Section 6-3 : Series Solutions

Before we get into finding series solutions to differential equations we need to determine when we can find series solutions to differential equations. So, let's start with the differential equation,

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = 0 \tag{1}$$

This time we really do mean nonconstant coefficients. To this point we've only dealt with constant coefficients. However, with series solutions we can now have nonconstant coefficient differential equations. Also, in order to make the problems a little nicer we will be dealing only with polynomial coefficients.

Now, we say that $x=x_0$ is an **ordinary point** if provided both

$$\frac{q(x)}{p(x)} \qquad \text{and} \qquad \frac{r(x)}{p(x)}$$

are analytic at $x=x_0$. That is to say that these two quantities have Taylor series around $x=x_0$. We are going to be only dealing with coefficients that are polynomials so this will be equivalent to saying that

$$p(x_0) \neq 0$$

for most of the problems.

If a point is not an ordinary point we call it a **singular point**.

The basic idea to finding a series solution to a differential equation is to assume that we can write the solution as a power series in the form,

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \tag{2}$$

and then try to determine what the a_n's need to be. We will only be able to do this if the point $x=x_0$, is an ordinary point. We will usually say that (2) is a series solution around $x=x_0$.

Let's start with a very basic example of this. In fact, it will be so basic that we will have constant coefficients. This will allow us to check that we get the correct solution.

Example 1 Determine a series solution for the following differential equation about $x_0 = 0$.

$$y'' + y = 0$$

Solution

Notice that in this case $p(x)=1$ and so every point is an ordinary point. We will be looking for a solution in the form,

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

We will need to plug this into our differential equation so we'll need to find a couple of derivatives.

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad\qquad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Recall from the power series review section on power series that we can start these at *n=0* if we need to, however it's almost always best to start them where we have here. If it turns out that it would have been easier to start them at *n=0* we can easily fix that up when the time comes around.

So, plug these into our differential equation. Doing this gives,

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

The next step is to combine everything into a single series. To do this requires that we get both series starting at the same point and that the exponent on the *x* be the same in both series.

We will always start this by getting the exponent on the *x* to be the same. It is usually best to get the exponent to be an *n*. The second series already has the proper exponent and the first series will need to be shifted down by 2 in order to get the exponent up to an *n*. If you don't recall how to do this take a quick look at the first review section where we did several of these types of problems.

Shifting the first power series gives us,

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^{n} + \sum_{n=0}^{\infty} a_n x^n = 0$$

Notice that in the process of the shift we also got both series starting at the same place. This won't always happen, but when it does we'll take it. We can now add up the two series. This gives,

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} + a_n\right] x^n = 0$$

Now recalling the fact from the power series review section we know that if we have a power series that is zero for all *x* (as this is) then all the coefficients must have been zero to start with. This gives us the following,

$$(n+2)(n+1)a_{n+2} + a_n = 0, \qquad n = 0,1,2,\ldots$$

This is called the **recurrence relation** and notice that we included the values of *n* for which it must be true. We will always want to include the values of *n* for which the recurrence relation is true since they won't always start at *n* = 0 as it did in this case.

Now let's recall what we were after in the first place. We wanted to find a series solution to the differential equation. In order to do this, we needed to determine the values of the a_n's. We are almost to the point where we can do that. The recurrence relation has two different a_n's in it so we can't just solve this for a_n and get a formula that will work for all *n*. We can however, use this to determine what all but two of the a_n's are.

To do this we first solve the recurrence relation for the a_n that has the largest subscript. Doing this gives,

$$a_{n+2} = -\frac{a_n}{(n+2)(n+1)} \qquad n = 0,1,2,\ldots$$

Now, at this point we just need to start plugging in some value of *n* and see what happens,

n=0 $a_2 = \frac{-a_0}{(2)(1)}$ *n=1* $a_3 = \frac{-a_1}{(3)(2)}$

n=2
$$\begin{aligned} a_4 &= -\frac{a_2}{(4)(3)} \\ &= \frac{a_0}{(4)(3)(2)(1)} \end{aligned}$$

n=3
$$\begin{aligned} a_5 &= -\frac{a_3}{(5)(4)} \\ &= \frac{a_1}{(5)(4)(3)(2)} \end{aligned}$$

n=4
$$\begin{aligned} a_6 &= -\frac{a_4}{(6)(5)} \\ &= \frac{-a_0}{(6)(5)(4)(3)(2)(1)} \end{aligned}$$

n=5
$$\begin{aligned} a_7 &= -\frac{a_5}{(7)(6)} \\ &= \frac{-a_1}{(7)(6)(5)(4)(3)(2)} \end{aligned}$$

$$\vdots \qquad\qquad \vdots$$

$$a_{2k} = \frac{(-1)^k a_0}{(2k)!}, \; k = 1,2,\ldots \qquad a_{2k+1} = \frac{(-1)^k a_1}{(2k+1)!}, \; k = 1,2,\ldots$$

Notice that at each step we always plugged back in the previous answer so that when the subscript was even we could always write the a_n in terms of a_0 and when the coefficient was odd we could always write the a_n in terms of a_1. Also notice that, in this case, we were able to find a general formula for a_n's with even coefficients and a_n's with odd coefficients. This won't always be possible to do.

There's one more thing to notice here. The formulas that we developed were only for *k=1,2,…* however, in this case again, they will also work for *k=0*. Again, this is something that won't always work, but does here.

Do not get excited about the fact that we don't know what a_0 and a_1 are. As you will see, we actually need these to be in the problem to get the correct solution.

Now that we've got formulas for the a_n's let's get a solution. The first thing that we'll do is write out the solution with a couple of the a_n's plugged in.

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_{2k} x^{2k} + a_{2k+1} x^{2k+1} + \cdots \\ &= a_0 + a_1 x - \frac{a_0}{2!} x^2 - \frac{a_1}{3!} x^3 + \cdots + \frac{(-1)^k a_0}{(2k)!} x^{2k} + \frac{(-1)^k a_1}{(2k+1)!} x^{2k+1} + \cdots \end{aligned}$$

The next step is to collect all the terms with the same coefficient in them and then factor out that coefficient.

$$y(x) = a_0\left\{1 - \frac{x^2}{2!}\cdots + \frac{(-1)^k x^{2k}}{(2k)!} + \cdots\right\} + a_1\left\{x - \frac{x^3}{3!} + \cdots + \frac{(-1)^k}{(2k+1)!}x^{2k+1} + \cdots\right\}$$
$$= a_0\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k)!} + a_1\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

In the last step we also used the fact that we knew what the general formula was to write both portions as a power series. This is also our solution. We are done.

Before working another problem let's take a look at the solution to the previous example. First, we started out by saying that we wanted a series solution of the form,

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

and we didn't get that. We got a solution that contained two different power series. Also, each of the solutions had an unknown constant in them. This is not a problem. In fact, it's what we want to have happen. From our work with second order constant coefficient differential equations we know that the solution to the differential equation in the last example is,

$$y(x) = c_1\cos(x) + c_2\sin(x)$$

Solutions to second order differential equations consist of two separate functions each with an unknown constant in front of them that are found by applying any initial conditions. So, the form of our solution in the last example is exactly what we want to get. Also recall that the following Taylor series,

$$\cos(x) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} \qquad \sin(x) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Recalling these we very quickly see that what we got from the series solution method was exactly the solution we got from first principles, with the exception that the functions were the Taylor series for the actual functions instead of the actual functions themselves.

Now let's work an example with nonconstant coefficients since that is where series solutions are most useful.

Example 2 Find a series solution around $x_0 = 0$ for the following differential equation.

$$y'' - xy = 0$$

Solution

As with the first example *p(x)=1* and so again for this differential equation every point is an ordinary point. Now we'll start this one out just as we did the first example. Let's write down the form of the solution and get its derivatives.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \qquad y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

Plugging into the differential equation gives,

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=0}^{\infty} a_n x^n = 0$$

Unlike the first example we first need to get all the coefficients moved into the series.

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

Now we will need to shift the first series down by 2 and the second series up by 1 to get both of the series in terms of x^n.

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

Next, we need to get the two series starting at the same value of *n*. The only way to do that for this problem is to strip out the *n=0* term.

$$(2)(1)a_2 x^0 + \sum_{n=1}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2} - a_{n-1}\right] x^n = 0$$

We now need to set all the coefficients equal to zero. We will need to be careful with this however. The *n=0* coefficient is in front of the series and the *n=1,2,3…* are all in the series. So, setting coefficient equal to zero gives,

$$n = 0: \qquad 2a_2 = 0$$
$$n = 1,2,3,\ldots \qquad (n+2)(n+1)a_{n+2} - a_{n-1} = 0$$

Solving the first as well as the recurrence relation gives,

$$n = 0: \qquad a_2 = 0$$
$$n = 1,2,3,\ldots \qquad a_{n+2} = \frac{a_{n-1}}{(n+2)(n+1)}$$

Now we need to start plugging in values of *n*.

$$a_3 = \frac{a_0}{(3)(2)} \qquad a_4 = \frac{a_1}{(4)(3)} \qquad a_5 = \frac{a_2}{(5)(4)} = 0$$

$$a_6 = \frac{a_3}{(6)(5)} = \frac{a_0}{(6)(5)(3)(2)} \qquad a_7 = \frac{a_4}{(7)(6)} = \frac{a_1}{(7)(6)(4)(3)} \qquad a_8 = \frac{a_5}{(8)(7)} = 0$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$a_{3k} = \frac{a_0}{(2)(3)(5)(6)\cdots(3k-1)(3k)} \qquad a_{3k+1} = \frac{a_1}{(3)(4)(6)(7)\cdots(3k)(3k+1)} \qquad a_{3k+2} = 0$$
$$k = 1,2,3,\cdots \qquad\qquad k = 1,2,3,\cdots \qquad\qquad k = 0,1,2,\cdots$$

There are a couple of things to note about these coefficients. First, every third coefficient is zero. Next, the formulas here are somewhat unpleasant and not all that easy to see the first time around. Finally, these formulas will not work for *k=0* unlike the first example.

Now, get the solution,

$$\begin{aligned} y(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots + a_{3k} x^{3k} + a_{3k+1} x^{3k+1} + \cdots \\ &= a_0 + a_1 x + \frac{a_0}{6} x^3 + \frac{a_1}{12} x^4 \cdots + \frac{a_0 x^{3k}}{(2)(3)(5)(6)\cdots(3k-1)(3k)} + \\ &\qquad \frac{a_1 x^{3k+1}}{(3)(4)(6)(7)\cdots(3k)(3k+1)} + \cdots \end{aligned}$$

Again, collect up the terms that contain the same coefficient, factor the coefficient out and write the results as a new series,

$$y(x) = a_0 \left\{ 1 + \sum_{k=1}^{\infty} \frac{x^{3k}}{(2)(3)(5)(6)\cdots(3k-1)(3k)} \right\} + a_1 \left\{ x + \sum_{k=1}^{\infty} \frac{x^{3k+1}}{(3)(4)(6)(7)\cdots(3k)(3k+1)} \right\}$$

We couldn't start our series at *k=0* this time since the general term doesn't hold for *k=0*.

Now, we need to work an example in which we use a point other that *x=0*. In fact, let's just take the previous example and rework it for a different value of x_0. We're also going to need to change up the instructions a little for this example.

Example 3 Find the first four terms in each portion of the series solution around $x_0 = -2$ for the following differential equation.

$$y'' - xy = 0$$

Solution

Unfortunately for us there is nothing from the first example that can be reused here. Changing to $x_0 = -2$ completely changes the problem. In this case our solution will be,

$$y(x) = \sum_{n=0}^{\infty} a_n (x+2)^n$$

The derivatives of the solution are,

$$y'(x) = \sum_{n=1}^{\infty} n a_n (x+2)^{n-1} \qquad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x+2)^{n-2}$$

Plug these into the differential equation.

$$\sum_{n=2}^{\infty} n(n-1)a_n(x+2)^{n-2} - x\sum_{n=0}^{\infty} a_n(x+2)^n = 0$$

We now run into our first real difference between this example and the previous example. In this case we can't just multiply the *x* into the second series since in order to combine with the series it must be *x+2*. Therefore, we will first need to modify the coefficient of the second series before multiplying it into the series.

$$\sum_{n=2}^{\infty} n(n-1)a_n(x+2)^{n-2} - (x+2-2)\sum_{n=0}^{\infty} a_n(x+2)^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n(x+2)^{n-2} - (x+2)\sum_{n=0}^{\infty} a_n(x+2)^n + 2\sum_{n=0}^{\infty} a_n(x+2)^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n(x+2)^{n-2} - \sum_{n=0}^{\infty} a_n(x+2)^{n+1} + \sum_{n=0}^{\infty} 2a_n(x+2)^n = 0$$

We now have three series to work with. This will often occur in these kinds of problems. Now we will need to shift the first series down by 2 and the second series up by 1 the get common exponents in all the series.

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}(x+2)^n - \sum_{n=1}^{\infty} a_{n-1}(x+2)^n + \sum_{n=0}^{\infty} 2a_n(x+2)^n = 0$$

In order to combine the series we will need to strip out the *n=0* terms from both the first and third series.

$$2a_2 + 2a_0 + \sum_{n=1}^{\infty} (n+2)(n+1)a_{n+2}(x+2)^n - \sum_{n=1}^{\infty} a_{n-1}(x+2)^n + \sum_{n=1}^{\infty} 2a_n(x+2)^n = 0$$

$$2a_2 + 2a_0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2} - a_{n-1} + 2a_n\right](x+2)^n = 0$$

Setting coefficients equal to zero gives,

$$\begin{array}{ll} n=0 & 2a_2 + 2a_0 = 0 \\ n=1,2,3,\ldots & (n+2)(n+1)a_{n+2} - a_{n-1} + 2a_n = 0 \end{array}$$

We now need to solve both of these. In the first case there are two options, we can solve for a_2 or we can solve for a_0. Out of habit I'll solve for a_0. In the recurrence relation we'll solve for the term with the largest subscript as in previous examples.

$$\begin{array}{ll} n=0 & a_2 = -a_0 \\ n=1,2,3,\ldots & a_{n+2} = \dfrac{a_{n-1} - 2a_n}{(n+2)(n+1)} \end{array}$$

Notice that in this example we won't be having every third term drop out as we did in the previous example.

At this point we'll also acknowledge that the instructions for this problem are different as well. We aren't going to get a general formula for the a_n's this time so we'll have to be satisfied with just getting the first couple of terms for each portion of the solution. This is often the case for series solutions. Getting general formulas for the a_n's is the exception rather than the rule in these kinds of problems.

