CALCULUS III

Assignment Problems

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Preface

Here are a set of assignment problems for the Calculus III notes. Please note that these problems do not have any solutions available. These are intended mostly for instructors who might want a set of problems to assign for turning in. Having solutions available (or even just final answers) would defeat the purpose the problems.

If you are looking for some practice problems (with solutions available) please check out the Practice Problems. There you will find a set of problems that should give you quite a bit practice.

Outline

Here is a listing of sections for which assignment problems have been written as well as a brief description of the material covered in the notes for that particular section.

cylindrical coordinates are really nothing more than a very natural extension of polar coordinates into a three dimensional setting.
Spherical Coordinates – In this section we will define the spherical coordinate system, yet another alternate coordinate system for the three dimensional coordinate system.

Partial Derivatives – In this chapter we'll take a brief look at limits of functions of more than one variable and then move into derivatives of functions of more than one variable. As we'll see if we can do derivatives of functions with one variable it isn't much more difficult to do derivatives of functions of more than one variable (with a very important subtlety). We will also discuss interpretations of partial derivatives, higher order partial derivatives and the chain rule as applied to functions of more than one variable. We will also define and discuss directional derivatives.
Limits – In the section we'll take a quick look at evaluating limits of functions of several variables. We will also see a fairly quick method that can be used, on occasion, for showing that some limits do not exist.
Partial Derivatives – In this section we will the idea of partial derivatives. We will give the formal definition of the partial derivative as well as the standard notations and how to compute them in practice (*i.e.* without the use of the definition). As you will see if you can do derivatives of functions of one variable you won't have much of an issue with partial derivatives. There is only one (very important) subtlety that you need to always keep in mind while computing partial derivatives.
Interpretations of Partial Derivatives – In the section we will take a look at a couple of important interpretations of partial derivatives. First, the always important, rate of change of the function. Although we now have multiple 'directions' in which the function can change (unlike in Calculus I). We will also see that partial derivatives give the slope of tangent lines to the traces of the function.
Higher Order Partial Derivatives – In the section we will take a look at higher order partial derivatives. Unlike Calculus I however, we will have multiple second order derivatives, multiple third order derivatives, *etc*. because we are now working with functions of multiple variables. We will also discuss Clairaut's Theorem to help with some of the work in finding higher order derivatives.
Differentials – In this section we extend the idea of differentials we first saw in Calculus I to functions of several variables.
Chain Rule – In the section we extend the idea of the chain rule to functions of several variables. In particular, we will see that there are multiple variants to the chain rule here all depending on how many variables our function is dependent on and how each of those variables can, in turn, be written in terms of different variables. We will also give a nice method for writing down the chain rule for pretty much an ysituation you might run into when dealing with functions of multiple variables. In addition, we will derive a very quick way of doing implicit differentiation so we no longer need to go through the process we first did back in Calculus I.
Directional Derivatives – In the section we introduce the concept of directional derivatives, including how to compute them and see a couple of nice facts pertaining to directional derivatives.

Applications of Partial Derivatives – In this chapter we will take a look at several applications of partial derivatives. We will find the equation of tangent planes to surfaces and we will revisit on of the more important applications of derivatives from earlier Calculus classes. We will spend a significant amount of time finding relative and absolute extrema of functions of multiple variables. We will also introduce Lagrange multipliers to find the absolute extrema of a function subject to one or more constraints.

Tangent Planes and Linear Approximations – In this section formally define just what a tangent plane to a surface is and how we use partial derivatives to find the equations of tangent planes to surfaces that can be written as $z = f(x,y)$. We will also see how tangent planes can be thought of as a linear approximation to the surface at a given point.
Gradient Vector, Tangent Planes and Normal Lines – In this section discuss how the gradient vector can be used to find tangent planes to a much more general function than in the previous section. We will also define the normal line and discuss how the gradient vector can be used to find the equation of the normal line.
Relative Minimums and Maximums – In this section we will define critical points for functions of two variables and discuss a method for determining if they are relative minimums, relative maximums or saddle points (*i.e.* neither a relative minimum or relative maximum).
Absolute Minimums and Maximums – In this section we will how to find the absolute extrema of a function of two variables when the independent variables are only allowed to come from a region that is bounded (*i.e.* no part of the region goes out to infinity) and closed (*i.e.* all of the points on the boundary are valid points that can be used in the process).
Lagrange Multipliers – In this section we'll see discuss how to use the method of Lagrange Multipliers to find the absolute minimums and maximums of functions of two or three variables in which the independent variables are subject to one or more constraints. We also give a brief justification for how/why the method works.

Multiple Integrals – In this chapter will be looking at double integrals, *i.e.* integrating functions of two variables in which the independent variables are from two dimensional regions, and triple integrals, *i.e.* integrating functions of three variables in which the independent variables are from three dimensional regions. Included will be double integrals in polar coordinates and triple integrals in cylindrical and spherical coordinates and more generally change in variables in double and triple integrals.
Double Integrals – In this section we will formally define the double integral as well as giving a quick interpretation of the double integral.
Iterated Integrals – In this section we will show how Fubini's Theorem can be used to evaluate double integrals where the region of integration is a rectangle.
Double Integrals over General Regions – In this section we will start evaluating double integrals over general regions, *i.e.* regions that aren't rectangles. We will illustrate how a double integral of a function can be interpreted as the net volume of the solid between the surface given by the function and the *xy*-plane.
Double Integrals in Polar Coordinates – In this section we will look at converting integrals (including *dA*) in Cartesian coordinates into Polar coordinates. The regions of integration in these cases will be all or portions of disks or rings and so we will also need to convert the original Cartesian limits for these regions into Polar coordinates.
Triple Integrals – In this section we will define the triple integral. We will also illustrate quite a few examples of setting up the limits of integration from the three dimensional region of integration. Getting the limits of integration is often the difficult part of these problems.
Triple Integrals in Cylindrical Coordinates – In this section we will look at converting integrals (including *dV*) in Cartesian coordinates into Cylindrical coordinates. We will also be converting the original Cartesian limits for these regions into Cylindrical coordinates.
Triple Integrals in Spherical Coordinates – In this section we will look at converting integrals (including *dV*) in Cartesian coordinates into Spherical coordinates. We will also be converting the original Cartesian limits for these regions into Spherical coordinates.

Change of Variables – In previous sections we've converted Cartesian coordinates in Polar, Cylindrical and Spherical coordinates. In this section we will generalize this idea and discuss how we convert integrals in Cartesian coordinates into alternate coordinate systems. Included will be a derivation of the *dV* conversion formula when converting to Spherical coordinates.
Surface Area – In this section we will show how a double integral can be used to determine the surface area of the portion of a surface that is over a region in two dimensional space.
Area and Volume Revisited – In this section we summarize the various area and volume formulas from this chapter.

Line Integrals – In this chapter we will introduce a new kind of integral : Line Integrals. With Line Integrals we will be integrating functions of two or more variables where the independent variables now are defined by curves rather than regions as with double and triple integrals. We will also investigate conservative vector fields and discuss Green's Theorem in this chapter.
Vector Fields – In this section we will start off with a quick review of parameterizing curves. This is a skill that will be required in a great many of the line integrals we evaluate and so needs to be understood. We will then formally define the first kind of line integral we will be looking at : line integrals with respect to arc length.
Line Integrals – Part I – In this section we will start looking at line integrals. In particular we will look at line integrals with respect to arc length.
Line Integrals – Part II – In this section we will continue looking at line integrals and define the second kind of line integral we'll be looking at : line integrals with respect to x, y, and/or z. We also introduce an alternate form of notation for this kind of line integral that will be useful on occasion.
Line Integrals of Vector Fields – In this section we will define the third type of line integrals we'll be looking at : line integrals of vector fields. We will also see that this particular kind of line integral is related to special cases of the line integrals with respect to x, y and z.
Fundamental Theorem for Line Integrals – In this section we will give the fundamental theorem of calculus for line integrals of vector fields. This will illustrate that certain kinds of line integrals can be very quickly computed. We will also give quite a few definitions and facts that will be useful.
Conservative Vector Fields – In this section we will take a more detailed look at conservative vector fields than we've done in previous sections. We will also discuss how to find potential functions for conservative vector fields.
Green's Theorem – In this section we will discuss Green's Theorem as well as an interesting application of Green's Theorem that we can use to find the area of a two dimensional region.

Surface Integrals – In this chapter we look at yet another kind on integral : Surface Integrals. With Surface Integrals we will be integrating functions of two or more variables where the independent variables are now on the surface of three dimensional solids. We will also look at Stokes' Theorem and the Divergence Theorem.
Curl and Divergence – In this section we will introduce the concepts of the curl and the divergence of a vector field. We will also give two vector forms of Green's Theorem and show how the curl can be used to identify if a three dimensional vector field is conservative field or not.
Parametric Surfaces – In this section we will take a look at the basics of representing a surface with parametric equations. We will also see how the parameterization of a surface can be used to find a normal vector for the surface (which will be very useful in a couple of sections) and how the parameterization can be used to find the surface area of a surface.

Surface Integrals – In this section we introduce the idea of a surface integral. With surface integrals we will be integrating over the surface of a solid. In other words, the variables will always be on the surface of the solid and will never come from inside the solid itself. Also, in this section we will be working with the first kind of surface integrals we'll be looking at in this chapter : surface integrals of functions.
Surface Integrals of Vector Fields – In this section we will introduce the concept of an oriented surface and look at the second kind of surface integral we'll be looking at : surface integrals of vector fields.
Stokes' Theorem – In this section we will discuss Stokes' Theorem.
Divergence Theorem – In this section we will discuss the Divergence Theorem.

Chapter 1 : 3-Dimensional Space

Here are a set of assignment problems for the 3-Dimensional Space chapter of the Calculus III notes. Please note that these problems do not have any solutions available. These are intended mostly for instructors who might want a set of problems to assign for turning in. Having solutions available (or even just final answers) would defeat the purpose the problems.

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The 3-D Coordinate System – In this section we will introduce the standard three dimensional coordinate system as well as some common notation and concepts needed to work in three dimensions.

Equations of Lines – In this section we will derive the vector form and parametric form for the equation of lines in three dimensional space. We will also give the symmetric equations of lines in three dimensional space. Note as well that while these forms can also be useful for lines in two dimensional space.

Equations of Planes – In this section we will derive the vector and scalar equation of a plane. We also show how to write the equation of a plane from three points that lie in the plane.

Quadric Surfaces – In this section we will be looking at some examples of quadric surfaces. Some examples of quadric surfaces are cones, cylinders, ellipsoids, and elliptic paraboloids.

Functions of Several Variables – In this section we will give a quick review of some important topics about functions of several variables. In particular we will discuss finding the domain of a function of several variables as well as level curves, level surfaces and traces.

Vector Functions – In this section we introduce the concept of vector functions concentrating primarily on curves in three dimensional space. We will however, touch briefly on surfaces as well. We will illustrate how to find the domain of a vector function and how to graph a vector function. We will also show a simple relationship between vector functions and parametric equations that will be very useful at times.

Calculus with Vector Functions – In this section here we discuss how to do basic calculus, *i.e.* limits, derivatives and integrals, with vector functions.

Tangent, Normal and Binormal Vectors – In this section we will define the tangent, normal and binormal vectors.

Arc Length with Vector Functions – In this section we will extend the arc length formula we used early in the material to include finding the arc length of a vector function. As we will see the new formula really is just an almost natural extension of one we've already seen.

Curvature – In this section we give two formulas for computing the curvature (*i.e.* how fast the function is changing at a given point) of a vector function.

Velocity and Acceleration – In this section we will revisit a standard application of derivatives, the velocity and acceleration of an object whose position function is given by a vector function. For the acceleration we give formulas for both the normal acceleration and the tangential acceleration.

Cylindrical Coordinates – In this section we will define the cylindrical coordinate system, an alternate coordinate system for the three dimensional coordinate system. As we will see cylindrical coordinates are really nothing more than a very natural extension of polar coordinates into a three dimensional setting.

Spherical Coordinates – In this section we will define the spherical coordinate system, yet another alternate coordinate system for the three dimensional coordinate system. This coordinates system is very useful for dealing with spherical objects. We will derive formulas to convert between cylindrical coordinates and spherical coordinates as well as between Cartesian and spherical coordinates (the more useful of the two).

Section 1-1 : The 3-D Coordinate System

1. Give the projection of $P = (-9,1,5)$ onto the three coordinate planes.

2. Give the projection of $P = (3,-2,-5)$ onto the three coordinate planes.

3. Which of the points $P = (8,-9,3)$ and $Q = (-6,4,-5)$ is closest to the *xz*-plane?

4. Which of the points $P = (8,-9,3)$ and $Q = (-6,4,-5)$ is closest to the *xy*-plane?

5. Which of the points $P = (5,-4,3)$ and $Q = (-6,3,9)$ is closest to the *x*-axis?

6. Which of the points $P = (5,-4,3)$ and $Q = (-6,3,9)$ is closest to the *y*-axis?

For problems 7 – 9 list all of the coordinates systems ($\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$) that the given equation will have a graph in. Do not sketch the graph.

7. $8z + \dfrac{x+1}{y^2+2} = 4x$

8. $\sqrt{y+2} = 6$

9. $7y^3 - \dfrac{2}{x+1} = xy$

Section 1-2 : Equations of Lines

For problems 1 – 4 give the equation of the line in vector form, parametric form and symmetric form.

1. The line through the points $\left(7,-3,1\right)$ and $\left(-2,1,4\right)$.

2. The line through the point $\left(1,-5,0\right)$ and parallel to the line given by $\vec{r}\left(t\right)=\left\langle 8-3t,-10+9t,-1-t\right\rangle$.

3. The line through the point $\left(1,-7,14\right)$ and parallel to the line given by $x=6t$, $y=9$, $z=8-16t$.

4. The line through the point $\left(-7,2,4\right)$ and orthogonal to both $\vec{v}=\left\langle 0,-9,1\right\rangle$ and $\vec{w}=3\vec{i}+\vec{j}-4\vec{k}$.

For problems 5 – 7 determine if the two lines are parallel, orthogonal or neither.

5. The line given by $\vec{r}\left(t\right)=\left\langle 4-7t,-10+5t,21-4t\right\rangle$ and the line given by $\vec{r}\left(t\right)=\left\langle -2+3t,7+5t,5+t\right\rangle$.

6. The line through the points $\left(10,-4,18\right)$ and $\left(5,6,-7\right)$ and the line given by $x=5+3t$, $y=-6t$, $z=1+15t$.

7. The line given by $x=29$, $y=-3-6t$, $z=12-t$ and the line given by $\vec{r}\left(t\right)=\left\langle 12-14t,2+7t,-10+3t\right\rangle$.

For problems 8 – 10 determine the intersection point of the two lines or show that they do not intersect.

8. The line passing through the points $\left(0,-9,-1\right)$ and $\left(1,6,-3\right)$ and the line given by $\vec{r}\left(t\right)=\left\langle -9-4t,10+6t,1-2t\right\rangle$.

9. The line given by $x=1+6t$, $y=-1-3t$, $z=4+12t$ and the line given by $x=4+t$, $y=-10-8t$, $z=3-5t$.

10. The line given by $\vec{r}\left(t\right)=\left\langle 14+5t,-3t,1+7t\right\rangle$ and the line given by $\vec{r}\left(t\right)=\left\langle 3-3t,5+2t,-2+4t\right\rangle$.

11. Does the line passing through $\left(-5,4,-1\right)$ and $\left(-3,-5,0\right)$ intersect the *yz*-plane? If so, give the point.

12. Does the line given by $\vec{r}\left(t\right)=\left\langle 6+t,-8+14t,4t\right\rangle$ intersect the *xz*-plane? If so, give the point.

13. Which of the three coordinate planes does the line given by $x = 16t$, $y = -4 - 9t$, $z = 34$ intersect?

Section 1-3 : Equations of Planes

For problems 1 – 5 write down the equation of the plane.

1. The plane containing the points $\left(6,-3,1\right)$, $\left(5,-4,1\right)$ and $\left(3,-4,0\right)$.

2. The plane containing the point $\left(1,-5,8\right)$ and orthogonal to the line given by $x=-3+15t$, $y=14-t$, $z=9-3t$.

3. The plane containing the point $\left(-8,3,7\right)$ and parallel to the plane given by $4x+8y-2z=45$.

4. The plane containing the point $\left(2,0,-8\right)$ and containing the line given by $\vec{r}\left(t\right)=\left\langle 8t,-1-5t,4-t\right\rangle$.

5. The plane containing the two lines given by $\vec{r}\left(t\right)=\left\langle 7+5t,2+t,6t\right\rangle$ and $\vec{r}\left(t\right)=\left\langle 7-6t,2-2t,10t\right\rangle$.

For problems 6 – 8 determine if the two planes are parallel, orthogonal or neither.

6. The plane given by $-5x+3y+2z=-8$ and the plane given by $6x-8z=15$.

7. The plane given by $3x+9y+7z=-1$ and the plane containing the points $\left(1,-1,9\right)$, $\left(4,-1,2\right)$ and $\left(-2,3,4\right)$.

8. The plane given by $-x-8y+3z=6$ and the plane given by $2x+2y+6z=-91$.

For problems 9 – 11 determine where the line intersects the plane or show that it does not intersect the plane.

9. The line given by $\vec{r}\left(t\right)=\left\langle 9+t,-4+t,2+5t\right\rangle$ and the plane given by $4x-9y+z=6$.

10. The line given by $\vec{r}\left(t\right)=\left\langle 2-3t,1+t,-4-2t\right\rangle$ and the plane given by $x-7y-4z=-1$.

