

# Mat 3310 Notes : Power Series solution to ODE's

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## 1 Introduction

Linear Ordinary Differential Equations (ODEs) can be broadly divided into two:

1. Linear ODEs with constant coefficients, and
2. Linear ODEs with variable coefficients.

Linear ODEs with constant coefficients can be solved by algebraic methods, and their solutions are elementary functions known from calculus. For ODEs with variable coefficients the situation is more complicated, and their solutions may be nonelementary functions.

The power series method is the standard method for solving linear ordinary differential equations with variable coefficients. It gives solutions in the form of power series.

## 2 Power Series

### 2.1 Definition

A power series (in powers of  $x - x_0$ ) is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots$$

In the above,  $x$  is a variable and  $a_0, a_1, a_2, \dots$  are constants called the coefficients of the series.

$x_0$  is a constant, called the center of the series. In particular, if  $x_0 = 0$ , we obtain a power series in powers of  $x$ .

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$$

We shall assume that all variables and constants are real.

Please note that our use of the term “power series” strictly refers to “infinite sums” that do not include negative or fractional powers of  $x$ .

## 2.2 Analytic Functions

A function is analytic if it can be represented by a power series.

Not every function is analytic, although the majority of the functions you have seen in calculus are.

An analytic function  $f(x)$  is equal to its Taylor series near a point  $x_0$ . That is, for  $x$  near  $x_0$  we have

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n.$$

where  $f^{(n)}(x_0)$  denotes the  $n^{\text{th}}$  derivative of  $f(x)$  at the point  $x_0$ . When  $x_0 = 0$ , we get what is called the Maclaurin series. That is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

Familiar Power Series are the Maclaurin series

1.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

2.

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots.$$

3.

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots.$$

## 2.3 Manipulating Power Series

1. **Termwise Differentiation.** A power series may be differentiated term by term. More precisely: if

$$y = \sum_{n=0}^{\infty} a_n x^n,$$

then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Similarly for the third and further derivatives.

2. **Termwise Addition.** Two power series may be added term by term. More precisely, if:

$$y = \sum_{n=0}^{\infty} a_n x^n \text{ and } z = \sum_{n=0}^{\infty} b_n x^n$$

then

$$y + z = \sum_{n=0}^{\infty} (a_n + b_n) x^n.$$

3. **Shifting summation indices.** It is often convenient or necessary in the power series method to shift the index so that the power under the summation sign is  $x^n$ . For example,

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n.$$

How do we see or convert this? In the series

$$\sum_{n=1}^{\infty} n a_n x^{n-1}$$

Let  $m = n - 1$ , then  $n = m + 1$ . Therefore, replacing  $m$  in that series gives us  $\sum_{m=0}^{\infty} (m+1) a_{m+1} x^m$ . Since  $m$  is just the letter of indexing, we can still go back to  $n$  and have

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

One can verify this by writing down the first few terms of each series explicitly.

### 3 Idea and Technique of the Power Series Method

The idea of the power series method for solving linear ODEs seems natural, once we know that the most important ODEs in applied mathematics have solutions of this form. We explain the idea by an ODE that can readily be solved otherwise.

#### 3.0.1 Example

Find a power series solution to the ordinary differential equation  $y' - y = 0$ .

**Solution:**

Assume first that  $y = \sum_{n=0}^{\infty} a_n x^n$ . Then  $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$ .

Insert these into the differential equation, we get

$$\sum_{n=1}^{\infty} na_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

Manipulating the first series, we get

$$\sum_{n=0}^{\infty} (n+1)a_{n+1}x^n - \sum_{n=0}^{\infty} a_n x^{n-1} = 0$$

Rewriting under one summation we get

$$\sum_{n=0}^{\infty} [(n+1)a_{n+1} - a_n] x^n = 0$$

Since the left hand side which is a series equals zero on the right hand side, it implies that the coefficients of the series on the left hand side must all equal zero.

That is  $(n+1)a_{n+1} - a_n = 0$ , implying that  $a_{n+1} = \frac{a_n}{n+1}$ .

value of $n$	implication
$n = 0$	$a_1 = a_0$
$n = 1$	$a_2 = \frac{a_1}{2} = \frac{a_0}{2}$
$n = 2$	$a_3 = \frac{a_2}{3} = \frac{a_0}{2 \cdot 3} = \frac{a_0}{3!}$
$n = 3$	$a_4 = \frac{a_3}{4} = \frac{a_0}{4 \cdot 3!} = \frac{a_0}{4!}$
$\dots$	$\dots$

From this, we see that all the coefficients are equal. That is,

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_0 x + \frac{a_0}{2!} x^2 + \frac{a_0}{3!} x^3 + \dots = a_0 \left( 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \right) = a_0 e^x.$$

## 4 Series Solutions of Linear Second Order Differential Equations