# MAT 3310 : Laplace Transforms

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## 1 Introduction

In this topic, we will be looking at how to use Laplace transforms to solve differential equations. Laplace transforms "transforms" a differential equation to an algebra equation, which can be solved using purely algebraic methods. Once the algebraic equation is solved, we "transform back" the equation using Laplace transforms again.

## 2 Notation

We will use the notation, $f(t)$, for a function instead of, $f(x)$. That is, $t$ our independent variable. The reason is that we will use $s$ as the independent variable of the Laplace transform of a function with independent variable $t$.

## 3 Definition of the Laplace Transform

We now begin going into detail by looking at the definition of the Laplace transform.

**Definition 1.** *Let $f(t)$ be a piecewise continuous function defined for $0 \leq t \leq \infty$. The Laplace transform of $f(t)$ is denoted by $\mathcal{L}\{f(t)\}$ and defined as*

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\ \mathrm{d}t = \lim_{x\to\infty}\int_0^x f(t)e^{-st}\ \mathrm{d}t$$

*provided this limit exists.*

We can simply say that a piecewise continuous function is a function that has a finite number of breaks in it.

There is an alternative notation for Laplace transforms. We write

$$\mathcal{L}\{f(t)\} = F(s).$$

Note that we are using small letter $f$ for a function which we want to find the transform and big letter $F$ for the transform.

***Example.*** 1. Let $f(t) = c$ be a constant function, that is $c$ is a constant. Then

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\ \mathrm{d}t = \int_0^\infty ce^{-st}\ \mathrm{d}t = c\int_0^\infty e^{-st}\ \mathrm{d}t = -\frac{c}{s}e^{-st}\Big|_0^\infty = \frac{c}{s}.$$

Therefore, the Laplace transform of $f(t) = 1$ is $\frac{1}{s}$.

2. Let us try finding the Laplace transform of the linear function $f(t) = t$.

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty te^{-st}\ \mathrm{d}t = \left[-\frac{te^{-st}}{s} - \frac{e^{-st}}{s^2}\right]_0^\infty = \frac{1}{s^2}.$$

Verify using integration by parts that

$$\int te^{-st}\ \mathrm{d}t = -\frac{te^{-st}}{s} - \frac{e^{-st}}{s^2}$$

**Homework**

For an integer $n \geq 1$, show $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$.

3. We now wish to find he Laplace transform of the exponential function $f(t) = e^{ct}$, where $c$ is a constant.

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{at}e^{-st}\ \mathrm{d}t = \int_0^\infty e^{-(s-a)t}\ \mathrm{d}t = -\frac{e^{-(s-a)t}}{s-a}\Bigg|_0^\infty = \frac{1}{s-a} \quad (\text{for } s > a).$$

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Before we can find the Laplace transforms of some more functions, we look at one property of the Laplace transform.

**Proposistion 1.** *Let $f(t)$ and $g(t)$ be functions and let $a, b$ be constants, then*

$$\mathcal{L}\{a\ f(t) + b\ g(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}.$$

*Proof.*

$$\mathcal{L}\{a\ f(t)+b\ g(t)\} = \int_0^\infty \{a\ f(t)+b\ g(t)\}e^{-st}\ \mathrm{d}t = a\int_0^\infty f(t)e^{-st}\ \mathrm{d}t + b\int_0^\infty g(t)e^{-st}\ \mathrm{d}t = a\mathcal{L}\{f(t)\}+b\mathcal{L}\{g(t)\}.$$

□

We now wish to find the Laplace transform of the trignonometric functions $\sin at$ and $\cos at$, where $a$ is a constant. There two ways we can do this. We can use do direct integration using integration by parts of we can use Euler equation and the above proposition. Here, we demonstrate the later and leave the former as an exercise.

Remember Euler's formula,

$$e^{it} = \cos t + i \sin t.$$

Applying Laplace on both sides of the equation gives us

$$\mathcal{L}\{e^{it}\} = \mathcal{L}\{\cos t + i \sin t\}.$$

Using the above proposition, we get

$$\mathcal{L}\{e^{it}\} = \mathcal{L}\{\cos t\} + i\mathcal{L}\{\sin t\}.$$

Evaluating the left hand side of the above equation, we get

$$\mathcal{L}\{e^{it}\} = \frac{1}{s-i} = \frac{s+i}{s^2+1} = \frac{s}{s^2+1} + i\frac{1}{s^2+1}$$

The first equality is given by the rule of finding the Laplace transform of exponential function as was established in example 3 above while the second equality is given by conjugation.

From this, we get the following equation.

$$\frac{s}{s^2+1} + i\frac{1}{s^2+1} = \mathcal{L}\{\cos t\} + i\mathcal{L}\{\sin t\}.$$

Thus, $\mathcal{L}\{\cos t\} = \frac{s}{s^2+1}$ and $\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$.

## 4 Inverse Laplace Transform

Now that we have an idea of how to find the Laplace transform of a function, we would like go the other way, finding the inverse Laplace transform. That is, if we are given a transform, $F(s)$, we want to find the function that gives us this transform.

