2+y^3}\,dA \qquad R = [0,3]\times[0,2]$

7. $\displaystyle\iint\limits_R xy\cos\left(yx^2\right)dA \qquad R = [-2,3]\times[-1,1]$

8. $\displaystyle\iint\limits_R xy\cos(y) - x^2\,dA \qquad R = [1,2]\times\left[\tfrac{\pi}{2},\pi\right]$

9. Determine the volume that lies under $f(x,y) = 9x^2 + 4xy + 4$ and above the rectangle given by $[-1,1]\times[0,2]$ in the *xy*-plane.

Section 4-3 : Double Integrals over General Regions

1. Evaluate $\iint\limits_D 42y^2 - 12x\,dA$ where $D = \left\{ (x,y) \mid 0 \le x \le 4, (x-2)^2 \le y \le 6 \right\}$

2. Evaluate $\iint\limits_D 2yx^2 + 9y^3\,dA$ where *D* is the region bounded by $y = \frac{2}{3}x$ and $y = 2\sqrt{x}$.

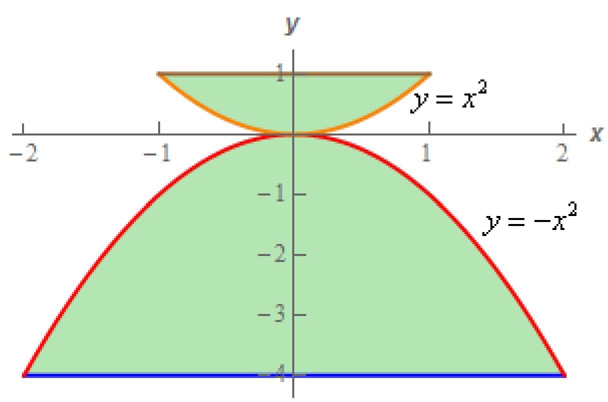
3. Evaluate $\iint\limits_D 10x^2y^3 - 6\,dA$ where *D* is the region bounded by $x = -2y^2$ and $x = y^3$.

4. Evaluate $\iint\limits_D x(y-1)\,dA$ where *D* is the region bounded by $y = 1 - x^2$ and $y = x^2 - 3$.

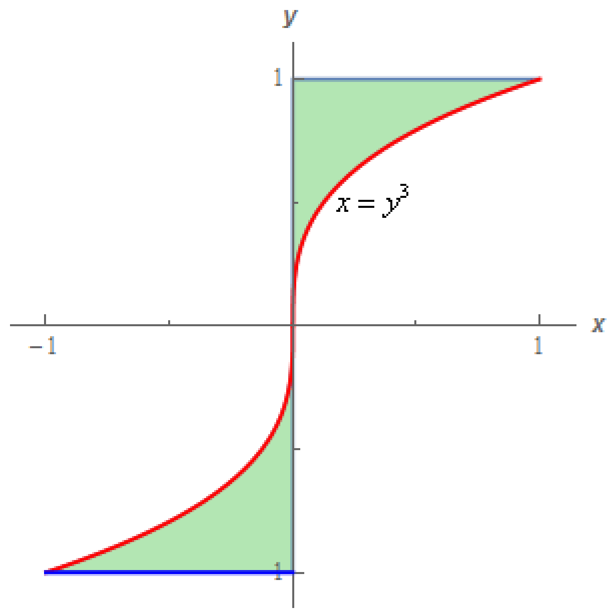
5. Evaluate $\iint\limits_D 5x^3\cos\left(y^3\right)dA$ where *D* is the region bounded by $y = 2$, $y = \frac{1}{4}x^2$ and the *y*-axis.

6. Evaluate $\iint\limits_D \frac{1}{y^{\frac{1}{3}}\left(x^3+1\right)}\,dA$ where *D* is the region bounded by $x = -y^{\frac{1}{3}}$, $x = 3$ and the *x*-axis.

7. Evaluate $\iint\limits_D 3 - 6xy\,dA$ where *D* is the region shown below.



8. Evaluate $\iint\limits_D \mathbf{e}^{y^4}\,dA$ where *D* is the region shown below.



9. Evaluate $\iint\limits_D 7x^2 + 14y\,dA$ where *D* is the region bounded by $x = 2y^2$ and $x = 8$ in the order given below.

(a) Integrate with respect to ***x*** first and then ***y***.

(b) Integrate with respect to ***y*** first and then ***x***.

For problems 10 & 11 evaluate the given integral by first reversing the order of integration.

10. $\int_0^3 \int_{2x}^6 \sqrt{y^2 + 2}\,dy\,dx$

11. $\int_0^1 \int_{-\sqrt{y}}^{y^2} 6x - y\,dx\,dy$

12. Use a double integral to determine the area of the region bounded by $y = 1 - x^2$ and $y = x^2 - 3$.

13. Use a double integral to determine the volume of the region that is between the *xy*-plane and $f(x,y) = 2 + \cos(x^2)$ and is above the triangle with vertices $(0,0)$, $(6,0)$ and $(6,2)$.

14. Use a double integral to determine the volume of the region bounded by $z = 6 - 5x^2$ and the planes $y = 2x$, $y = 2$, $x = 0$ and the *xy*-plane.

15. Use a double integral to determine the volume of the region formed by the intersection of the two cylinders $x^2 + y^2 = 4$ and $x^2 + z^2 = 4$.

Section 4-4 : Double Integrals in Polar Coordinates

1. Evaluate $\iint\limits_D y^2 + 3x\,dA$ where *D* is the region in the 3rd quadrant between $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$.

2. Evaluate $\iint\limits_D \sqrt{1 + 4x^2 + 4y^2}\,dA$ where *D* is the bottom half of $x^2 + y^2 = 16$.

3. Evaluate $\iint\limits_D 4xy - 7\,dA$ where *D* is the portion of $x^2 + y^2 = 2$ in the 1st quadrant.

4. Use a double integral to determine the area of the region that is inside $r = 4 + 2\sin\theta$ and outside $r = 3 - \sin\theta$.

5. Evaluate the following integral by first converting to an integral in polar coordinates.

$$\int_0^3 \int_{-\sqrt{9-x^2}}^0 \mathbf{e}^{x^2+y^2}\,dy\,dx$$

6. Use a double integral to determine the volume of the solid that is inside the cylinder $x^2 + y^2 = 16$, below $z = 2x^2 + 2y^2$ and above the *xy*-plane.

7. Use a double integral to determine the volume of the solid that is bounded by $z = 8 - x^2 - y^2$ and $z = 3x^2 + 3y^2 - 4$.

Section 4-5 : Triple Integrals

1. Evaluate $\int_{2}^{3}\int_{-1}^{4}\int_{1}^{0} 4x^2 y - z^3 \, dz\, dy\, dx$

2. Evaluate $\int_{0}^{1}\int_{0}^{z^2}\int_{0}^{3} y\cos\left(z^5\right) dx\, dy\, dz$

3. Evaluate $\iiint\limits_E 6z^2 \, dV$ where *E* is the region below $4x + y + 2z = 10$ in the first octant.

4. Evaluate $\iiint\limits_E 3 - 4x \, dV$ where *E* is the region below $z = 4 - xy$ and above the region in the *xy*-plane defined by $0 \le x \le 2$, $0 \le y \le 1$.