11. The line given by $x=8$, $y=-9t$, $z=1+10t$ and the plane given by $8x+9y+2z=17$.

For problems 12 & 13 find the line of intersection of the two planes.

12. Find the line of intersection of the plane given by $4x+y+10z=-2$ and the plane given by $-8x+2y+3z=-8$.

13. Find the line of intersection of the plane given by $x-10y-2z=3$ and the plane given by $2x-y+z=-13$.

14. Determine if the line given by $x = 4 + 3t$, $y = -2$, $z = 1 + 6t$ and the plane given by $8x - y + 4z = -3$ are parallel, orthogonal or neither.

Section 1-4 : Quadric Surfaces

Sketch each of the following quadric surfaces.

1. $z^2 = x^2 + \frac{y^2}{2}$

2. $z = 2 + 4x^2 + 6y^2$

3. $4x^2 + y^2 + 3z^2 = 1$

4. $\frac{x^2}{9} + \frac{y^2}{16} = 1$

5. $y = \frac{x^2}{2} + \frac{z^2}{3} - 7$

6. $6x^2 + 2z^2 = 1$

7. $x = 12 - \frac{y^2}{4} - 3z^2$

8. $x^2 = 4y^2 + 9z^2$

Section 1-5 : Functions of Several Variables

For problems 1 – 6 find the domain of the given function.

1. $f\left(x,y\right)=\sqrt{2x+4y-1}$

2. $f\left(x,y\right)=\ln\left(\dfrac{1}{x-y}\right)$

3. $f\left(x,y\right)=\sqrt{\dfrac{1}{x^2}-\dfrac{1}{y^2}}$

4. $f\left(x,y,z\right)=\dfrac{1}{x+1}+\dfrac{1}{y-1}+\dfrac{1}{x+y-z}$

5. $f\left(x,y,z\right)=\ln\left(x^2+y^2-8z\right)$

6. $f\left(x,y\right)=\sqrt{x+y}-\sqrt{x-3}$

For problems 7 – 11 identify and sketch the level curves (or contours) for the given function.

7. $x^2-4z-y=2$

8. $x-4z-y^2=2$

9. $z^2+4x^2=1-4y^2$

10. $z+4x^2=1-4y^2$

11. $2x-6y+z=-2$

For problems 12 – 14 identify and sketch the traces for the given curves.

12. $x^2-4z-y=2$

13. $z^2+4x^2=1-4y^2$

14. $2x-6y+z=-2$

Section 1-6 : Vector Functions

For problems 1 – 3 find the domain of the given vector function.

1. $\vec r\left(t\right)=\left\langle \frac{1}{t^2-1},\frac{1}{t+3},\frac{1}{t-6}\right\rangle$

2. $\vec r\left(t\right)=\left\langle \sqrt{t},\sqrt{t+1},\sqrt{t+2}\right\rangle$

3. $\vec r\left(t\right)=\left\langle \ln\left(t+7\right),\ln\left(t-3\right)\right\rangle$

For problems 4 – 8 sketch the graph of the given vector function.

4. $\vec r\left(t\right)=\left\langle -4,t+1\right\rangle$

5. $\vec r\left(t\right)=\left\langle -2\cos\left(t\right),5\sin\left(t\right)\right\rangle$

6. $\vec r\left(t\right)=\left\langle \sqrt{t+2},1-t\right\rangle$

7. $\vec r\left(t\right)=\left\langle 2t+1,t^2-1\right\rangle$

8. $\vec r\left(t\right)=\left\langle t^2+4,6-t^2\right\rangle$

For problems 9 – 12 identify the graph of the vector function without sketching the graph.

9. $\vec r\left(t\right)=\left\langle 6,2+8t,-1+10t\right\rangle$

10. $\vec r\left(t\right)=\left\langle 12t,6-8t,4+7t\right\rangle$

11. $\vec r\left(t\right)=\left\langle 2,6\cos\left(t\right),6\sin\left(t\right)\right\rangle$

12. $\vec r\left(t\right)=\left\langle -2t,6\cos\left(t\right),6\sin\left(t\right)\right\rangle$

For problems 13 – 16 write down the equation of the line segment between the two points.

13. The line segment starting at $\left(4,-7\right)$ and ending at $\left(2,0\right)$.

14. The line segment starting at $\left(-1,2\right)$ and ending at $\left(7,-2\right)$.

15. The line segment starting at $(4,1,-3)$ and ending at $(-1,2,6)$.

16. The line segment starting at $(1,-1,9)$ and ending at $(4,-7,10)$.

Section 1-7 : Calculus with Vector Functions

For problems 1 – 6 evaluate the given limit.

1. $\lim\limits_{t\to 0}\left(\cos\left(2t\right)\vec{i} - \mathbf{e}^{4-t}\vec{j} + \left(t^2+3t-9\right)\vec{k}\right)$

2. $\lim\limits_{t\to 4}\left\langle \dfrac{t-4}{t^2-3t-4}, \dfrac{t^2-4t}{16-t^2}\right\rangle$

3. $\lim\limits_{t\to 0}\left\langle \dfrac{\sin\left(t\right)}{2t}, \dfrac{1-\cos\left(t\right)}{t}, -3\right\rangle$

4. $\lim\limits_{t\to -8}\left(\dfrac{\mathbf{e}^{t^2-64}-1}{t+8}\vec{i} + \dfrac{\sin\left(t+8\right)}{t+8}\vec{j} - \vec{k}\right)$

5. $\lim\limits_{t\to -\infty}\left\langle \dfrac{5t^2-8t+1}{12+5t^2}, \dfrac{2+t^3}{1+t^2+t^4}\right\rangle$

6. $\lim\limits_{t\to \infty}\left\langle \ln\left(1-\dfrac{4}{t}\right), \mathbf{e}^{\frac{1}{t^2}}, 2\right\rangle$

For problems 7 – 11 compute the derivative of the given vector function.

7. $\vec{r}\left(t\right) = \left\langle \sqrt{3t}, \dfrac{1}{t^4}, \dfrac{1}{2t}\right\rangle$

8. $\vec{r}\left(t\right) = \cos\left(2t\right)\vec{i} - \sin\left(2t\right)\vec{j} + \ln\left(2t\right)\vec{k}$

9. $\vec{r}\left(t\right) = \left\langle \mathbf{e}^{t^2-1}, 4-\sec\left(2t\right), 7\right\rangle$

10. $\vec{r}\left(t\right) = \sin\left(t\right)\cos\left(t\right)\,\vec{i} - t^3\ln\left(t^2\right)\vec{j}$

11. $\vec{r}\left(t\right) = \left\langle \dfrac{1}{\left(t^2-4\right)^2}, \dfrac{t^3}{t^2+2}, \dfrac{t^2+2}{t^3}\right\rangle$

For problems 11 – 14 evaluate the given integral.

11. $\int \vec{r}\left(t\right)dt$, where $\vec{r}\left(t\right)=\left\langle \frac{1}{t^3}-t^3,5t,\frac{1}{6t}-\frac{8}{t^4}\right\rangle$

12. $\int \vec{r}\left(t\right)dt$, where $\vec{r}\left(t\right)=\left(t^2-5\right)\mathbf{e}^{t^3-15t}\,\vec{i}+4t\sqrt{t^2+1}\,\vec{j}-\sin^2\left(5t\right)\vec{k}$

13. $\int_0^1 \vec{r}\left(t\right)dt$ where $\vec{r}\left(t\right)=\left\langle t\cos\left(\pi t\right),8t-2,12t^3-\mathbf{e}^{2t}\right\rangle$

14. $\int_1^{-1} \vec{r}\left(t\right)dt$ where $\vec{r}\left(t\right)=\tan\left(t\right)\vec{i}-\sin^3\left(t\right)\cos^2\left(t\right)\vec{j}-8t$

Section 1-8 : Tangent, Normal and Binormal Vectors

For problems 1 – 3 find the unit tangent vector for the given vector function.

1. $\vec r\left(t\right) = t^2\,\vec i - \cos\left(8t\right)\vec j + \sin\left(8t\right)\vec k$

2. $\vec r\left(t\right) = \left\langle 8t, 2 - t^6, t^4\right\rangle$

3. $\vec r\left(t\right) = \left\langle \ln\left(6t\right), \mathbf{e}^{1-t}, 5t\right\rangle$

For problems 4 & 5 find the tangent line to the vector function at the given point.

4. $\vec r\left(t\right) = \left\langle 3 + t^2, t^4, 6\right\rangle$ at $t = -1$.

5. $\vec r\left(t\right) = \left\langle 2t, \cos^2\left(t\right), \mathbf{e}^{6t}\right\rangle$ at $t = 0$.

For problems 5 & 6 find the unit normal and the binormal vectors for the given vector function.

5. $\vec r\left(t\right) = \left\langle \mathbf{e}^{4t}\sin\left(t\right), \mathbf{e}^{4t}\cos\left(t\right), 2\right\rangle$

6. $\vec r\left(t\right) = 2t\,\vec i + \frac{1}{2}t^2\,\vec j + \ln\left(t^2\right)\vec k$

Section 1-9 : Arc Length with Vector Functions

1. $\vec{r}(t) = 4\cos(2t)\vec{i} + 3t\,\vec{j} - 4\sin(2t)\vec{k}$ from $0 \le t \le 3\pi$.

2. $\vec{r}(t) = \left\langle 9 - 2t, 4 + 2t, \sqrt{2}\,t^2 \right\rangle$ from $0 \le t \le 1$.

3. $\vec{r}(t) = 2t\,\vec{i} + \frac{1}{2}t^2\,\vec{j} + \ln\left(t^2\right)\vec{k}$ from $1 \le t \le 3$.

For problems 4 – 6 find the arc length function for the given vector function.

4. $\vec{r}(t) = \langle 8t, 6 + t, -7t \rangle$

5. $\vec{r}(t) = \left\langle 8t, 4t^{\frac{3}{2}}, 3 \right\rangle$

6. $\vec{r}(t) = \left\langle \mathbf{e}^{4t}\sin(t), \mathbf{e}^{4t}\cos(t), 2 \right\rangle$

7. Determine where on the curve given by $\vec{r}(t) = \left\langle 8t, 4t^{\frac{3}{2}}, 3 \right\rangle$ we are after traveling a distance of 4.

8. Determine where on the curve given by $\vec{r}(t) = \left\langle \mathbf{e}^{4t}\sin(t), \mathbf{e}^{4t}\cos(t), 2 \right\rangle$ we are after traveling a distance of 15.

Section 1-10 : Curvature

Find the curvature for each the following vector functions.

1. $\vec{r}\left(t\right) = \left\langle 5t, 1-2t, 4t^{\frac{3}{2}} \right\rangle$

2. $\vec{r}\left(t\right) = \left\langle 6, \mathbf{e}^{-5t}, 3t\mathbf{e}^{-5t} \right\rangle$

3. $\vec{r}\left(t\right) = \left\langle \cos\left(\omega t\right), t, \sin\left(\omega t\right) \right\rangle$

Section 1-11 : Velocity and Acceleration

1. An objects acceleration is given by $\vec{a} = \cos\left(2t\right)\vec{i} + 4t^3\,\vec{j} + 6\sin\left(3t\right)\vec{k}$. The objects initial velocity is $\vec{v}\left(0\right) = 6\vec{i} + 2\vec{j} + 7\vec{k}$ and the objects initial position is $\vec{r}\left(0\right) = \vec{i} - 9\vec{j} + 6\vec{k}$. Determine the objects velocity and position functions.

2. An objects acceleration is given by $\vec{a} = 10t\,\vec{i} - 6\,\vec{j} + t\cos\left(t\right)\vec{k}$. The objects initial velocity is $\vec{v}\left(0\right) = -\vec{i} + 11\vec{j} - \vec{k}$ and the objects initial position is $\vec{r}\left(0\right) = 4\vec{i} + \vec{j} + 10\vec{k}$. Determine the objects velocity and position functions.

3. Determine the tangential and normal components of acceleration for the object whose position is given by $\vec{r}\left(t\right) = \left\langle 5t, 1-2t, 4t^{\frac{3}{2}} \right\rangle$.

4. Determine the tangential and normal components of acceleration for the object whose position is given by $\vec{r}\left(t\right) = \left\langle 6, \mathbf{e}^{-5t}, 3t\mathbf{e}^{-5t} \right\rangle$.

Section 1-12 : Cylindrical Coordinates

For problems 1 & 2 convert the Cartesian coordinates for the point into Cylindrical coordinates.

1. $\left(-3,5,-8\right)$

2. $\left(4,1,7\right)$

3. Convert the following equation written in Cartesian coordinates into an equation in Cylindrical coordinates.

$$\frac{x-y}{x^2+y^2+1}=xyz$$

For problems 4 – 6 convert the equation written in Cylindrical coordinates into an equation in Cartesian coordinates.

4. $zr^3\cos\left(\theta\right)=4r+8$

5. $r^2-3\sin\left(\theta\right)=z^3+\sqrt{r^2+1}$

6. $\tan\left(\theta\right)+2z=1-r^2$

For problems 7 – 9 identify the surface generated by the given equation.

7. $z=-4r,\ \ z<0$

8. $2r+6\cos\left(\theta\right)+18\sin\left(\theta\right)=\dfrac{51}{r}$

9. $\theta=\dfrac{\pi}{3}$

Section 1-13 : Spherical Coordinates

For problems 1 – 3 convert the Cartesian coordinates for the point into Spherical coordinates.

1. $\left(6,2,-8\right)$

2. $\left(-1,5,2\right)$

3. $\left(-3,-2,1\right)$

4. Convert the Cylindrical coordinates for the point $\left(5,1.294,6\right)$ into Spherical coordinates.

5. Convert the following equation written in Cartesian coordinates into an equation in Spherical coordinates.

$$\frac{xz}{y} = 2 - x$$

For problems 6 – 8 convert the equation written in Spherical coordinates into an equation in Cartesian coordinates.

6. $\rho\cos\varphi\sin\varphi\sin\theta = 3$

7. $\rho - \cos\varphi = 2 + \cos^2\varphi$

8. $\tan\varphi\left(\cos\theta - \sin\theta\right) = 4$

For problems 9 & 10 identify the surface generated by the given equation.

9. $\cos^2\varphi - \sin^2\varphi = 0$

10. $\sin\varphi\cos\theta + \sin\varphi\sin\theta + \cos\varphi = \dfrac{1}{\rho}$

Chapter 2 : Partial Derivatives

Here are a set of assignment problems for the Partial Derivatives chapter of the Calculus III notes. Please note that these problems do not have any solutions available. These are intended mostly for instructors who might want a set of problems to assign for turning in. Having solutions available (or even just final answers) would defeat the purpose the problems.

If you are looking for some practice problems (with solutions available) please check out the Practice Problems. There you will find a set of problems that should give you quite a bit practice.

Here is a list of all the sections for which assignment problems have been written as well as a brief description of the material covered in the notes for that particular section.

Limits – In the section we'll take a quick look at evaluating limits of functions of several variables. We will also see a fairly quick method that can be used, on occasion, for showing that some limits do not exist.

Partial Derivatives – In this section we will the idea of partial derivatives. We will give the formal definition of the partial derivative as well as the standard notations and how to compute them in practice (*i.e.* without the use of the definition). As you will see if you can do derivatives of functions of one variable you won't have much of an issue with partial derivatives. There is only one (very important) subtlety that you need to always keep in mind while computing partial derivatives.

Interpretations of Partial Derivatives – In the section we will take a look at a couple of important interpretations of partial derivatives. First, the always important, rate of change of the function. Although we now have multiple 'directions' in which the function can change (unlike in Calculus I). We will also see that partial derivatives give the slope of tangent lines to the traces of the function.

Higher Order Partial Derivatives – In the section we will take a look at higher order partial derivatives. Unlike Calculus I however, we will have multiple second order derivatives, multiple third order derivatives, *etc*. because we are now working with functions of multiple variables. We will also discuss Clairaut's Theorem to help with some of the work in finding higher order derivatives.

Differentials – In this section we extend the idea of differentials we first saw in Calculus I to functions of several variables.

Chain Rule – In the section we extend the idea of the chain rule to functions of several variables. In particular, we will see that there are multiple variants to the chain rule here all depending on how many variables our function is dependent on and how each of those variables can, in turn, be written in terms of different variables. We will also give a nice method for writing down the chain rule for pretty much any situation you might run into when dealing with functions of multiple variables. In addition, we will derive a very quick way of doing implicit differentiation so we no longer need to go through the process we first did back in Calculus I.

Directional Derivatives – In the section we introduce the concept of directional derivatives. With directional derivatives we can now ask how a function is changing if we allow all the independent variables to change rather than holding all but one constant as we had to do with partial derivatives. In addition, we will define the gradient vector to help with some of the notation and work here. The gradient vector will be very useful in some later sections as well. We will also give a nice fact that will allow us to determine the direction in which a given function is changing the fastest.

Section 2-1 : Limits

Evaluate each of the following limits.

1. $\displaystyle \lim_{(x,y,z)\to(2,1,0)} \frac{\left(4y - z^3\right)\mathbf{e}^{3x-6}}{4z - yx^2}$

2. $\displaystyle \lim_{(x,y)\to(3,-7)} \frac{6x - y + xy}{2x^3 + y^3}$

3. $\displaystyle \lim_{(x,y)\to(-3,4)} \frac{4x^2 - xy - 3y^2}{12x^2 + 17xy + 6y^2}$

4. $\displaystyle \lim_{(x,y)\to(-1,10)} \frac{10x^2 + 11xy + y^2}{10x^2 - 39xy - 4y^2}$

5. $\displaystyle \lim_{(x,y)\to(0,0)} \frac{2x^2 + 7y^2}{4y^2 + x^2}$

6. $\displaystyle \lim_{(x,y)\to(0,0)} \frac{6\sqrt[3]{x} - 3y^{10}}{9y^{30} + 2x}$

7. $\displaystyle \lim_{(x,y)\to(0,0)} \frac{2x^4 y}{x^8 + 6y^2}$

Section 2-2 : Partial Derivatives

For problems 1 – 13 find all the 1st order partial derivatives.