**Definition 2.** *Let $F(s)$ be a function with independent variable $s$. We say that a function $f(t)$ is the inverse Laplace transform of $F(s)$ if $\mathcal{L}\{\{(\sqcup)\} = F(s)$. If this is the case, we write*

$$\mathcal{L}^{-1}\{F(s)\} = f(t).$$

We leave as an exercise for the students to prove that, as with Laplace transforms, if $F(s)$ and $G(s)$ are Laplace transforms then,

$$\mathcal{L}^{-1}\left[a\ F(s) + b\ G(s)\right] = a\mathcal{L}^{-1}\{F(s)\} + b\mathcal{L}^{-1}\{G(s)\}.$$

***Example.*** Find the inverse transform of each of the following.

1. $F(s) = \frac{3}{s^2+4}$
2. $F(s) = \frac{s+1}{s^2+4}$
3. $F(s) = \frac{s+1}{s^2-4}$

**Solutions:**

1. $\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{3}{s^2+4}\right\} = \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = \frac{3}{2}\sin 2t.$
2. $\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{s+1}{s^2+4}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = \cos 2t + \frac{1}{2}\sin 2t.$
3. Using our knowledge of partial-fraction decomposition, we have that

$$\frac{s+1}{s^2-4} = \frac{s+1}{(s-2)(s+2)} = \frac{\frac{3}{4}}{s-2} + \frac{\frac{1}{4}}{s+2}$$

   Students are expected to verify that this is true.

   Thus, $\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{s+1}{s^2-4}\right\} = \mathcal{L}^{-1}\left\{\frac{\frac{3}{4}}{s-2}\right\} + \mathcal{L}^{-1}\left\{\frac{\frac{1}{4}}{s+2}\right\} = \frac{3}{4}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = \frac{3}{4}e^{2t} + \frac{1}{4}e^{-2t}$

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## 5 Laplace Transform of Derivative

We are almost ready to illustrate how we can use Laplace transforms to solve ODEs. Before we delve into it, we need the following property of Laplace transform of derivatives of a functions.

**Proposistion 2.** *Let $f(t)$ be a function with first derivative. Then*

*1.*
$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$

*2.*
$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$$

*Proof.* 1.

$$\mathcal{L}\{f'(t)\} = \int_0^\infty f'(t)e^{-st}\ \mathrm{d}t.$$

Using integration by parts, let $u = e^{-st}$ and $\mathrm{d}v = f'(t)\mathrm{d}t$. Then $\mathrm{d}u = -se^{-st}\mathrm{d}t$ and $v = f(t)$. Thus,

$$\begin{aligned}\mathcal{L}\{f'(t)\} &= \int_0^\infty f'(t)e^{-st}\ \mathrm{d}t. \\ &= uv - \int_0^\infty v\mathrm{d}u \\ &= e^{-st}f(t)\Big|_0^\infty - \int_0^\infty f(t)\{-se^{-st}\}\mathrm{d}t \\ &= s\int_0^\infty f(t)e^{-st}\mathrm{d}t - f(0) \\ &= s\mathcal{L}\{f(t)\} - f(0)\end{aligned}$$

2. To prove this, notice that we can use what we have in 1 to get

$$\mathcal{L}\{f''(t)\} = s\mathcal{L}\{f'(t)\} - f'(0)$$

But we know from 1 that $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$ . Therefore,

$$\begin{aligned}\mathcal{L}\{f''(t)\} &= s\mathcal{L}\{f'(t)\} - f'(0) \\ &= s\left[s\mathcal{L}\{f(t)\} - f(0)\right] - f'(0) \\ &= s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0) \\ &= s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0).\end{aligned}$$

We also leave to the students to prove, using induction, that

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - s^2f^{(n-3)} - sf^{(n-2)} - f^{(n-1)}.$$

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# 6 Solution of Ordinary Differential Equations with Initial Conditions using Laplace transforms

To solve a linear differential equation using Laplace transforms, we will need the following three steps.

1. Take the Laplace transforms of both sides of an equation.
2. Simplify algebraically the result to solve for $\mathcal{L}\{y\} = Y(s)$ in terms of $s$.
3. Find the inverse transform of $Y(s)$. This inverse transform, $y(t)$, is the solution of the given differential equation.

We now give an example illustrating how to use Laplace transform to solve initial value problems.

***Example***. Solve the following initial value problems using Laplace transforms.

$$y'' - 6y' + 5y = 0, \quad y(0) = 1, y'(0) = -3.$$

**Solution:** We apply Laplace transform to both sides of the equation.

$$\begin{aligned}
&\mathcal{L}\{y'' - 6y' + 5y\} = \mathcal{L}\{0\} \\
&\mathcal{L}\{y''\} - 6\mathcal{L}\{y'\} + 5\mathcal{L}\{y\} = 0 \\
&\left(s^2\mathcal{L}\{y\} - sy(0) - y'(0)\right) - 6\left(s\mathcal{L}\{y\} - y(0)\right) + 5\mathcal{L}\{y\} = 0 \\
&\left(s^2\mathcal{L}\{y\} - s + 3\right) - 6\left(s\mathcal{L}\{y\} - 1\right) + 5\mathcal{L}\{y\} = 0 \\
&s^2\mathcal{L}\{y\} - s + 3 - 6s\mathcal{L}\{y\} + 6 + 5\mathcal{L}\{y\} = 0 \\
&\left(s^2 - 6s + 5\right)\mathcal{L}\{y\} = s - 9 \\
&\mathcal{L}\{y\} = \frac{s-9}{s^2 - 6s + 5}.
\end{aligned}$$

Therefore, our solution

$$y(t) = \mathcal{L}^{-1}\left(\frac{s-9}{s^2 - 6s + 5}\right)$$

Now,

$$\frac{s-9}{s^2 - 6s + 5} = \frac{2}{s-1} - \frac{1}{s-5}$$

This has been left for students to verify using the method of partial-fraction decomposition.

Therefore,

$$y(t) = \mathcal{L}^{-1}\left(\frac{s-9}{s^2 - 6s + 5}\right) = \mathcal{L}^{-1}\left(\frac{2}{s-1} - \frac{1}{s-5}\right) = \mathcal{L}^{-1}\left(\frac{2}{s-1}\right) - \mathcal{L}^{-1}\left(\frac{1}{s-5}\right) = 2e^t - e^5 t$$

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