5. Evaluate $\iiint\limits_E 12y - 8x \, dV$ where *E* is the region behind $y = 10 - 2z$ and in front of the region in the *xz*-plane bounded by $z = 2x$, $z = 5$ and $x = 0$.

6. Evaluate $\iiint\limits_E yz \, dV$ where *E* is the region bounded by $x = 2y^2 + 2z^2 - 5$ and the plane $x = 1$.

7. Evaluate $\iiint\limits_E 15z \, dV$ where *E* is the region between $2x + y + z = 4$ and $4x + 4y + 2z = 20$ that is in front of the region in the *yz*-plane bounded by $z = 2y^2$ and $z = \sqrt{4y}$.

8. Use a triple integral to determine the volume of the region below $z = 4 - xy$ and above the region in the *xy*-plane defined by $0 \le x \le 2$, $0 \le y \le 1$.

9. Use a triple integral to determine the volume of the region that is below $z = 8 - x^2 - y^2$ above $z = -\sqrt{4x^2 + 4y^2}$ and inside $x^2 + y^2 = 4$.

Section 4-6 : Triple Integrals in Cylindrical Coordinates

1. Evaluate $\displaystyle\iiint\limits_E 4xy\,dV$ where *E* is the region bounded by $z = 2x^2 + 2y^2 - 7$ and $z = 1$.

2. Evaluate $\displaystyle\iiint\limits_E \mathbf{e}^{-x^2-z^2}\,dV$ where *E* is the region between the two cylinders $x^2 + z^2 = 4$ and $x^2 + z^2 = 9$ with $1 \le y \le 5$ and $z \le 0$.

3. Evaluate $\displaystyle\iiint\limits_E z\,dV$ where *E* is the region between the two planes $x + y + z = 2$ and $x = 0$ and inside the cylinder $y^2 + z^2 = 1$.

4. Use a triple integral to determine the volume of the region below $z = 6 - x$, above $z = -\sqrt{4x^2 + 4y^2}$ inside the cylinder $x^2 + y^2 = 3$ with $x \le 0$.

5. Evaluate the following integral by first converting to an integral in cylindrical coordinates.

$$\int_0^{\sqrt{5}} \int_{-\sqrt{5-x^2}}^{0} \int_{x^2+y^2-11}^{9-3x^2-3y^2} 2x - 3y \;\; dz\,dy\,dx$$

Section 4-7 : Triple Integrals in Spherical Coordinates

1. Evaluate $\iiint\limits_E 10xz + 3\,dV$ where *E* is the region portion of $x^2 + y^2 + z^2 = 16$ with $z \ge 0$.

2. Evaluate $\iiint\limits_E x^2 + y^2\,dV$ where *E* is the region portion of $x^2 + y^2 + z^2 = 4$ with $y \ge 0$.

3. Evaluate $\iiint\limits_E 3z\,dV$ where *E* is the region below $x^2 + y^2 + z^2 = 1$ and inside $z = \sqrt{x^2 + y^2}$.

4. Evaluate $\iiint\limits_E x^2\,dV$ where *E* is the region above $x^2 + y^2 + z^2 = 36$ and inside $z = -\sqrt{3x^2 + 3y^2}$.

5. Evaluate the following integral by first converting to an integral in spherical coordinates.

$$\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{6x^2+6y^2}}^{\sqrt{7-x^2-y^2}} 18y\;\;dz\,dy\,dx$$

Section 4-8 : Change of Variables

For problems 1 – 3 compute the Jacobian of each transformation.

1. $x = 4u - 3v^2 \qquad y = u^2 - 6v$

2. $x = u^2 v^3 \qquad y = 4 - 2\sqrt{u}$

3. $x = \dfrac{v}{u} \qquad y = u^2 - 4v^2$

4. If *R* is the region inside $\dfrac{x^2}{4} + \dfrac{y^2}{36} = 1$ determine the region we would get applying the transformation $x = 2u$, $y = 6v$ to *R*.

5. If *R* is the parallelogram with vertices $(1,0)$, $(4,3)$, $(1,6)$ and $(-2,3)$ determine the region we would get applying the transformation $x = \frac{1}{2}(v - u)$, $y = \frac{1}{2}(v + u)$ to *R*.

6. If *R* is the region bounded by $xy = 1$, $xy = 3$, $y = 2$ and $y = 6$ determine the region we would get applying the transformation $x = \dfrac{v}{6u}$, $y = 2u$ to *R*.

7. Evaluate $\displaystyle\iint\limits_R xy^3\, dA$ where *R* is the region bounded by $xy = 1$, $xy = 3$, $y = 2$ and $y = 6$ using the transformation $x = \dfrac{v}{6u}$, $y = 2u$.

8. Evaluate $\displaystyle\iint\limits_R 6x - 3y\, dA$ where *R* is the parallelogram with vertices $(2,0)$, $(5,3)$, $(6,7)$ and $(3,4)$ using the transformation $x = \frac{1}{3}(v - u)$, $y = \frac{1}{3}(4v - u)$ to *R*.

9. Evaluate $\displaystyle\iint\limits_R x + 2y\, dA$ where *R* is the triangle with vertices $(0,3)$, $(4,1)$ and $(2,6)$ using the transformation $x = \frac{1}{2}(u - v)$, $y = \frac{1}{4}(3u + v + 12)$ to *R*.

10. Derive the transformation used in problem 8.

11. Derive a transformation that will convert the triangle with vertices $(1,0)$, $(6,0)$ and $(3,8)$ into a right triangle with the right angle occurring at the origin of the *uv* system.

Section 4-9 : Surface Area

1. Determine the surface area of the portion of $2x+3y+6z=9$ that is in the 1st octant.

2. Determine the surface area of the portion of $z=13-4x^2-4y^2$ that is above $z=1$ with $x\le 0$ and $y\le 0$.

3. Determine the surface area of the portion of $z=3+2y+\frac{1}{4}x^4$ that is above the region in the *xy*-plane bounded by $y=x^5$, $x=1$ and the *x*-axis.

4. Determine the surface area of the portion of $y=2x^2+2z^2-7$ that is inside the cylinder $x^2+z^2=4$.

5. Determine the surface area region formed by the intersection of the two cylinders $x^2+y^2=4$ and $x^2+z^2=4$.

Section 4-10 : Area and Volume Revisited

The intent of the section was just to “recap” the various area and volume formulas from this chapter and so no problems have been (or likely will be in the near future) written.

Chapter 5 : Line Integrals

Here are a set of practice problems for the Line Integrals chapter of the Calculus II notes. If you are viewing the pdf version of this document (as opposed to viewing it on the web) this document contains only the problems themselves and no solutions are included in this document. Solutions can be found in a couple of places on the site.

11. If you'd like a pdf document containing the solutions the download tab on the website contains links to pdf's containing the solutions for the full book, chapter and section. At this time, I do not offer pdf's for solutions to individual problems.