1. $f(x,y,z) = x^3\sqrt{y} + 4z^3y^2 - xyz + x^2 - \sqrt[3]{z^5}$

2. $W(a,b,c,d) = a^2 + b^3 - c^2d^4 - 5a^3c - d^6b^2a$

3. $A(p,t,u) = \dfrac{1}{pt^2} - \dfrac{t^3}{u^2} + \dfrac{4u\,p}{t^4}$

4. $g(x,y,z) = \sqrt{x^2 + z^{-2}} + \sin(xy - x^2)$

5. $f(s,t) = \cos\left(s\mathbf{e}^{t^2}\right) + \cos\left(s + \mathbf{e}^{t^2}\right)$

6. $f(x,y) = \ln\left(\dfrac{y}{x}\right) + \ln\left(\dfrac{1}{x+y}\right) - \ln\left(\dfrac{x}{6}\right)$

7. $A(y,z) = \dfrac{1}{y-4z^5} + \tan\left(yz^2 - y^3\right)$

8. $g(u,v) = \dfrac{u}{v}\cos\left(\dfrac{v}{u}\right) + 4u - v^2u$

9. $w = (x-y)\mathbf{e}^{4x+z^6} - \sin(2x+7z)\sec\left(yz^3\right)$

10. $f(u,v,w) = (uw+4)\sin^{-1}\left(u^2+v^2\right) - \ln\left(\dfrac{w^2}{v^4}\right)$

11. $f(x,y,z) = \sin\left(\dfrac{z}{z^2+x}\right) - \dfrac{6x^2+y}{y^2-z^2}$

12. $g(s,t,p) = \dfrac{p^3t^2}{s^2+1} + \dfrac{(4s-1)t^2}{6-s}$

13. $f(x,y,z,w) = x^2\sin(4y) + z^3(6x-y) + y^4$

For problems 14 & 15 find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for the given function.

14. $z^4 - y^2 + x^2 = 6x^2 y^3 z^7$

15. $x^2 \sin\left(z\right) + \left(x^2 - 1\right) y^4 = z^6$

Section 2-3 : Interpretations of Partial Derivatives

1. Determine if $f\left(x,y\right)=10-x^2-y^2$ is increasing or decreasing at $\left(7,-3\right)$ if

   (a) we allow *x* to vary and hold *y* fixed.
   (b) we allow *y* to vary and hold *x* fixed.

2. Determine if $f\left(x,y\right)=x\mathbf{e}^{x-y}+100y$ is increasing or decreasing at $\left(-2,1\right)$ if

   (a) we allow *x* to vary and hold *y* fixed.
   (b) we allow *y* to vary and hold *x* fixed.

3. Determine if $f\left(x,y\right)=\dfrac{x+y}{y-x}$ is increasing or decreasing at $\left(0,7\right)$ if

   (a) we allow *x* to vary and hold *y* fixed.
   (b) we allow *y* to vary and hold *x* fixed.

4. Write down the vector equations of the tangent lines to the traces for $f\left(x,y\right)=\sin\left(x\right)\cos\left(y\right)$ at $\left(\frac{\pi}{3},-\frac{\pi}{4}\right)$.

5. Write down the vector equations of the tangent lines to the traces for $f\left(x,y\right)=\ln\left(\dfrac{x}{y^2}\right)$ at $\left(6,2\right)$.

Section 2-4 : Higher Order Partial Derivatives

For problems 1 – 3 verify Clairaut's Theorem for the given function.

1. $Q\left(s,t\right)=\ln\left(st\right)-s^4\sin\left(6t\right)+st$

2. $f\left(u,w\right)=\sin\left(uw\right)+4u^2w^{-2}$

3. $f\left(x,y\right)=\mathbf{e}^{xy}\sin\left(y\right)$

For problems 4 – 9 find all 2nd order derivatives for the given function.

4. $h\left(x,y\right)=x^4y^{-2}-4xy+\mathbf{e}^{7y}+\ln\left(2x\right)$

5. $A\left(u,v\right)=u^2\cos\left(3v\right)+\ln\left(\dfrac{u}{4v^2}\right)$

6. $g\left(v,w\right)=\ln\left(v\sin\left(w\right)\right)+\sin\left(v\ln\left(w\right)\right)$

7. $f\left(x,y\right)=\cos\left(x^2+y^2\right)-\sin\left(xy\right)$

8. $h\left(x,y,z\right)=7x^3y^2z^4+8$

9. $Q\left(u,v,w\right)=u^4\sin\left(w^2\right)-\dfrac{2v}{u^4}+\ln\left(v^2w\right)$

For problems 10 & 11 find all 3rd order derivatives for the given function.

10. $h\left(x,y\right)=x^4y^5-5\sqrt{x}+8y^2$

11. $A\left(u,v\right)=u^3\sin\left(2v\right)-\dfrac{u^3}{v^2}$

12. Given $f\left(x,y,z\right)=\mathbf{e}^{-z}\cos\left(4y\right)\ln\left(2x\right)$ find f_{xyyzxz}.

13. Given $w=\ln\left(\dfrac{xy}{z}\right)+8x^4y^3\sqrt{z}$ find $\dfrac{\partial^5 w}{\partial x\partial z^2\partial y\partial x}$.

14. Given $h\left(u,v\right)=\cos\left(u^4+u^2+1\right)-\dfrac{\sqrt{u}}{v^3}$ find $\dfrac{\partial^7 h}{\partial u^2\partial v\partial u^4}$.

15. Given $f\left(x,y\right)=\dfrac{x^6}{1+6y}-\cos\left(x^2\right)+6\mathbf{e}^x\sin\left(y\right)$ find $f_{x\,x\,y\,x\,y\,x}$.

Section 2-5 : Differentials

Compute the differential of each of the following functions.

1. $w = \dfrac{x^4 z^8}{y}$

2. $f\left(x, y\right) = \tan\left(xy^2\right)$

3. $A\left(x, y, z\right) = z^3 \mathbf{e}^{xz} \sec\left(y\right)$

Section 2-6 : Chain Rule

1. Given the following information use the Chain Rule to determine $\dfrac{dz}{dt}$.

$$z = \mathbf{e}^{x^2 - y} \qquad x = \sin(4t), \quad y = t^3 - 9$$

2. Given the following information use the Chain Rule to determine $\dfrac{dw}{dt}$.

$$w = x^4 - 4xy^2 + z^3 \qquad x = \sqrt{t}, \quad y = \mathbf{e}^{2t}, \quad z = \frac{1}{t}$$

3. Given the following information use the Chain Rule to determine $\dfrac{dw}{dt}$.

$$w = \frac{4x}{yz^3} \qquad x = 7t - 1, \quad y = 1 - 2t, \quad z = t^4$$

4. Given the following information use the Chain Rule to determine $\dfrac{dz}{dx}$.

$$z = 2x^3\mathbf{e}^{4y} \qquad y = \cos(6x)$$

5. Given the following information use the Chain Rule to determine $\dfrac{dz}{dx}$.

$$z = \tan\left(\frac{x}{y}\right) \qquad y = \mathbf{e}^{x^2}$$

6. Given the following information use the Chain Rule to determine $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$.

$$z = x\sin\left(y^2 - x\right) \qquad x = 3u - v^2, \quad y = u^6$$

7. Given the following information use the Chain Rule to determine w_u and w_v.

$$w = x^4 y^{-3} z^2 \qquad x = u^2 v, \quad y = 3 - uv, \quad z = 7u^2 - 10v$$

8. Given the following information use the Chain Rule to determine $\dfrac{\partial z}{\partial t}$ and $\dfrac{\partial z}{\partial s}$.

$$z = 6x + y^2\tan(x) \qquad x = p^2 - 3t, \quad y = s^2 - \mathrm{t}^2, \quad p = \mathbf{e}^{3s}$$

9. Given the following information use the Chain Rule to determine w_p and w_t.

$$w = x^2 y^4 z^6 - 2xy \qquad x = 2p, \quad y = 3tq, \quad z = 3tp^2, \quad q = 2t$$

10. Given the following information use the Chain Rule to determine $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$.

$$w = \frac{\sqrt{y}}{x^2 z^3} \qquad x = uv, \quad y = u^2 - p^3, \quad z = 4qp, \quad p = 2u - 3v, \quad q = v^2$$

11. Determine formulas for w_u and w_t for the following situation.

$$w = w(x, y) \qquad x = x(y,z), \quad y = y(u,v), \quad z = z(u,t), \quad v = v(t)$$

12. Determine formulas for $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$ for the following situation.

$$w = w(x, y, z) \qquad x = x(u,v,t), \quad y = y(p), \quad z = z(u,t), \quad v = v(p,t), \quad p = p(s,t)$$

13. Compute $\frac{dy}{dx}$ for the following equation.

$$\cos(2x + 3y) = x^5 - 8y^2$$

14. Compute $\frac{dy}{dx}$ for the following equation.

$$\cos(2x)\sin(3y) - xy = y^4 + 9$$

15. Compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for the following equation.

$$z^3 y^4 - x^2 \cos(2y - 4z) = 4z$$

16. Compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for the following equation.

$$\sin(x)\mathbf{e}^{4xz} + 2z^2 y = \cos(z)$$

17. Determine f_{uu} and f_{vv} for the following situation.

$$f = f(x, y) \qquad x = \mathbf{e}^u \sin(v), \quad y = \mathbf{e}^u \cos(v)$$

18. Determine f_{uu} and f_{vv} for the following situation.

$$f = f(x, y) \qquad x = u^2 - v^2, \quad y = \frac{u}{v}$$

Section 2-7 : Directional Derivatives

For problems 1 – 4 determine the gradient of the given function.

1. $f\left(x,y\right)=y^3x^5+\ln\left(xy\right)$

2. $f\left(x,y\right)=\mathbf{e}^{\frac{x}{y}}+y^4\sin\left(xy\right)$

3. $f\left(x,y,z\right)=4z-\dfrac{y^4}{2z^3}+\sqrt{x^3}\left(z-1\right)$

4. $f\left(x,y,z\right)=\cos\left(\dfrac{xy}{z}\right)+z^3y^2x$

For problems 5 – 8 determine $D_{\vec{u}}f$ for the given function in the indicated direction.

5. $f\left(x,y\right)=\ln\left(2xy\right)-\sin\left(x^2+y^2\right)$ in the direction of $\vec{v}=\left\langle 7,-3\right\rangle$

6. $f\left(x,y\right)=4x^2y^3-\sqrt{2x+5y}$ in the direction of $\vec{v}=\left\langle -1,4\right\rangle$

7. $f\left(x,y,z\right)=8xy^2-\dfrac{5z^2}{x}+y^4$ in the direction of $\vec{v}=\left\langle -4,1,2\right\rangle$

8. $f\left(x,y,z\right)=\dfrac{3x}{y^2-z^3}+5x^2-8y$ in the direction of $\vec{v}=\left\langle 0,3,-2\right\rangle$

9. Determine $D_{\vec{u}}f\left(-1,4,6\right)$ for $f\left(x,y,z\right)=\mathbf{e}^{xy^2}+4zy^3$ direction of $\vec{v}=\left\langle 2,-3,6\right\rangle$.

10. Determine $D_{\vec{u}}f\left(8,1,2\right)$ for $f\left(x,y,z\right)=\ln\left(\dfrac{x}{z}\right)+\ln\left(\dfrac{z}{y}\right)+y^2x$ direction of $\vec{v}=\left\langle 1,5,2\right\rangle$.

For problems 11 – 13 find the maximum rate of change of the function at the indicated point and the direction in which this maximum rate of change occurs.

11. $f\left(x,y\right)=\mathbf{e}^{4xy}$ at $\left(6,-2\right)$

12. $f\left(x,y,z\right)=x^2y^4-3z^2x$ at $\left(1,-6,3\right)$

13. $f\left(x,y,z\right)=\ln\left(\frac{2x+3y}{z}\right)$ at $\left(2,7,4\right)$

14. Given $\vec{u}=\left\langle -\frac{3}{5},-\frac{4}{5}\right\rangle$, $\vec{v}=\left\langle \frac{4}{\sqrt{18}},\frac{2}{\sqrt{18}}\right\rangle$, $\vec{w}=\left\langle -\frac{3}{\sqrt{11}},-\frac{2}{\sqrt{11}}\right\rangle$, $D_{\vec{u}}\left(-1,4\right)=\frac{14}{5}$ and $D_{\vec{v}}\left(-1,4\right)=-\frac{22}{\sqrt{18}}$ determine the value of $D_{\vec{w}}\left(-1,4\right)$.

15. Given $\vec{u}=\left\langle \frac{1}{\sqrt{15}},\frac{4}{\sqrt{15}}\right\rangle$, $\vec{v}=\left\langle -\frac{3}{\sqrt{34}},-\frac{5}{\sqrt{34}}\right\rangle$, $\vec{w}=\left\langle -\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right\rangle$, $D_{\vec{u}}\left(0,1\right)=\frac{18}{\sqrt{15}}$ and $D_{\vec{v}}\left(0,1\right)=-\frac{40}{\sqrt{34}}$ determine the value of $D_{\vec{w}}\left(0,1\right)$.

Chapter 3 : Applications of Partial Derivatives

Here are a set of assignment problems for the Applications of Partial Derivatives chapter of the Calculus III notes. Please note that these problems do not have any solutions available. These are intended mostly for instructors who might want a set of problems to assign for turning in. Having solutions available (or even just final answers) would defeat the purpose the problems.

If you are looking for some practice problems (with solutions available) please check out the Practice Problems. There you will find a set of problems that should give you quite a bit practice.

Here is a list of all the sections for which assignment problems have been written as well as a brief description of the material covered in the notes for that particular section.

Tangent Planes and Linear Approximations – In this section formally define just what a tangent plane to a surface is and how we use partial derivatives to find the equations of tangent planes to surfaces that can be written as $z = f\left(x, y\right)$. We will also see how tangent planes can be thought of as a linear approximation to the surface at a given point.

Gradient Vector, Tangent Planes and Normal Lines – In this section discuss how the gradient vector can be used to find tangent planes to a much more general function than in the previous section. We will also define the normal line and discuss how the gradient vector can be used to find the equation of the normal line.

Relative Minimums and Maximums – In this section we will define critical points for functions of two variables and discuss a method for determining if they are relative minimums, relative maximums or saddle points (*i.e.* neither a relative minimum or relative maximum).

Absolute Minimums and Maximums – In this section we will how to find the absolute extrema of a function of two variables when the independent variables are only allowed to come from a region that is bounded (*i.e.* no part of the region goes out to infinity) and closed (*i.e.* all of the points on the boundary are valid points that can be used in the process).

Lagrange Multipliers – In this section we'll see discuss how to use the method of Lagrange Multipliers to find the absolute minimums and maximums of functions of two or three variables in which the independent variables are subject to one or more constraints. We also give a brief justification for how/why the method works.

Section 3-1 : Tangent Planes and Linear Approximations

1. Find the equation of the tangent plane to $z = x^2 y^4 - \frac{12x}{y}$ at $(-1,6)$.

2. Find the equation of the tangent plane to $z = \ln\left(x^2 y\right) - x\sqrt{y}$ at $\left(-\frac{1}{2},4\right)$.

3. Find the equation of the tangent plane to $z = \mathbf{e}^{xy} + y^2\mathbf{e}^{1-y}$ at $(0,1)$.

4. Find the linear approximation to $z = \cos\left(\sin\left(y\right) - x\right)$ at $(-2,0)$.

5. Find the linear approximation to $z = \frac{10x^2}{x-y}$ at $(4,-1)$.

Section 3-2 : Gradient Vector, Tangent Planes and Normal Lines

1. Find the tangent plane and normal line to $z\,y + 4x\sqrt{z} - x^3 y^2 = 221$ at $\left(-2,5,9\right)$.

2. Find the tangent plane and normal line to $\mathbf{e}^{x\,y^2} + zy^4 = 61 + \dfrac{z^2}{x+1}$ at $\left(0,-2,6\right)$.

3. Find the tangent plane and normal line to $9yz - \sqrt{x^2 - 8z} = xy^2 - 26$ at $\left(3,1,-2\right)$.

4. Find the point(s) on $6x^2 + y^2 - 3z^2 = 4$ where the tangent plane to the surface is parallel to the plane given by $2x + 7y - z = 6$.

5. Find the point(s) on $x^2 - 8y^2 - 2z^2 = -3$ where the tangent plane to the surface is parallel to the plane given by $-4x - y + 8z = 1$.

Section 3-3 : Relative Minimums and Maximums

Find and classify all the critical points of the following functions.

1. $f\left(x,y\right)=2y-9x-xy+5x^2+y^2$

2. $f\left(x,y\right)=x^3-y^3+8xy$

3. $f\left(x,y\right)=\left(y-x\right)\left(1-2x-3y\right)$

4. $f\left(x,y\right)=\frac{1}{2}x^4-4xy^2-2x^2+8y^2$

5. $f\left(x,y\right)=xy\,\mathbf{e}^{-8\left(x^2+y^2\right)}$

6. $f\left(x,y\right)=8x-x\sqrt{y-1}+x^3+\frac{1}{2}y-12x^2$

Section 3-4 : Absolute Minimums and Maximums

1. Find the absolute minimum and absolute maximum of $f\left(x,y\right)=18x^2+4y^2-y^2x-2$ on the triangle with vertices $\left(-1,-1\right)$, $\left(5,-1\right)$ and $\left(5,17\right)$.