To get the first four terms we'll just start plugging in terms until we've got the required number of terms. Note that we will already be starting with an a_0 and an a_1 from the first two terms of the solution so all we will need are three more terms with an a_0 in them and three more terms with an a_1 in them.

$$n = 0 \qquad a_2 = -a_0$$

We've got two a_0's and one a_1.

$$n = 1 \qquad a_3 = \frac{a_0 - 2a_1}{(3)(2)} = \frac{a_0}{6} - \frac{a_1}{3}$$

We've got three a_0's and two a_1's.

$$n = 2 \qquad a_4 = \frac{a_1 - 2a_2}{(4)(3)} = \frac{a_1 - 2(-a_0)}{(4)(3)} = \frac{a_0}{6} + \frac{a_1}{12}$$

We've got four a_0's and three a_1's. We've got all the a_0's that we need, but we still need one more a_1. So, we'll need to do one more term it looks like.

$$n = 3 \qquad a_5 = \frac{a_2 - 2a_3}{(5)(4)} = -\frac{a_0}{20} - \frac{1}{10}\left(\frac{a_0}{6} - \frac{a_1}{3}\right) = -\frac{a_0}{15} + \frac{a_1}{30}$$

We've got five a_0's and four a_1's. We've got all the terms that we need.

Now, all that we need to do is plug into our solution.

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_n (x+2)^n \\ &= a_0 + a_1(x+2) + a_2(x+2)^2 + a_3(x+2)^3 + a_4(x+2)^4 + a_5(x+2)^5 + \cdots \\ &= a_0 + a_1(x+2) - a_0(x+2)^2 + \left(\frac{a_0}{6} - \frac{a_1}{3}\right)(x+2)^3 + \\ &\qquad \left(\frac{a_0}{6} + \frac{a_1}{12}\right)(x+2)^4 + \left(-\frac{a_0}{15} + \frac{a_1}{30}\right)(x+2)^5 + \cdots \end{aligned}$$

Finally collect all the terms up with the same coefficient and factor out the coefficient to get,

$$\begin{aligned} y(x) = a_0 &\left\{1 - (x+2)^2 + \frac{1}{6}(x+2)^3 + \frac{1}{6}(x+2)^4 - \frac{1}{15}(x+2)^5 + \cdots\right\} + \\ a_1 &\left\{(x+2) - \frac{1}{3}(x+2)^3 + \frac{1}{12}(x+2)^4 + \frac{1}{30}(x+2)^5 + \cdots\right\} \end{aligned}$$

That's the solution for this problem as far as we're concerned. Notice that this solution looks nothing like the solution to the previous example. It's the same differential equation but changing x_0 completely changed the solution.

Let's work one final problem.

Example 4 Find the first four terms in each portion of the series solution around $x_0 = 0$ for the following differential equation.

$$\left(x^2+1\right)y'' - 4xy' + 6y = 0$$

Solution

We finally have a differential equation that doesn't have a constant coefficient for the second derivative.

$$p\left(x\right) = x^2 + 1 \qquad\qquad p\left(0\right) = 1 \neq 0$$

So $x_0 = 0$ is an ordinary point for this differential equation. We first need the solution and its derivatives,

$$y\left(x\right) = \sum_{n=0}^{\infty} a_n x^n \qquad y'\left(x\right) = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad y''\left(x\right) = \sum_{n=2}^{\infty} n\left(n-1\right) a_n x^{n-2}$$

Plug these into the differential equation.

$$\left(x^2+1\right)\sum_{n=2}^{\infty} n\left(n-1\right) a_n x^{n-2} - 4x\sum_{n=1}^{\infty} n a_n x^{n-1} + 6\sum_{n=0}^{\infty} a_n x^n = 0$$

Now, break up the first term into two so we can multiply the coefficient into the series and multiply the coefficients of the second and third series in as well.

$$\sum_{n=2}^{\infty} n\left(n-1\right) a_n x^n + \sum_{n=2}^{\infty} n\left(n-1\right) a_n x^{n-2} - \sum_{n=1}^{\infty} 4n a_n x^n + \sum_{n=0}^{\infty} 6 a_n x^n = 0$$

We will only need to shift the second series down by two to get all the exponents the same in all the series.

$$\sum_{n=2}^{\infty} n\left(n-1\right) a_n x^n + \sum_{n=0}^{\infty} \left(n+2\right)\left(n+1\right) a_{n+2} x^n - \sum_{n=1}^{\infty} 4n a_n x^n + \sum_{n=0}^{\infty} 6 a_n x^n = 0$$

At this point we could strip out some terms to get all the series starting at *n=2*, but that's actually more work than is needed. Let's instead note that we could start the third series at *n=0* if we wanted to because that term is just zero. Likewise, the terms in the first series are zero for both *n=1* and *n=0* and so we could start that series at *n=0*. If we do this all the series will now start at *n=0* and we can add them up without stripping terms out of any series.

$$\sum_{n=0}^{\infty} \left[n\left(n-1\right) a_n + \left(n+2\right)\left(n+1\right) a_{n+2} - 4n a_n + 6 a_n\right] x^n = 0$$

$$\sum_{n=0}^{\infty} \left[\left(n^2 - 5n + 6\right) a_n + \left(n+2\right)\left(n+1\right) a_{n+2}\right] x^n = 0$$

$$\sum_{n=0}^{\infty} \left[\left(n-2\right)\left(n-3\right) a_n + \left(n+2\right)\left(n+1\right) a_{n+2}\right] x^n = 0$$

Now set coefficients equal to zero.

$$\left(n-2\right)\left(n-3\right)a_n+\left(n+2\right)\left(n+1\right)a_{n+2}=0, \qquad n=0,1,2,\ldots$$

Solving this gives,

$$a_{n+2}=-\frac{\left(n-2\right)\left(n-3\right)a_n}{\left(n+2\right)\left(n+1\right)}, \qquad n=0,1,2,\ldots$$

Now, we plug in values of *n*.

$$n=0: \qquad a_2=-3a_0$$

$$n=1: \qquad a_3=-\frac{1}{3}a_1$$

$$n=2: \qquad a_4=-\frac{0}{12}a_2=0$$

$$n=3: \qquad a_5=-\frac{0}{20}a_3=0$$

Now, from this point on all the coefficients are zero. In this case both of the series in the solution will terminate. This won't always happen, and often only one of them will terminate.

The solution in this case is,

$$y\left(x\right)=a_0\left\{1-3x^2\right\}+a_1\left\{x-\frac{1}{3}x^3\right\}$$

Section 6-4 : Euler Equations

In this section we want to look for solutions to

$$ax^2y'' + bxy' + cy = 0 \qquad (1)$$

around $x_0 = 0$. These types of differential equations are called **Euler Equations**.

Recall from the previous section that a point is an ordinary point if the quotients,

$$\frac{bx}{ax^2} = \frac{b}{ax} \qquad \text{and} \qquad \frac{c}{ax^2}$$

have Taylor series around $x_0 = 0$. However, because of the *x* in the denominator neither of these will have a Taylor series around $x_0 = 0$ and so $x_0 = 0$ is a singular point. So, the method from the previous section won't work since it required an ordinary point.

However, it is possible to get solutions to this differential equation that aren't series solutions. Let's start off by assuming that *x>0* (the reason for this will be apparent after we work the first example) and that all solutions are of the form,

$$y(x) = x^r \qquad (2)$$

Now plug this into the differential equation to get,

$$\begin{aligned} ax^2(r)(r-1)x^{r-2} + bx(r)x^{r-1} + cx^r &= 0 \\ ar(r-1)x^r + b(r)x^r + cx^r &= 0 \\ \left(ar(r-1) + b(r) + c\right)x^r &= 0 \end{aligned}$$

Now, we assumed that *x>0* and so this will only be zero if,

$$ar(r-1) + b(r) + c = 0 \qquad (3)$$

So solutions will be of the form (2) provided *r* is a solution to (3). This equation is a quadratic in *r* and so we will have three cases to look at : Real, Distinct Roots, Double Roots, and Complex Roots.

Real, Distinct Roots

There really isn't a whole lot to do in this case. We'll get two solutions that will form a fundamental set of solutions (we'll leave it to you to check this) and so our general solution will be,

$$y(x) = c_1x^{r_1} + c_2x^{r_2}$$

Example 1 Solve the following IVP

$$2x^2 y'' + 3xy' - 15y = 0, \qquad y(1) = 0 \quad y'(1) = 1$$

Solution

We first need to find the roots to (3).

$$2r(r-1) + 3r - 15 = 0$$

$$2r^2 + r - 15 = (2r-5)(r+3) = 0 \qquad \Rightarrow \qquad r_1 = \frac{5}{2},\ r_2 = -3$$

The general solution is then,

$$y(x) = c_1 x^{\frac{5}{2}} + c_2 x^{-3}$$

To find the constants we differentiate and plug in the initial conditions as we did back in the second order differential equations chapter.

$$y'(x) = \frac{5}{2} c_1 x^{\frac{3}{2}} - 3c_2 x^{-4}$$

$$\left.\begin{aligned} 0 &= y(1) = c_1 + c_2 \\ 1 &= y'(1) = \frac{5}{2}c_1 - 3c_2 \end{aligned}\right\} \qquad \Rightarrow \qquad c_1 = \frac{2}{11},\ c_2 = -\frac{2}{11}$$

The actual solution is then,

$$y(x) = \frac{2}{11} x^{\frac{5}{2}} - \frac{2}{11} x^{-3}$$

With the solution to this example we can now see why we required *x>0*. The second term would have division by zero if we allowed *x=0* and the first term would give us square roots of negative numbers if we allowed *x<0*.

Double Roots

This case will lead to the same problem that we've had every other time we've run into double roots (or double eigenvalues). We only get a single solution and will need a second solution. In this case it can be shown that the second solution will be,

$$y_2(x) = x^r \ln x$$

and so the general solution in this case is,

$$y(x) = c_1 x^r + c_2 x^r \ln x = x^r \left(c_1 + c_2 \ln x\right)$$

We can again see a reason for requiring *x>0*. If we didn't we'd have all sorts of problems with that logarithm.

Example 2 Find the general solution to the following differential equation.

$$x^2 y'' - 7xy' + 16y = 0$$

Solution

First the roots of (3).

$$\begin{aligned} r(r-1) - 7r + 16 &= 0 \\ r^2 - 8r + 16 &= 0 \\ (r-4)^2 &= 0 \qquad \Rightarrow \qquad r = 4 \end{aligned}$$

So, the general solution is then,

$$y(x) = c_1 x^4 + c_2 x^4 \ln x$$

Complex Roots

In this case we'll be assuming that our roots are of the form,

$$r_{1,2} = \lambda \pm \mu i$$

If we take the first root we'll get the following solution.

$$x^{\lambda + \mu i}$$

This is a problem since we don't want complex solutions, we only want real solutions. We can eliminate this by recalling that,

$$x^r = \mathbf{e}^{\ln x^r} = \mathbf{e}^{r \ln x}$$

Plugging the root into this gives,

$$\begin{aligned} x^{\lambda + \mu i} &= \mathbf{e}^{(\lambda + \mu i)\ln x} \\ &= \mathbf{e}^{\lambda \ln x} \mathbf{e}^{\mu i \ln x} \\ &= \mathbf{e}^{\ln x^\lambda} \left(\cos(\mu \ln x) + i \sin(\mu \ln x) \right) \\ &= x^\lambda \cos(\mu \ln x) + i x^\lambda \sin(\mu \ln x) \end{aligned}$$

Note that we had to use Euler formula as well to get to the final step. Now, as we've done every other time we've seen solutions like this we can take the real part and the imaginary part and use those for our two solutions.

So, in the case of complex roots the general solution will be,

$$y(x) = c_1 x^\lambda \cos(\mu \ln x) + c_2 x^\lambda \sin(\mu \ln x) = x^\lambda \left(c_1 \cos(\mu \ln x) + c_2 \sin(\mu \ln x) \right)$$

Once again, we can see why we needed to require $x > 0$.

Example 3 Find the solution to the following differential equation.

$$x^2 y'' + 3xy' + 4y = 0$$

Solution

Get the roots to (3) first as always.

$$r(r-1) + 3r + 4 = 0$$

$$r^2 + 2r + 4 = 0 \qquad \Rightarrow \qquad r_{1,2} = -1 \pm \sqrt{3}\, i$$

The general solution is then,

$$y(x) = c_1 x^{-1} \cos\left(\sqrt{3} \ln x\right) + c_2 x^{-1} \sin\left(\sqrt{3} \ln x\right)$$

We should now talk about how to deal with $x < 0$ since that is a possibility on occasion. To deal with this we need to use the variable transformation,

$$\eta = -x$$

In this case since $x < 0$ we will get $\eta > 0$. Now, define,

$$u(\eta) = y(x) = y(-\eta)$$

Then using the chain rule we can see that,

$$u'(\eta) = -y'(x) \qquad \text{and} \qquad u''(\eta) = y''(x)$$

With this transformation the differential equation becomes,

$$a(-\eta)^2 u'' + b(-\eta)(-u') + cu = 0$$

$$a\eta^2 u'' + b\eta u' + cu = 0$$

In other words, since $\eta > 0$ we can use the work above to get solutions to this differential equation. We'll also go back to *x*'s by using the variable transformation in reverse.

$$\eta = -x$$

Let's just take the real, distinct case first to see what happens.

$$u(\eta) = c_1 \eta^{r_1} + c_2 \eta^{r_2}$$

$$y(x) = c_1 (-x)^{r_1} + c_2 (-x)^{r_2}$$

Now, we could do this for the rest of the cases if we wanted to, but before doing that let's notice that if we recall the definition of absolute value,

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

we can combine both of our solutions to this case into one and write the solution as,

$$y(x) = c_1 |x|^{r_1} + c_2 |x|^{r_2}, \qquad x \neq 0$$

Note that we still need to avoid $x = 0$ since we could still get division by zero. However, this is now a solution for any interval that doesn't contain $x = 0$.

We can do likewise for the other two cases and the following solutions for any interval not containing $x = 0$.

$$y(x) = c_1 |x|^r + c_2 |x|^r \ln|x|$$
$$y(x) = c_1 |x|^\lambda \cos(\mu \ln|x|) + c_2 |x|^\lambda \sin(\mu \ln|x|)$$

We can make one more generalization before working one more example. A more general form of an Euler Equation is,

$$a(x - x_0)^2 y'' + b(x - x_0) y' + cy = 0$$

and we can ask for solutions in any interval not containing $x = x_0$. The work for generating the solutions in this case is identical to all the above work and so isn't shown here.