12. If you'd like to view the solutions on the web go to the problem set web page, click the solution link for any problem and it will take you to the solution to that problem.

Note that some sections will have more problems than others and some will have more or less of a variety of problems. Most sections should have a range of difficulty levels in the problems although this will vary from section to section.

Here is a list of all the sections for which practice problems have been written as well as a brief description of the material covered in the notes for that particular section.

Vector Fields – In this section we introduce the concept of a vector field and give several examples of graphing them. We also revisit the gradient that we first saw a few chapters ago.

Line Integrals – Part I – In this section we will start off with a quick review of parameterizing curves. This is a skill that will be required in a great many of the line integrals we evaluate and so needs to be understood. We will then formally define the first kind of line integral we will be looking at : line integrals with respect to arc length.

Line Integrals – Part II – In this section we will continue looking at line integrals and define the second kind of line integral we'll be looking at : line integrals with respect to *x*, *y*, and/or *z*. We also introduce an alternate form of notation for this kind of line integral that will be useful on occasion.

Line Integrals of Vector Fields – In this section we will define the third type of line integrals we'll be looking at : line integrals of vector fields. We will also see that this particular kind of line integral is related to special cases of the line integrals with respect to *x*, *y* and *z*.

Fundamental Theorem for Line Integrals – In this section we will give the fundamental theorem of calculus for line integrals of vector fields. This will illustrate that certain kinds of line integrals can be very quickly computed. We will also give quite a few definitions and facts that will be useful.

Conservative Vector Fields – In this section we will take a more detailed look at conservative vector fields than we've done in previous sections. We will also discuss how to find potential functions for conservative vector fields.

Green's Theorem – In this section we will discuss Green's Theorem as well as an interesting application of Green's Theorem that we can use to find the area of a two dimensional region.

Section 5-1 : Vector Fields

1. Sketch the vector field for $\vec{F}\left(x,y\right)=2x\vec{i}-2\vec{j}$.

2. Sketch the vector field for $\vec{F}\left(x,y\right)=\left(y-1\right)\vec{i}+\left(x+y\right)\vec{j}$.

3. Compute the gradient vector field for $f\left(x,y\right)=y^2\cos\left(2x-y\right)$.

4. Compute the gradient vector field for $f\left(x,y,z\right)=z^2\mathbf{e}^{x^2+4y}+\ln\left(\frac{xy}{z}\right)$.

Section 5-2 : Line Integrals - Part I

For problems 1 – 7 evaluate the given line integral. Follow the direction of *C* as given in the problem statement.

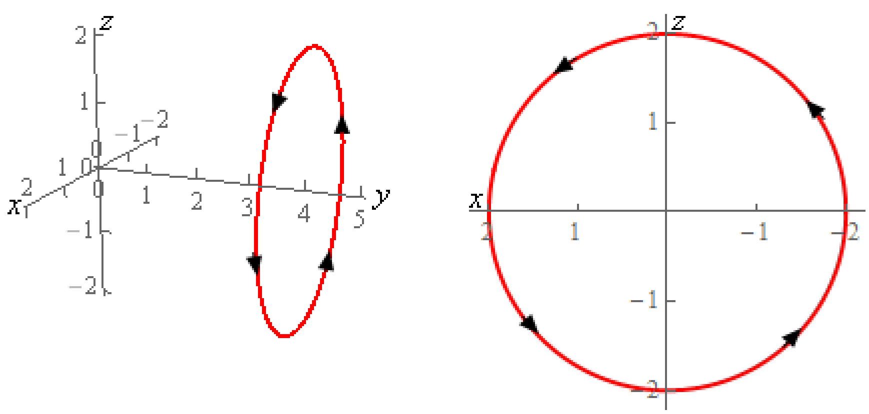
1. Evaluate $\int\limits_C 3x^2 - 2y\,ds$ where *C* is the line segment from $(3,6)$ to $(1,-1)$.

2. Evaluate $\int\limits_C 2yx^2 - 4x\,ds$ where *C* is the lower half of the circle centered at the origin of radius 3 with clockwise rotation.

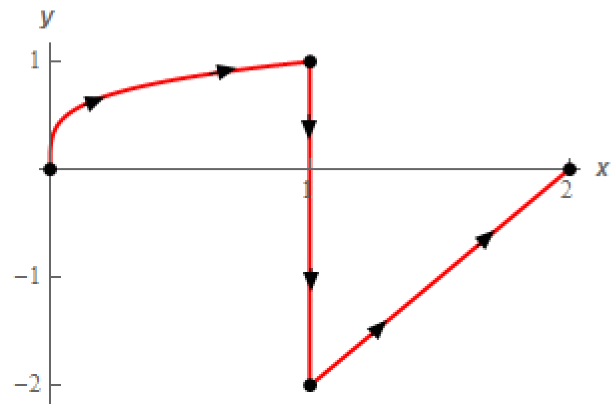
3. Evaluate $\int\limits_C 6x\,ds$ where *C* is the portion of $y = x^2$ from $x = -1$ to $x = 2$. The direction of *C* is in the direction of increasing *x*.

4. Evaluate $\int\limits_C xy - 4z\,ds$ where *C* is the line segment from $(1,1,0)$ to $(2,3,-2)$.

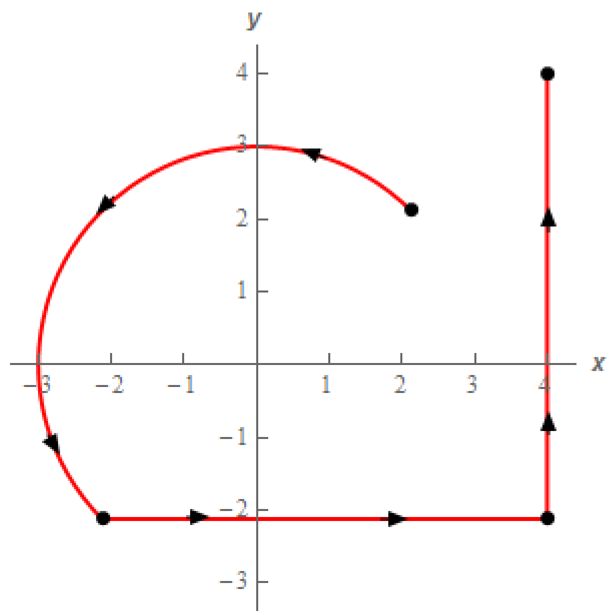
5. Evaluate $\int\limits_C x^2y^2\,ds$ where *C* is the circle centered at the origin of radius 2 centered on the *y*-axis at $y = 4$. See the sketches below for orientation. Note the “odd” axis orientation on the 2D circle is intentionally that way to match the 3D axis the direction.



6. Evaluate $\int\limits_C 16y^5\,ds$ where *C* is the portion of $x = y^4$ from $y = 0$ to $y = 1$ followed by the line segment form $(1,1)$ to $(1,-2)$ which in turn is followed by the line segment from $(1,-2)$ to $(2,0)$. See the sketch below for the direction.