2. Find the absolute minimum and absolute maximum of $f\left(x,y\right)=2x^3-4y^3+24xy$ on the rectangle given by $0\le x\le 5$, $-3\le y\le -1$.

3. Find the absolute minimum and absolute maximum of $f\left(x,y\right)=x^2-y^2+xy-5x$ on the region bounded by $y=5-x^2$ and the *x*-axis.

Section 3-5 : Lagrange Multipliers

1. Find the maximum and minimum values of $f\left(x,y\right)=10y^2-4x^2$ subject to the constraint $x^4+y^4=1$.

2. Find the maximum and minimum values of $f\left(x,y\right)=3x-6y$ subject to the constraint $4x^2+2y^2=25$.

3. Find the maximum and minimum values of $f\left(x,y\right)=xy$ subject to the constraint $x^2-y=12$. Assume that $y\le 0$ for this problem. Why is this assumption needed?

4. Find the maximum and minimum values of $f\left(x,y,z\right)=x^2+3y^2$ subject to the constraint $x^2+4y^2+z^2=36$.

5. Find the maximum and minimum values of $f\left(x,y,z\right)=xyz$ subject to the constraint $x^2+2y^2+4z^2=24$.

6. Find the maximum and minimum values of $f\left(x,y,z\right)=2x+4y+z^2$ subject to the constraints $y^2+z^2=1$ and $x^2+z^2=1$.

7. Find the maximum and minimum values of $f\left(x,y,z\right)=x+y+z^2$ subject to the constraints $x+y+z=1$ and $x^2+z^2=1$.

Chapter 4 : Multiple Integrals

Here are a set of assignment problems for the Multiple Integrals chapter of the Calculus III notes. Please note that these problems do not have any solutions available. These are intended mostly for instructors who might want a set of problems to assign for turning in. Having solutions available (or even just final answers) would defeat the purpose the problems.

If you are looking for some practice problems (with solutions available) please check out the Practice Problems. There you will find a set of problems that should give you quite a bit practice.

Here is a list of all the sections for which assignment problems have been written as well as a brief description of the material covered in the notes for that particular section.

Double Integrals – In this section we will define the double integral.

Iterated Integrals – In this section we will show how Fubini's Theorem can be used to evaluate double integrals where the region of integration is a rectangle.

Double Integrals over General Regions – In this section we will start evaluating double integrals over general regions, *i.e.* regions that aren't rectangles. We will illustrate how a double integral of a function can be interpreted as the net volume of the solid between the surface given by the function and the *xy*-plane.

Double Integrals in Polar Coordinates – In this section we will look at converting integrals (including *dA*) in Cartesian coordinates into Polar coordinates. The regions of integration in these cases will be all or portions of disks or rings and so we will also need to convert the original Cartesian limits for these regions into Polar coordinates.

Triple Integrals – In this section we will define the triple integral. We will also illustrate quite a few examples of setting up the limits of integration from the three dimensional region of integration. Getting the limits of integration is often the difficult part of these problems.

Triple Integrals in Cylindrical Coordinates – In this section we will look at converting integrals (including *dV*) in Cartesian coordinates into Cylindrical coordinates. We will also be converting the original Cartesian limits for these regions into Cylindrical coordinates.

Triple Integrals in Spherical Coordinates – In this section we will look at converting integrals (including *dV*) in Cartesian coordinates into Spherical coordinates. We will also be converting the original Cartesian limits for these regions into Spherical coordinates.

Change of Variables – In previous sections we've converted Cartesian coordinates in Polar, Cylindrical and Spherical coordinates. In this section we will generalize this idea and discuss how we convert integrals in Cartesian coordinates into alternate coordinate systems. Included will be a derivation of the *dV* conversion formula when converting to Spherical coordinates.

Surface Area – In this section we will show how a double integral can be used to determine the surface area of the portion of a surface that is over a region in two dimensional space.

Area and Volume Revisited – In this section we summarize the various area and volume formulas from this chapter.

Section 4-1 : Double Integrals

1. Use the Midpoint Rule to estimate the volume under $f\left(x,y\right)=4x+8y$ and above the rectangle given by $0\le x\le 4$, $2\le y\le 6$ in the *xy*-plane. Use 4 subdivisions in the *x* direction and 4 subdivisions in the *y* direction.

2. Use the Midpoint Rule to estimate the volume under $f\left(x,y\right)=4x-y^2$ and above the rectangle given by $0\le x\le 2$, $-2\le y\le 1$ in the *xy*-plane. Use 2 subdivisions in the *x* direction and 3 subdivisions in the *y* direction.

Section 4-2 : Iterated Integrals

1. Compute the following double integral over the indicated rectangle **(a)** by integrating with respect to *x* first and **(b)** by integrating with respect to *y* first.

$$\iint\limits_R 16xy - 9x^2 + 1\,dA \qquad R = [2,3]\times[-1,1]$$

2. Compute the following double integral over the indicated rectangle **(a)** by integrating with respect to *x* first and **(b)** by integrating with respect to *y* first.

$$\iint\limits_R \cos(x)\sin(y)\,dA \qquad R = \left[\tfrac{\pi}{6},\tfrac{\pi}{4}\right]\times\left[\tfrac{\pi}{4},\tfrac{\pi}{3}\right]$$

For problems 3 – 16 compute the given double integral over the indicated rectangle.

3. $\iint\limits_R 8x^3 - 4\,dA \qquad R = [-3,-1]\times[0,4]$

4. $\iint\limits_R 15y^4 + \dfrac{2}{x^2}\,dA \qquad R = [1,2]\times[1,4]$

5. $\iint\limits_R 4y\sec^2(x) + \dfrac{2x}{y}\,dA \qquad R = \left[0,\tfrac{\pi}{4}\right]\times[1,5]$

6. $\iint\limits_R y^5 - x^2\mathbf{e}^y\,dA \qquad R = [-1,2]\times[-3,3]$

7. $\iint\limits_R \dfrac{x^3}{1+x^4} - \dfrac{1}{\mathbf{e}^{3y}}\,dA \qquad R = [-1,0]\times[0,4]$

8. $\iint\limits_R x\mathbf{e}^{x^2} - 12x^3\sin(\pi y)\,dA \qquad R = [-2,0]\times\left[\tfrac{1}{2},1\right]$

9. $\iint\limits_R x\cos\left(4y+3x^2\right)dA \qquad R = \left[0,\sqrt{\pi}\right]\times\left[\tfrac{\pi}{2},\pi\right]$

10. $\iint\limits_R \dfrac{\ln(4xy)}{xy}\,dA \qquad R = [1,2]\times[3,4]$

11. $\displaystyle\iint\limits_R x^2 y^3 \mathbf{e}^{x^3 - y^4}\, dA$ $\quad R = [0,1]\times[-1,0]$

12. $\displaystyle\iint\limits_R 42 y x^3 \left(1 + x^2 y^2\right)^6 dA$ $\quad R = [0,1]\times[0,2]$

13. $\displaystyle\iint\limits_R \frac{\cos\left(\frac{x}{y}\right)}{y^3} dA$ $\quad R = \left[\frac{\pi}{2}, \pi\right]\times[1,2]$

14. $\displaystyle\iint\limits_R \frac{x\cos\left(\frac{x}{y}\right)}{y^2} dA$ $\quad R = \left[\frac{\pi}{2}, \pi\right]\times[1,2]$

15. $\displaystyle\iint\limits_R 2y\ln(x) - 20x^3y^3\, dA$ $\quad R = [1,2]\times[0,4]$

16. $\displaystyle\iint\limits_R xy\mathbf{e}^x \cos(y)\, dA$ $\quad R = [0,1]\times[0,\pi]$

17. Determine the volume that lies under $f(x,y) = 20 - 3x^3 - 3y^2$ and above the rectangle given by $[-2,2]\times[-1,1]$ in the *xy*-plane.

18. Determine the volume that lies under $f(x,y) = 10 + xy\sin\left(x^2 - y^2\right)$ and above the rectangle given by $[-3,0]\times[1,3]$ in the *xy*-plane.

Section 4-3 : Double Integrals over General Regions

1. Evaluate $\iint\limits_D 8yx^3\,dA$ where $D = \left\{(x,y)\,|\,-1 \le y \le 2, -1 \le x \le 1+y^2\right\}$

2. Evaluate $\iint\limits_D 12x^2y - y^2\,dA$ where $D = \left\{(x,y)\,|\,-2 \le x \le 2, -x^2 \le y \le x^2\right\}$

3. Evaluate $\iint\limits_D 9 - \frac{6y^2}{x^2}\,dA$ where *D* is the region in the 1st quadrant bounded by $y = x^3$ and $y = 4x$.

4. Evaluate $\iint\limits_D 15x^2 - 6y\,dA$ where *D* is the region bounded by $x = \frac{1}{2}y^2$ and $x = 4\sqrt{y}$.

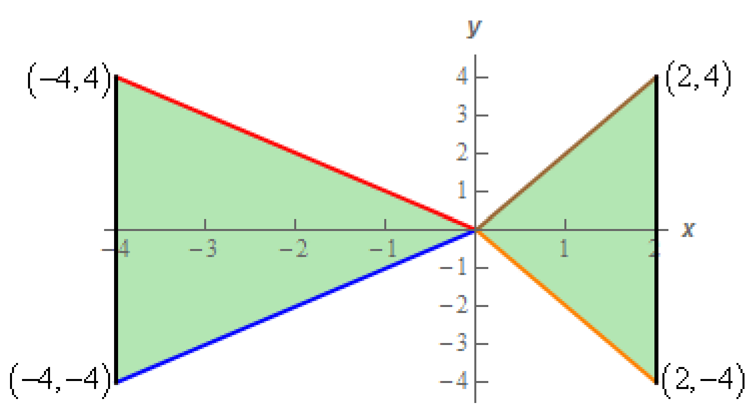
5. Evaluate $\iint\limits_D 6y(x+6)^2\,dA$ where *D* is the region bounded by $x = -y^2$ and $x = y-6$.

6. Evaluate $\iint\limits_D \mathbf{e}^{y^2+1}\,dA$ where *D* is the triangle with vertices $(0,0)$, $(-2,4)$ and $(8,4)$.

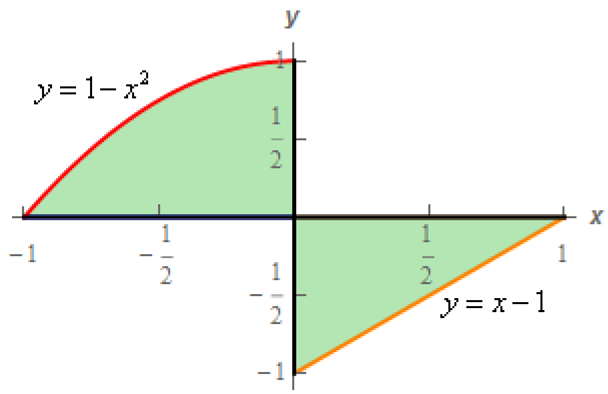
7. Evaluate $\iint\limits_D 7y^3\mathbf{e}^{x^2+1}\,dA$ where *D* is the region bounded by $y = 2\sqrt[4]{x}$, $x = 9$ and the *x*-axis.

8. Evaluate $\iint\limits_D x^5\sin\left(y^4\right)dA$ where *D* is the region in the 2nd quadrant bounded by $y = 3x^2$, $y = 12$ and the *y*-axis.

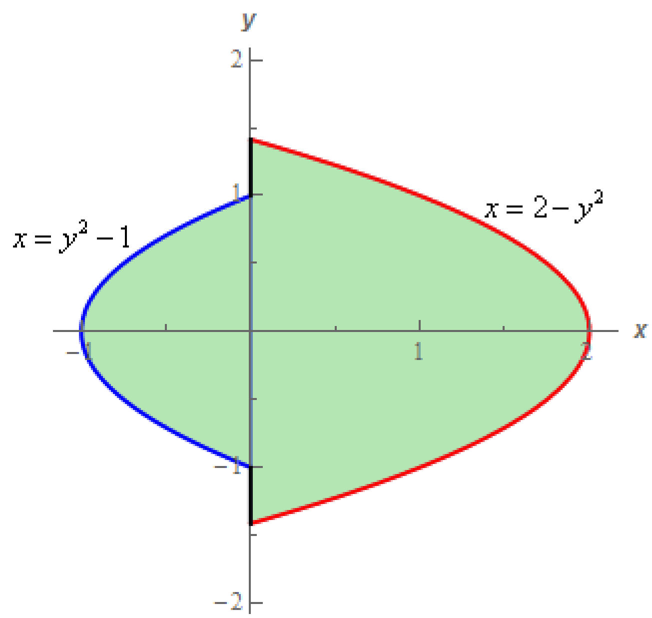
9. Evaluate $\iint\limits_D xy - y^2\,dA$ where *D* is the region shown below.



10. Evaluate $\iint\limits_D 12x^3 - 3\,dA$ where *D* is the region shown below.



11. Evaluate $\iint\limits_D 6y^2 + 10yx^4 \, dA$ where *D* is the region shown below.



12. Evaluate $\iint\limits_D \frac{x^3}{y^2} \, dA$ where *D* is the region bounded by $y = \frac{1}{x^2}$, $x = 1$ and $y = \frac{1}{4}$ in the order given below.

(a) Integrate with respect to ***x*** first and then ***y***.
(b) Integrate with respect to ***y*** first and then ***x***.

13. Evaluate $\iint\limits_D xy - y^3 \, dA$ where *D* is the region bounded by $y = x^2$, $y = -x^2$ and $x = 2$ in the order given below.

(a) Integrate with respect to ***x*** first and then ***y***.
(b) Integrate with respect to ***y*** first and then ***x***.

For problems 14 – 16 evaluate the given integral by first reversing the order of integration.

14. $\displaystyle\int_0^8 \int_{y^{\frac{1}{3}}}^2 \frac{y}{x^7+1}\,dx\,dy$

15. $\displaystyle\int_{-4}^0 \int_{\sqrt{-x}}^2 x^{-\frac{2}{3}}\sqrt{y^{\frac{5}{3}}+1}\,dy\,dx$

16. $\displaystyle\int_0^2 \int_{-x}^{3x} 5y^2x^3+2\,dy\,dx$

17. Use a double integral to determine the area of the region bounded by $x=-y^2$ and $x=y-6$.

18. Use a double integral to determine the area of the region bounded by $y=x^2+1$ and $y=\frac{1}{2}x^2+3$.

19. Use a double integral to determine the volume of the region that is between the *xy*-plane and $f\left(x,y\right)=2-xy^2$ and is above the region in the *xy*-plane that is bounded by $y=x^2$ and $x=1$.

20. Use a double integral to determine the volume of the region that is between the *xy*-plane and $f\left(x,y\right)=1+y^5\sqrt{x^4+1}$ and is above the region in the *xy*-plane that is bounded by $y=\sqrt{x}$, $x=2$ and the *x*-axis.

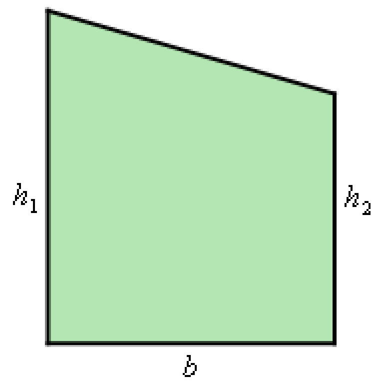
21. Use a double integral to determine the volume of the region in the first octant that is below the plane given by $2x+6y+4z=8$.

22. Use a double integral to determine the volume of the region bounded by $z=3-2y$, the surface $y=1-x^2$ and the planes $y=0$ and $z=0$.

23. Use a double integral to determine the volume of the region bounded by the planes $z=4-2x-2y$, $y=2x$, $y=0$ and $z=0$.

24. Use a double integral to determine the formula for the area of a right triangle with base, *b* and height *h*.

25. Use a double integral to determine a formula for the figure below.


h_1
h_2
b

Section 4-4 : Double Integrals in Polar Coordinates

1. Evaluate $\iint\limits_D 3xy^2 - 2\,dA$ where *D* is the unit circle centered at the origin.

2. Evaluate $\iint\limits_D 4x - 2y\,dA$ where *D* is the top half of region between $x^2 + y^2 = 4$ and $x^2 + y^2 = 25$.

3. Evaluate $\iint\limits_D 6xy + 4x^2\,dA$ where *D* is the portion of $x^2 + y^2 = 9$ in the 2nd quadrant.

4. Evaluate $\iint\limits_D \sin\left(3x^2 + 3y^2\right)dA$ where *D* is the region between $x^2 + y^2 = 1$ and $x^2 + y^2 = 7$.

5. Evaluate $\iint\limits_D \mathbf{e}^{1-x^2-y^2}\,dA$ where *D* is the region in the 4th quadrant between $x^2 + y^2 = 16$ and $x^2 + y^2 = 36$.