The solutions in this general case for any interval not containing $x = a$ are,

$$y(x) = c_1 |x - a|^{r_1} + c_2 |x - a|^{r_2}$$
$$y(x) = |x - a|^r (c_1 + c_2 \ln|x - a|)$$
$$y(x) = |x - a|^\lambda (c_1 \cos(\mu \ln|x - a|) + c_2 \sin(\mu \ln|x - a|))$$

Where the roots are solutions to

$$ar(r - 1) + b(r) + c = 0$$

Example 4 Find the solution to the following differential equation on any interval not containing $x = -6$.

$$3(x + 6)^2 y'' + 25(x + 6) y' - 16y = 0$$

Solution

So, we get the roots from the identical quadratic in this case.

$$3r(r - 1) + 25r - 16 = 0$$
$$3r^2 + 22r - 16 = 0$$
$$(3r - 2)(r + 8) = 0 \qquad \Rightarrow \qquad r_1 = \frac{2}{3},\ r_2 = -8$$

The general solution is then,

$$y(x) = c_1 |x + 6|^{\frac{2}{3}} + c_2 |x + 6|^{-8}$$

Section 5-3 : Review : Eigenvalues & Eigenvectors

If you get nothing out of this quick review of linear algebra you must get this section. Without this section you will not be able to do any of the differential equations work that is in this chapter.

So, let's start with the following. If we multiply an *n* x *n* matrix by an *n* x 1 vector we will get a new *n* x 1 vector back. In other words,

$$A\vec{\eta} = \vec{y}$$

What we want to know is if it is possible for the following to happen. Instead of just getting a brand new vector out of the multiplication is it possible instead to get the following,

$$A\vec{\eta} = \lambda\vec{\eta} \qquad \textbf{(1)}$$

In other words, is it possible, at least for certain λ and $\vec{\eta}$, to have matrix multiplication be the same as just multiplying the vector by a constant? Of course, we probably wouldn't be talking about this if the answer was no. So, it is possible for this to happen, however, it won't happen for just any value of λ or $\vec{\eta}$. If we do happen to have a λ and $\vec{\eta}$ for which this works (and they will always come in pairs) then we call λ an **eigenvalue** of *A* and $\vec{\eta}$ an **eigenvector** of *A*.

So, how do we go about finding the eigenvalues and eigenvectors for a matrix? Well first notice that if $\vec{\eta} = \vec{0}$ then (1) is going to be true for any value of λ and so we are going to make the assumption that $\vec{\eta} \neq \vec{0}$. With that out of the way let's rewrite (1) a little.

$$\begin{aligned} A\vec{\eta} - \lambda\vec{\eta} &= \vec{0} \\ A\vec{\eta} - \lambda I_n\vec{\eta} &= \vec{0} \\ \left(A - \lambda I_n\right)\vec{\eta} &= \vec{0} \end{aligned}$$

Notice that before we factored out the $\vec{\eta}$ we added in the appropriately sized identity matrix. This is equivalent to multiplying things by a one and so doesn't change the value of anything. We needed to do this because without it we would have had the difference of a matrix, *A*, and a constant, λ, and this can't be done. We now have the difference of two matrices of the same size which can be done.

So, with this rewrite we see that

$$\left(A - \lambda I_n\right)\vec{\eta} = \vec{0} \qquad \textbf{(2)}$$

is equivalent to (1). In order to find the eigenvectors for a matrix we will need to solve a homogeneous system. Recall the fact from the previous section that we know that we will either have exactly one solution ($\vec{\eta} = \vec{0}$) or we will have infinitely many nonzero solutions. Since we've already said that we don't want $\vec{\eta} = \vec{0}$ this means that we want the second case.

Knowing this will allow us to find the eigenvalues for a matrix. Recall from this fact that we will get the second case only if the matrix in the system is singular. Therefore, we will need to determine the values of λ for which we get,

$$\det\left(A-\lambda I\right)=0$$

Once we have the eigenvalues we can then go back and determine the eigenvectors for each eigenvalue. Let's take a look at a couple of quick facts about eigenvalues and eigenvectors.

Fact

If *A* is an *n* x *n* matrix then $\det\left(A-\lambda I\right)=0$ is an n^{th} degree polynomial. This polynomial is called the **characteristic polynomial**.

To find eigenvalues of a matrix all we need to do is solve a polynomial. That's generally not too bad provided we keep *n* small. Likewise this fact also tells us that for an *n* x *n* matrix, *A*, we will have *n* eigenvalues if we include all repeated eigenvalues.

Fact

If $\lambda_1,\lambda_2,\ldots,\lambda_n$ is the complete list of eigenvalues for *A* (including all repeated eigenvalues) then,

1. If λ occurs only once in the list then we call λ **simple**.
2. If λ occurs *k>1* times in the list then we say that λ has **multiplicity *k***.
3. If $\lambda_1,\lambda_2,\ldots,\lambda_k$ ($k\le n$) are the simple eigenvalues in the list with corresponding eigenvectors $\vec{\eta}^{(1)}$, $\vec{\eta}^{(2)}$, …, $\vec{\eta}^{(k)}$ then the eigenvectors are all linearly independent.
4. If λ is an eigenvalue of multiplicity *k > 1* then λ will have anywhere from 1 to *k* linearly independent eigenvectors.

The usefulness of these facts will become apparent when we get back into differential equations since in that work we will want linearly independent solutions.

Let's work a couple of examples now to see how we actually go about finding eigenvalues and eigenvectors.

Example 1 Find the eigenvalues and eigenvectors of the following matrix.

$$A=\begin{pmatrix}2 & 7\\ -1 & -6\end{pmatrix}$$

Solution

The first thing that we need to do is find the eigenvalues. That means we need the following matrix,

$$A-\lambda I=\begin{pmatrix}2 & 7\\ -1 & -6\end{pmatrix}-\lambda\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}2-\lambda & 7\\ -1 & -6-\lambda\end{pmatrix}$$

In particular we need to determine where the determinant of this matrix is zero.

$$\det\left(A-\lambda I\right)=\left(2-\lambda\right)\left(-6-\lambda\right)+7=\lambda^2+4\lambda-5=\left(\lambda+5\right)\left(\lambda-1\right)$$

So, it looks like we will have two simple eigenvalues for this matrix, $\lambda_1=-5$ and $\lambda_2=1$. We will now need to find the eigenvectors for each of these. Also note that according to the fact above, the two eigenvectors should be linearly independent.

To find the eigenvectors we simply plug in each eigenvalue into (2) and solve. So, let's do that.

$\lambda_1 = -5$:

In this case we need to solve the following system.

$$\begin{pmatrix} 7 & 7 \\ -1 & -1 \end{pmatrix} \vec{\eta} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Recall that officially to solve this system we use the following augmented matrix.

$$\begin{pmatrix} 7 & 7 & 0 \\ -1 & -1 & 0 \end{pmatrix} \begin{matrix} \frac{1}{7}R_1 + R_2 \\ \Rightarrow \end{matrix} \begin{pmatrix} 7 & 7 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Upon reducing down we see that we get a single equation

$$7\eta_1 + 7\eta_2 = 0 \qquad \Rightarrow \qquad \eta_1 = -\eta_2$$

that will yield an infinite number of solutions. This is expected behavior. Recall that we picked the eigenvalues so that the matrix would be singular and so we would get infinitely many solutions.

Notice as well that we could have identified this from the original system. This won't always be the case, but in the 2 x 2 case we can see from the system that one row will be a multiple of the other and so we will get infinite solutions. From this point on we won't be actually solving systems in these cases. We will just go straight to the equation and we can use either of the two rows for this equation.

Now, let's get back to the eigenvector, since that is what we were after. In general then the eigenvector will be any vector that satisfies the following,

$$\vec{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -\eta_2 \\ \eta_2 \end{pmatrix} \qquad ,\eta_2 \neq 0$$

To get this we used the solution to the equation that we found above.

We really don't want a general eigenvector however so we will pick a value for η_2 to get a specific eigenvector. We can choose anything (except $\eta_2 = 0$), so pick something that will make the eigenvector "nice". Note as well that since we've already assumed that the eigenvector is not zero we must choose a value that will not give us zero, which is why we want to avoid $\eta_2 = 0$ in this case. Here's the eigenvector for this eigenvalue.

$$\vec{\eta}^{(1)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \qquad \text{using } \eta_2 = 1$$

Now we get to do this all over again for the second eigenvalue.

$\lambda_2 = 1$:

We'll do much less work with this part than we did with the previous part. We will need to solve the following system.

$$\begin{pmatrix} 1 & 7 \\ -1 & -7 \end{pmatrix}\vec{\eta} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Clearly both rows are multiples of each other and so we will get infinitely many solutions. We can choose to work with either row. We'll run with the first because to avoid having too many minus signs floating around. Doing this gives us,

$$\eta_1 + 7\eta_2 = 0 \qquad\qquad \eta_1 = -7\eta_2$$

Note that we can solve this for either of the two variables. However, with an eye towards working with these later on let's try to avoid as many fractions as possible. The eigenvector is then,

$$\vec{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -7\eta_2 \\ \eta_2 \end{pmatrix} \qquad ,\eta_2 \neq 0$$

$$\vec{\eta}^{(2)} = \begin{pmatrix} -7 \\ 1 \end{pmatrix}, \qquad\qquad \text{using } \eta_2 = 1$$

Summarizing we have,

$$\lambda_1 = -5 \qquad\qquad \vec{\eta}^{(1)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 1 \qquad\qquad \vec{\eta}^{(2)} = \begin{pmatrix} -7 \\ 1 \end{pmatrix}$$

Note that the two eigenvectors are linearly independent as predicted.

Example 2 Find the eigenvalues and eigenvectors of the following matrix.

$$A = \begin{pmatrix} 1 & -1 \\ \frac{4}{9} & -\frac{1}{3} \end{pmatrix}$$

Solution

This matrix has fractions in it. That's life so don't get excited about it. First, we need the eigenvalues.

$$\begin{aligned} \det\left(A - \lambda I\right) &= \begin{vmatrix} 1-\lambda & -1 \\ \frac{4}{9} & -\frac{1}{3}-\lambda \end{vmatrix} \\ &= \left(1-\lambda\right)\left(-\frac{1}{3}-\lambda\right) + \frac{4}{9} \\ &= \lambda^2 - \frac{2}{3}\lambda + \frac{1}{9} \\ &= \left(\lambda - \frac{1}{3}\right)^2 \qquad \Rightarrow \qquad \lambda_{1,2} = \frac{1}{3} \end{aligned}$$

So, it looks like we've got an eigenvalue of multiplicity 2 here. Remember that the power on the term will be the multiplicity.

Now, let's find the eigenvector(s). This one is going to be a little different from the first example. There is only one eigenvalue so let's do the work for that one. We will need to solve the following system,

$$\begin{pmatrix} \frac{2}{3} & -1 \\ \frac{4}{9} & -\frac{2}{3} \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \Rightarrow \qquad R_1 = \frac{3}{2} R_2$$

So, the rows are multiples of each other. We'll work with the first equation in this example to find the eigenvector.

$$\frac{2}{3}\eta_1 - \eta_2 = 0 \qquad\qquad \eta_2 = \frac{2}{3}\eta_1$$

Recall in the last example we decided that we wanted to make these as "nice" as possible and so should avoid fractions if we can. Sometimes, as in this case, we simply can't so we'll have to deal with it. In this case the eigenvector will be,

$$\vec{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \eta_1 \\ \frac{2}{3}\eta_1 \end{pmatrix}, \qquad \eta_1 \neq 0$$

$$\vec{\eta}^{(1)} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \qquad \eta_1 = 3$$

Note that by careful choice of the variable in this case we were able to get rid of the fraction that we had. This is something that in general doesn't much matter if we do or not. However, when we get back to differential equations it will be easier on us if we don't have any fractions so we will usually try to eliminate them at this step.

Also, in this case we are only going to get a single (linearly independent) eigenvector. We can get other eigenvectors, by choosing different values of η_1. However, each of these will be linearly dependent with the first eigenvector. If you're not convinced of this try it. Pick some values for η_1 and get a different vector and check to see if the two are linearly dependent.

Recall from the fact above that an eigenvalue of multiplicity *k* will have anywhere from 1 to *k* linearly independent eigenvectors. In this case we got one. For most of the 2 x 2 matrices that we'll be working with this will be the case, although it doesn't have to be. We can, on occasion, get two.

Example 3 Find the eigenvalues and eigenvectors of the following matrix.

$$A = \begin{pmatrix} -4 & -17 \\ 2 & 2 \end{pmatrix}$$

Solution

So, we'll start with the eigenvalues.

$$\begin{aligned} \det\left(A - \lambda I\right) &= \begin{vmatrix} -4-\lambda & -17 \\ 2 & 2-\lambda \end{vmatrix} \\ &= \left(-4-\lambda\right)\left(2-\lambda\right) + 34 \\ &= \lambda^2 + 2\lambda + 26 \end{aligned}$$

This doesn't factor, so upon using the quadratic formula we arrive at,

$$\lambda_{1,2} = -1 \pm 5i$$

In this case we get complex eigenvalues which are definitely a fact of life with eigenvalue/eigenvector problems so get used to them.

Finding eigenvectors for complex eigenvalues is identical to the previous two examples, but it will be somewhat messier. So, let's do that.

$\lambda_1 = -1 + 5i$:

The system that we need to solve this time is

$$\begin{pmatrix} -4-(-1+5i) & -17 \\ 2 & 2-(-1+5i) \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -3-5i & -17 \\ 2 & 3-5i \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Now, it's not super clear that the rows are multiples of each other, but they are. In this case we have,

$$R_1 = -\frac{1}{2}(3+5i)R_2$$

This is not something that you need to worry about, we just wanted to make the point. For the work that we'll be doing later on with differential equations we will just assume that we've done everything correctly and we've got two rows that are multiples of each other. Therefore, all that we need to do here is pick one of the rows and work with it.

We'll work with the second row this time.

$$2\eta_1 + (3-5i)\eta_2 = 0$$

Now we can solve for either of the two variables. However, again looking forward to differential equations, we are going to need the "*i*" in the numerator so solve the equation in such a way as this will happen. Doing this gives,

$$2\eta_1 = -(3-5i)\eta_2$$

$$\eta_1 = -\frac{1}{2}(3-5i)\eta_2$$

So, the eigenvector in this case is

$$\vec{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}(3-5i)\eta_2 \\ \eta_2 \end{pmatrix}, \qquad \eta_2 \neq 0$$

$$\vec{\eta}^{(1)} = \begin{pmatrix} -3+5i \\ 2 \end{pmatrix}, \qquad \eta_2 = 2$$

As with the previous example we choose the value of the variable to clear out the fraction.