7. Evaluate $\int\limits_C 4y - x\,ds$ where *C* is the upper portion of the circle centered at the origin of radius 3 from $\left(\frac{3}{\sqrt{2}},\frac{3}{\sqrt{2}}\right)$ to $\left(-\frac{3}{\sqrt{2}},-\frac{3}{\sqrt{2}}\right)$ in the counter clockwise rotation followed by the line segment form $\left(-\frac{3}{\sqrt{2}},-\frac{3}{\sqrt{2}}\right)$ to $\left(4,-\frac{3}{\sqrt{2}}\right)$ which in turn is followed by the line segment from $\left(4,-\frac{3}{\sqrt{2}}\right)$ to $(4,4)$. See the sketch below for the direction.



8. Evaluate $\int\limits_C y^3 - x^2\,ds$ for each of the following curves.

(a) *C* is the line segment from $(3,6)$ to $(0,0)$ followed by the line segment from $(0,0)$ to $(3,-6)$.

(b) *C* is the line segment from $(3,6)$ to $(3,-6)$.

9. Evaluate $\int\limits_C 4x^2\,ds$ for each of the following curves.

(a) C is the portion of the circle centered at the origin of radius 2 in the 1st quadrant rotating in the clockwise direction.

(b) C is the line segment from $\left(0,2\right)$ to $\left(2,0\right)$.

10. Evaluate $\int\limits_C 2x^3\,ds$ for each of the following curves.

(a) C is the portion $y = x^3$ from $x = -1$ to $x = 2$.

(b) C is the portion $y = x^3$ from $x = 2$ to $x = -1$.

Section 5-3 : Line Integrals - Part II

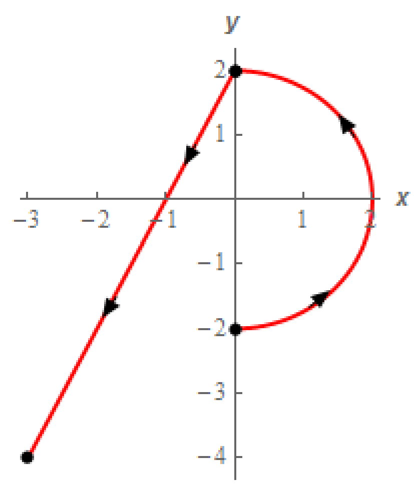
For problems 1 – 5 evaluate the given line integral. Follow the direction of *C* as given in the problem statement.

1. Evaluate $\int\limits_C \sqrt{1+y}\,dy$ where *C* is the portion of $y = \mathbf{e}^{2x}$ from $x = 0$ to $x = 2$.

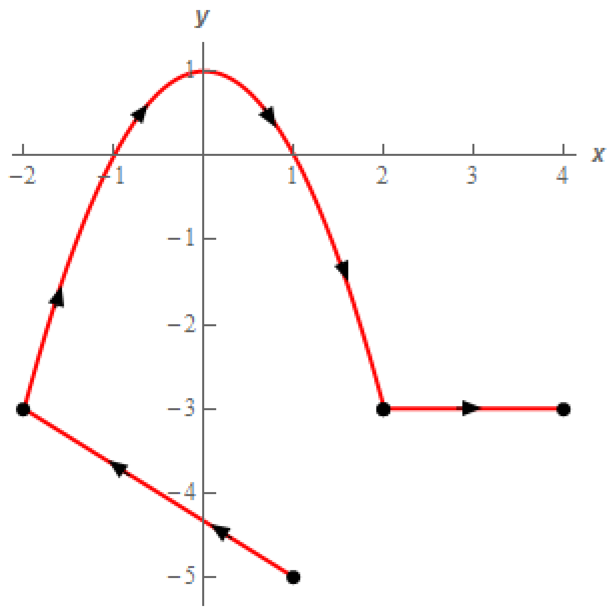
2. Evaluate $\int\limits_C 2y\,dx + (1-x)\,dy$ where *C* is portion of $y = 1 - x^3$ from $x = -1$ to $x = 2$.

3. Evaluate $\int\limits_C x^2\,dy - yz\,dz$ where *C* is the line segment from $(4,-1,2)$ to $(1,7,-1)$.

4. Evaluate $\int\limits_C 1 + x^3\,dx$ where *C* is the right half of the circle of radius 2 with counter clockwise rotation followed by the line segment from $(0,2)$ to $(-3,-4)$. See the sketch below for the direction.



5. Evaluate $\int\limits_C 2x^2\,dy - xy\,dx$ where *C* is the line segment from $(1,-5)$ to $(-2,-3)$ followed by the portion of $y = 1 - x^2$ from $x = -2$ to $x = 2$ which in turn is followed by the line segment from $(2,-3)$ to $(4,-3)$. See the sketch below for the direction.



6. Evaluate $\int_C (x-y)\,dx - yx^2\,dy$ for each of the following curves.

   (a) *C* is the portion of the circle of radius 6 in the 1st, 2nd and 3rd quadrant with clockwise rotation.

   (b) *C* is the line segment from $(0,-6)$ to $(6,0)$.

7. Evaluate $\int_C x^3\,dy - (y+1)\,dx$ for each of the following curves.

   (a) *C* is the line segment from $(1,7)$ to $(-2,4)$.

   (b) *C* is the line segment from $(-2,4)$ to $(1,7)$.

Section 5-4 : Line Integrals of Vector Fields

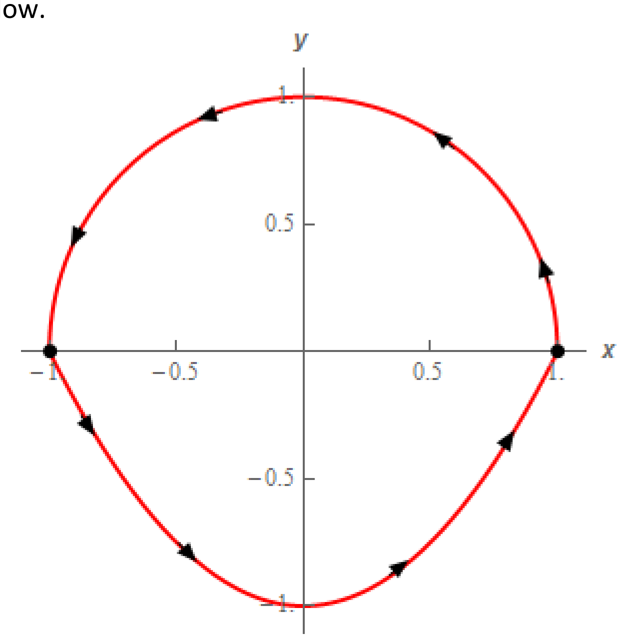
1. Evaluate $\int\limits_C \vec F \bullet d\vec r$ where $\vec F\left(x,y\right) = y^2\,\vec i + \left(3x - 6y\right)\vec j$ and *C* is the line segment from $\left(3,7\right)$ to $\left(0,12\right)$.

2. Evaluate $\int\limits_C \vec F \bullet d\vec r$ where $\vec F\left(x,y\right) = \left(x + y\right)\vec i + \left(1 - x\right)\vec j$ and *C* is the portion of $\frac{x^2}{4} + \frac{y^2}{9} = 1$ that is in the 4th quadrant with the counter clockwise rotation.