6. Use a double integral to determine the area of the region that is inside $r = 6 - 4\cos\theta$.

7. Use a double integral to determine the area of the region that is inside $r = 4$ and outside $r = 8 + 6\sin\theta$.

8. Evaluate the following integral by first converting to an integral in polar coordinates.

$$\int_{-2}^{0}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} x^2\,dx\,dy$$

9. Evaluate the following integral by first converting to an integral in polar coordinates.

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} \sqrt{x^2 + y^2}\,dy\,dx$$

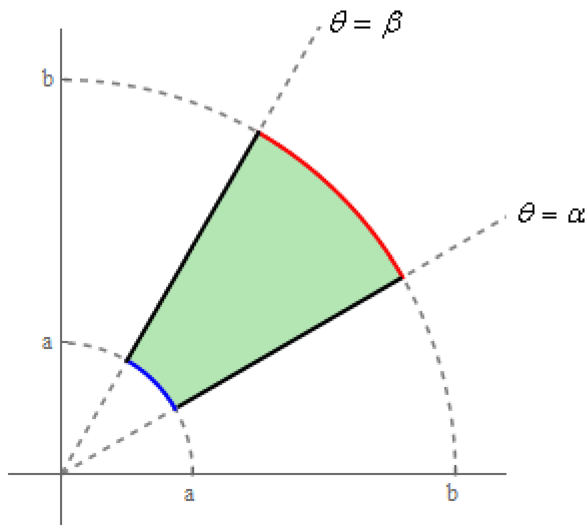
10. Use a double integral to determine the volume of the solid that is below $z = 9 - 4x^2 - 4y^2$ and above the *xy*-plane.

11. Use a double integral to determine the volume of the solid that is bounded by $z = 12 - 3x^2 - 3y^2$ and $z = x^2 + y^2 - 8$.

12. Use a double integral to determine the volume of the solid that is inside both the cylinder $x^2 + y^2 = 9$ and the sphere $x^2 + y^2 + z^2 = 16$.

13. Use a double integral to derive the area of a circle of radius *a*.

14. Use a double integral to derive the area of the region between circles of radius *a* and *b* with $\alpha \le \theta \le \beta$. See the image below for a sketch of the region.



Section 4-5 : Triple Integrals

1. Evaluate $\int_{1}^{2}\int_{0}^{2}\int_{-1}^{1} 2+z^2-xy\,dz\,dx\,dy$

2. Evaluate $\int_{2}^{0}\int_{x^2}^{1}\int_{0}^{xz} y^2-6z\,dy\,dz\,dx$

3. Evaluate $\int_{-1}^{2}\int_{0}^{1}\int_{0}^{2z} 3x-\sqrt{1+z^2}\,dx\,dz\,dy$

4. Evaluate $\iiint\limits_E 12y\,dV$ where *E* is the region below $6x+4y+3z=12$ in the first octant.

5. Evaluate $\iiint\limits_E 5x^2\,dV$ where *E* is the region below $x+2y+4z=8$ in the first octant.

6. Evaluate $\iiint\limits_E 10z^2-x\,dV$ where *E* is the region below $z=8-y$ and above the region in the *xy*-plane bounded by $y=2x$, $x=3$ and $y=0$.

7. Evaluate $\iiint\limits_E 4y^2\,dV$ where *E* is the region below $z=-3x^2-3y^2$ and above $z=-12$.

8. Evaluate $\iiint\limits_E 2y-9z\,dV$ where *E* is the region behind $6x+3y+3z=15$ front of the triangle in the *xz*-plane with vertices, in (x,z) form : $(0,0)$, $(0,4)$ and $(2,4)$.

9. Evaluate $\iiint\limits_E 18x\,dV$ where *E* is the region behind the surface $y=4-x^2$ that is in front of the region in the *xz*-plane bounded by $z=-3x$, $z=2x$ and $z=2$.

10. Evaluate $\iiint\limits_E 20x^3\,dV$ where *E* is the region bounded by $x=2-y^2-z^2$ and $x=5y^2+5z^2-6$.

11. Evaluate $\iiint\limits_E 6z^2\,dV$ where *E* is the region behind $x+6y+2z=8$ that is in front of the region in the *yz*-plane bounded by $z=2y$ and $z=\sqrt{4y}$.

12. Evaluate $\iiint\limits_E 8y\,dV$ where *E* is the region between $x+y+z=6$ and $x+y+z=10$ above the triangle in the *xy*-plane with vertices, in (x,y) form : $(0,0)$, $(1,2)$ and $(1,4)$.

13. Evaluate $\iiint\limits_E 8y\,dV$ where *E* is the region between $x+y+z=6$ and $x+y+z=10$ in front of the triangle in the *xz*-plane with vertices, in $\left(x,z\right)$ form : $\left(0,0\right)$, $\left(1,2\right)$ and $\left(1,4\right)$.

14. Evaluate $\iiint\limits_E 8y\,dV$ where *E* is the region between $x+y+z=6$ and $x+y+z=10$ in front of the triangle in the *yz*-plane with vertices, in $\left(y,z\right)$ form : $\left(0,0\right)$, $\left(1,2\right)$ and $\left(1,4\right)$.

15. Use a triple integral to determine the volume of the region below $z=8-y$ and above the region in the *xy*-plane bounded by $y=2x$, $x=3$ and $y=0$.

16. Use a triple integral to determine the volume of the region in the 1st octant that is below $4x+8y+z=16$.

17. Use a triple integral to determine the volume of the region behind $6x+3y+3z=15$ front of the triangle in the *xz*-plane with vertices, in $\left(x,z\right)$ form :$\left(0,0\right)$, $\left(0,4\right)$and $\left(2,4\right)$.

18. Use a triple integral to determine the volume of the region bounded by $y=x^2+z^2$ and $y=\sqrt{x^2+z^2}$.

19. Use a triple integral to determine the volume of the region behind $x+6y+2z=8$ that is in front of the region in the *yz*-plane bounded by $z=2y$ and $z=\sqrt{4y}$.

Section 4-6 : Triple Integrals in Cylindrical Coordinates

1. Evaluate $\iiint\limits_E 8z\,dV$ where *E* is the region bounded by $z = 2x^2 + 2y^2 - 4$ and $z = 5 - x^2 - y^2$ in the 1st octant.

2. Evaluate $\iiint\limits_E 6xy\,dV$ where *E* is the region above $z = 2x - 10$, below $z = 2$ and inside the cylinder $x^2 + z^2 = 4$.

3. Evaluate $\iiint\limits_E 9yz^3\,dV$ where *E* is the region between $x = -\sqrt{9y^2 + 9z^2}$ and $x = \sqrt{y^2 + z^2}$ inside the cylinder $y^2 + z^2 = 1$.

4. Evaluate $\iiint\limits_E x + 2\,dV$ where *E* is the region bounded by $x = 18 - 4y^2 - 4z^2$ and $x = 2$ with $z \ge 0$.

5. Evaluate $\iiint\limits_E x + 2\,dV$ where *E* is the region between the two planes $2x + y + z = 6$ and $6x + 3y + 3z = 12$ inside the cylinder $x^2 + z^2 = 16$.

6. Evaluate $\iiint\limits_E x^2\,dV$ where *E* is the region bounded by $y = x^2 + z^2 - 4$ and $y = 8 - 5x^2 - 5z^2$ with $x \le 0$.

7. Use a triple integral to determine the volume of the region bounded by $z = \sqrt{x^2 + y^2}$, and $z = x^2 + y^2$ in the 1st octant.

8. Use a triple integral to determine the volume of the region bounded by $y = \sqrt{9x^2 + 9z^2}$, and $y = -3x^2 - 3z^2$ in the 1st octant.

9. Use a triple integral to determine the volume of the region behind $x = z + 3$, in front of $x = -z - 6$ and inside the cylinder $y^2 + z^2 = 4$.

10. Evaluate the following integral by first converting to an integral in cylindrical coordinates.

$$\int_{-4}^{4}\int_{0}^{\sqrt{16-y^2}}\int_{0}^{6+x} 6yx^2\,dz\,dx\,dy$$

11. Evaluate the following integral by first converting to an integral in cylindrical coordinates.

$$\int_{0}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{-\sqrt{2x^2+2y^2}}^{6+x^2+y^2} 15z\,dz\,dy\,dx$$

12. Use a triple integral in cylindrical coordinates to derive the volume of a cylinder of height h and radius a.

Section 4-7 : Triple Integrals in Spherical Coordinates

1. Evaluate $\iiint\limits_E 4y^2\,dV$ where *E* is the sphere $x^2+y^2+z^2=9$.

2. Evaluate $\iiint\limits_E 3x-2y\,dV$ where *E* is the region between the spheres $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=4$ with $z\le 0$.

3. Evaluate $\iiint\limits_E 2yz\,dV$ where *E* is the region below $x^2+y^2+z^2=16$ and inside $z=\sqrt{3x^2+3y^2}$ that is in the 1st octant.

4. Evaluate $\iiint\limits_E z^2\,dV$ where *E* is the region between the spheres $x^2+y^2+z^2=4$ and $x^2+y^2+z^2=25$ and inside $z=-\sqrt{\frac{1}{3}x^2+\frac{1}{3}y^2}$.

5. Evaluate $\iiint\limits_E 5y^2\,dV$ where *E* is the portion of $x^2+y^2+z^2=4$ with $x\le 0$.

6. Evaluate $\iiint\limits_E 2+16x\,dV$ where *E* is the region between the spheres $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=4$ with $y\ge 0$ and $z\le 0$.

7. Evaluate the following integral by first converting to an integral in cylindrical coordinates.

$$\int_0^2\int_{-\sqrt{4-x^2}}^0\int_{\sqrt{5x^2+5y^2}}^{\sqrt{9-x^2-y^2}} 7x\;\;dz\,dy\,dx$$

8. Evaluate the following integral by first converting to an integral in cylindrical coordinates.

$$\int_{-\sqrt{5}}^{\sqrt{5}}\int_0^{\sqrt{5-y^2}}\int_{-\sqrt{10-x^2-y^2}}^{-\sqrt{x^2+y^2}} 3xz^2\;\;dz\,dx\,dy$$

9. Use a triple integral in spherical coordinates to derive the volume of a sphere with radius *a*.

Section 4-8 : Change of Variables

For problems 1 – 4 compute the Jacobian of each transformation.

1. $x = 4u^2 v \qquad y = 6v - 7u$

2. $x = \sqrt{u} \qquad y = 10u + v$

3. $x = v^3 u \qquad y = \dfrac{u^2}{v}$

4. $x = \mathbf{e}^u \cos v \qquad y = \mathbf{e}^u \sin v$

5. If *R* is the region inside $\dfrac{x^2}{25} + 49y^2 = 1$ determine the region we would get applying the transformation $x = 5u$, $y = \dfrac{1}{7}v$ to *R*.

6. If *R* is the triangle with vertices $(2,0)$, $(6,4)$ and $(1,4)$ determine the region we would get applying the transformation $x = \dfrac{1}{5}(u - v)$, $y = \dfrac{1}{5}(u + 4v)$ to *R*.

7. If *R* is the parallelogram with vertices $(0,0)$, $(4,2)$, $(0,4)$ and $(-4,2)$ determine the region we would get applying the transformation $x = u - v$, $y = \dfrac{1}{2}(u + v)$ to *R*.

8. If *R* is the square defined by $0 \le x \le 3$ and $0 \le y \le 3$ determine the region we would get applying the transformation $x = 3u$, $y = v\left(2 + u^2\right)$ to *R*.

9. If *R* is the parallelogram with vertices $(1,1)$, $(5,3)$, $(8,8)$ and $(4,6)$ determine the region we would get applying the transformation $x = \dfrac{6}{7}(u - v)$, $y = \dfrac{1}{7}(10u - 3v)$ to *R*.

10. If *R* is the region bounded by $xy = 4$, $xy = 10$, $y = x$ and $y = 6x$ determine the region we would get applying the transformation $x = 2\sqrt{\dfrac{u}{v}}$, $y = 4\sqrt{uv}$ to *R*.

11. Evaluate $\iint\limits_R x^2 y^4 \, dA$ where *R* is the region bounded by $xy = 4$, $xy = 10$, $y = x$ and $y = 6x$ using the transformation $x = 2\sqrt{\frac{u}{v}}$, $y = 4\sqrt{uv}$.

12. Evaluate $\iint\limits_R 1 - y \, dA$ where *R* is the triangle with vertices $(0,4)$, $(1,1)$ and $(2,5)$ using the transformation $x = \frac{1}{7}(7 + u - v)$, $y = \frac{1}{7}(7 + 4u + 3v)$ to *R*.

13. Evaluate $\iint\limits_R 121x \, dA$ where *R* is the parallelogram with vertices $(0,0)$, $(6,2)$, $(7,6)$ and $(1,4)$ using the transformation $x = \frac{1}{11}(v - 3u)$, $y = \frac{1}{11}(4v - u)$ to *R*.

14. Evaluate $\iint\limits_R \frac{15y}{x} \, dA$ where *R* is the region bounded by $xy = 2$, $xy = 6$, $y = 4$ and $y = 10$ using the transformation $x = v$, $y = \frac{2u}{3v}$.

15. Evaluate $\iint\limits_R 2y - 8x \, dA$ where *R* is the parallelogram with vertices $(6,0)$, $(8,4)$, $(6,8)$ and $(4,4)$ using the transformation $x = \frac{1}{4}(u - v)$, $y = \frac{1}{2}(u + v)$ to *R*.

16. Derive a transformation that will transform the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ into a unit circle.

17. Derive the transformation used in problem 12.

18. Derive the transformation used in problem 13.

19. Derive a transformation that will convert the parallelogram with vertices $(4,1)$, $(7,4)$, $(6,8)$ and $(3,5)$ into a rectangle in the *uv* system.

20. Derive a transformation that will convert the parallelogram with vertices $(4,1)$, $(7,4)$, $(6,8)$ and $(3,5)$ into a rectangle with one corner occurring at the origin of the *uv* system.

Section 4-9 : Surface Area

1. Determine the surface area of the portion of $6x + y + 2z = 10$ that is in the 1st octant.

2. Determine the surface area of the portion of $4x + 3y + 5z = 8$ that is inside the cylinder $x^2 + y^2 = 49$.

3. Determine the surface area of the portion of $z = 9x^2 + 9y^2 - 1$ that is below the *xy*-plane with $x \le 0$.

4. Determine the surface area of the portion of $z = 6y + 2x^2$ that is above the triangle in the *xy*-plane with vertices $(0,0)$, $(8,0)$ and $(8,2)$.

5. Determine the surface area of the portion of $y = 8z + 2x^3 + 1$ that is in front of the region in the *xz*-plane bounded by $z = x^3$, $x = 2$ and the *x*-axis.

6. Determine the surface area of the portion of $x = 6 - y^2 - z^2$ that is in front of $x = 2$ with $y \ge 0$.

7. Determine the surface area of the portion of $y = 4x + 3z^2$ that is in front of the triangle in the *xz*-plane with vertices $(0,0)$, $(2,6)$ and $(0,6)$.

8. Determine the surface area of the portion of $y = 3x^2 + 3z^2$ that is inside the cylinder $x^2 + z^2 = 1$.

9. Determine the surface area of the portion of the sphere of radius 4 that is inside the cylinder $x^2 + y^2 = 3$.

Section 4-10 : Area and Volume Revisited

The intent of the section was just to "recap" the various area and volume formulas from this chapter and so no problems have been (or likely will be in the near future) written.

Chapter 5 : Line Integrals

Here are a set of assignment problems for the Line Integrals chapter of the Calculus III notes. Please note that these problems do not have any solutions available. These are intended mostly for instructors who might want a set of problems to assign for turning in. Having solutions available (or even just final answers) would defeat the purpose the problems.

If you are looking for some practice problems (with solutions available) please check out the Practice Problems. There you will find a set of problems that should give you quite a bit practice.

Here is a list of all the sections for which assignment problems have been written as well as a brief description of the material covered in the notes for that particular section.

Vector Fields – In this section we introduce the concept of a vector field and give several examples of graphing them. We also revisit the gradient that we first saw a few chapters ago.

Line Integrals – Part I – In this section we will start off with a quick review of parameterizing curves. This is a skill that will be required in a great many of the line integrals we evaluate and so needs to be understood. We will then formally define the first kind of line integral we will be looking at : line integrals with respect to arc length.

Line Integrals – Part II – In this section we will continue looking at line integrals and define the second kind of line integral we'll be looking at : line integrals with respect to x, y, and/or z. We also introduce an alternate form of notation for this kind of line integral that will be useful on occasion.

Line Integrals of Vector Fields – In this section we will define the third type of line integrals we'll be looking at : line integrals of vector fields. We will also see that this particular kind of line integral is related to special cases of the line integrals with respect to x, y and z.

Fundamental Theorem for Line Integrals – In this section we will give the fundamental theorem of calculus for line integrals of vector fields. This will illustrate that certain kinds of line integrals can be very quickly computed. We will also give quite a few definitions and facts that will be useful.

Conservative Vector Fields – In this section we will take a more detailed look at conservative vector fields than we've done in previous sections. We will also discuss how to find potential functions for conservative vector fields.

Green's Theorem – In this section we will discuss Green's Theorem as well as an interesting application of Green's Theorem that we can use to find the area of a two dimensional region.

Section 5-1 : Vector Fields

1. Sketch the vector field for $\vec{F} = -y^2\,\vec{i} + x\,\vec{j}$.

2. Sketch the vector field for $\vec{F} = \vec{i} + xy\,\vec{j}$.

3. Sketch the vector field for $\vec{F} = 4y\,\vec{i} + \left(x+2\right)\vec{j}$.

4. Compute the gradient vector field for $f\left(x,y\right) = 6x^2 - 9y + x^3\sqrt{y}$.

5. Compute the gradient vector field for $f\left(x,y\right) = \sin\left(2x\right)\cos\left(3x\right)$.

6. Compute the gradient vector field for $f\left(x,y,z\right) = z\,\mathbf{e}^{xy} + y^3\tan\left(4x\right)$.

7. Compute the gradient vector field for $f\left(x,y,z\right) = x\,y^2z^3 + 4x\mathbf{e}^{y^2} - \ln\left(x-z\right)$.

Section 5-2 : Line Integrals - Part I

For problems 1 – 10 evaluate the given line integral. Follow the direction of *C* as given in the problem statement.