Now, the work for the second eigenvector is almost identical and so we'll not dwell on that too much.

$\lambda_2 = -1-5i$:

The system that we need to solve here is

$$\begin{pmatrix} -4-(-1-5i) & -17 \\ 2 & 2-(-1-5i) \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -3+5i & -17 \\ 2 & 3+5i \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Working with the second row again gives,

$$2\eta_1 + (3+5i)\eta_2 = 0 \qquad \Rightarrow \qquad \eta_1 = -\frac{1}{2}(3+5i)\eta_2$$

The eigenvector in this case is

$$\vec{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}(3+5i)\eta_2 \\ \eta_2 \end{pmatrix}, \qquad \eta_2 \neq 0$$

$$\vec{\eta}^{(2)} = \begin{pmatrix} -3-5i \\ 2 \end{pmatrix}, \qquad \eta_2 = 2$$

Summarizing,

$$\lambda_1 = -1+5i \qquad \vec{\eta}^{(1)} = \begin{pmatrix} -3+5i \\ 2 \end{pmatrix}$$

$$\lambda_2 = -1-5i \qquad \vec{\eta}^{(2)} = \begin{pmatrix} -3-5i \\ 2 \end{pmatrix}$$

There is a nice fact that we can use to simplify the work when we get complex eigenvalues. We need a bit of terminology first however.

If we start with a complex number,

$$z = a + bi$$

then the **complex conjugate** of *z* is

$$\bar{z} = a - bi$$

To compute the complex conjugate of a complex number we simply change the sign on the term that contains the "*i*". The complex conjugate of a vector is just the conjugate of each of the vector's components.

We now have the following fact about complex eigenvalues and eigenvectors.

Fact

If *A* is an *n* x *n* matrix with only real numbers and if $\lambda_1 = a + bi$ is an eigenvalue with eigenvector $\vec{\eta}^{(1)}$. Then $\lambda_2 = \overline{\lambda_1} = a - bi$ is also an eigenvalue and its eigenvector is the conjugate of $\vec{\eta}^{(1)}$.

This fact is something that you should feel free to use as you need to in our work.

Now, we need to work one final eigenvalue/eigenvector problem. To this point we've only worked with 2 x 2 matrices and we should work at least one that isn't 2 x 2. Also, we need to work one in which we get an eigenvalue of multiplicity greater than one that has more than one linearly independent eigenvector.

Example 4 Find the eigenvalues and eigenvectors of the following matrix.

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Solution

Despite the fact that this is a 3 x 3 matrix, it still works the same as the 2 x 2 matrices that we've been working with. So, start with the eigenvalues

$$\begin{aligned} \det\left(A - \lambda I\right) &= \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} \\ &= -\lambda^3 + 3\lambda + 2 \\ &= \left(\lambda - 2\right)\left(\lambda + 1\right)^2 \qquad \lambda_1 = 2,\ \lambda_{2,3} = -1 \end{aligned}$$

So, we've got a simple eigenvalue and an eigenvalue of multiplicity 2. Note that we used the same method of computing the determinant of a 3 x 3 matrix that we used in the previous section. We just didn't show the work.

Let's now get the eigenvectors. We'll start with the simple eigenvector.

$\lambda_1 = 2$:

Here we'll need to solve,

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This time, unlike the 2 x 2 cases we worked earlier, we actually need to solve the system. So let's do that.

$$\begin{pmatrix} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{pmatrix} \begin{matrix} R_1 \leftrightarrow R_2 \\ \Rightarrow \end{matrix} \begin{pmatrix} 1 & -2 & 1 & 0 \\ -2 & 1 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{pmatrix} \begin{matrix} R_2 + 2R_1 \\ R_3 - R_1 \\ \Rightarrow \end{matrix} \begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{pmatrix}$$

$$\begin{matrix} -\frac{1}{3}R_2 \\ \Rightarrow \end{matrix} \begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & -3 & 0 \end{pmatrix} \begin{matrix} R_3 - 3R_2 \\ R_1 + 2R_2 \\ \Rightarrow \end{matrix} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Going back to equations gives,

$$\eta_1 - \eta_3 = 0 \qquad \Rightarrow \qquad \eta_1 = \eta_3$$
$$\eta_2 - \eta_3 = 0 \qquad \Rightarrow \qquad \eta_2 = \eta_3$$

So, again we get infinitely many solutions as we should for eigenvectors. The eigenvector is then,

$$\vec{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} \eta_3 \\ \eta_3 \\ \eta_3 \end{pmatrix}, \qquad \eta_3 \neq 0$$

$$\vec{\eta}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \eta_3 = 1$$

Now, let's do the other eigenvalue.

$\lambda_2 = -1$:

Here we'll need to solve,

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Okay, in this case is clear that all three rows are the same and so there isn't any reason to actually solve the system since we can clear out the bottom two rows to all zeroes in one step. The equation that we get then is,

$$\eta_1 + \eta_2 + \eta_3 = 0 \Rightarrow \qquad \eta_1 = -\eta_2 - \eta_3$$

So, in this case we get to pick two of the values for free and will still get infinitely many solutions. Here is the general eigenvector for this case,

$$\vec{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} -\eta_2 - \eta_3 \\ \eta_2 \\ \eta_3 \end{pmatrix}, \qquad \eta_2 \neq 0 \text{ and } \eta_3 \neq 0 \text{ at the same time}$$

Notice the restriction this time. Recall that we only require that the eigenvector not be the zero vector. This means that we can allow one or the other of the two variables to be zero, we just can't allow both of them to be zero at the same time!

What this means for us is that we are going to get two linearly independent eigenvectors this time. Here they are.

$$\vec{\eta}^{(2)} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \qquad \eta_2 = 0 \text{ and } \eta_3 = 1$$

$$\vec{\eta}^{(3)} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \qquad \eta_2 = 1 \text{ and } \eta_3 = 0$$

Now when we talked about linear independent vectors in the last section we only looked at *n* vectors each with *n* components. We can still talk about linear independence in this case however. Recall back with we did linear independence for functions we saw at the time that if two functions were linearly dependent then they were multiples of each other. Well the same thing holds true for vectors. Two vectors will be linearly dependent if they are multiples of each other. In this case there is no way to get $\vec{\eta}^{(2)}$ by multiplying $\vec{\eta}^{(3)}$ by a constant. Therefore, these two vectors must be linearly independent.

So, summarizing up, here are the eigenvalues and eigenvectors for this matrix

$$\lambda_1 = 2 \qquad \vec{\eta}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = -1 \qquad \vec{\eta}^{(2)} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda_3 = -1 \qquad \vec{\eta}^{(3)} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

Section 5-4 : Systems of Differential Equations

In the introduction to this section we briefly discussed how a system of differential equations can arise from a population problem in which we keep track of the population of both the prey and the predator. It makes sense that the number of prey present will affect the number of the predator present. Likewise, the number of predator present will affect the number of prey present. Therefore the differential equation that governs the population of either the prey or the predator should in some way depend on the population of the other. This will lead to two differential equations that must be solved simultaneously in order to determine the population of the prey and the predator.

The whole point of this is to notice that systems of differential equations can arise quite easily from naturally occurring situations. Developing an effective predator-prey system of differential equations is not the subject of this chapter. However, systems can arise from n^{th} order linear differential equations as well. Before we get into this however, let's write down a system and get some terminology out of the way.

We are going to be looking at first order, linear systems of differential equations. These terms mean the same thing that they have meant up to this point. The largest derivative anywhere in the system will be a first derivative and all unknown functions and their derivatives will only occur to the first power and will not be multiplied by other unknown functions. Here is an example of a system of first order, linear differential equations.

$$\begin{aligned} x_1' &= x_1 + 2x_2 \\ x_2' &= 3x_1 + 2x_2 \end{aligned}$$

We call this kind of system a **coupled** system since knowledge of x_2 is required in order to find x_1 and likewise knowledge of x_1 is required to find x_2. We will worry about how to go about solving these later. At this point we are only interested in becoming familiar with some of the basics of systems.

Now, as mentioned earlier, we can write an n^{th} order linear differential equation as a system. Let's see how that can be done.

Example 1 Write the following 2^{nd} order differential equation as a system of first order, linear differential equations.

$$2y'' - 5y' + y = 0 \qquad y(3) = 6 \qquad y'(3) = -1$$

Solution

We can write higher order differential equations as a system with a very simple change of variable. We'll start by defining the following two new functions.

$$\begin{aligned} x_1(t) &= y(t) \\ x_2(t) &= y'(t) \end{aligned}$$

Now notice that if we differentiate both sides of these we get,

$$x_1' = y' = x_2$$
$$x_2' = y'' = -\frac{1}{2}y + \frac{5}{2}y' = -\frac{1}{2}x_1 + \frac{5}{2}x_2$$

Note the use of the differential equation in the second equation. We can also convert the initial conditions over to the new functions.

$$x_1(3) = y(3) = 6$$
$$x_2(3) = y'(3) = -1$$

Putting all of this together gives the following system of differential equations.

$$x_1' = x_2 \qquad x_1(3) = 6$$
$$x_2' = -\frac{1}{2}x_1 + \frac{5}{2}x_2 \qquad x_2(3) = -1$$

We will call the system in the above example an **Initial Value Problem** just as we did for differential equations with initial conditions.

Let's take a look at another example.

Example 2 Write the following 4th order differential equation as a system of first order, linear differential equations.

$$y^{(4)} + 3y'' - \sin(t)\,y' + 8y = t^2 \qquad y(0) = 1 \quad y'(0) = 2 \quad y''(0) = 3 \quad y'''(0) = 4$$

Solution

Just as we did in the last example we'll need to define some new functions. This time we'll need 4 new functions.

$$x_1 = y \qquad \Rightarrow \qquad x_1' = y' = x_2$$
$$x_2 = y' \qquad \Rightarrow \qquad x_2' = y'' = x_3$$
$$x_3 = y'' \qquad \Rightarrow \qquad x_3' = y''' = x_4$$
$$x_4 = y''' \qquad \Rightarrow \qquad x_4' = y^{(4)} = -8y + \sin(t)\,y' - 3y'' + t^2 = -8x_1 + \sin(t)\,x_2 - 3x_3 + t^2$$

The system along with the initial conditions is then,

$$x_1' = x_2 \qquad x_1(0) = 1$$
$$x_2' = x_3 \qquad x_2(0) = 2$$
$$x_3' = x_4 \qquad x_3(0) = 3$$
$$x_4' = -8x_1 + \sin(t)\,x_2 - 3x_3 + t^2 \qquad x_4(0) = 4$$

Now, when we finally get around to solving these we will see that we generally don't solve systems in the form that we've given them in this section. Systems of differential equations can be converted to **matrix form** and this is the form that we usually use in solving systems.

Example 3 Convert the following system to matrix from.

$$x_1' = 4x_1 + 7x_2$$
$$x_2' = -2x_1 - 5x_2$$

Solution

First write the system so that each side is a vector.

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 4x_1 + 7x_2 \\ -2x_1 - 5x_2 \end{pmatrix}$$

Now the right side can be written as a matrix multiplication,

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 4 & 7 \\ -2 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Now, if we define,

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

then,

$$\vec{x}' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}$$

The system can then be written in the matrix form,

$$\vec{x}' = \begin{pmatrix} 4 & 7 \\ -2 & -5 \end{pmatrix} \vec{x}$$

Example 4 Convert the systems from Examples 1 and 2 into matrix form.

Solution

We'll start with the system from Example 1.

$$x_1' = x_2 \qquad x_1(3) = 6$$
$$x_2' = -\frac{1}{2}x_1 + \frac{5}{2}x_2 \qquad x_2(3) = -1$$

First define,

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The system is then,

$$\vec{x}' = \begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & \frac{5}{2} \end{pmatrix} \vec{x} \qquad \vec{x}(3) = \begin{pmatrix} x_1(3) \\ x_2(3) \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \end{pmatrix}$$

Now, let's do the system from Example 2.

$$\begin{aligned} x_1' &= x_2 & x_1(0) &= 1 \\ x_2' &= x_3 & x_2(0) &= 2 \\ x_3' &= x_4 & x_3(0) &= 3 \\ x_4' &= -8x_1 + \sin(t)\,x_2 - 3x_3 + t^2 & x_4(0) &= 4 \end{aligned}$$

In this case we need to be careful with the t^2 in the last equation. We'll start by writing the system as a vector again and then break it up into two vectors, one vector that contains the unknown functions and the other that contains any known functions.

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ -8x_1 + \sin(t)\,x_2 - 3x_3 + t^2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ -8x_1 + \sin(t)\,x_2 - 3x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ t^2 \end{pmatrix}$$

Now, the first vector can now be written as a matrix multiplication and we'll leave the second vector alone.

$$\vec{x}' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -8 & \sin(t) & -3 & 0 \end{pmatrix}\vec{x} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ t^2 \end{pmatrix} \qquad \vec{x}(0) = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

where,

$$\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix}$$

Note that occasionally for "large" systems such as this we will go one step farther and write the system as,

$$\vec{x}' = A\vec{x} + \vec{g}(t)$$

The last thing that we need to do in this section is get a bit of terminology out of the way. Starting with

$$\vec{x}' = A\vec{x} + \vec{g}(t)$$

we say that the system is **homogeneous** if $\vec{g}(t) = \vec{0}$ and we say the system is **nonhomogeneous** if $\vec{g}(t) \neq \vec{0}$.

Section 5-5 : Solutions to Systems

Now that we've got some of the basics out of the way for systems of differential equations it's time to start thinking about how to solve a system of differential equations. We will start with the homogeneous system written in matrix form,

$$\vec{x}\,' = A\,\vec{x} \qquad \textbf{(1)}$$

where, *A* is an *n* x *n* matrix and $\vec{x}$ is a vector whose components are the unknown functions in the system.

Now, if we start with *n* = 1 then the system reduces to a fairly simple linear (or separable) first order differential equation.

$$x' = ax$$

and this has the following solution,

$$x\left(t\right) = c\,\mathbf{e}^{a\,t}$$

So, let's use this as a guide and for a general *n* let's see if

$$\vec{x}\left(t\right) = \vec{\eta}\,\mathbf{e}^{r\,t} \qquad \textbf{(2)}$$

will be a solution. Note that the only real difference here is that we let the constant in front of the exponential be a vector. All we need to do then is plug this into the differential equation and see what we get. First notice that the derivative is,

$$\vec{x}\,'\left(t\right) = r\vec{\eta}\,\mathbf{e}^{r\,t}$$

So, upon plugging the guess into the differential equation we get,

$$\begin{aligned} r\vec{\eta}\mathbf{e}^{r\,t} &= A\vec{\eta}\mathbf{e}^{r\,t} \\ \left(A\vec{\eta} - r\vec{\eta}\right)\mathbf{e}^{r\,t} &= \vec{0} \\ \left(A - rI\right)\vec{\eta}\mathbf{e}^{r\,t} &= \vec{0} \end{aligned}$$

Now, since we know that exponentials are not zero we can drop that portion and we then see that in order for (2) to be a solution to (1) then we must have

$$\left(A - rI\right)\vec{\eta} = \vec{0}$$

Or, in order for (2) to be a solution to (1), *r* and $\vec{\eta}$ must be an eigenvalue and eigenvector for the matrix *A*.