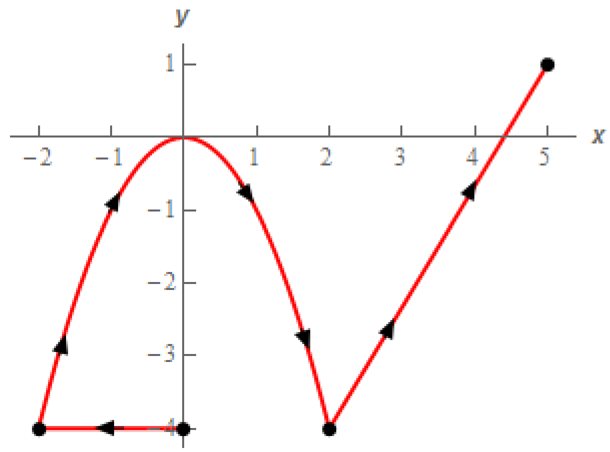
3. Evaluate $\int\limits_C \vec F \bullet d\vec r$ where $\vec F\left(x,y\right) = y^2\,\vec i + \left(x^2 - 4\right)\vec j$ and *C* is the portion of $y = \left(x - 1\right)^2$ from $x = 0$ to $x = 3$.

4. Evaluate $\int\limits_C \vec F \bullet d\vec r$ where $\vec F\left(x,y,z\right) = \mathbf{e}^{2x}\,\vec i + z\left(y + 1\right)\vec j + z^3\,\vec k$ and *C* is given by $\vec r\left(t\right) = t^3\,\vec i + \left(1 - 3t\right)\vec j + \mathbf{e}^t\,\vec k$ for $0 \le t \le 2$.

5. Evaluate $\int\limits_C \vec F \bullet d\vec r$ where $\vec F\left(x,y\right) = 3y\,\vec i + \left(x^2 - y\right)\vec j$ and *C* is the upper half of the circle centered at the origin of radius 1 with counter clockwise rotation and the portion of $y = x^2 - 1$ from $x = -1$ to $x = 1$. See the sketch below.



6. Evaluate $\int\limits_C \vec{F} \bullet d\vec{r}$ where $\vec{F}(x,y) = xy\,\vec{i} + (1+3y)\,\vec{j}$ and *C* is the line segment from $(0,-4)$ to $(-2,-4)$ followed by portion of $y = -x^2$ from $x = -2$ to $x = 2$ which is in turn followed by the line segment from $(2,-4)$ to $(5,1)$. See the sketch below.



7. Evaluate $\int\limits_C \vec{F} \bullet d\vec{r}$ where $\vec{F}(x,y) = (6x-2y)\vec{i} + x^2\vec{j}$ for each of the following curves.

(a) *C* is the line segment from $(6,-3)$ to $(0,0)$ followed by the line segment from $(0,0)$ to $(6,3)$.

(b) *C* is the line segment from $(6,-3)$ to $(6,3)$.

8. Evaluate $\int\limits_C \vec{F} \bullet d\vec{r}$ where $\vec{F}(x,y) = 3\vec{i} + (xy-2x)\vec{j}$ for each of the following curves.

(a) *C* is the upper half of the circle centered at the origin of radius 4 with counter clockwise rotation.

(b) *C* is the upper half of the circle centered at the origin of radius 4 with clockwise rotation.

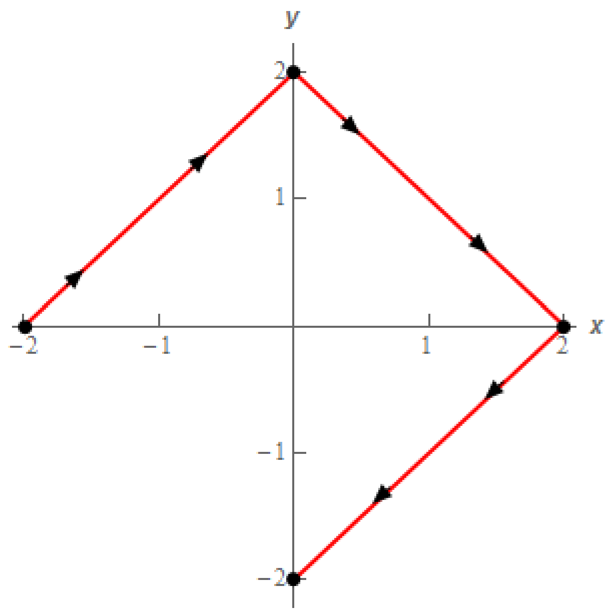
Section 5-5 : Fundamental Theorem for Line Integrals

1. Evaluate $\int\limits_C \nabla f \bullet d\vec{r}$ where $f(x,y) = x^3(3-y^2)+4y$ and C is given by $\vec{r}(t) = \langle 3-t^2, 5-t \rangle$ with $-2 \le t \le 3$.

2. Evaluate $\int\limits_C \nabla f \bullet d\vec{r}$ where $f(x,y) = y\mathbf{e}^{x^2-1} + 4x\sqrt{y}$ and C is given by $\vec{r}(t) = \langle 1-t, 2t^2 - 2t \rangle$ with $0 \le t \le 2$.

3. Given that $\int\limits_C \vec{F} \bullet d\vec{r}$ is independent of path compute $\int\limits_C \vec{F} \bullet d\vec{r}$ where C is the ellipse given by $\frac{(x-5)^2}{4} + \frac{y^2}{9} = 1$ with the counter clockwise rotation.

4. Evaluate $\int\limits_C \nabla f \bullet d\vec{r}$ where $f(x,y) = \mathbf{e}^{xy} - x^2 + y^3$ and C is the curve shown below.



Section 5-6 : Conservative Vector Fields

For problems 1 – 3 determine if the vector field is conservative.

1. $\vec{F} = \left(x^3 - 4xy^2 + 2\right)\vec{i} + \left(6x - 7y + x^3y^3\right)\vec{j}$

2. $\vec{F} = \left(2x\sin\left(2y\right) - 3y^2\right)\vec{i} + \left(2 - 6xy + 2x^2\cos\left(2y\right)\right)\vec{j}$

3. $\vec{F} = \left(6 - 2xy + y^3\right)\vec{i} + \left(x^2 - 8y + 3xy^2\right)\vec{j}$

For problems 4 – 8 find the potential function for the vector field.