1. Evaluate $\int\limits_C 3y\,ds$ where *C* is the portion of $x = 9 - y^2$ from $y = -1$ and $y = 2$.

2. Evaluate $\int\limits_C \sqrt{x} + 2xy\,ds$ where *C* is the line segment from $(7,3)$ to $(0,6)$.

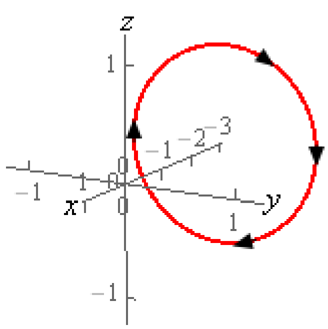
3. Evaluate $\int\limits_C y^2 - 10xy\,ds$ where *C* is the left half of the circle centered at the origin of radius 6 with counter clockwise rotation.

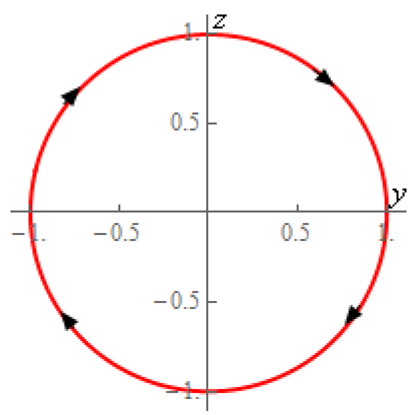
4. Evaluate $\int\limits_C x^2 - 2y\,ds$ where *C* is given by $\vec r\left(t\right) = \left\langle 4t^4, t^4 \right\rangle$ for $-1 \le t \le 0$.

5. Evaluate $\int\limits_C z^3 - 4x + 2y\,ds$ where *C* is the line segment from $(2,4,-1)$ to $(1,-1,0)$.

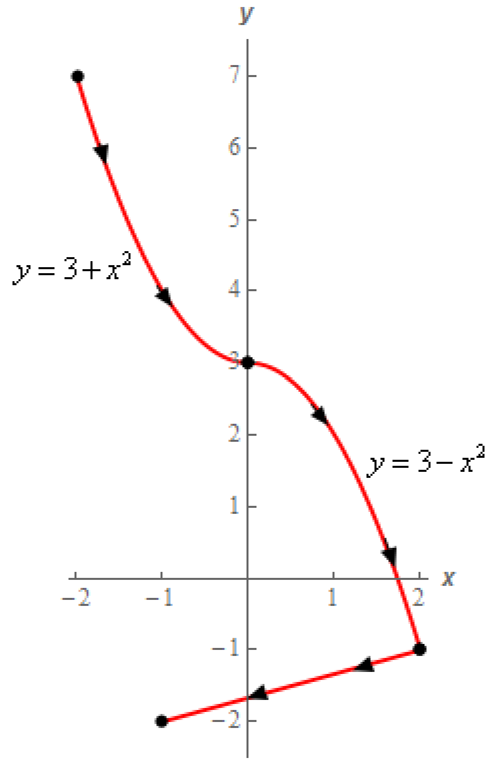
6. Evaluate $\int\limits_C x + 12xz\,ds$ where *C* is given by $\vec r\left(t\right) = \left\langle t, \frac{1}{2}t^2, \frac{1}{4}t^4 \right\rangle$ for $-2 \le t \le 1$.

7. Evaluate $\int\limits_C z^3\left(x + 7\right) - 2y\,ds$ where *C* is the circle centered at the origin of radius 1 centered on the *x*-axis at $x = -3$. See the sketches below for the direction.

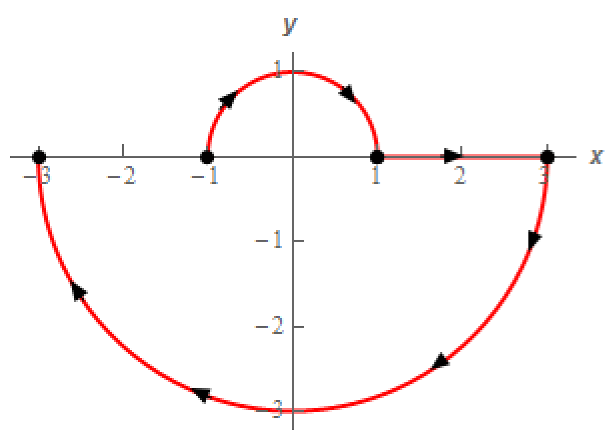




8. Evaluate $\int\limits_C 6x\,ds$ where C is the portion of $y = 3 + x^2$ from $x = -2$ to $x = 0$ followed by the portion of $y = 3 - x^2$ form $x = 0$ to $x = 2$ which in turn is followed by the line segment from $(2,-1)$ to $(-1,-2)$. See the sketch below for the direction.



9. Evaluate $\int\limits_C 2 - xy\,ds$ where C is the upper half of the circle centered at the origin of radius 1 with the clockwise rotation followed by the line segment form $(1,0)$ to $(3,0)$ which in turn is followed by the lower half of the circle centered at the origin of radius 3 with the clockwise rotation. See the sketch below for the direction.



10. Evaluate $\int\limits_C 3xy + (x-1)^2 \, ds$ where *C* is the triangle with vertices $(0,3)$, $(6,0)$ and $(0,0)$ with the clockwise rotation.

11. Evaluate $\int\limits_C x^5 \, ds$ for each of the following curves.

(a) *C* is the line segment from $(-1,3)$ to $(0,0)$ followed by the line segment from $(0,0)$ to $(0,4)$.

(b) *C* is the portion of $y = 4 - x^4$ from $x = -1$ to $x = 0$.

12. Evaluate $\int\limits_C 3x - 6y \, ds$ for each of the following curves.

(a) *C* is the line segment from $(6,0)$ to $(0,3)$ followed by the line segment from $(0,3)$ to $(6,6)$.

(b) *C* is the line segment from $(6,0)$ to $(6,6)$.

13. Evaluate $\int\limits_C y^2 - 3z + 2 \, ds$ for each of the following curves.

(a) *C* is the line segment from $(1,0,4)$ to $(2,-1,1)$.

(b) *C* is the line segment from $(2,-1,1)$ to $(1,0,4)$.

Section 5-3 : Line Integrals - Part II

For problems 1 – 7 evaluate the given line integral. Follow the direction of *C* as given in the problem statement.

1. Evaluate $\int\limits_C xy\,dx + (x - y)\,dy$ where *C* is the line segment from $(0,-3)$ to $(-4,1)$.

2. Evaluate $\int\limits_C \mathbf{e}^{3x}\,dx$ where *C* is portion of $x = \sin(4y)$ from $y = \frac{\pi}{8}$ to $y = \pi$.

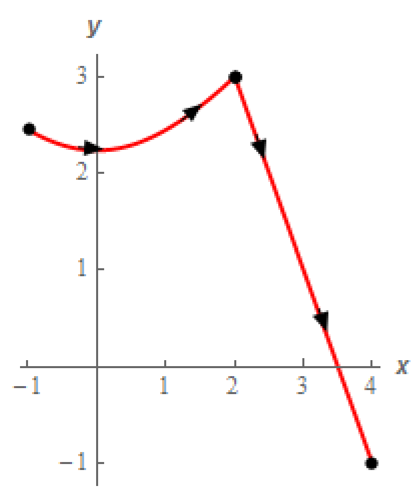
3. Evaluate $\int\limits_C x\,dy - (x^2 + y)\,dx$ where *C* is portion of the circle centered at the origin of radius 3 in the 2nd quadrant with clockwise rotation.

4. Evaluate $\int\limits_C dx - 3y^3\,dy$ where *C* is given by $\vec r(t) = 4\sin(\pi t)\,\vec i + (t-1)^2\,\vec j$ with $0 \le t \le 1$.

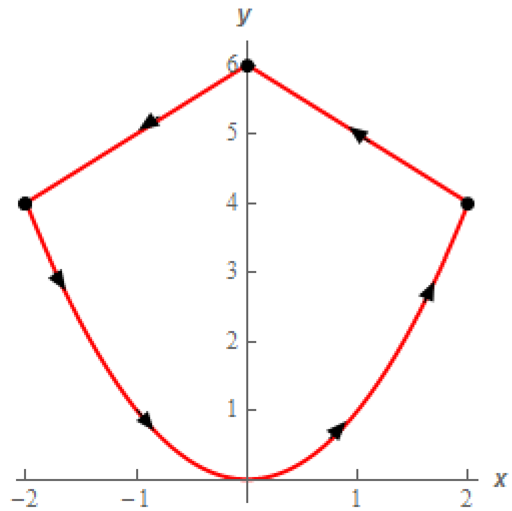
5. Evaluate $\int\limits_C 4y^2\,dx + 3x\,dy + 2z\,dz$ where *C* is the line segment from $(4,-1,2)$ to $(1,7,-1)$.

6. Evaluate $\int\limits_C (yz + x)\,dx + yz\,dy - (y + z)\,dz$ where *C* is given by $\vec r(t) = 3t\,\vec i + 4\sin(t)\,\vec j + 4\cos(t)\,\vec k$ with $0 \le t \le \pi$.

7. Evaluate $\int\limits_C 7xy\,dy$ where *C* is the portion of $y = \sqrt{x^2 + 5}$ from $x = -1$ to $x = 2$ followed by the line segment from $(2,3)$ to $(4,-1)$. See the sketch below for the direction.



8. Evaluate $\int_C \left(y^2 - x\right) dx - 4y\,dy$ where *C* is the portion of $y = x^2$ from $x = -2$ to $x = 2$ followed by the line segment from $\left(2,4\right)$ to $\left(0,6\right)$ which in turn is followed by the line segment from $\left(0,6\right)$ to $\left(-2,4\right)$. See the sketch below for the direction.



9. Evaluate $\int_C \left(x^2 - 2\right) dx + 7xy^2\,dy$ for each of the following curves.

(a) *C* is the portion of $x = -y^2$ from $y = -1$ to $y = 1$.

(b) *C* is the line segment from $\left(-1,-1\right)$ to $\left(1,1\right)$.

10. Evaluate $\int_C x^3 + 9y\,dy$ for each of the following curves.

(a) *C* is the portion of $y = 1 - x^2$ from $x = -1$ to $x = 1$.

(b) *C* is the line segment from $\left(-1,0\right)$ to $\left(0,-1\right)$ followed by the line segment from $\left(0,-1\right)$ to $\left(1,0\right)$.

11. Evaluate $\int_C xy^3\,dx - 4x\,dy$ for each of the following curves.

(a) *C* is the portion of the circle centered at the origin of radius 7 in the 1st quadrant with counter clockwise rotation.

(b) *C* is the portion of the circle centered at the origin of radius 7 in the 1st quadrant with clockwise rotation.

Section 5-4 : Line Integrals of Vector Fields

1. Evaluate $\int\limits_C \vec{F} \bullet d\vec{r}$ where $\vec{F}(x,y) = 2x^2\,\vec{i} + (y^2 - 1)\,\vec{j}$ and *C* is the portion of $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$ that is in the 1st, 4th and 3rd quadrant with the clockwise orientation.

2. Evaluate $\int\limits_C \vec{F} \bullet d\vec{r}$ where $\vec{F}(x,y) = xy\,\vec{i} + (4x - 2y)\,\vec{j}$ and *C* is the line segment from $(4,-3)$ to $(7,0)$.

3. Evaluate $\int\limits_C \vec{F} \bullet d\vec{r}$ where $\vec{F}(x,y) = (x^3 - y)\vec{i} + (x^2 + 7x)\,\vec{j}$ and *C* is the portion of $y = x^3 + 2$ from $x = -1$ to $x = 2$.

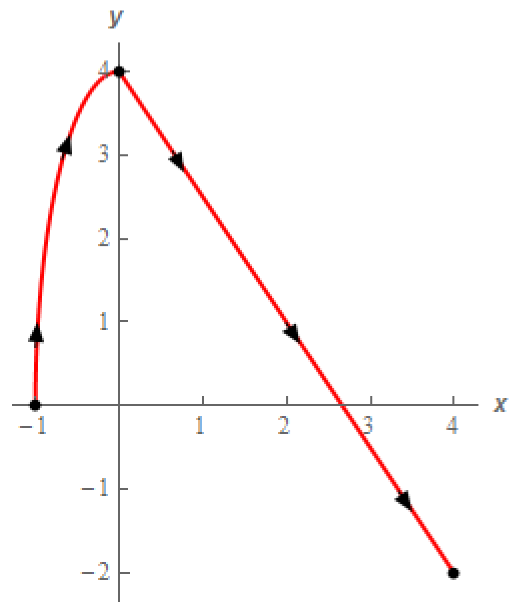
4. Evaluate $\int\limits_C \vec{F} \bullet d\vec{r}$ where $\vec{F}(x,y) = xy\,\vec{i} + (1 + x^2)\,\vec{j}$ and *C* is given by $\vec{r}(t) = \mathbf{e}^{6t}\,\vec{i} + (4 - \mathbf{e}^{2t})\,\vec{j}$ for $-2 \le t \le 0$.

5. Evaluate $\int\limits_C \vec{F} \bullet d\vec{r}$ where $\vec{F}(x,y,z) = (3x - 3y)\vec{i} + (y^3 - 10)\,\vec{j} + yz\,\vec{k}$ and *C* is the line segment from $(1,4,-2)$ to $(3,4,6)$.

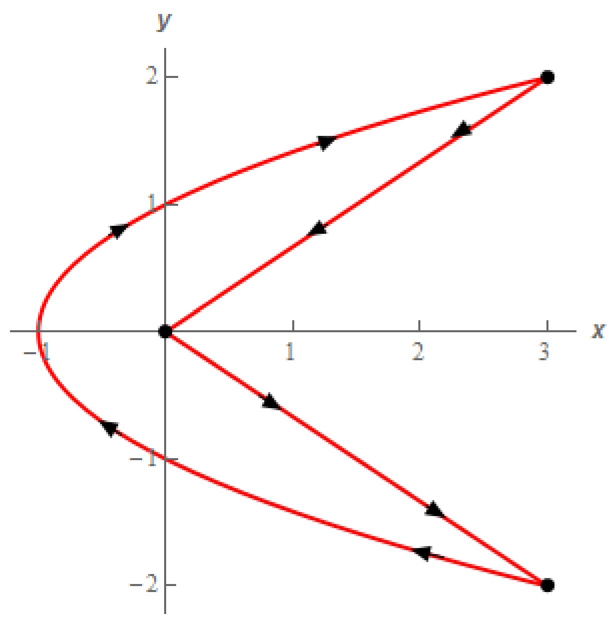
6. Evaluate $\int\limits_C \vec{F} \bullet d\vec{r}$ where $\vec{F}(x,y,z) = (x + z)\vec{i} + y^3\vec{j} + (1 - x)\vec{k}$ and *C* is the portion of the spiral on the *y*-axis given by $\vec{r}(t) = \cos(2t)\vec{i} - t\,\vec{j} + \sin(2t)\vec{k}$ for $-\pi \le t \le 2\pi$.

7. Evaluate $\int\limits_C \vec{F} \bullet d\vec{r}$ where $\vec{F}(x,y) = x^2\,\vec{i} + (y^2 - x)\,\vec{j}$ and *C* is the line segment from $(2,4)$ to $(0,4)$ followed by the line segment form $(0,4)$ to $(3,-1)$.

8. Evaluate $\int\limits_C \vec{F} \bullet d\vec{r}$ where $\vec{F}(x,y) = xy\,\vec{i} - 3\,\vec{j}$ and *C* is the portion of $x^2 + \dfrac{y^2}{4} = 1$ in the 2nd quadrant with clockwise rotation followed by the line segment from $(0,4)$ to $(4,-2)$. See the sketch below.



9. Evaluate $\int\limits_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x,y) = xy^2\,\vec{i} + (2y+3x)\,\vec{j}$ and *C* is the portion of $x = y^2 - 1$ from $y = -2$ to $y = 2$ followed by the line segment from $(3,2)$ to $(0,0)$ which in turn is followed by the line segment from $(0,0)$ to $(3,-2)$. See the sketch below.



10. Evaluate $\int\limits_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x,y) = (1 - y^2)\vec{i} - x\,\vec{j}$ for each of the following curves.

(a) *C* is the top half of the circle centered at the origin of radius 1 with the counter clockwise rotation.

(b) C is the bottom half of $x^2 + \frac{y^2}{36}$ with clockwise rotation.

11. Evaluate $\int\limits_C \vec{F} \bullet d\vec{r}$ where $\vec{F}(x,y) = (x^2 + y + 2)\vec{i} + x\,y\,\vec{j}$ for each of the following curves.

(a) C is the portion of $y = x^2 - 2$ from $x = -3$ to $x = 3$.

(b) C is the line segment from $(-3,5)$ to $(3,5)$.

12. Evaluate $\int\limits_C \vec{F} \bullet d\vec{r}$ where $\vec{F}(x,y) = y^2\,\vec{i} + (1-3x)\,\vec{j}$ for each of the following curves.

(a) C is the line segment from $(1,4)$ to $(-2,3)$.

(b) C is the line segment from $(-2,3)$ to $(1,4)$.

13. Evaluate $\int\limits_C \vec{F} \bullet d\vec{r}$ where $\vec{F}(x,y) = -2x\,\vec{i} + (x + 2y)\,\vec{j}$ for each of the following curves.

(a) C is the portion of $\frac{x^2}{16} + \frac{y^2}{4} = 1$ in the 1st quadrant with counter clockwise rotation.

(b) C is the portion of $\frac{x^2}{16} + \frac{y^2}{4} = 1$ in the 1st quadrant with clockwise rotation.