Therefore, in order to solve (1) we first find the eigenvalues and eigenvectors of the matrix *A* and then we can form solutions using (2). There are going to be three cases that we'll need to look at. The cases are real, distinct eigenvalues, complex eigenvalues and repeated eigenvalues.

None of this tells us how to completely solve a system of differential equations. We'll need the following couple of facts to do this.

Fact

1. If $\vec{x}_1(t)$ and $\vec{x}_2(t)$ are two solutions to a homogeneous system, (1), then
$$c_1\vec{x}_1(t)+c_2\vec{x}_2(t)$$
is also a solution to the system.
2. Suppose that *A* is an *n* x *n* matrix and suppose that $\vec{x}_1(t)$, $\vec{x}_2(t)$, …, $\vec{x}_n(t)$ are solutions to a homogeneous system, (1). Define,
$$X=\left(\vec{x}_1\ \vec{x}_2\ \cdots\ \vec{x}_n\right)$$
In other words, *X* is a matrix whose *ith* column is the i^{th} solution. Now define,
$$W=\det(X)$$
We call *W* the **Wronskian**. If $W\neq 0$ then the solutions form a **fundamental set of solutions** and the general solution to the system is,
$$\vec{x}(t)=c_1\vec{x}_1(t)+c_2\vec{x}_2(t)+\cdots+c_n\vec{x}_n(t)$$

Note that if we have a fundamental set of solutions then the solutions are also going to be linearly independent. Likewise, if we have a set of linearly independent solutions then they will also be a fundamental set of solutions since the Wronskian will not be zero.

Section 5-7 : Real Eigenvalues

It's now time to start solving systems of differential equations. We've seen that solutions to the system,

$$\vec{x}\,' = A\vec{x}$$

will be of the form

$$\vec{x} = \vec{\eta}\,\mathbf{e}^{\lambda t}$$

where λ and $\vec{\eta}$ are eigenvalues and eigenvectors of the matrix *A*. We will be working with 2 x 2 systems so this means that we are going to be looking for two solutions, $\vec{x}_1\left(t\right)$ and $\vec{x}_2\left(t\right)$, where the determinant of the matrix,

$$X = \left(\vec{x}_1 \;\; \vec{x}_2\right)$$

is nonzero.

We are going to start by looking at the case where our two eigenvalues, λ_1 and λ_2 are real and distinct. In other words, they will be real, simple eigenvalues. Recall as well that the eigenvectors for simple eigenvalues are linearly independent. This means that the solutions we get from these will also be linearly independent. If the solutions are linearly independent the matrix *X* must be nonsingular and hence these two solutions will be a fundamental set of solutions. The general solution in this case will then be,

$$\vec{x}\left(t\right) = c_1\mathbf{e}^{\lambda_1 t}\vec{\eta}^{(1)} + c_2\mathbf{e}^{\lambda_2 t}\vec{\eta}^{(2)}$$

Note that each of our examples will actually be broken into two examples. The first example will be solving the system and the second example will be sketching the phase portrait for the system. Phase portraits are not always taught in a differential equations course and so we'll strip those out of the solution process so that if you haven't covered them in your class you can ignore the phase portrait example for the system.

Example 1 Solve the following IVP.

$$\vec{x}\,' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\vec{x}, \qquad \vec{x}\left(0\right) = \begin{pmatrix} 0 \\ -4 \end{pmatrix}$$

Solution

So, the first thing that we need to do is find the eigenvalues for the matrix.

$$\begin{aligned} \det\left(A - \lambda I\right) &= \begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} \\ &= \lambda^2 - 3\lambda - 4 \\ &= \left(\lambda+1\right)\left(\lambda-4\right) \qquad \Rightarrow \qquad \lambda_1 = -1,\ \lambda_2 = 4 \end{aligned}$$

Now let's find the eigenvectors for each of these.

$\lambda_1 = -1$:

We'll need to solve,

$$\begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \Rightarrow \qquad 2\eta_1 + 2\eta_2 = 0 \qquad \Rightarrow \qquad \eta_1 = -\eta_2$$

The eigenvector in this case is,

$$\vec{\eta} = \begin{pmatrix} -\eta_2 \\ \eta_2 \end{pmatrix} \qquad \Rightarrow \qquad \vec{\eta}^{(1)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \qquad \eta_2 = 1$$

$\lambda_2 = 4$:

We'll need to solve,

$$\begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \Rightarrow \qquad -3\eta_1 + 2\eta_2 = 0 \qquad \Rightarrow \qquad \eta_1 = \frac{2}{3}\eta_2$$

The eigenvector in this case is,

$$\vec{\eta} = \begin{pmatrix} \frac{2}{3}\eta_2 \\ \eta_2 \end{pmatrix} \qquad \Rightarrow \qquad \vec{\eta}^{(2)} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \qquad \eta_2 = 3$$

Then general solution is then,

$$\vec{x}(t) = c_1 \mathbf{e}^{-t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \mathbf{e}^{4t}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Now, we need to find the constants. To do this we simply need to apply the initial conditions.

$$\begin{pmatrix} 0 \\ -4 \end{pmatrix} = \vec{x}(0) = c_1\begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

All we need to do now is multiply the constants through and we then get two equations (one for each row) that we can solve for the constants. This gives,

$$\left.\begin{aligned} -c_1 + 2c_2 &= 0 \\ c_1 + 3c_2 &= -4 \end{aligned}\right\} \qquad \Rightarrow \qquad c_1 = -\frac{8}{5}, \; c_2 = -\frac{4}{5}$$

The solution is then,

$$\vec{x}(t) = -\frac{8}{5}\mathbf{e}^{-t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} - \frac{4}{5}\mathbf{e}^{4t}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Now, let's take a look at the phase portrait for the system.

Example 2 Sketch the phase portrait for the following system.

$$\vec{x}' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\vec{x}$$

Solution

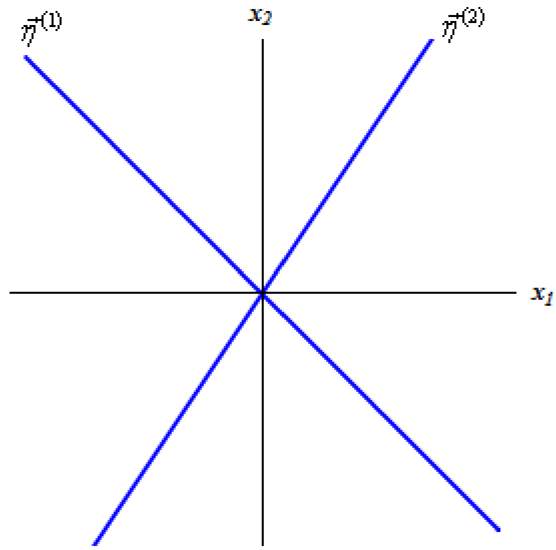
From the last example we know that the eigenvalues and eigenvectors for this system are,

$$\lambda_1 = -1 \qquad \vec{\eta}^{(1)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 4 \qquad \vec{\eta}^{(2)} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

It turns out that this is all the information that we will need to sketch the direction field. We will relate things back to our solution however so that we can see that things are going correctly.

We'll start by sketching lines that follow the direction of the two eigenvectors. This gives,



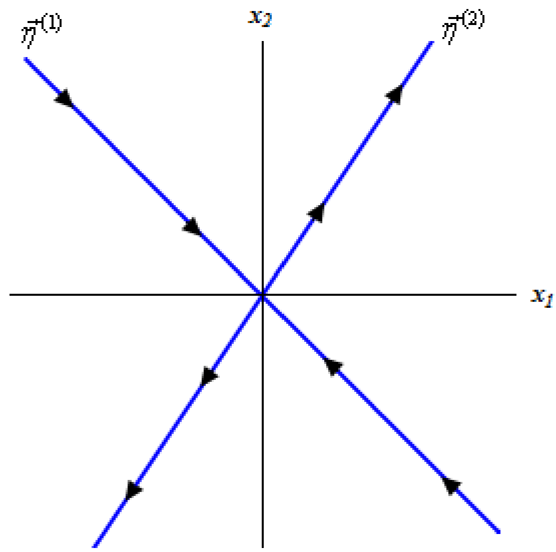
Now, from the first example our general solution is

$$\vec{x}(t) = c_1 \mathbf{e}^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \mathbf{e}^{4t} \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

If we have $c_2 = 0$ then the solution is an exponential times a vector and all that the exponential does is affect the magnitude of the vector and the constant c_1 will affect both the sign and the magnitude of the vector. In other words, the trajectory in this case will be a straight line that is parallel to the vector, $\vec{\eta}^{(1)}$. Also notice that as *t* increases the exponential will get smaller and smaller and hence the trajectory will be moving in towards the origin. If $c_1 > 0$ the trajectory will be in Quadrant II and if $c_1 < 0$ the trajectory will be in Quadrant IV.

So, the line in the graph above marked with $\vec{\eta}^{(1)}$ will be a sketch of the trajectory corresponding to $c_2 = 0$ and this trajectory will approach the origin as *t* increases.

If we now turn things around and look at the solution corresponding to having $c_1 = 0$ we will have a trajectory that is parallel to $\vec{\eta}^{(2)}$. Also, since the exponential will increase as *t* increases and so in this case the trajectory will now move away from the origin as *t* increases. We will denote this with arrows on the lines in the graph above.

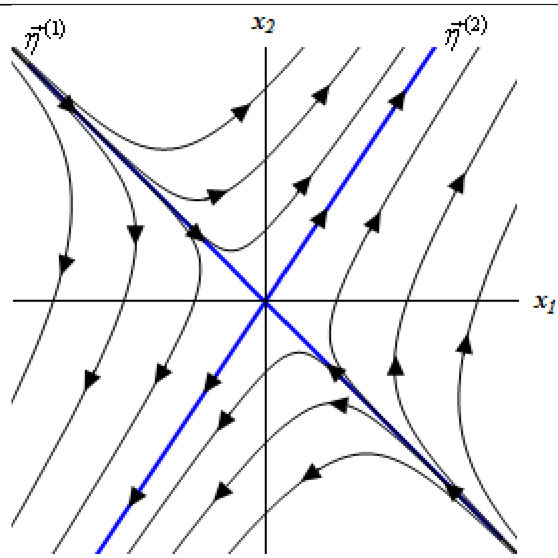


Notice that we could have gotten this information with actually going to the solution. All we really need to do is look at the eigenvalues. Eigenvalues that are negative will correspond to solutions that will move towards the origin as *t* increases in a direction that is parallel to its eigenvector. Likewise, eigenvalues that are positive move away from the origin as *t* increases in a direction that will be parallel to its eigenvector.

If both constants are in the solution we will have a combination of these behaviors. For large negative *t*'s the solution will be dominated by the portion that has the negative eigenvalue since in these cases the exponent will be large and positive. Trajectories for large negative *t*'s will be parallel to $\vec{\eta}^{(1)}$ and moving in the same direction.

Solutions for large positive *t*'s will be dominated by the portion with the positive eigenvalue. Trajectories in this case will be parallel to $\vec{\eta}^{(2)}$ and moving in the same direction.

In general, it looks like trajectories will start "near" $\vec{\eta}^{(1)}$, move in towards the origin and then as they get closer to the origin they will start moving towards $\vec{\eta}^{(2)}$ and then continue up along this vector. Sketching some of these in will give the following phase portrait. Here is a sketch of this with the trajectories corresponding to the eigenvectors marked in blue.



In this case the equilibrium solution $(0,0)$ is called a **saddle point** and is unstable. In this case unstable means that solutions move away from it as *t* increases.

So, we've solved a system in matrix form, but remember that we started out without the systems in matrix form. Now let's take a quick look at an example of a system that isn't in matrix form initially.

Example 3 Find the solution to the following system.

$$x_1' = x_1 + 2x_2 \qquad x_1(0) = 0$$
$$x_2' = 3x_1 + 2x_2 \qquad x_2(0) = -4$$

Solution

We first need to convert this into matrix form. This is easy enough. Here is the matrix form of the system.

$$\vec{x}\,' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\vec{x}, \qquad \vec{x}(0) = \begin{pmatrix} 0 \\ -4 \end{pmatrix}$$

This is just the system from the first example and so we've already got the solution to this system. Here it is.

$$\vec{x}(t) = -\frac{8}{5}\mathbf{e}^{-t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} - \frac{4}{5}\mathbf{e}^{4t}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Now, since we want the solution to the system not in matrix form let's go one step farther here. Let's multiply the constants and exponentials into the vectors and then add up the two vectors.

$$\vec{x}(t) = \begin{pmatrix} \frac{8}{5}\mathbf{e}^{-t} \\ -\frac{8}{5}\mathbf{e}^{-t} \end{pmatrix} - \begin{pmatrix} \frac{8}{5}\mathbf{e}^{4t} \\ \frac{12}{5}\mathbf{e}^{4t} \end{pmatrix} = \begin{pmatrix} \frac{8}{5}\mathbf{e}^{-t} - \frac{8}{5}\mathbf{e}^{4t} \\ -\frac{8}{5}\mathbf{e}^{-t} - \frac{12}{5}\mathbf{e}^{4t} \end{pmatrix}$$

Now, recall,

$$\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

So, the solution to the system is then,

$$x_1(t) = \frac{8}{5}\mathbf{e}^{-t} - \frac{8}{5}\mathbf{e}^{4t}$$
$$x_2(t) = -\frac{8}{5}\mathbf{e}^{-t} - \frac{12}{5}\mathbf{e}^{4t}$$

Let's work another example.

Example 4 Solve the following IVP.

$$\vec{x}' = \begin{pmatrix} -5 & 1 \\ 4 & -2 \end{pmatrix}\vec{x}, \qquad \vec{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Solution
So, the first thing that we need to do is find the eigenvalues for the matrix.