4. $\vec{F} = \left(6x^2 - 2xy^2 + \dfrac{y}{2\sqrt{x}}\right)\vec{i} - \left(2x^2y - 4 - \sqrt{x}\right)\vec{j}$

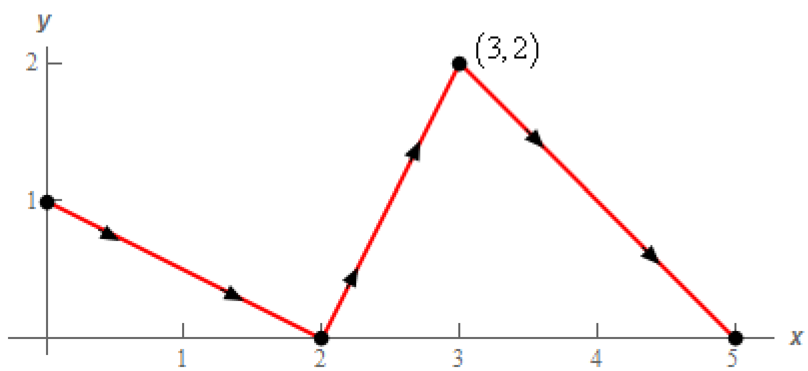
5. $\vec{F} = y^2\left(1 + \cos\left(x + y\right)\right)\vec{i} + \left(2xy - 2y + y^2\cos\left(x + y\right) + 2y\sin\left(x + y\right)\right)\vec{j}$

6. $\vec{F} = \left(2z^4 - 2y - y^3\right)\vec{i} + \left(z - 2x - 3xy^2\right)\vec{j} + \left(6 + y + 8xz^3\right)\vec{k}$

7. $\vec{F} = \dfrac{2xy}{z^3}\vec{i} + \left(2y - z^2 + \dfrac{x^2}{z^3}\right)\vec{j} - \left(4z^3 + 2yz + \dfrac{3x^2y}{z^4}\right)\vec{k}$

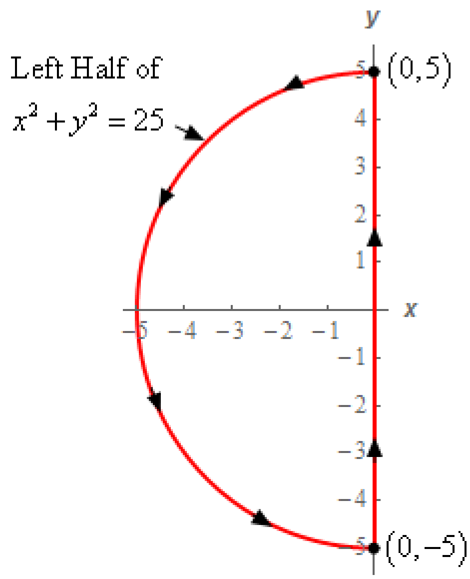
8. Evaluate $\int\limits_C \vec{F} \cdot d\vec{r}$ where C is the portion of the circle centered at the origin with radius 2 in the 1st quadrant with counter clockwise rotation and $\vec{F}\left(x,y\right) = \left(2xy - 4 - \frac{1}{2}\sin\left(\frac{1}{2}x\right)\sin\left(\frac{1}{2}y\right)\right)\vec{i} + \left(x^2 + \frac{1}{2}\cos\left(\frac{1}{2}x\right)\cos\left(\frac{1}{2}y\right)\right)\vec{j}$.

9. Evaluate $\int\limits_C \vec{F} \cdot d\vec{r}$ where $\vec{F}\left(x,y\right) = \left(2y\mathbf{e}^{xy} + 2x\mathbf{e}^{x^2-y^2}\right)\vec{i} + \left(2x\mathbf{e}^{xy} - 2y\mathbf{e}^{x^2-y^2}\right)\vec{j}$ and C is the curve shown below.

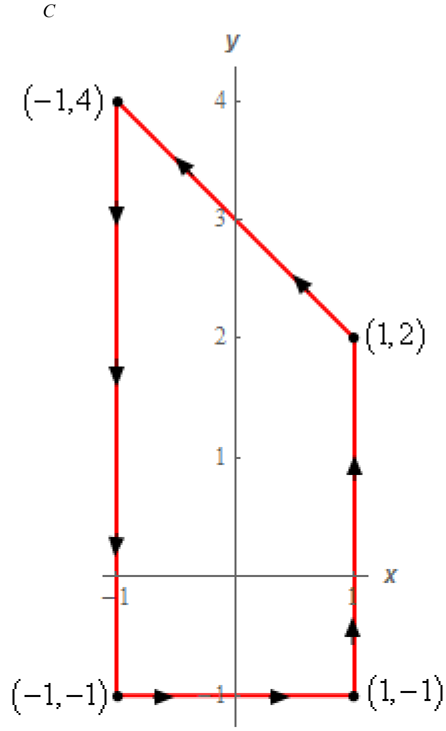


Section 5-7 : Green's Theorem

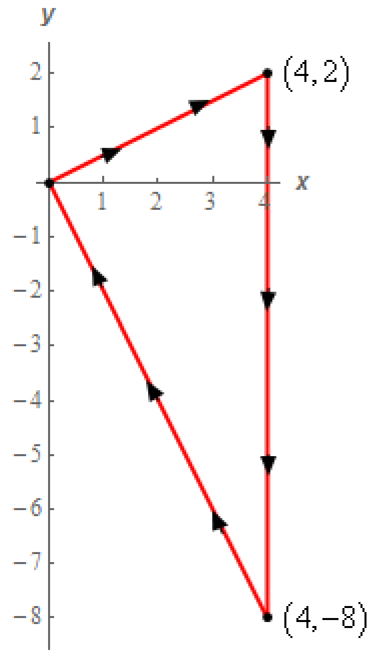
1. Use Green's Theorem to evaluate $\int_C yx^2\,dx - x^2\,dy$ where *C* is shown below.



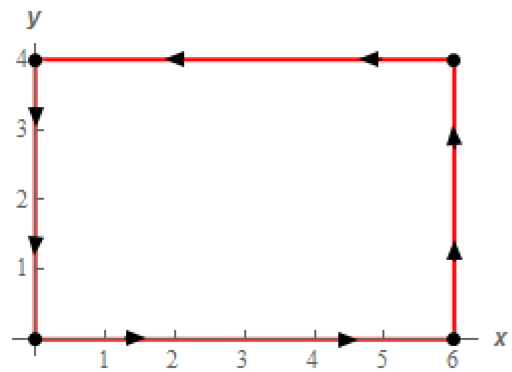
2. Use Green's Theorem to evaluate $\int_C (6y-9x)\,dy - (yx - x^3)\,dx$ where *C* is shown below.



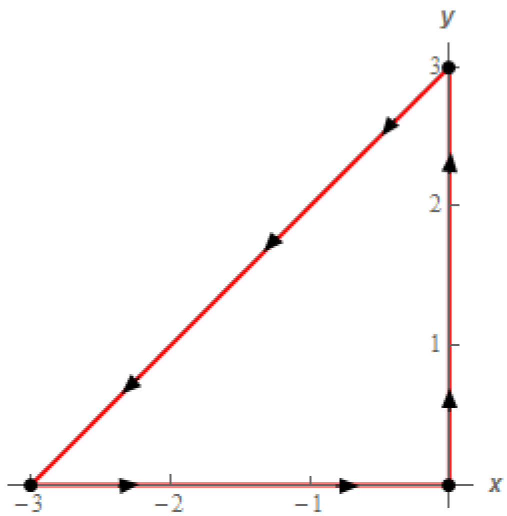
3. Use Green's Theorem to evaluate $\int_C x^2y^2\,dx + (yx^3 + y^2)\,dy$ where *C* is shown below.