Section 5-5 : Fundamental Theorem for Line Integrals

1. Evaluate $\int_C \nabla f \bullet d\vec{r}$ where $f(x,y) = 5x - y^2 + 10xy + 9$ and *C* is given by $\vec{r}(t) = \left\langle \frac{2t}{t^2+1}, 1-8t \right\rangle$ with $-2 \le t \le 0$.

2. Evaluate $\int_C \nabla f \bullet d\vec{r}$ where $f(x,y,z) = \frac{3x-8y}{z-6}$ and *C* is given by $\vec{r}(t) = 6t\,\vec{i} + 4\,\vec{j} + (9-t^3)\vec{k}$ with $-1 \le t \le 3$.

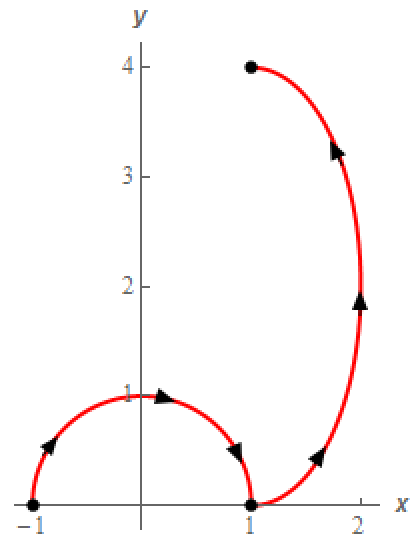
3. Evaluate $\int_C \nabla f \bullet d\vec{r}$ where $f(x,y) = 20y\cos(x+3) - yx^3$ and *C* is right half of the ellipse given by $(x+3)^2 + \frac{(y-1)^2}{16} = 1$ with clockwise rotation.

4. Compute $\int_C \vec{F} \bullet d\vec{r}$ where $\vec{F} = 2x\,\vec{i} + 4y\,\vec{j}$ and *C* is the circle centered at the origin of radius 5 with the counter clockwise rotation. Is $\int_C \vec{F} \bullet d\vec{r}$ independent of path? If it is not possible to determine if $\int_C \vec{F} \bullet d\vec{r}$ is independent of path clearly explain why not.

5. Compute $\int_C \vec{F} \bullet d\vec{r}$ where $\vec{F} = y\,\vec{i} + x^2\,\vec{j}$ and *C* is the circle centered at the origin of radius 5 with the counter clockwise rotation. Is $\int_C \vec{F} \bullet d\vec{r}$ independent of path? If it is not possible to determine if $\int_C \vec{F} \bullet d\vec{r}$ is independent of path clearly explain why not.

6. Evaluate $\int_C \nabla f \bullet d\vec{r}$ where $f(x,y,z) = zx^2 + x(y-2)^2$ and *C* is the line segment from $(1,2,0)$ to $(-3,10,9)$ followed by the line segment from $(-3,10,9)$ to $(6,0,2)$.

7. Evaluate $\int_C \nabla f \bullet d\vec{r}$ where $f(x,y) = 4x + 3xy^2 - \ln(x^2+y^2)$ and *C* is the upper half of $x^2 + y^2 = 1$ with clockwise rotation followed by the right half of $(x-1)^2 + \frac{(y-2)^2}{4} = 1$ with counter clockwise rotation. See the sketch below.


y
4
3
2
1
−1
1
2
x

Section 5-6 : Conservative Vector Fields

For problems 1 – 4 determine if the vector field is conservative.

1. $\vec{F} = \left(2xy^3 + \mathbf{e}^x \cos(y)\right)\vec{i} + \left(\mathbf{e}^x \sin(y) - 3x^2y^2\right)\vec{j}$

2. $\vec{F} = \left(xy^2 - 3y^4 + 2\right)\vec{i} + \left(xy^2 + x^2y^2 - x\right)\vec{j}$

3. $\vec{F} = \left(2 + 12xy^2 - 3x^2\sqrt{y}\right)\vec{i} - \left(\dfrac{x^3}{2\sqrt{y}} - 12x^2y\right)\vec{j}$

4. $\vec{F} = \left(8 - \dfrac{3x^2}{y} + 5x^4y^2\right)\vec{i} + \left(6 + \dfrac{x^3}{y^2} - 3y^2 + 2x^5y\right)\vec{j}$

For problems 5 – 11 find the potential function for the vector field.

5. $\vec{F} = \left(4x^3 + 3y + \dfrac{2y^3}{x^3}\right)\vec{i} + \left(3x - 3y^2 - \dfrac{3y^2}{x^2}\right)\vec{j}$

6. $\vec{F} = \left(3x^2\mathbf{e}^{2y} + 4y\mathbf{e}^{4x}\right)\vec{i} - \left(7 - 2x^3\mathbf{e}^{2y} - \mathbf{e}^{4x}\right)\vec{j}$

7. $\vec{F} = \left(\cos(x)\cos(x+y) - 2y^2 - \sin(x)\sin(x+y)\right)\vec{i} - \left(4xy + \sin(x)\sin(x+y)\right)\vec{j}$

8. $\vec{F} = \left(\dfrac{4}{x^2} + \dfrac{2x}{y} + \dfrac{2}{x^2y^3}\right)\vec{i} + \left(\dfrac{6}{xy^4} - \dfrac{1+x^2}{y^2}\right)\vec{j}$

9. $\vec{F} = \left(2x\mathbf{e}^{x^2-z}\sin\left(y^2\right) - 3y^3\right)\vec{i} + \left(2y\mathbf{e}^{x^2-z}\cos\left(y^2\right) - 9xy^2\right)\vec{j} + \left(12z - \mathbf{e}^{x^2-z}\sin\left(y^2\right)\right)\vec{k}$

10. $\vec{F} = \left(12x - 5z^2\right)\vec{i} + \ln\left(1+z^2\right)\vec{j} - \left(10xz - \dfrac{2yz}{1+z^2}\right)\vec{k}$

11. $\vec{F} = \left(zy^2\mathbf{e}^{y-x} - xy^2z\mathbf{e}^{y-x}\right)\vec{i} + \left(2xyz\mathbf{e}^{y-x} + xy^2z\mathbf{e}^{y-x}\right)\vec{j} + \left(xy^2\mathbf{e}^{y-x} - 24z\right)\vec{k}$

12. Evaluate $\int\limits_C \vec{F} \bullet d\vec{r}$ where $\vec{F}(x,y) = \left(\dfrac{3x^2}{y-1} - 3x^2y\right)\vec{i} + \left(8y - x^3 - \dfrac{x^3}{(y-1)^2}\right)\vec{j}$ and *C* is the line segment from $(1,2)$ to $(4,3)$.

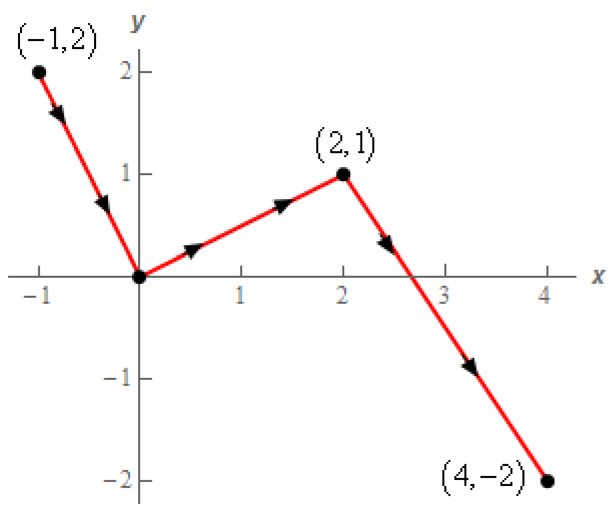
13. Evaluate $\int_C \vec{F}\bullet d\vec{r}$ where $\vec{F}(x,y) = (y^2 - 4y + 5)\vec{i} + (2xy - 4x - 9)\vec{j}$ and *C* the upper half of $\dfrac{x^2}{36} + \dfrac{y^2}{16} = 1$ with clockwise rotation.

14. Evaluate $\int_C \vec{F}\bullet d\vec{r}$ where $\vec{F}(x,y) = -\left(3 - (1+2y)\mathbf{e}^{x-1}\right)\vec{i} + \left(3y^2 + 2\mathbf{e}^{x-1}\right)\vec{j}$ and *C* is the portion of $y = x^3 + 1$ from $x = -2$ to $x = 1$.

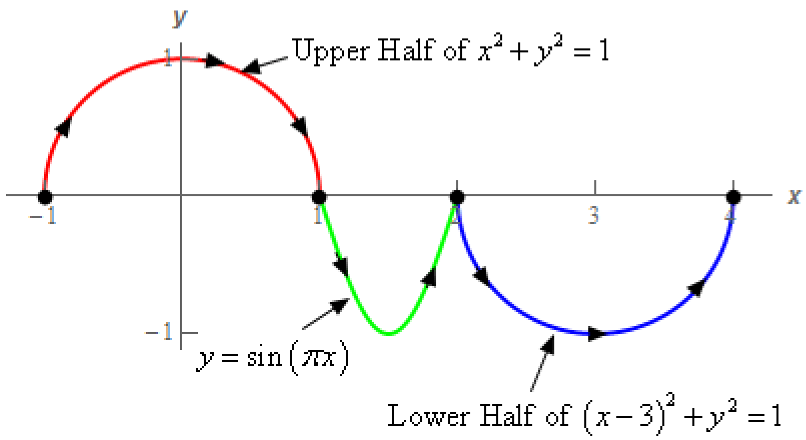
15. Evaluate $\int_C \vec{F}\bullet d\vec{r}$ where $\vec{F}(x,y,z) = \dfrac{x}{\sqrt{x^2+z^2}}\vec{i} + (2yz - 6y)\vec{j} + \left(y^2 + \dfrac{z}{\sqrt{x^2+z^2}}\right)\vec{k}$ and *C* is the line segment from $(1,0,-1)$ to $(2,-4,3)$.

16. Evaluate $\int_C \vec{F}\bullet d\vec{r}$ where $\vec{F}(x,y) = (12xy - 2x)\vec{i} + (6x^2 - 8yz)\vec{j} + (8 - 4y^2)\vec{k}$ and *C* is the spiral given by $\vec{r}(t) = \langle \sin(\pi t), \cos(\pi t), 3t\rangle$ for $0 \le t \le 6$.

17. Evaluate $\int_C \vec{F}\bullet d\vec{r}$ where $\vec{F}(x,y) = \left(8 - 14xy^2 + 2y\mathbf{e}^{2x}\right)\vec{i} + \left(\mathbf{e}^{2x} - 14x^2 y\right)\vec{j}$ and *C* is the curve shown below.

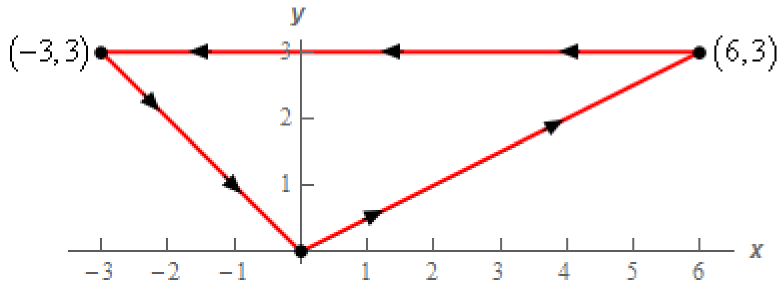


18. Evaluate $\int_C \vec{F}\bullet d\vec{r}$ where $\vec{F}(x,y) = \left(6x - 5y^2 + 2xy^3 - 10\right)\vec{i} + \left(3x^2y^2 - 10xy\right)\vec{j}$ and *C* is the curve shown below.

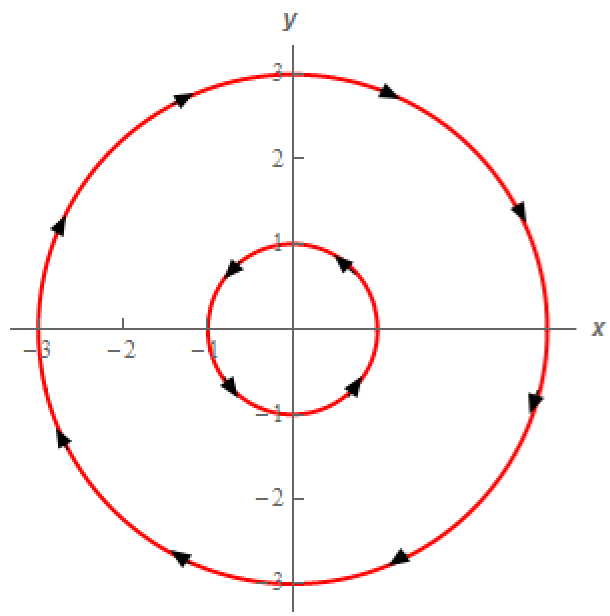

y
Upper Half of $x^2 + y^2 = 1$
1
−1
1
3
4
x
−1
$y = \sin(\pi x)$
Lower Half of $(x-3)^2 + y^2 = 1$

Section 5-7 : Green's Theorem

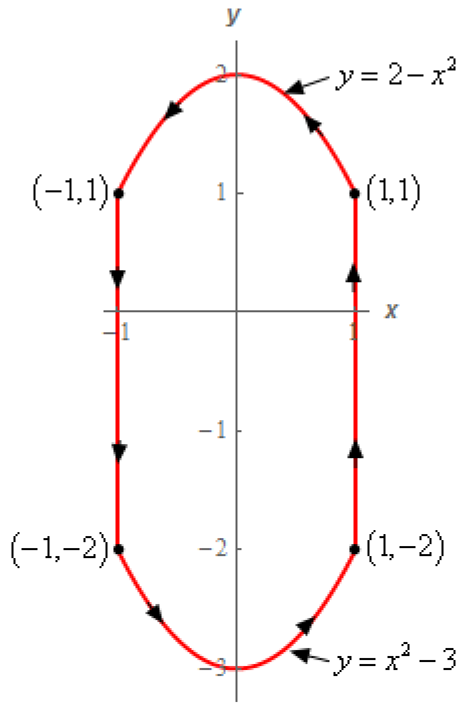
1. Use Green's Theorem to evaluate $\int_C \left(yx^2 - y\right)dx + \left(x^3 + 4\right)dy$ where C is shown below.



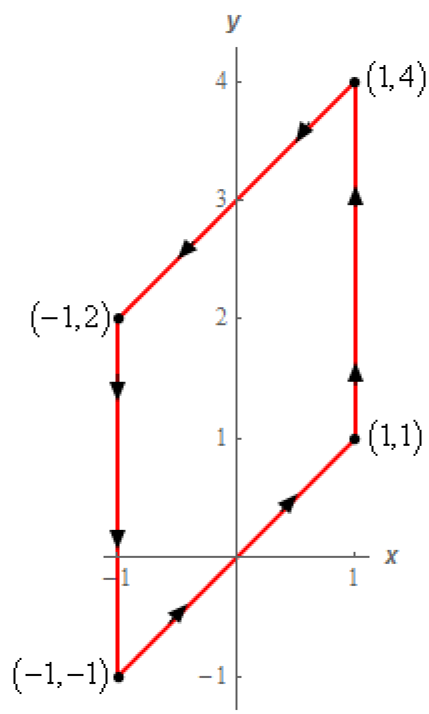
2. Use Green's Theorem to evaluate $\int_C \left(7x + y^2\right)dy - \left(x^2 - 2y\right)dx$ where C is are the two circles as shown below.



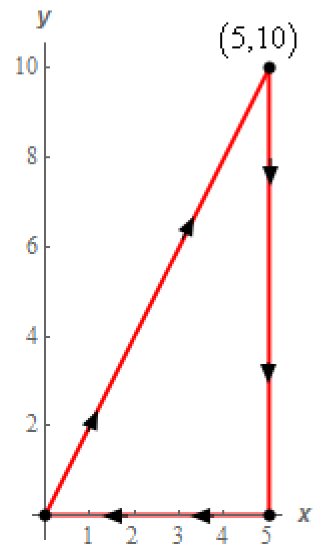
3. Use Green's Theorem to evaluate $\int_C \left(y^2 - 6y\right)dx + \left(y^3 + 10y^2\right)dy$ where C is shown below.



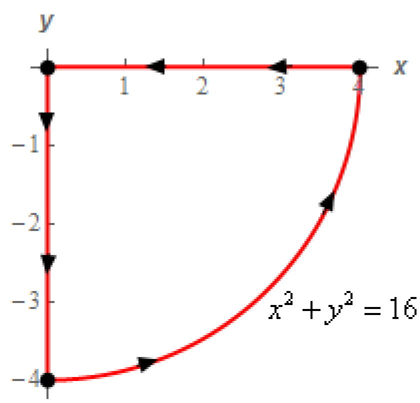
4. Use Green's Theorem to evaluate $\int_C xy^2\,dx + \left(1 - xy^3\right)dy$ where *C* is shown below.



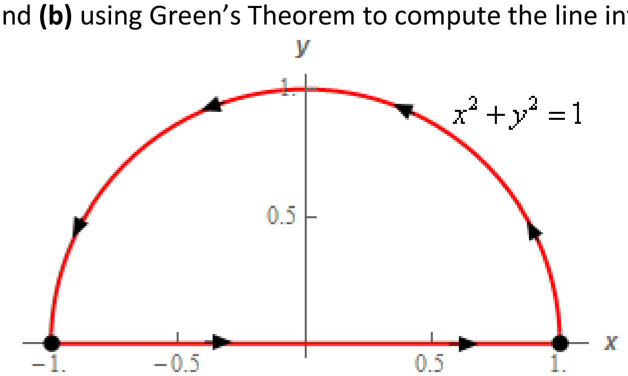
5. Use Green's Theorem to evaluate $\oint_C \left(y^2 - 4x\right)dx - \left(2 + x^2y^2\right)dy$ where *C* is shown below.