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} -5-\lambda & 1 \\ 4 & -2-\lambda \end{vmatrix} \\ &= \lambda^2 + 7\lambda + 6 \\ &= (\lambda+1)(\lambda+6) \qquad \Rightarrow \qquad \lambda_1 = -1,\ \lambda_2 = -6 \end{aligned}$$

Now let's find the eigenvectors for each of these.

$\lambda_1 = -1$:
We'll need to solve,

$$\begin{pmatrix} -4 & 1 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad -4\eta_1 + \eta_2 = 0 \quad \Rightarrow \quad \eta_2 = 4\eta_1$$

The eigenvector in this case is,

$$\vec{\eta} = \begin{pmatrix} \eta_1 \\ 4\eta_1 \end{pmatrix} \quad \Rightarrow \quad \vec{\eta}^{(1)} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \quad \eta_1 = 1$$

$\lambda_2 = -6$:
We'll need to solve,

$$\begin{pmatrix} 1 & 1 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \eta_1 + \eta_2 = 0 \quad \Rightarrow \quad \eta_1 = -\eta_2$$

The eigenvector in this case is,

$$\vec{\eta} = \begin{pmatrix} -\eta_2 \\ \eta_2 \end{pmatrix} \quad \Rightarrow \quad \vec{\eta}^{(2)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad \eta_2 = 1$$

Then general solution is then,

$$\vec{x}(t) = c_1 \mathbf{e}^{-t} \begin{pmatrix} 1 \\ 4 \end{pmatrix} + c_2 \mathbf{e}^{-6t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Now, we need to find the constants. To do this we simply need to apply the initial conditions.

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \vec{x}(0) = c_1 \begin{pmatrix} 1 \\ 4 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Now solve the system for the constants.

$$\left.\begin{aligned} c_1 - c_2 &= 1 \\ 4c_1 + c_2 &= 2 \end{aligned}\right\} \qquad \Rightarrow \qquad c_1 = \frac{3}{5}, \; c_2 = -\frac{2}{5}$$

The solution is then,

$$\vec{x}(t) = \frac{3}{5} \mathbf{e}^{-t} \begin{pmatrix} 1 \\ 4 \end{pmatrix} - \frac{2}{5} \mathbf{e}^{-6t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Now let’s find the phase portrait for this system.

Example 5 Sketch the phase portrait for the following system.

$$\vec{x}\,' = \begin{pmatrix} -5 & 1 \\ 4 & -2 \end{pmatrix} \vec{x}$$

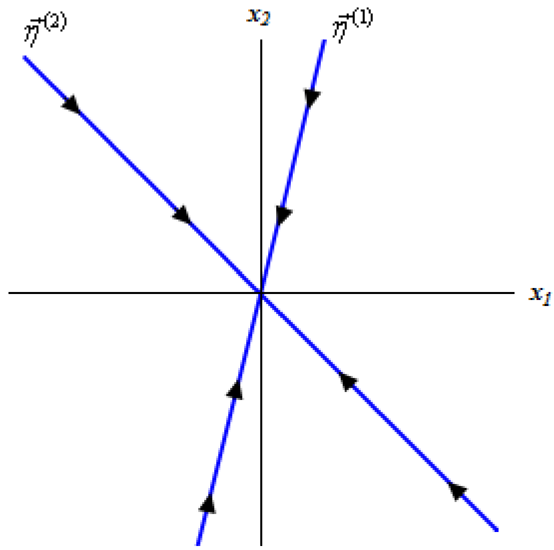
Solution

From the last example we know that the eigenvalues and eigenvectors for this system are,

$$\lambda_1 = -1 \qquad \vec{\eta}^{(1)} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

$$\lambda_2 = -6 \qquad \vec{\eta}^{(2)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

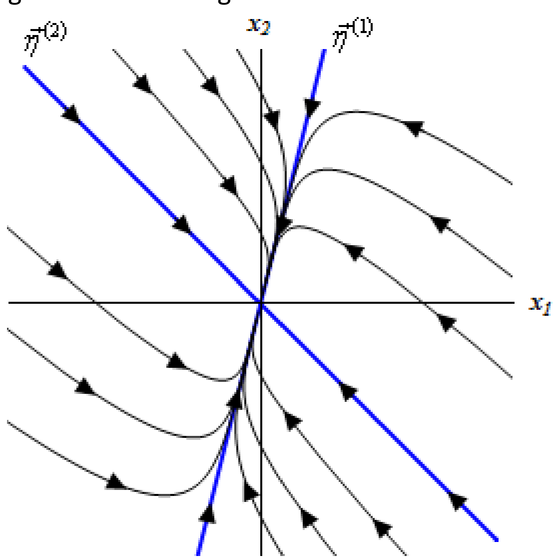
This one is a little different from the first one. However, it starts in the same way. We’ll first sketch the trajectories corresponding to the eigenvectors. Notice as well that both of the eigenvalues are negative and so trajectories for these will move in towards the origin as *t* increases. When we sketch the trajectories we’ll add in arrows to denote the direction they take as *t* increases. Here is the sketch of these trajectories.



Now, here is where the slight difference from the first phase portrait comes up. All of the trajectories will move in towards the origin as *t* increases since both of the eigenvalues are negative. The issue that we need to decide upon is just how they do this. This is actually easier than it might appear to be at first.

The second eigenvalue is larger than the first. For large and positive *t*'s this means that the solution for this eigenvalue will be smaller than the solution for the first eigenvalue. Therefore, as *t* increases the trajectory will move in towards the origin and do so parallel to $\vec{\eta}^{(1)}$. Likewise, since the second eigenvalue is larger than the first this solution will dominate for large and negative *t*'s. Therefore, as we decrease *t* the trajectory will move away from the origin and do so parallel to $\vec{\eta}^{(2)}$.

Adding in some trajectories gives the following sketch.



In these cases we call the equilibrium solution $(0,0)$ a **node** and it is asymptotically stable. Equilibrium solutions are asymptotically stable if all the trajectories move in towards it as *t* increases.

Note that nodes can also be unstable. In the last example if both of the eigenvalues had been positive all the trajectories would have moved away from the origin and in this case the equilibrium solution would have been unstable.

Before moving on to the next section we need to do one more example. When we first started talking about systems it was mentioned that we can convert a higher order differential equation into a system. We need to do an example like this so we can see how to solve higher order differential equations using systems.

Example 6 Convert the following differential equation into a system, solve the system and use this solution to get the solution to the original differential equation.

$$2y'' + 5y' - 3y = 0, \qquad y(0) = -4 \quad y'(0) = 9$$

Solution

So, we first need to convert this into a system. Here's the change of variables,

$$x_1 = y \qquad x_1' = y' = x_2$$

$$x_2 = y' \qquad x_2' = y'' = \frac{3}{2}y - \frac{5}{2}y' = \frac{3}{2}x_1 - \frac{5}{2}x_2$$

The system is then,

$$\vec{x}' = \begin{pmatrix} 0 & 1 \\ \frac{3}{2} & -\frac{5}{2} \end{pmatrix}\vec{x} \qquad \vec{x}(0) = \begin{pmatrix} -4 \\ 9 \end{pmatrix}$$

where,

$$\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}$$

Now we need to find the eigenvalues for the matrix.

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} -\lambda & 1 \\ \frac{3}{2} & -\frac{5}{2} - \lambda \end{vmatrix} \\ &= \lambda^2 + \tfrac{5}{2}\lambda - \tfrac{3}{2} \\ &= \tfrac{1}{2}(\lambda + 3)(2\lambda - 1) \qquad \lambda_1 = -3, \ \lambda_2 = \frac{1}{2} \end{aligned}$$

Now let's find the eigenvectors.

$\lambda_1 = -3$:

We'll need to solve,

$$\begin{pmatrix} 3 & 1 \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad 3\eta_1 + \eta_2 = 0 \quad \Rightarrow \quad \eta_2 = -3\eta_1$$

The eigenvector in this case is,

$$\vec{\eta} = \begin{pmatrix} \eta_1 \\ -3\eta_1 \end{pmatrix} \qquad \Rightarrow \qquad \vec{\eta}^{(1)} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}, \quad \eta_1 = 1$$

$\lambda_2 = \frac{1}{2}$:

We'll need to solve,

$$\begin{pmatrix} -\frac{1}{2} & 1 \\ \frac{3}{2} & -3 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \qquad -\frac{1}{2}\eta_1 + \eta_2 = 0 \qquad \Rightarrow \qquad \eta_2 = \frac{1}{2}\eta_1$$

The eigenvector in this case is,

$$\vec{\eta} = \begin{pmatrix} \eta_1 \\ \frac{1}{2}\eta_1 \end{pmatrix} \qquad \Rightarrow \qquad \vec{\eta}^{(2)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \eta_1 = 2$$

The general solution is then,

$$\vec{x}(t) = c_1 \mathbf{e}^{-3t}\begin{pmatrix} 1 \\ -3 \end{pmatrix} + c_2 \mathbf{e}^{\frac{t}{2}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Apply the initial condition.

$$\begin{pmatrix} -4 \\ 9 \end{pmatrix} = \vec{x}(0) = c_1\begin{pmatrix} 1 \\ -3 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

This gives the system of equations that we can solve for the constants.

$$\left.\begin{aligned} c_1 + 2c_2 &= -4 \\ -3c_1 + c_2 &= 9 \end{aligned}\right\} \qquad \Rightarrow \qquad c_1 = -\frac{22}{7}, \; c_2 = -\frac{3}{7}$$

The actual solution to the system is then,

$$\vec{x}(t) = -\frac{22}{7}\mathbf{e}^{-3t}\begin{pmatrix} 1 \\ -3 \end{pmatrix} - \frac{3}{7}\mathbf{e}^{\frac{t}{2}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Now recalling that,

$$\vec{x}(t) = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}$$

we can see that the solution to the original differential equation is just the top row of the solution to the matrix system. The solution to the original differential equation is then,

$$y(t) = -\frac{22}{7}\mathbf{e}^{-3t} - \frac{6}{7}\mathbf{e}^{\frac{t}{2}}$$

Notice that as a check, in this case, the bottom row should be the derivative of the top row.

Section 5-8 : Complex Eigenvalues

In this section we will look at solutions to

$$\vec{x}' = A\vec{x}$$

where the eigenvalues of the matrix *A* are complex. With complex eigenvalues we are going to have the same problem that we had back when we were looking at second order differential equations. We want our solutions to only have real numbers in them, however since our solutions to systems are of the form,

$$\vec{x} = \vec{\eta}\mathbf{e}^{\lambda t}$$

we are going to have complex numbers come into our solution from both the eigenvalue and the eigenvector. Getting rid of the complex numbers here will be similar to how we did it back in the second order differential equation case but will involve a little more work this time around. It's easiest to see how to do this in an example.

Example 1 Solve the following IVP.

$$\vec{x}' = \begin{pmatrix} 3 & 9 \\ -4 & -3 \end{pmatrix}\vec{x} \qquad\qquad \vec{x}(0) = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$$

Solution

We first need the eigenvalues and eigenvectors for the matrix.

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 9 \\ -4 & -3-\lambda \end{vmatrix}$$

$$= \lambda^2 + 27 \qquad\qquad \lambda_{1,2} = \pm 3\sqrt{3}\,i$$

So, now that we have the eigenvalues recall that we only need to get the eigenvector for one of the eigenvalues since we can get the second eigenvector for free from the first eigenvector.

$\lambda_1 = 3\sqrt{3}\,i$:

We need to solve the following system.

$$\begin{pmatrix} 3-3\sqrt{3}\,i & 9 \\ -4 & -3-3\sqrt{3}\,i \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Using the first equation we get,

$$\left(3 - 3\sqrt{3}\,i\right)\eta_1 + 9\eta_2 = 0$$

$$\eta_2 = -\frac{1}{3}\left(1 - \sqrt{3}\,i\right)\eta_1$$

So, the first eigenvector is,

$$\vec{\eta} = \begin{pmatrix} \eta_1 \\ -\frac{1}{3}\left(1-\sqrt{3}\,i\right)\eta_1 \end{pmatrix}$$

$$\vec{\eta}^{(1)} = \begin{pmatrix} 3 \\ -1+\sqrt{3}\,i \end{pmatrix} \qquad \eta_1 = 3$$

When finding the eigenvectors in these cases make sure that the complex number appears in the numerator of any fractions since we'll need it in the numerator later on. Also try to clear out any fractions by appropriately picking the constant. This will make our life easier down the road.

Now, the second eigenvector is,

$$\vec{\eta}^{(2)} = \begin{pmatrix} 3 \\ -1-\sqrt{3}\,i \end{pmatrix}$$

However, as we will see we won't need this eigenvector.

The solution that we get from the first eigenvalue and eigenvector is,

$$\vec{x}_1(t) = \mathbf{e}^{3\sqrt{3}\,i\,t}\begin{pmatrix} 3 \\ -1+\sqrt{3}\,i \end{pmatrix}$$

So, as we can see there are complex numbers in both the exponential and vector that we will need to get rid of in order to use this as a solution. Recall from the complex roots section of the second order differential equation chapter that we can use Euler's formula to get the complex number out of the exponential. Doing this gives us,

$$\vec{x}_1(t) = \left(\cos\left(3\sqrt{3}t\right) + i\sin\left(3\sqrt{3}t\right)\right)\begin{pmatrix} 3 \\ -1+\sqrt{3}\,i \end{pmatrix}$$

The next step is to multiply the cosines and sines into the vector.

$$\vec{x}_1(t) = \begin{pmatrix} 3\cos\left(3\sqrt{3}t\right) + 3i\sin\left(3\sqrt{3}t\right) \\ -\cos\left(3\sqrt{3}t\right) - i\sin\left(3\sqrt{3}t\right) + \sqrt{3}\,i\cos\left(3\sqrt{3}t\right) - \sqrt{3}\sin\left(3\sqrt{3}t\right) \end{pmatrix}$$

Now combine the terms with an "*i*" in them and split these terms off from those terms that don't contain an "*i*". Also factor the "*i*" out of this vector.

$$\begin{aligned}\vec{x}_1(t) &= \begin{pmatrix} 3\cos\left(3\sqrt{3}t\right) \\ -\cos\left(3\sqrt{3}t\right) - \sqrt{3}\sin\left(3\sqrt{3}t\right) \end{pmatrix} + i\begin{pmatrix} 3\sin\left(3\sqrt{3}t\right) \\ -\sin\left(3\sqrt{3}t\right) + \sqrt{3}\cos\left(3\sqrt{3}t\right) \end{pmatrix} \\ &= \vec{u}(t) + i\,\vec{v}(t)\end{aligned}$$

Now, it can be shown (we'll leave the details to you) that $\vec{u}(t)$ and $\vec{v}(t)$ are two linearly independent solutions to the system of differential equations. This means that we can use them to form a general solution and they are both real solutions.