4. Use Green's Theorem to evaluate $\int\limits_C \left(y^4 - 2y\right)dx - \left(6x - 4xy^3\right)dy$ where *C* is shown below.



5. Verify Green's Theorem for $\oint\limits_C \left(xy^2 + x^2\right)dx + \left(4x - 1\right)dy$ where *C* is shown below by **(a)** computing the line integral directly and **(b)** using Green's Theorem to compute the line integral.


y
3
2
1
x
−3
−2
−1

Chapter 6 : Surface Integrals

Here are a set of practice problems for the Surface Integrals chapter of the Calculus II notes. If you are viewing the pdf version of this document (as opposed to viewing it on the web) this document contains only the problems themselves and no solutions are included in this document. Solutions can be found in a couple of places on the site.

13. If you'd like a pdf document containing the solutions the download tab on the website contains links to pdf's containing the solutions for the full book, chapter and section. At this time, I do not offer pdf's for solutions to individual problems.

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Note that some sections will have more problems than others and some will have more or less of a variety of problems. Most sections should have a range of difficulty levels in the problems although this will vary from section to section.

Here is a list of all the sections for which practice problems have been written as well as a brief description of the material covered in the notes for that particular section.

Curl and Divergence – In this section we will introduce the concepts of the curl and the divergence of a vector field. We will also give two vector forms of Green's Theorem and show how the curl can be used to identify if a three dimensional vector field is conservative field or not.

Parametric Surfaces – In this section we will take a look at the basics of representing a surface with parametric equations. We will also see how the parameterization of a surface can be used to find a normal vector for the surface (which will be very useful in a couple of sections) and how the parameterization can be used to find the surface area of a surface.

Surface Integrals – In this section we introduce the idea of a surface integral. With surface integrals we will be integrating over the surface of a solid. In other words, the variables will always be on the surface of the solid and will never come from inside the solid itself. Also, in this section we will be working with the first kind of surface integrals we'll be looking at in this chapter : surface integrals of functions.

Surface Integrals of Vector Fields – In this section we will introduce the concept of an oriented surface and look at the second kind of surface integral we'll be looking at : surface integrals of vector fields.

Stokes' Theorem – In this section we will discuss Stokes' Theorem.

Divergence Theorem – In this section we will discuss the Divergence Theorem.

Section 6-1 : Curl and Divergence

For problems 1 & 2 compute $\mathrm{div}\,\vec{F}$ and $\mathrm{curl}\,\vec{F}$.

1. $\vec{F} = x^2 y\,\vec{i} - \left(z^3 - 3x\right)\vec{j} + 4y^2\vec{k}$

2. $\vec{F} = \left(3x + 2z^2\right)\vec{i} + \dfrac{x^3 y^2}{z}\vec{j} - \left(z - 7x\right)\vec{k}$

For problems 3 & 4 determine if the vector field is conservative.

3. $\vec{F} = \left(4y^2 + \dfrac{3x^2 y}{z^2}\right)\vec{i} + \left(8xy + \dfrac{x^3}{z^2}\right)\vec{j} + \left(11 - \dfrac{2x^3 y}{z^3}\right)\vec{k}$

4. $\vec{F} = 6x\,\vec{i} + \left(2y - y^2\right)\vec{j} + \left(6z - x^3\right)\vec{k}$

Section 6-2 : Parametric Surfaces

For problems 1 – 6 write down a set of parametric equations for the given surface.

1. The plane $7x+3y+4z=15$.

2. The portion of the plane $7x+3y+4z=15$ that lies in the 1st octant.

3. The cylinder $x^2+y^2=5$ for $-1\le z\le 6$.

4. The portion of $y=4-x^2-z^2$ that is in front of $y=-6$.

5. The portion of the sphere of radius 6 with $x\ge 0$.

6. The tangent plane to the surface given by the following parametric equation at the point $\left(8,14,2\right)$.

$$\vec r\left(u,v\right)=\left(u^2+2u\right)\vec i+\left(3v-2u\right)\vec j+\left(6v-10\right)\vec k$$

7. Determine the surface area of the portion of $2x+3y+6z=9$ that is inside the cylinder $x^2+y^2=7$.

8. Determine the surface area of the portion of $x^2+y^2+z^2=25$ with $z\le 0$.

9. Determine the surface area of the portion of $z=3+2y+\frac{1}{4}x^4$ that is above the region in the *xy*-plane bounded by $y=x^5$, $x=1$ and the *y*-axis.

10. Determine the surface area of the portion of the surface given by the following parametric equation that lies inside the cylinder $u^2+v^2=4$.

$$\vec r\left(u,v\right)=\left\langle 2u,vu,1-2v\right\rangle$$

Section 6-3 : Surface Integrals

1. Evaluate $\iint\limits_S z + 3y - x^2 \, dS$ where *S* is the portion of $z = 2 - 3y + x^2$ that lies over the triangle in the *xy*-plane with vertices $(0,0)$, $(2,0)$ and $(2,-4)$.

2. Evaluate $\iint\limits_S 40y \, dS$ where *S* is the portion of $y = 3x^2 + 3z^2$ that lies behind $y = 6$.

3. Evaluate $\iint\limits_S 2y \, dS$ where *S* is the portion of $y^2 + z^2 = 4$ between $x = 0$ and $x = 3 - z$.

4. Evaluate $\iint\limits_S xz \, dS$ where *S* is the portion of the sphere of radius 3 with $x \le 0$, $y \ge 0$ and $z \ge 0$.

5. Evaluate $\iint\limits_S yz + 4xy \, dS$ where *S* is the surface of the solid bounded by $4x + 2y + z = 8$, $z = 0$, $y = 0$ and $x = 0$. Note that all four surfaces of this solid are included in *S*.

6. Evaluate $\iint\limits_S x - z \, dS$ where *S* is the surface of the solid bounded by $x^2 + y^2 = 4$, $z = x - 3$, and $z = x + 2$. Note that all three surfaces of this solid are included in *S*.

Section 6-4 : Surface Integrals of Vector Fields

1. Evaluate $\iint\limits_S \vec{F} \bullet d\vec{S}$ where $\vec{F} = 3x\,\vec{i} + 2z\,\vec{j} + \left(1 - y^2\right)\vec{k}$ and *S* is the portion of $z = 2 - 3y + x^2$ that lies over the triangle in the *xy*-plane with vertices $\left(0,0\right)$, $\left(2,0\right)$ and $\left(2,-4\right)$ oriented in the negative *z*-axis direction.

2. Evaluate $\iint\limits_S \vec{F} \bullet d\vec{S}$ where $\vec{F} = -x\,\vec{i} + 2y\,\vec{j} - z\,\vec{k}$ and *S* is the portion of $y = 3x^2 + 3z^2$ that lies behind $y = 6$ oriented in the positive *y*-axis direction.

3. Evaluate $\iint\limits_S \vec{F} \bullet d\vec{S}$ where $\vec{F} = x^2\,\vec{i} + 2z\,\vec{j} - 3y\,\vec{k}$ and *S* is the portion of $y^2 + z^2 = 4$ between $x = 0$ and $x = 3 - z$ oriented outwards (*i.e.* away from the *x*-axis).