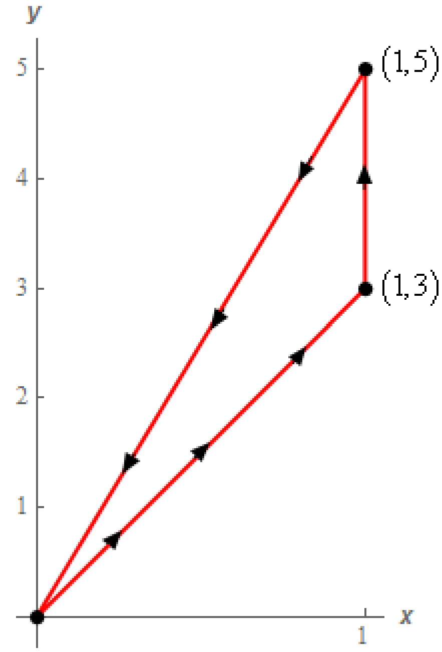
6. Use Green's Theorem to evaluate $\oint_C \left(y^3 - xy^2\right)dx + \left(2 - x^3\right)dy$ where *C* is shown below.



7. Verify Green's Theorem for $\oint_C \left(6 + x^2\right)dx + \left(1 - 2xy\right)dy$ where *C* is shown below by **(a)** computing the line integral directly and **(b)** using Green's Theorem to compute the line integral.



8. Verify Green's Theorem for $\oint\limits_C \left(6y - 3y^2 + x\right) dx + yx^3 dy$ where *C* is shown below by **(a)** computing the line integral directly and **(b)** using Green's Theorem to compute the line integral.



Chapter 6 : Surface Integrals

Here are a set of assignment problems for the Surface Integrals chapter of the Calculus III notes. Please note that these problems do not have any solutions available. These are intended mostly for instructors who might want a set of problems to assign for turning in. Having solutions available (or even just final answers) would defeat the purpose the problems.

If you are looking for some practice problems (with solutions available) please check out the Practice Problems. There you will find a set of problems that should give you quite a bit practice.

Here is a list of all the sections for which assignment problems have been written as well as a brief description of the material covered in the notes for that particular section.

Curl and Divergence – In this section we will introduce the concepts of the curl and the divergence of a vector field. We will also give two vector forms of Green's Theorem and show how the curl can be used to identify if a three dimensional vector field is conservative field or not.

Parametric Surfaces – In this section we will take a look at the basics of representing a surface with parametric equations. We will also see how the parameterization of a surface can be used to find a normal vector for the surface (which will be very useful in a couple of sections) and how the parameterization can be used to find the surface area of a surface.

Surface Integrals – In this section we introduce the idea of a surface integral. With surface integrals we will be integrating over the surface of a solid. In other words, the variables will always be on the surface of the solid and will never come from inside the solid itself. Also, in this section we will be working with the first kind of surface integrals we'll be looking at in this chapter : surface integrals of functions.

Surface Integrals of Vector Fields – In this section we will introduce the concept of an oriented surface and look at the second kind of surface integral we'll be looking at : surface integrals of vector fields.

Stokes' Theorem – In this section we will discuss Stokes' Theorem.

Divergence Theorem – In this section we will discuss the Divergence Theorem.

Section 6-1 : Curl and Divergence

For problems 1 – 3 compute $\operatorname{div}\vec{F}$ and $\operatorname{curl}\vec{F}$.

1. $\vec{F} = \left(2y - \cos\left(x\right)\right)\vec{i} - z^2\mathbf{e}^{3x}\vec{j} + \left(x^2 - 7z\right)\vec{k}$

2. $\vec{F} = -\left(4y - 1\right)\vec{i} + xy^2\vec{j} + \left(x - 3y\right)\vec{k}$

3. $\vec{F} = z^2\left(y - x\right)\vec{i} + \dfrac{4y^2}{z^3}\vec{j} + \left(x^2 - 3z\right)\vec{k}$

For problems 4 – 6 determine if the vector field is conservative.

4. $\vec{F} = \left(2xy^2 - 16x\right)\vec{i} + 2y\left(x^2 - 1\right)\vec{j} + 9\vec{k}$

5. $\vec{F} = \left(y - 3z\right)\vec{i} + \left(x^2 + y^4\right)\vec{j} - 4z^2\vec{k}$

6. $\vec{F} = \left(18x^2 + 4z^3\right)\vec{i} - 12yz\,\vec{j} - \left(6y^2 - 12xz^2\right)\vec{k}$

Section 6-2 : Parametric Surfaces

For problems 1 – 10 write down a set of parametric equations for the given surface.

1. The plane containing the three points $(1,4,-2)$, $(-3,0,1)$ and $(2,4,-5)$.

2. The portion of the plane $x+9y+3z=8$ that lies in the 1st octant.

3. The portion of $x=2y^2+2z^2-7$ that is behind $x=5$.

4. The portion of $y=10-3x^2-3z^2$ that is in front of the *xz*-plane.

5. The cylinder $x^2+z^2=121$.

6. The cylinder $y^2+z^2=6$ for $2\le x\le 9$.

7. The sphere $x^2+y^2+z^2=17$.

8. The portion of the sphere of radius 3 with $y\ge 0$ and $z\le 0$.

9. The tangent plane to the surface given by the following parametric equation at the point $(-5,4,-12)$.

$$\vec{r}(u,v)=(u+2v)\vec{i}+(u^2+3)\vec{j}-3v^2\vec{k}$$

10. The tangent plane to the surface given by the following parametric equation at the point $(1,-11,19)$.

$$\vec{r}(u,v)=\left\langle \mathbf{e}^{6-2v},u^2-15,1-uv^2\right\rangle$$

11. Determine the surface area of the portion of $3x+3y+4z=16$ that is in the 1st octant.

12. Determine the surface area of the portion of $x+4y+8z=4$ that is inside the cylinder $x^2+y^2=16$.

13. Determine the surface area of the portion of $z=6y+2x^2$ that is above the triangle in the *xy*-plane with vertices $(0,0)$, $(8,0)$ and $(8,2)$.

14. Determine the surface area of the portion of $x=6-y^2-z^2$ that is in front of $x=2$ with $y\ge 0$.

15. Determine the surface area of the portion of $x^2+y^2+z^2=11$ with $x\ge 0$, $y\le 0$ and $z\ge 0$.

16. Determine the surface area of the portion of the surface given by the following parametric equation that lies above the triangle in the *uv*-plane with vertices $\left(0,0\right)$, $\left(10,2\right)$ and $\left(0,2\right)$.

$$\vec r\left(u,v\right) = \left\langle v^2, 3v, 2u\right\rangle$$

17. Determine the surface area of the portion of the surface given by the following parametric equation that lies above the region in the *uv*-plane bounded by $v = \frac{3}{2}u^2$, $u = 2$ and the *u*-axis.

$$\vec r\left(u,v\right) = \left\langle uv, 3uv, v\right\rangle$$

18. Determine the surface area of the portion of the surface given by the following parametric equation that lies inside the cylinder $u^2 + v^2 = 16$.

$$\vec r\left(u,v\right) = \left\langle uv, 1-3v, 2+3u\right\rangle$$

Section 6-3 : Surface Integrals

1. Evaluate $\iint\limits_S 2x - 3y + z\,dS$ where *S* is the portion of $x + y + z = 2$ that is in the 1st octant.

2. Evaluate $\iint\limits_S x + y^2 + z^2\,dS$ where *S* is the portion of $x = 4 - y^2 - z^2$ that lies in front of $x = -2$.

3. Evaluate $\iint\limits_S 6\,dS$ where *S* is the portion of $y = 4z + x^3 + 6$ that lies over the region in the *xz*-plane with bounded by $z = x^3$, $x = 1$ and the *x*-axis.

4. Evaluate $\iint\limits_S xyz\,dS$ where *S* is the portion of $x^2 + y^2 = 36$ between $z = -3$ and $z = 1$.

5. Evaluate $\iint\limits_S z^2 + x\,dS$ where *S* is the portion of $x^2 + y^2 + z^2 = 4$ with $z \ge 0$.

6. Evaluate $\iint\limits_S 4y\,dS$ where *S* is the portion of $x^2 + z^2 = 9$ between $y = 2$ and $y = 10 - x$.

7. Evaluate $\iint\limits_S z + 3\,dS$ where *S* is the surface of the solid bounded by $z = 2x^2 + 2y^2 - 3$ and $z = 1$. Note that both surfaces of this solid are included in *S*.

8. Evaluate $\iint\limits_S z\,dS$ where *S* is the surface of the solid bounded by $y^2 + z^2 = 4$, $x = y - 3$, and $x = 6 - z$. Note that all three surfaces of this solid are included in *S*.

9. Evaluate $\iint\limits_S 4 + z\,dS$ where *S* is the portion of the sphere of radius 1 with $z \ge 0$ and $x \le 0$. Note that all three surfaces of this solid are included in *S*.

Section 6-4 : Surface Integrals of Vector Fields

1. Evaluate $\iint\limits_S \vec{F} \bullet d\vec{S}$ where $\vec{F} = (z - y)\vec{i} + x\vec{j} + 4y\vec{k}$ and *S* is the portion of $x + y + z = 2$ that is in the 1st octant oriented in the positive *z*-axis direction.

2. Evaluate $\iint\limits_S \vec{F} \bullet d\vec{S}$ where $\vec{F} = (x - 4)\vec{i} + z\vec{j} - y\vec{k}$ and *S* is the portion of $x = 4 - y^2 - z^2$ that lies in front of $x = -2$ oriented in the negative *x*-axis direction.

3. Evaluate $\iint\limits_S \vec{F} \bullet d\vec{S}$ where $\vec{F} = \vec{i} + 4z\vec{j} + (z - y)\vec{k}$ and *S* is the portion of $y = 4z + x^3 + 6$ that lies over the region in the *xz*-plane with bounded by $z = x^3$, $x = 1$ and the *x*-axis oriented in the positive *y*-axis direction.

4. Evaluate $\iint\limits_S \vec{F} \bullet d\vec{S}$ where $\vec{F} = (x + y)\vec{i} + x\vec{j} + zx^2\vec{k}$ and *S* is the portion of $x^2 + y^2 = 36$ between $z = -3$ and $z = 1$ oriented outward (*i.e.* away from the *z*-axis).

5. Evaluate $\iint\limits_S \vec{F} \bullet d\vec{S}$ where $\vec{F} = z\vec{i} + 3\vec{k}$ and *S* is the portion of $x^2 + y^2 + z^2 = 4$ with $z \ge 0$ oriented outwards (*i.e.* away from the origin).

6. Evaluate $\iint\limits_S \vec{F} \bullet d\vec{S}$ where $\vec{F} = -x\vec{i} + (4 + y)\vec{j} - z\vec{k}$ and *S* is the portion of $x^2 + z^2 = 9$ between $y = 2$ and $y = 10 - x$ oriented inward (*i.e.* towards from the *y*-axis).

7. Evaluate $\iint\limits_S \vec{F} \bullet d\vec{S}$ where $\vec{F} = y\vec{i} + 2\vec{j} + (z + 3)^2\vec{k}$ and *S* is the surface of the solid bounded by $z = 2x^2 + 2y^2 - 3$ and $z = 1$ with the negative orientation. Note that both surfaces of this solid are included in *S*.

8. Evaluate $\iint\limits_S \vec{F} \bullet d\vec{S}$ where $\vec{F} = (x - y)\vec{i} + z\vec{j} + y\vec{k}$ and *S* is the surface of the solid bounded by $y^2 + z^2 = 4$, $x = y - 3$, and $x = 6 - z$ with the positive orientation. Note that all three surfaces of this solid are included in *S*.

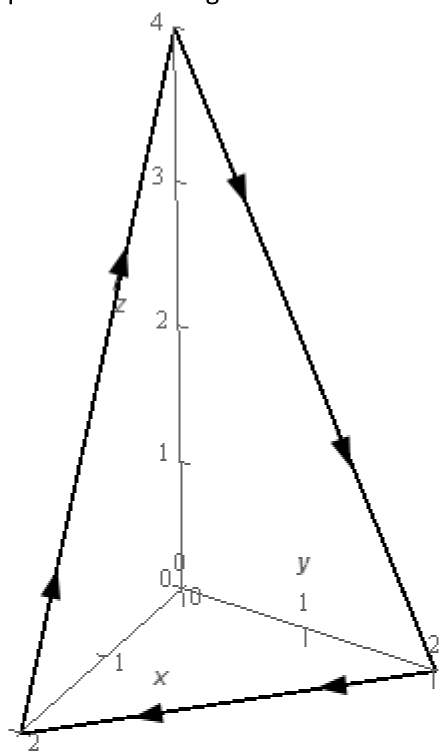
9. Evaluate $\iint\limits_S \vec{F} \bullet d\vec{S}$ where $\vec{F} = y\vec{i} - 2\vec{k}$ and *S* is the portion of the sphere of radius 1 with $z \ge 0$ and $x \le 0$ with the positive orientation. Note that all three surfaces of this solid are included in *S*.

Section 6-5 : Stokes' Theorem

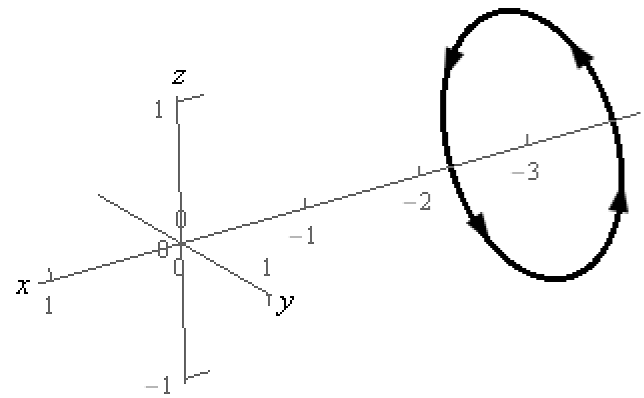
1. Use Stokes' Theorem to evaluate $\iint\limits_S \operatorname{curl}\vec F \bullet d\vec S$ where $\vec F = x^3\,\vec i + \left(4y - z^3y^3\right)\vec j + 2x\,\vec k$ and *S* is the portion of $z = x^2 + y^2 - 3$ below $z = 1$ with orientation in the negative *z*-axis direction.

2. Use Stokes' Theorem to evaluate $\iint\limits_S \operatorname{curl}\vec F \bullet d\vec S$ where $\vec F = 2y\,\vec i + 3x\,\vec j + \left(z - x\right)\vec k$ and *S* is the portion of $y = 11 - 3x^2 - 3z^2$ in front of $y = 5$ with orientation in the positive *y*-axis direction.

3. Use Stokes' Theorem to evaluate $\int\limits_C \vec F \bullet d\vec r$ where $\vec F = \left(zx^3 - 2z\right)\vec i + xz\,\vec j + yx\,\vec k$ and *C* is the triangle with vertices $\left(0,0,4\right)$, $\left(0,2,0\right)$ and $\left(2,0,0\right)$. *C* has a clockwise rotation if you are above the triangle and looking down towards the *xy*-plane. See the figure below for a sketch of the curve.



4. Use Stokes' Theorem to evaluate $\int\limits_C \vec F \bullet d\vec r$ where $\vec F = x^2\,\vec i - 4z\,\vec j + xy\,\vec k$ and *C* is is the circle of radius 1 at $x = -3$ and perpendicular to the *x*-axis. *C* has a counter clockwise rotation if you are looking down the *x*-axis from the positive *x*-axis to the negative *x*-axis. See the figure below for a sketch of the curve.


z
1
0
0
0
-1
-2
-3
x
1
1
y
-1

Section 6-6 : Divergence Theorem

1. Use the Divergence Theorem to evaluate $\iint\limits_S \vec{F}\bullet d\vec{S}$ where $\vec{F} = \left(3x - zx^2\right)\vec{i} + \left(x^3 - 1\right)\vec{j} + \left(4y^2 + x^2z^2\right)\vec{k}$ and *S* is the surface of the box with $0 \le x \le 1$, $-3 \le y \le 0$ and $-2 \le z \le 1$. Note that all six sides of the box are included in *S*.

2. Use the Divergence Theorem to evaluate $\iint\limits_S \vec{F}\bullet d\vec{S}$ where $\vec{F} = 4x\,\vec{i} + \left(1 - 6y\right)\vec{j} + z^3\,\vec{k}$ and *S* is the surface of the sphere of radius 2 with $z \ge 0$, $y \le 0$ and $x \ge 0$. Note that all four surfaces of this solid are included in *S*.

3. Use the Divergence Theorem to evaluate $\iint\limits_S \vec{F}\bullet d\vec{S}$ where $\vec{F} = -xy\,\vec{i} + \left(z - 1\right)\vec{j} + z^3\,\vec{k}$ and *S* is the surface of the solid bounded by $y = 4x^2 + 4z^2 - 1$ and the plane $y = 7$. Note that both of the surfaces of this solid included in *S*.

4. Use the Divergence Theorem to evaluate $\iint\limits_S \vec{F}\bullet d\vec{S}$ where $\vec{F} = \left(4x - z^2\right)\vec{i} + \left(x + 3z\right)\vec{j} + \left(6 - z\right)\vec{k}$ and *S* is the surface of the solid bounded by the cylinder $x^2 + y^2 = 36$ and the planes $z = -2$ and $z = 3$. Note that both of the surfaces of this solid included in *S*.