So, the general solution to a system with complex roots is

$$\vec{x}(t) = c_1\vec{u}(t) + c_2\vec{v}(t)$$

where $\vec{u}(t)$ and $\vec{v}(t)$ are found by writing the first solution as

$$\vec{x}(t) = \vec{u}(t) + i\,\vec{v}(t)$$

For our system then, the general solution is,

$$\vec{x}(t) = c_1\begin{pmatrix} 3\cos\left(3\sqrt{3}t\right) \\ -\cos\left(3\sqrt{3}t\right) - \sqrt{3}\sin\left(3\sqrt{3}t\right) \end{pmatrix} + c_2\begin{pmatrix} 3\sin\left(3\sqrt{3}t\right) \\ -\sin\left(3\sqrt{3}t\right) + \sqrt{3}\cos\left(3\sqrt{3}t\right) \end{pmatrix}$$

We now need to apply the initial condition to this to find the constants.

$$\begin{pmatrix} 2 \\ -4 \end{pmatrix} = \vec{x}(0) = c_1\begin{pmatrix} 3 \\ -1 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ \sqrt{3} \end{pmatrix}$$

This leads to the following system of equations to be solved,

$$\left.\begin{aligned} 3c_1 &= 2 \\ -c_1 + \sqrt{3}c_2 &= -4 \end{aligned}\right\} \qquad \Rightarrow \qquad c_1 = \frac{2}{3},\; c_2 = \frac{-10}{3\sqrt{3}}$$

The actual solution is then,

$$\vec{x}(t) = \frac{2}{3}\begin{pmatrix} 3\cos\left(3\sqrt{3}t\right) \\ -\cos\left(3\sqrt{3}t\right) - \sqrt{3}\sin\left(3\sqrt{3}t\right) \end{pmatrix} - \frac{10}{3\sqrt{3}}\begin{pmatrix} 3\sin\left(3\sqrt{3}t\right) \\ -\sin\left(3\sqrt{3}t\right) + \sqrt{3}\cos\left(3\sqrt{3}t\right) \end{pmatrix}$$

As we did in the last section we'll do the phase portraits separately from the solution of the system in case phase portraits haven't been taught in your class.

Example 2 Sketch the phase portrait for the system.

$$\vec{x}\,' = \begin{pmatrix} 3 & 9 \\ -4 & -3 \end{pmatrix}\vec{x}$$

Solution

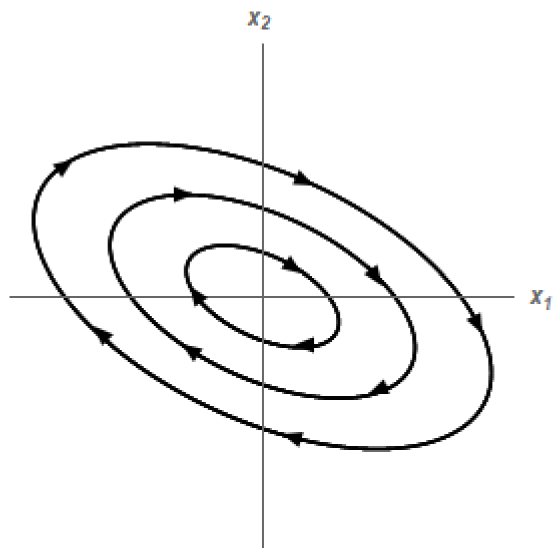
When the eigenvalues of a matrix *A* are purely complex, as they are in this case, the trajectories of the solutions will be circles or ellipses that are centered at the origin. The only thing that we really need to concern ourselves with here are whether they are rotating in a clockwise or counterclockwise direction.

This is easy enough to do. Recall when we first looked at these phase portraits a couple of sections ago that if we pick a value of $\vec{x}\left(t\right)$ and plug it into our system we will get a vector that will be tangent to the trajectory at that point and pointing in the direction that the trajectory is traveling. So, let's pick the following point and see what we get.

$$\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \Rightarrow \qquad \vec{x}' = \begin{pmatrix} 3 & 9 \\ -4 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$$

Therefore, at the point $\left(1,0\right)$ in the phase plane the trajectory will be pointing in a downwards direction. The only way that this can be is if the trajectories are traveling in a clockwise direction.

Here is the sketch of some of the trajectories for this problem.



The equilibrium solution in the case is called a **center** and is stable.

Note in this last example that the equilibrium solution is stable and not asymptotically stable. Asymptotically stable refers to the fact that the trajectories are moving in toward the equilibrium solution as *t* increases. In this example the trajectories are simply revolving around the equilibrium solution and not moving in towards it. The trajectories are also not moving away from the equilibrium solution and so they aren't unstable. Therefore, we call the equilibrium solution stable.

Not all complex eigenvalues will result in centers so let's take a look at an example where we get something different.

Example 3 Solve the following IVP.

$$\vec{x}' = \begin{pmatrix} 3 & -13 \\ 5 & 1 \end{pmatrix}\vec{x} \qquad\qquad \vec{x}(0) = \begin{pmatrix} 3 \\ -10 \end{pmatrix}$$

Solution

Let's get the eigenvalues and eigenvectors for the matrix.

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -13 \\ 5 & 1-\lambda \end{vmatrix}$$

$$= \lambda^2 - 4\lambda + 68 \qquad\qquad \lambda_{1,2} = 2 \pm 8i$$

Now get the eigenvector for the first eigenvalue.

$\lambda_1 = 2 + 8i$:

We need to solve the following system.

$$\begin{pmatrix} 1-8i & -13 \\ 5 & -1-8i \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Using the second equation we get,

$$5\eta_1 + (-1-8i)\eta_2 = 0$$

$$\eta_1 = \frac{1}{5}(1+8i)\eta_2$$

So, the first eigenvector is,

$$\vec{\eta} = \begin{pmatrix} \frac{1}{5}(1+8i)\eta_2 \\ \eta_2 \end{pmatrix}$$

$$\vec{\eta}^{(1)} = \begin{pmatrix} 1+8i \\ 5 \end{pmatrix} \qquad\qquad \eta_2 = 5$$

The solution corresponding to this eigenvalue and eigenvector is

$$\vec{x}_1(t) = \mathbf{e}^{(2+8i)t}\begin{pmatrix} 1+8i \\ 5 \end{pmatrix}$$

$$= \mathbf{e}^{2t}\mathbf{e}^{8it}\begin{pmatrix} 1+8i \\ 5 \end{pmatrix}$$

$$= \mathbf{e}^{2t}\left(\cos(8t) + i\sin(8t)\right)\begin{pmatrix} 1+8i \\ 5 \end{pmatrix}$$

As with the first example multiply cosines and sines into the vector and split it up. Don't forget about the exponential that is in the solution this time.

$$\vec{x}_1(t) = \mathbf{e}^{2t}\begin{pmatrix}\cos(8t) - 8\sin(8t) \\ 5\cos(8t)\end{pmatrix} + i\mathbf{e}^{2t}\begin{pmatrix}8\cos(8t) + \sin(8t) \\ 5\sin(8t)\end{pmatrix}$$
$$= \vec{u}(t) + i\,\vec{v}(t)$$

The general solution to this system then,

$$\vec{x}(t) = c_1\mathbf{e}^{2t}\begin{pmatrix}\cos(8t) - 8\sin(8t) \\ 5\cos(8t)\end{pmatrix} + c_2\mathbf{e}^{2t}\begin{pmatrix}8\cos(8t) + \sin(8t) \\ 5\sin(8t)\end{pmatrix}$$

Now apply the initial condition and find the constants.

$$\begin{pmatrix}3 \\ -10\end{pmatrix} = \vec{x}(0) = c_1\begin{pmatrix}1 \\ 5\end{pmatrix} + c_2\begin{pmatrix}8 \\ 0\end{pmatrix}$$

$$\left.\begin{aligned} c_1 + 8c_2 &= 3 \\ 5c_1 &= -10 \end{aligned}\right\} \qquad \Rightarrow \qquad c_1 = -2,\ c_2 = \frac{5}{8}$$

The actual solution is then,

$$\vec{x}(t) = -2\mathbf{e}^{2t}\begin{pmatrix}\cos(8t) - 8\sin(8t) \\ 5\cos(8t)\end{pmatrix} + \frac{5}{8}\mathbf{e}^{2t}\begin{pmatrix}8\cos(8t) + \sin(8t) \\ 5\sin(8t)\end{pmatrix}$$

Let's take a look at the phase portrait for this problem.

Example 4 Sketch the phase portrait for the system.

$$\vec{x}' = \begin{pmatrix}3 & -13 \\ 5 & 1\end{pmatrix}\vec{x}$$

Solution

When the eigenvalues of a system are complex with a real part the trajectories will spiral into or out of the origin. We can determine which one it will be by looking at the real portion. Since the real portion will end up being the exponent of an exponential function (as we saw in the solution to this system) if the real part is positive the solution will grow very large as *t* increases. Likewise, if the real part is negative the solution will die out as *t* increases.

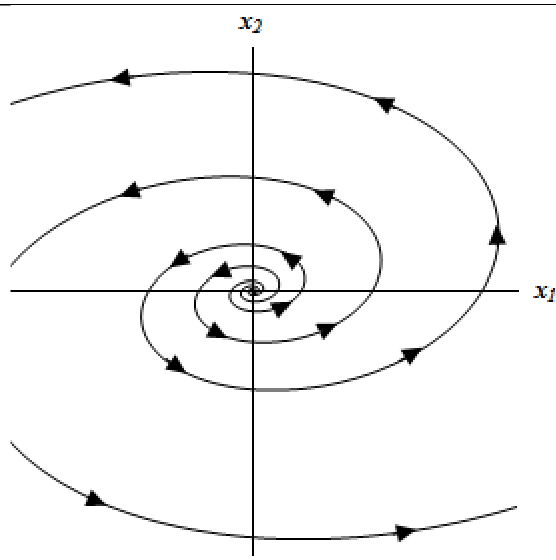
So, if the real part is positive the trajectories will spiral out from the origin and if the real part is negative they will spiral into the origin. We determine the direction of rotation (clockwise vs. counterclockwise) in the same way that we did for the center.

In our case the trajectories will spiral out from the origin since the real part is positive and

$$\vec{x}' = \begin{pmatrix}3 & -13 \\ 5 & 1\end{pmatrix}\begin{pmatrix}1 \\ 0\end{pmatrix} = \begin{pmatrix}3 \\ 5\end{pmatrix}$$

will rotate in the counterclockwise direction as the last example did.

Here is a sketch of some of the trajectories for this system.



Here we call the equilibrium solution a **spiral** (oddly enough…) and in this case it's unstable since the trajectories move away from the origin.

If the real part of the eigenvalue is negative the trajectories will spiral into the origin and in this case the equilibrium solution will be asymptotically stable.

Section 5-9 : Repeated Eigenvalues

This is the final case that we need to take a look at. In this section we are going to look at solutions to the system,

$$\vec{x}' = A\vec{x}$$

where the eigenvalues are repeated eigenvalues. Since we are going to be working with systems in which A is a 2 x 2 matrix we will make that assumption from the start. So, the system will have a double eigenvalue, λ.

This presents us with a problem. We want two linearly independent solutions so that we can form a general solution. However, with a double eigenvalue we will have only one,

$$\vec{x}_1 = \vec{\eta}\mathbf{e}^{\lambda t}$$

So, we need to come up with a second solution. Recall that when we looked at the double root case with the second order differential equations we ran into a similar problem. In that section we simply added a t to the solution and were able to get a second solution. Let's see if the same thing will work in this case as well. We'll see if

$$\vec{x} = t\,\mathbf{e}^{\lambda t}\vec{\eta}$$

will also be a solution.

To check all we need to do is plug into the system. Don't forget to product rule the proposed solution when you differentiate!

$$\vec{\eta}\mathbf{e}^{\lambda t} + \lambda\vec{\eta}t\mathbf{e}^{\lambda t} = A\vec{\eta}t\mathbf{e}^{\lambda t}$$

Now, we got two functions here on the left side, an exponential by itself and an exponential times a t. So, in order for our guess to be a solution we will need to require,

$$A\vec{\eta} = \lambda\vec{\eta} \qquad \Rightarrow \qquad \left(A - \lambda I\right)\vec{\eta} = \vec{0}$$

$$\vec{\eta} = \vec{0}$$

The first requirement isn't a problem since this just says that λ is an eigenvalue and it's eigenvector is $\vec{\eta}$. We already knew this however so there's nothing new there. The second however is a problem. Since $\vec{\eta}$ is an eigenvector we know that it can't be zero, yet in order to satisfy the second condition it would have to be.

So, our guess was incorrect. The problem seems to be that there is a lone term with just an exponential in it so let's see if we can't fix up our guess to correct that. Let's try the following guess.

$$\vec{x} = t\,\mathbf{e}^{\lambda t}\vec{\eta} + \mathbf{e}^{\lambda t}\vec{\rho}$$

where $\vec{\rho}$ is an unknown vector that we'll need to determine.

As with the first guess let's plug this into the system and see what we get.

$$\vec{\eta}\mathbf{e}^{\lambda t} + \lambda\vec{\eta}t\mathbf{e}^{\lambda t} + \lambda\vec{\rho}\mathbf{e}^{\lambda t} = A\left(\vec{\eta}t\mathbf{e}^{\lambda t} + \vec{\rho}\mathbf{e}^{\lambda t}\right)$$

$$\left(\vec{\eta} + \lambda\vec{\rho}\right)\mathbf{e}^{\lambda t} + \lambda\vec{\eta}t\mathbf{e}^{\lambda t} = A\vec{\eta}t\mathbf{e}^{\lambda t} + A\vec{\rho}\mathbf{e}^{\lambda t}$$

Now set coefficients equal again,

$$\lambda\vec{\eta} = A\vec{\eta} \qquad \Rightarrow \qquad (A-\lambda I)\vec{\eta} = \vec{0}$$

$$\vec{\eta} + \lambda\vec{\rho} = A\vec{\rho} \qquad \Rightarrow \qquad (A-\lambda I)\vec{\rho} = \vec{\eta}$$

As with our first guess the first equation tells us nothing that we didn't already know. This time the second equation is not a problem. All the second equation tells us is that $\vec{\rho}$ must be a solution to this equation.