4. Evaluate $\iint\limits_S \vec{F} \bullet d\vec{S}$ where $\vec{F} = \vec{i} + z\,\vec{j} + 6x\,\vec{k}$ and *S* is the portion of the sphere of radius 3 with $x \le 0$, $y \ge 0$ and $z \ge 0$ oriented inward (*i.e.* towards the origin).

5. Evaluate $\iint\limits_S \vec{F} \bullet d\vec{S}$ where $\vec{F} = y\,\vec{i} + 2x\,\vec{j} + \left(z - 8\right)\vec{k}$ and *S* is the surface of the solid bounded by $4x + 2y + z = 8$, $z = 0$, $y = 0$ and $x = 0$ with the positive orientation. Note that all four surfaces of this solid are included in *S*.

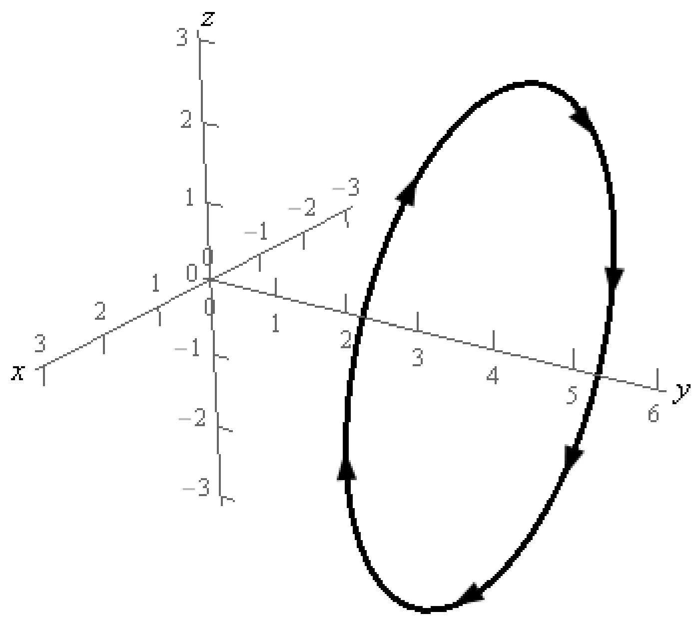
6. Evaluate $\iint\limits_S \vec{F} \bullet d\vec{S}$ where $\vec{F} = yz\,\vec{i} + x\,\vec{j} + 3y^2\,\vec{k}$ and *S* is the surface of the solid bounded by $x^2 + y^2 = 4$, $z = x - 3$, and $z = x + 2$ with the negative orientation. Note that all three surfaces of this solid are included in *S*.

Section 6-5 : Stokes' Theorem

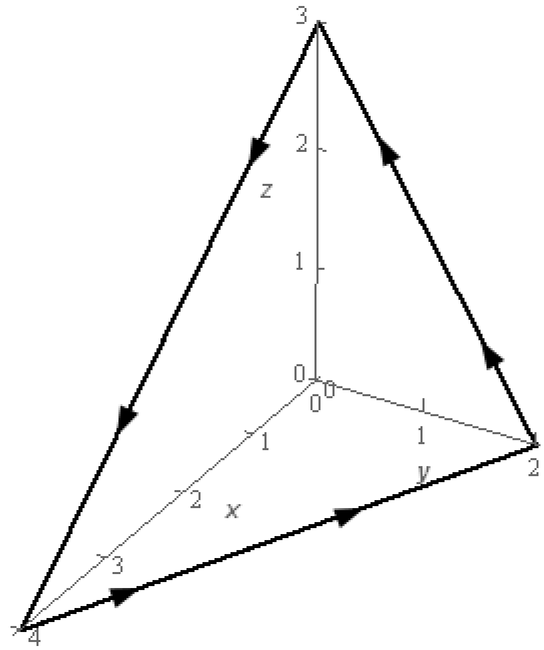
1. Use Stokes' Theorem to evaluate $\iint\limits_S \text{curl}\,\vec{F}\cdot d\vec{S}$ where $\vec{F} = y\,\vec{i} - x\,\vec{j} + yx^3\,\vec{k}$ and *S* is the portion of the sphere of radius 4 with $z \ge 0$ and the upwards orientation.

2. Use Stokes' Theorem to evaluate $\iint\limits_S \text{curl}\,\vec{F}\cdot d\vec{S}$ where $\vec{F} = \left(z^2 - 1\right)\vec{i} + \left(z + xy^3\right)\vec{j} + 6\vec{k}$ and *S* is the portion of $x = 6 - 4y^2 - 4z^2$ in front of $x = -2$ with orientation in the negative *x*-axis direction.

3. Use Stokes' Theorem to evaluate $\int\limits_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = -yz\,\vec{i} + \left(4y + 1\right)\vec{j} + xy\,\vec{k}$ and *C* is is the circle of radius 3 at $y = 4$ and perpendicular to the *y*-axis. *C* has a clockwise rotation if you are looking down the *y*-axis from the positive *y*-axis to the negative *y*-axis. See the figure below for a sketch of the curve.



4. Use Stokes' Theorem to evaluate $\int\limits_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = \left(3yx^2 + z^3\right)\vec{i} + y^2\,\vec{j} + 4yx^2\,\vec{k}$ and *C* is is triangle with vertices $\left(0,0,3\right)$, $\left(0,2,0\right)$ and $\left(4,0,0\right)$. *C* has a counter clockwise rotation if you are above the triangle and looking down towards the *xy*-plane. See the figure below for a sketch of the curve.


3
2
z
1
0
0
0
1
1
2
y
2
x
3
4

Section 6-6 : Divergence Theorem

1. Use the Divergence Theorem to evaluate $\iint\limits_S \vec{F} \bullet d\vec{S}$ where $\vec{F} = yx^2\,\vec{i} + \left(xy^2 - 3z^4\right)\vec{j} + \left(x^3 + y^2\right)\vec{k}$ and *S* is the surface of the sphere of radius 4 with $z \le 0$ and $y \le 0$. Note that all three surfaces of this solid are included in *S*.

2. Use the Divergence Theorem to evaluate $\iint\limits_S \vec{F} \bullet d\vec{S}$ where $\vec{F} = \sin\left(\pi x\right)\vec{i} + zy^3\,\vec{j} + \left(z^2 + 4x\right)\vec{k}$ and *S* is the surface of the box with $-1 \le x \le 2$, $0 \le y \le 1$ and $1 \le z \le 4$. Note that all six sides of the box are included in *S*.

3. Use the Divergence Theorem to evaluate $\iint\limits_S \vec{F} \bullet d\vec{S}$ where $\vec{F} = 2xz\vec{i} + \left(1 - 4xy^2\right)\vec{j} + \left(2z - z^2\right)\vec{k}$ and *S* is the surface of the solid bounded by $z = 6 - 2x^2 - 2y^2$ and the plane $z = 0$. Note that both of the surfaces of this solid included in *S*.