It looks like our second guess worked. Therefore,

$$\vec{x}_2 = t\,\mathbf{e}^{\lambda t}\vec{\eta} + \mathbf{e}^{\lambda t}\vec{\rho}$$

will be a solution to the system provided $\vec{\rho}$ is a solution to

$$(A-\lambda I)\vec{\rho} = \vec{\eta}$$

Also, this solution and the first solution are linearly independent and so they form a fundamental set of solutions and so the general solution in the double eigenvalue case is,

$$\vec{x} = c_1\mathbf{e}^{\lambda t}\vec{\eta} + c_2\left(t\,\mathbf{e}^{\lambda t}\vec{\eta} + \mathbf{e}^{\lambda t}\vec{\rho}\right)$$

Let's work an example.

Example 1 Solve the following IVP.

$$\vec{x}\,' = \begin{pmatrix} 7 & 1 \\ -4 & 3 \end{pmatrix}\vec{x} \qquad \vec{x}(0) = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$$

Solution

First find the eigenvalues for the system.

$$\det(A-\lambda I) = \begin{vmatrix} 7-\lambda & 1 \\ -4 & 3-\lambda \end{vmatrix}$$

$$= \lambda^2 - 10\lambda + 25$$

$$= (\lambda-5)^2 \qquad \Rightarrow \qquad \lambda_{1,2} = 5$$

So, we got a double eigenvalue. Of course, that shouldn't be too surprising given the section that we're in. Let's find the eigenvector for this eigenvalue.

$$\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \Rightarrow \qquad 2\eta_1 + \eta_2 = 0 \qquad \eta_2 = -2\eta_1$$

The eigenvector is then,

$$\vec{\eta} = \begin{pmatrix} \eta_1 \\ -2\eta_1 \end{pmatrix} \qquad \eta_1 \neq 0$$

$$\vec{\eta}^{(1)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \qquad \eta_1 = 1$$

The next step is find $\vec{\rho}$. To do this we'll need to solve,

$$\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}\begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \qquad \Rightarrow \qquad 2\rho_1 + \rho_2 = 1 \qquad \rho_2 = 1 - 2\rho_1$$

Note that this is almost identical to the system that we solve to find the eigenvalue. The only difference is the right hand side. The most general possible $\vec{\rho}$ is

$$\vec{\rho} = \begin{pmatrix} \rho_1 \\ 1 - 2\rho_1 \end{pmatrix} \qquad \Rightarrow \qquad \vec{\rho} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \text{if } \rho_1 = 0$$

In this case, unlike the eigenvector system we can choose the constant to be anything we want, so we might as well pick it to make our life easier. This usually means picking it to be zero.

We can now write down the general solution to the system.

$$\vec{x}(t) = c_1 \mathbf{e}^{5t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2\left(\mathbf{e}^{5t} t \begin{pmatrix} 1 \\ -2 \end{pmatrix} + \mathbf{e}^{5t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

Applying the initial condition to find the constants gives us,

$$\begin{pmatrix} 2 \\ -5 \end{pmatrix} = \vec{x}(0) = c_1\begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\left.\begin{aligned} c_1 &= 2 \\ -2c_1 + c_2 &= -5 \end{aligned}\right\} \qquad \Rightarrow \qquad c_1 = 2, \ c_2 = -1$$

The actual solution is then,

$$\begin{aligned} \vec{x}(t) &= 2\mathbf{e}^{5t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} - \left(t\mathbf{e}^{5t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + \mathbf{e}^{5t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \\ &= \mathbf{e}^{5t}\begin{pmatrix} 2 \\ -4 \end{pmatrix} - \mathbf{e}^{5t} t\begin{pmatrix} 1 \\ -2 \end{pmatrix} - \mathbf{e}^{5t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \mathbf{e}^{5t}\begin{pmatrix} 2 \\ -5 \end{pmatrix} - \mathbf{e}^{5t} t\begin{pmatrix} 1 \\ -2 \end{pmatrix} \end{aligned}$$

Note that we did a little combining here to simplify the solution up a little.

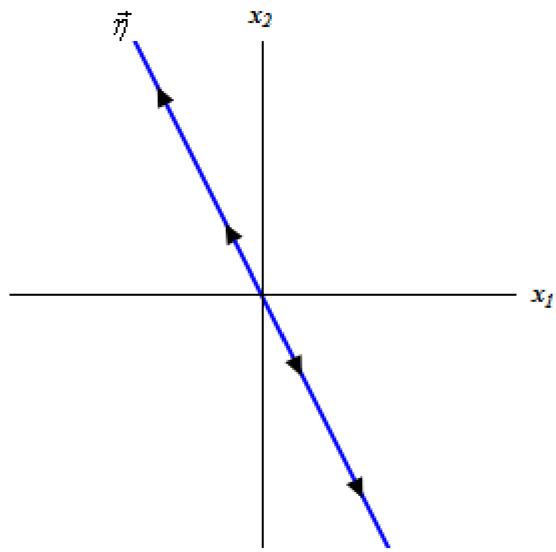
So, the next example will be to sketch the phase portrait for this system.

Example 2 Sketch the phase portrait for the system.

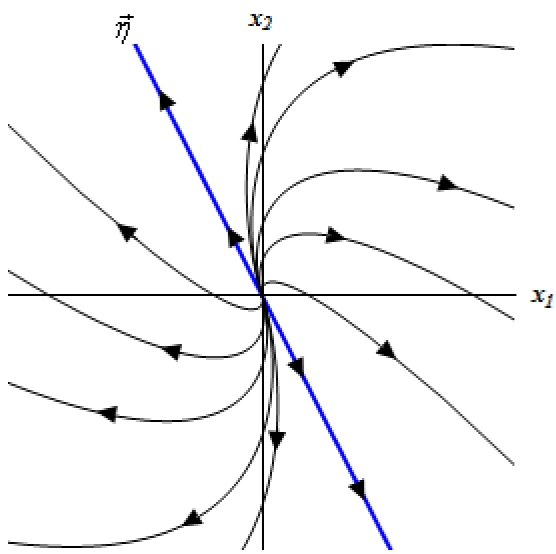
$$\vec{x}\,' = \begin{pmatrix} 7 & 1 \\ -4 & 3 \end{pmatrix}\vec{x}$$

Solution

These will start in the same way that real, distinct eigenvalue phase portraits start. We'll first sketch in a trajectory that is parallel to the eigenvector and note that since the eigenvalue is positive the trajectory will be moving away from the origin.



Now, it will be easier to explain the remainder of the phase portrait if we actually have one in front of us. So here is the full phase portrait with some more trajectories sketched in.



Trajectories in these cases always emerge from (or move into) the origin in a direction that is parallel to the eigenvector. Likewise, they will start in one direction before turning around and moving off into the other direction. The directions in which they move are opposite depending on which side of the trajectory corresponding to the eigenvector we are on. Also, as the trajectories move away from the origin it should start becoming parallel to the trajectory corresponding to the eigenvector.

So, how do we determine the direction? We can do the same thing that we did in the complex case. We'll plug in $(1,0)$ into the system and see which direction the trajectories are moving at that point. Since this point is directly to the right of the origin the trajectory at that point must have already turned around and so this will give the direction that it will traveling after turning around.

Doing that for this problem to check our phase portrait gives,

$$\begin{pmatrix} 7 & 1 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 7 \\ -4 \end{pmatrix}$$

This vector will point down into the fourth quadrant and so the trajectory must be moving into the fourth quadrant as well. This does match up with our phase portrait.

In these cases, the equilibrium is called a **node** and is unstable in this case. Note that sometimes you will hear nodes for the repeated eigenvalue case called **degenerate nodes** or **improper nodes**.

Let's work one more example.

Example 3 Solve the following IVP.

$$\vec{x}\,' = \begin{pmatrix} -1 & \frac{3}{2} \\ -\frac{1}{6} & -2 \end{pmatrix}\vec{x} \qquad \vec{x}\left(2\right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Solution

First the eigenvalue for the system.

$$\begin{aligned} \det\left(A - \lambda I\right) &= \begin{vmatrix} -1-\lambda & \frac{3}{2} \\ -\frac{1}{6} & -2-\lambda \end{vmatrix} \\ &= \lambda^2 + 3\lambda + \frac{9}{4} \\ &= \left(\lambda + \frac{3}{2}\right)^2 \qquad \Rightarrow \qquad \lambda_{1,2} = -\frac{3}{2} \end{aligned}$$

Now let's get the eigenvector.

$$\begin{pmatrix} \frac{1}{2} & \frac{3}{2} \\ -\frac{1}{6} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \Rightarrow \qquad \frac{1}{2}\eta_1 + \frac{3}{2}\eta_2 = 0 \qquad \eta_1 = -3\eta_2$$

$$\vec{\eta} = \begin{pmatrix} -3\eta_2 \\ \eta_2 \end{pmatrix} \qquad \eta_2 \neq 0$$

$$\vec{\eta}^{\,(1)} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \qquad \eta_2 = 1$$

Now find $\vec{\rho}$,

$$\begin{pmatrix} \frac{1}{2} & \frac{3}{2} \\ -\frac{1}{6} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \qquad \Rightarrow \qquad \frac{1}{2}\rho_1 + \frac{3}{2}\rho_2 = -3 \qquad \rho_1 = -6 - 3\rho_2$$

$$\vec{\rho} = \begin{pmatrix} -6-3\rho_2 \\ \rho_2 \end{pmatrix} \qquad \Rightarrow \qquad \vec{\rho} = \begin{pmatrix} -6 \\ 0 \end{pmatrix} \qquad \text{if } \rho_2 = 0$$

The general solution for the system is then,

$$\vec{x}(t) = c_1 \mathbf{e}^{-\frac{3t}{2}} \begin{pmatrix} -3 \\ 1 \end{pmatrix} + c_2 \left(t\mathbf{e}^{-\frac{3t}{2}} \begin{pmatrix} -3 \\ 1 \end{pmatrix} + \mathbf{e}^{-\frac{3t}{2}} \begin{pmatrix} -6 \\ 0 \end{pmatrix} \right)$$

Applying the initial condition gives,

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \vec{x}(2) = c_1 \mathbf{e}^{-3} \begin{pmatrix} -3 \\ 1 \end{pmatrix} + c_2 \left(2\mathbf{e}^{-3} \begin{pmatrix} -3 \\ 1 \end{pmatrix} + \mathbf{e}^{-3} \begin{pmatrix} -6 \\ 0 \end{pmatrix} \right)$$

Note that we didn't use *t=0* this time! We now need to solve the following system,

$$\left. \begin{aligned} -3\mathbf{e}^{-3} c_1 - 12\mathbf{e}^{-3} c_2 &= 1 \\ \mathbf{e}^{-3} c_1 + 2\mathbf{e}^{-3} c_2 &= 0 \end{aligned} \right\} \qquad \Rightarrow \qquad c_1 = \frac{\mathbf{e}^3}{3}, \; c_2 = -\frac{\mathbf{e}^3}{6}$$

The actual solution is then,

$$\begin{aligned} \vec{x}(t) &= \frac{\mathbf{e}^3}{3} \mathbf{e}^{-\frac{3t}{2}} \begin{pmatrix} -3 \\ 1 \end{pmatrix} - \frac{\mathbf{e}^3}{6} \left(t\mathbf{e}^{-\frac{3t}{2}} \begin{pmatrix} -3 \\ 1 \end{pmatrix} + \mathbf{e}^{-\frac{3t}{2}} \begin{pmatrix} -6 \\ 0 \end{pmatrix} \right) \\ &= \mathbf{e}^{-\frac{3t}{2}+3} \begin{pmatrix} 0 \\ \frac{1}{3} \end{pmatrix} + t\mathbf{e}^{-\frac{3t}{2}+3} \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{6} \end{pmatrix} \end{aligned}$$

And just to be consistent with all the other problems that we've done let's sketch the phase portrait.

Example 4 Sketch the phase portrait for the system.

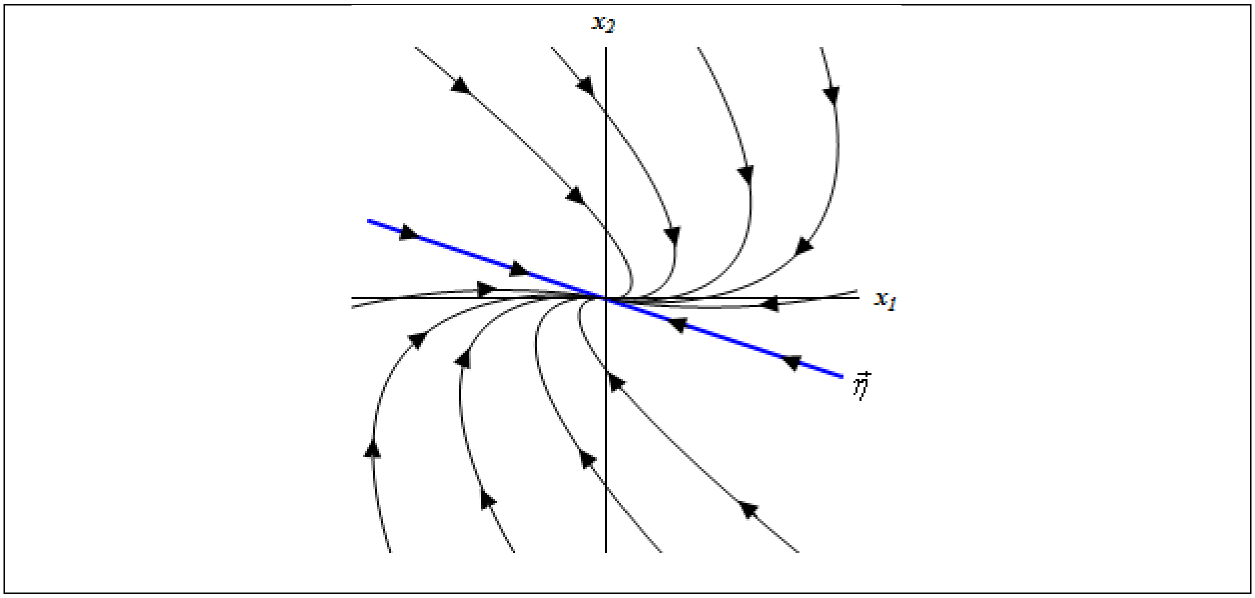
$$\vec{x}\,' = \begin{pmatrix} -1 & \frac{3}{2} \\ -\frac{1}{6} & -2 \end{pmatrix} \vec{x}$$

Solution

Let's first notice that since the eigenvalue is negative in this case the trajectories should all move in towards the origin. Let's check the direction of the trajectories at $(1,0)$.

$$\begin{pmatrix} -1 & \frac{3}{2} \\ -\frac{1}{6} & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -\frac{1}{6} \end{pmatrix}$$

So, it looks like the trajectories should be pointing into the third quadrant at $(1,0)$. This gives the following phase portrait.



x_2
x_1
$\vec{\eta}$